PERRON-FROBENIUS THEORY FOR POSITIVE MAPS ON TRACE IDEALS

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ABSTRACT. This article provides sufficient conditions for positive maps on the Schatten classes $\mathcal{J}_p,\ 1 \le p < \infty$ of bounded operators on a separable Hilbert space such that a corresponding Perron-Frobenius theorem holds. With applications in quantum information theory in mind sufficient conditions are given for a trace preserving, positive map on $\mathcal{J}_1$, the space of trace class operators, to have a unique, strictly positive density matrix which is left invariant under the map. Conversely to any given strictly positive density matrix there are trace preserving, positive maps for which the density matrix is the unique Perron-Frobenius vector.

Dedicated to S. Doplicher and J.E. Roberts on the occasion of their 60th birthday

1. INTRODUCTION

In the theory of quantum information the transmission through noisy channels plays an important role. Usually it is described by what physicists either call a quantum operation (see e.g. [25]) or a stochastic map ([1], see also [19]) or a super-operator (see e.g. [28]) and what mathematicians call a completely positive (trace preserving) map. In mathematics positive maps were first studied by Kadison in the context of $C^\star$-algebras [18] and completely positive maps by Stinespring [33]. In quantum physics it was first introduced by Haag and Kastler, who called it an operation [15]. They were then studied in more detail in [16]. [21, 22] contains an extensive discussion, how this concept naturally arises in quantum physics.

Because they preserve positivity such maps naturally fit into a context where it makes sense to ask about a formulation of a corresponding Perron-Frobenius theory.

As the name indicates, the earliest results relating positivity of a finite dimensional linear map to the non-degeneracy of an eigenvalue and the positivity of the resulting (Perron-Frobenius) eigenvector is due to Perron and Frobenius [27, 8, 9]. The first extension to the infinite dimensional was given by Jentzsch [17]. Since then there has been an extensive development (see e.g. [3, 23, 30]).

The interest in quantum physics and quantum field theory originates from the observation, that very often the ground state of a quantum system with Hamilton operator $H \ge 0$ is non-degenerate and nowhere vanishing. Indeed, under suitable circumstances such a ground state can be viewed as the Perron-Frobenius vector for $\exp -tH, t > 0$, an observation first made by Glimm and Jaffe [12] (see also [29, 13] for an overview and with references to other articles).

The Perron-Frobenius theory and related topics for positive linear maps on linear spaces of operators was first studied in [7, 36, 14, 2], where in the first article the space of operators was the $C^\star$-algebra of all linear operators on a finite dimensional Hilbert space. In the second and third articles the discussion was extended to the infinite dimensional case and in the fourth article the analysis was carried out in the context of a von Neumann algebra.

The present article can be viewed as an extension of the discussion in [7] to the infinite dimensional case. Indeed, we will consider positive maps on the Schatten classes $\mathcal{J}_p,\ 1 \le p \le \infty$ and provide a Perron-Frobenius theory on such spaces. As in the usual context the notions of ergodicity and positivity improving will play an essential role. Since density matrices are elements of $\mathcal{J}_1$ our main focus will be on the case $p = 1$. In particular we will show that any compact, ergodic, completely positive and trace preserving map in $\mathcal{J}_1$ has a unique density matrix invariant under this map (i.e. the eigenvalue equals 1) and which therefore can be viewed as the Perron-Frobenius vector for this map.

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The article is organized as follows. In Section 2 we will briefly review the concepts needed for our discussion. In Section 3 we will prove a Perron-Frobenius theorem (non-degeneracy of a certain eigenvalue), provided such an eigenvalue exists, a typical condition needed in infinite dimensional contexts. Section 4 will provide sufficient conditions for such an eigenvalue to exist. In Section 5 we will give examples. Also we will prove a converse in the sense that to any density matrix > 0 there are completely positive maps for which this density matrix is the Perron-Frobenius eigenvector.

2. Positive and Completely Positive Maps

We start by recalling and establishing some definitions and facts related to trace ideals and to the concept of positive and completely positive maps (for more details on trace ideals see e.g. [31, 32] and on positive and completely positive maps e.g. [33, 34, 5, 6, 35, 26]). Let $\mathcal{H}$ be any complex, separable Hilbert space with scalar product denoted by $\langle\ ,\rangle$. By $\mathcal{B}(\mathcal{H})$ we denote the $C^\star$-algebra of all bounded operators on $\mathcal{H}$ equipped with the norm $||\ ||$. By $\mathbb{I}\in\mathcal{B}(\mathcal{H})$ we denote the identity map on $\mathcal{H}$.

For any $A\in\mathcal{B}(\mathcal{H})$ we set $|A|=(A^\star A)^{1/2}\in\mathcal{B}(\mathcal{H})$ where * denotes the adjoint. Also we let Tr be the trace operation on $\mathcal{H}$. For $1\leq p<\infty$ let $\mathcal{J}_p=\mathcal{J}_p(\mathcal{H})$ be the Schatten class consisting of all elements $A\in\mathcal{B}(\mathcal{H})$ such that $|A|^p$ is trace class. $\mathcal{J}_p$ is equipped with the norm $||A||_p=(\operatorname{Tr}(|A|^p))^{1/p}$ making each $\mathcal{J}_p$ a complete Banach space. One has $A^\star\in\mathcal{J}_p$ if $A\in\mathcal{J}_p$, i.e. the space $\mathcal{J}_p$ is self-adjoint in $\mathcal{B}(\mathcal{H})$, such that $||A^\star||_p=||A||_p$. Also we set $\mathcal{J}_\infty=\mathcal{B}(\mathcal{H})$ again with the same norm, i.e. $||\ ||_\infty=||\ ||$. We will frequently make use of the trivial identity $||A||_p=||\,|A|\,||_p$ valid for all $A\in\mathcal{J}_p$ and all $1\leq p\leq\infty$. Via the trace $\mathcal{J}_q$ is the dual of $\mathcal{J}_p$ with $1/p+1/q=1$ for all $1\leq p<\infty$, i. e. each continuous linear functional on $\mathcal{J}_p$ is given by an $A\in\mathcal{J}_q$ in the the form $\operatorname{Tr}(A^\star B)$, $B\in\mathcal{J}_p$. In particular the spaces $\mathcal{J}_p$ are reflexive for all $1<p<\infty$. We prefer to use the adjoint in describing linear functionals since for $p=2$, in which case $\mathcal{J}_2$ is the space of Hilbert-Schmidt operators on $\mathcal{H}$, the scalar product making it a Hilbert space is just given by $\langle A,B\rangle_2=\operatorname{Tr}(A^\star B)$. Also one has the Hölder inequality $|\operatorname{Tr}(A^\star B)|\leq||A||_q||B||_p$. Let $\mathcal{C}=\mathcal{C}(\mathcal{H})=\mathcal{C}_\infty$ be the closed positive cone of all elements $A\geq 0$ in $\mathcal{B}(\mathcal{H})$. As usual we write $A\geq 0$ and $A>0$ if the relations $\langle\varphi,A\varphi\rangle\ \geq\ 0$ and $\langle\varphi,A\varphi\rangle\ >\ 0$ respectively hold for all $0\neq\varphi\in\mathcal{H}$. Correspondingly we write $A>B$ (or $B<A$) and $A\geq B$ (or $B\leq A$) if $A-B>0$ and $A-B\geq 0$ respectively. Obviously $A\geq B>0$ or $A>B\geq 0$ implies $A>0$. As is common, A is said to be positive definite if $A>0$. Set $\mathcal{C}_p=\mathcal{C}_p(\mathcal{H})=\mathcal{C}\cap\mathcal{J}_p$, a closed set in $\mathcal{J}_p$. Also $A\geq 0$ in $\mathcal{J}_p$ if and only if $\operatorname{Tr}(BA)\geq 0$ for all $0\leq B\in\mathcal{J}_q$. Correspondingly $A>0$ in $\mathcal{J}_p$ if and only if $\operatorname{Tr}(BA)>0$ for all $0\leq B\neq 0$ in $\mathcal{J}_q$ $(1/p+1/q=1,\ 1\leq p\leq\infty)$. In this sense the cones $\mathcal{C}_p$ and $\mathcal{C}_q$ are dual to each other. The closed set $\mathcal{C}_{1,1}=\{A\in\mathcal{C}_1|\operatorname{Tr}A=||A||_1=1\}$ in $\mathcal{J}_1$ is the set of all density matrices in $\mathcal{H}$.

By definition a positive map ϕ in $\mathcal{J}_p$ is a linear map from $\mathcal{J}_1$ into itself, which leaves $\mathcal{C}_p$ invariant. ϕ is called n-positive if the induced map $\phi_n=\phi\otimes\mathbb{I}_n$ in $\mathcal{J}_p\otimes\mathcal{B}(\mathcal{H}_n)$ also leaves the corresponding cone $\mathcal{C}_p(\mathcal{H}\otimes\mathcal{H}_n)$ of non-negative elements invariant. Here $\mathcal{H}_n$ is any Hilbert space of dimension $1\leq n<\infty$. Obviously if ϕ is n-positive, then it is also n'-positive for any $1\leq n'\leq n$ as is $\lambda\phi$, $\lambda>0$. If ϕ and ϕ' are both n-positive on $\mathcal{J}_p$ then so is their composite $\phi\circ\phi'$ and their sum $\phi+\phi'$. So the n-positive maps in $\mathcal{J}_p$ form a cone in the linear space of all linear maps in $\mathcal{J}_p$.

If ϕ is n-positive for all n, then ϕ is called completely positive. Thus the map $\phi_\alpha:A\to\alpha A\alpha^\star$ for any $\alpha\in\mathcal{B}(\mathcal{H})$ is completely positive on $\mathcal{J}_p$ for all $1\leq p\leq\infty$ as is any finite linear combination $\phi_{\underline{\alpha}}=\sum_i\phi_{\alpha_i}$ with $\alpha_i\in\mathcal{B}(\mathcal{H})$. If α has an inverse α^{-1} in $\mathcal{B}(\mathcal{H})$ then $\phi_\alpha^{-1}=\phi_{\alpha^{-1}}$. Also $\phi_\alpha\circ\phi_{\alpha'}=\phi_{\alpha\,\alpha'}$ and $\phi_{\lambda\alpha}=|\lambda|^2\phi_\alpha$ for α, $\alpha'\in\mathcal{B}(\mathcal{H})$ and $\lambda\in\mathbb{C}$. As it turns out at least for $p=1$ the induced linear map $\phi\otimes\mathbb{I}_{\mathcal{H}'}$ from $\mathcal{J}_{p=1}\otimes\mathcal{B}(\mathcal{H}')$ then also leaves the corresponding cone invariant for any separable Hilbert space $\mathcal{H}'$ (see e.g. [22]). Since density matrices are in $\mathcal{J}_1$, the case $p=1$ is of most interest in quantum physics. Then there are good physical reasons to consider completely positive maps rather than only positive maps (see e.g, [22]).

Although the following lemma is well known in similar contexts, we still will provide the short proof.

Lemma 2.1. *The following relation holds if the map ϕ in $\mathcal{J}_p$ $(1 \le p \le \infty)$ is positive.*

$$\phi(A)^\star = \phi(A^\star),\ A \in \mathcal{J}_p. \tag{1}$$

Proof. We first consider the case when A is self-adjoint, i.e. $A^\star = A$. Write $A = A_+ - A_-$ with $A_\pm \in \mathcal{C}_p$, $A_+A_- = A_-A_+ = 0$ and $|A| = A_+ + A_-$. Here $\pm A_\pm$ are the positive and negative parts of A, obtainable from the spectral representation of A or more explicitly as $A_\pm = 1/2(|A| \pm A)$. Obviously $||A_\pm||_p \le ||A||_p$. Since $\phi(A_\pm) \ge 0$, $\phi(A) = \phi(A_+) - \phi(A_-)$ is self-adjoint. For arbitrary $A \in \mathcal{B}(\mathcal{H})$ write $A = \Re A + i\Im A \in \mathcal{B}(\mathcal{H})$ with $\Re A = 1/2(A + A^\star)$, $\Im A = 1/2i(A - A^\star)$, such that $\Re A, \Im A \in \mathcal{B}(\mathcal{H})$ are self-adjoint with $\Re A, \Im A \in \mathcal{J}_p$ whenever $A \in \mathcal{J}_p$. More precisely we have the a priori bound $||\Re A||_p \le ||A||_p$, $||\Im A||_p \le ||A||_p$. This gives the decomposition

$$A = (\Re A)_+ - (\Re A)_- + i((\Im A)_+ - (\Im A)_-)$$

and hence by the linearity of ϕ

$$\phi(A) = \phi((\Re A)_+) - \phi((\Re A)_-) + i\phi((\Im A)_+) - i\phi((\Im A)_-) \tag{2}$$

with $||\phi(A)||_p \le ||\phi((\Re A)_+)||_p + ||\phi((\Re A)_-)||_p + ||\phi((\Im A)_+)||_p + ||\phi((\Im A)_-)||_p$. In particular (2) shows that indeed (1) holds. □

We also note the following. The relation $-|A| \le A \le |A|$ for $A = A^\star \in \mathcal{J}_p$ implies $-\phi(|A|) \le \phi(A) \le \phi(|A|)$, whenever the map ϕ in $\mathcal{J}_p$ is positive. This in turn gives

$$|\phi(A)| \le \phi(|A|) \tag{3}$$

valid for any $A = A^\star \in \mathcal{J}_p$ and any positive map ϕ in $\mathcal{J}_p$. We do not know whether (or when) this relation continues to hold when the condition $A = A^\star$ is dropped (see, however, (23) below for a weaker result, which will suffice for our purposes). Observe by comparison that for any $n \times n$ matrix S with non-negative entries, the classical context for the Perron-Frobenius theorem, one has $|S\underline{z}| \le S|\underline{z}|$ for $\underline{z} \in \mathbb{C}^n$. Here $|\underline{z}|$ is the vector whose components are the absolute values of the corresponding components of $\underline{z}$. Also for real vectors $\underline{x} = \{x_i\}$ and $\underline{y} = \{y_i\}$ by definition $\underline{x} \le \underline{y}$ if and only if $x_i \le y_i$ holds for all i.

Furthermore one does not necessarily have $\phi(A)_\pm = \phi(A_\pm)$ for any $A = A^\star \in \mathcal{J}_p$. However, since $-A_- \le A \le A_+$ implies $-\phi(A_-) \le \phi(A) \le \phi(A_+)$, the inequalities $\phi(A)_\pm \le \phi(A_\pm)$ hold.

Lemma 2.2. *Any positive map ϕ in $\mathcal{J}_p$ $(1 \le p \le \infty)$ is continuous, i.e. it satisfies $||\phi||_p < \infty$.*

Here we denote by $||\phi||_p$ the norm of any continuous linear map ϕ in $\mathcal{J}_p$, such that

$$||\phi||_p = \sup_{||A||_p \le 1} ||\phi(A)||_p. \tag{4}$$

In particular

$$||\phi||_1 = \sup_{A \in \mathcal{C}_{1,1}} \mathrm{Tr}(\phi(A)), \tag{5}$$

if ϕ is completely positive on $\mathcal{J}_1$ (see e.g. [6, 22]).

Proof. We adapt a standard proof (see e.g. [26], p. 19) used in the context of positive maps on non-unital $C^\star$-algebras. We first claim that it suffices to prove boundedness on $\mathcal{C}_p$. Indeed, this follows from the decomposition (2) and the related bounds. Assume that ϕ is not bounded on $\mathcal{C}_p$. Then there are $A_n \in \mathcal{C}_p$ with $||A_n||_p \le 1$ and $||\phi(A_n)||_p \ge n^3$. Let $A = \sum_n 1/n^2 A_n \in \mathcal{C}_p$, such that $0 \le 1/n^2 A_n \le A$ holds for all n.

We need the fact that

$$||B||_p \le ||B'||_p \tag{6}$$

holds for any $0 \le B \le B' \in \mathcal{C}_p$. The case $p = \infty$ is trivial. When $1 \le p \le \infty$ take φ_n to be a complete orthonormal basis in $\mathcal{H}$ diagonalizing B. Then we have

$$\begin{aligned} ||B||_p^p &= \sum_n \langle \varphi_n, B\varphi_n\rangle^p \le \sum_n \langle \varphi_n, B'\varphi_n\rangle^p \\ &\le \sum_n \langle \varphi_n, B'^p \varphi_n\rangle = ||B'||_p^p, \end{aligned}$$

proving (6). Here we have used the estimate

$$|\langle \varphi, A\varphi\rangle|^p \le \langle \varphi, |A|^p \varphi\rangle, \tag{7}$$

valid for any $A = A^\star \in \mathcal{B}(\mathcal{H})$, any normalized $\varphi \in \mathcal{H}$ and any $1 \le p$ (see e.g. [32], p.21). Since ϕ is positive (6) gives $n \le 1/n^2 ||\phi(A_n)||_p \le ||\phi(A)||_p$, which is a contradiction. □

For any continuous linear map ϕ in $\mathcal{J}_p$, $1 \le p < \infty$ let $\phi^\star$ be the "adjoint" continuous linear map from $\mathcal{J}_q$, $1/q + 1/p = 1$ into itself given by the relation $\mathrm{Tr}((\phi^\star(A))^\star B) = \mathrm{Tr}(A^\star \phi(B))$ for all $A \in \mathcal{J}_q$ and $B \in \mathcal{J}_p$. In particular $\phi^{\star\star} = \phi$ holds whenever $1 < p < \infty$. Also $\phi^\star_{\underline{\alpha}} = \phi_{\underline{\alpha}^\star}$ with $\underline{\alpha}^\star = (\alpha_1^\star, ...\alpha_i^\star, ...)$ by the cyclicity of the trace.

By definition a continuous linear map ϕ in $\mathcal{J}_1$ is called trace preserving if $\mathrm{Tr}(\phi(A)) = \mathrm{Tr}(A)$ holds for all $A \in \mathcal{J}_1$ and this in turn is equivalent to the relation $\phi^\star(\mathbb{I}) = \mathbb{I}$. Obviously trace preserving maps leave the set $\mathcal{C}_{1,1}$ of density matrices invariant. If ϕ in $\mathcal{J}_1$ is completely positive and trace invariant, then by (5) $||\phi||_1 = 1$.

All completely positive maps on $\mathcal{J}_1$ have the Kraus representation [22], a consequence of a theorem of Stinespring [33] (see also [5] for a proof in the finite dimensional case): Given such a ϕ there is an at most denumerable set of elements $\underline{\alpha} = \{\alpha_i\}_{i\in\mathbb{N}}$ in $\mathcal{B}(\mathcal{H})$ satisfying

$$\sum_{i\in K} \alpha_i^\star \alpha_i \le ||\phi||_1 \mathbb{I} \tag{8}$$

for any finite subset $K \subset \mathbb{N}$ such that $\phi = \phi_{\underline{\alpha}}$, again with

$$\phi_{\underline{\alpha}}(A) = \sum_{i\in\mathbb{N}} \alpha_i A \alpha_i^\star, \tag{9}$$

which now may be an infinite sum. This representation is not unique. Conversely each such $\phi_{\underline{\alpha}}$ is completely positive. More precisely $\phi_{\underline{\alpha}}$ is defined as follows. Let $\underline{\alpha}_N = (\alpha_1, ..., \alpha_N)$. Then the $\phi_{\underline{\alpha}_N}$ form a Cauchy sequence with respect to the norm $||\ ||_1$ and $\phi_{\underline{\alpha}}$ is defined as the limit (see e.g. [22]). If $\sum_i \alpha_i^\star \alpha_i = \mathbb{I}$, such that $||\phi||_1 = 1$, then $\phi_{\underline{\alpha}}$ is trace preserving in $\mathcal{J}_1$. In the finite dimensional case ($\dim \mathcal{H} < \infty$) the representation may always be chosen such that the index i runs through a finite set of the order at most $(\dim \mathcal{H})^2$.

An important and interesting feature of completely positive maps in the context of quantum physics is that they not necessarily map pure states, i.e. one-dimensional orthogonal projections, into pure states.

Again we have the following representation for the adjoint of $\phi_{\underline{\alpha}}$

$$\phi^\star_{\underline{\alpha}}(A) = \sum_{i\in\mathbb{N}} \alpha_i^\star A \alpha_i, \tag{10}$$

i.e. $\phi^\star_{\underline{\alpha}} = \phi_{\underline{\alpha}^\star}$. So $\phi_{\underline{\alpha}}$ is trace preserving if and only if $\phi^\star_{\underline{\alpha}}(\mathbb{I}) = \mathbb{I}$ and then $||\phi_{\underline{\alpha}}||_1 = 1$. Observe that to any completely positive ϕ with $||\phi||_1 \le 1$ or equivalently $\phi^\star(\mathbb{I}) \le \mathbb{I}$ (this condition is again natural in the context of quantum physics, see e.g. [22]) we may in a natural way associate a trace preserving, completely positive map $\hat{\phi}$ given as $\hat{\phi}(A) = \phi(A) + \tilde{\alpha} A \tilde{\alpha}$ where $\tilde{\alpha} = (\mathbb{I} - \phi^\star(\mathbb{I}))^{1/2} \ge 0$.

An example to which we shall return below is when the α_i's are orthogonal projection operators which are pairwise orthogonal, i.e. satisfy

$$\alpha_i \alpha_j = \delta_{ij} \alpha_i = \delta_{ij} \alpha_i^\star. \tag{11}$$

Then $\phi^\star_{\underline{\alpha}}(\mathbb{I}) = \mathbb{I}$ if and only if the α_i provide a decomposition of unity, i.e. if $\sum_{i\in\mathbb{N}} \alpha_i = \mathbb{I}$ holds. As already remarked $\phi_{\underline{\alpha}}$ is then also trace preserving. More generally if the α_i are all selfadjoint with $\sum_{i\in\mathbb{N}} \alpha_i^2 = \mathbb{I}$, then $\phi_{\underline{\alpha}}$ is trace preserving.

We remark that there are some situations (see e.g. [20] and which actually was the motivation for the present discussion) where one starts with completely positive maps of the following form. Let (Ω, μ) be a measure space, i.e. μ is countably additive. Suppose we are given a measurable map $\omega \to \alpha(\omega)$ from Ω into $\mathcal{B}(\mathcal{H})$ such that

$$\int ||\alpha(\omega)||^2 d\mu(\omega) < \infty. \tag{12}$$

We then set

$$\phi(A) = \int \alpha(\omega) A \alpha(\omega)^\star d\mu(\omega), \tag{13}$$

such that its adjoint takes the form

$$\phi^\star(A) = \int \alpha(\omega)^\star A \alpha(\omega) d\mu(\omega), \tag{14}$$

The condition corresponding to (8) is given as

$$\phi^\star(\mathbb{I}) = \int \alpha(\omega)^\star \alpha(\omega) d\mu(\omega) \leq c\,\mathbb{I} \tag{15}$$

for some $0 < c < \infty$. To cast ϕ in the form (9) such that (15) gives (8), let χ_i be any orthonormal basis in the separable Hilbert space $L^2(\Omega, d\mu)$ and set

$$\alpha_i = \int \overline{\chi_i}(\omega) \alpha(\omega) d\mu(\omega).$$

As an example one may choose Ω to be a compact group, μ its Haar measure and $\omega \to \alpha(\omega)$ a unitary representation. Then the image of ϕ consists of all trace class operators which commute with all $\alpha(\omega)$. Also ϕ is trace preserving and an idempotent, i.e. $\phi^2 = \phi$. Moreover ϕ is a conditional expectation when $\dim \mathcal{H} < \infty$, i.e. $\phi(A\phi(B)) = \phi(A)\phi(B) = \phi(\phi(A)B)$ holds for each A, B.

3. The Perron-Frobenius Theorem

Before we turn to a discussion of the Perron-Frobenius theorem we make some general remarks on $\sigma_p(\phi)$ for the map ϕ in $\mathcal{J}_p$, in particular when ϕ is positive. Here $\sigma_p(\phi) \subseteq \mathbb{C}$ is the spectrum of ϕ, i.e. the set of all λ for which $(\lambda - \phi)$ does not have a bounded inverse in $\mathcal{J}_p$. Let $r_p(\phi)$ be the spectral radius of any bounded linear map ϕ in $\mathcal{J}_p$, i.e.

$$r_p(\phi) = \lim_{n\to\infty} ||\phi^n||_p^{1/n} \leq ||\phi||_p. \tag{16}$$

By a celebrated general result of Gelfand [11], one has $\sup|\sigma_p(\phi)| = r_p(\phi)$. If the map ϕ in $\mathcal{J}_1$ is trace preserving and completely positive then $r_1(\phi) = ||\phi||_1 = 1$. Indeed, ϕ^n is then also completely positive and trace preserving such that by (5) $||\phi^n||_1 = 1$ holds for all n.

Lemma 2.1 allows us to draw the following conclusions on $\sigma_p(\phi)$ for positive maps ϕ in $\mathcal{J}_p$. First we observe that in general $\sigma_p(\phi)$ is not contained in the real axis. In fact, in the finite dimensional case $\sigma_p(\phi) \subseteq \mathbb{R}$ if and only if $\phi^\star = \phi$ and in the infinite dimensional case the same statement is valid when $p = 2$. Let $\rho_p(\phi)$ be the resolvent set of the map ϕ in $\mathcal{J}_p$, i.e. the complement in $\mathbb{C}$ of $\sigma_p(\phi)$. If $\lambda \in \rho_p(\phi)$, then to each $A \in \mathcal{J}_p$ there is a unique $A' \in \mathcal{J}_p$, such that $(\lambda - \phi)(A') = A$. Taking the adjoint and using (1) gives $(\overline{\lambda} - \phi)(A'^\star) = A^\star$. Since A can be chosen arbitrary also $A^\star$ can be chosen arbitrary giving a unique $A^{\star'}$, this shows that both sets $\rho_p(\phi)$ and $\sigma_p(\phi)$ lie symmetric with respect to the real axis. Let $\lambda \in \sigma(\phi)$ be a real eigenvalue with eigenvector A: $\phi(A) = \lambda\, A$. So by (1) $A^\star$ is also an eigenvector with the same eigenvalue. With the decomposition $A = A_1 + iA_2$ we see that both self-adjoint operators A_1 and A_2 are eigenvectors, so any eigenspace for a real eigenvalue is spanned by self-adjoint elements. If in addition the map ϕ in $\mathcal{J}_1$ is trace preserving and $\lambda \neq 1$ an eigenvalue and A a corresponding eigenvector, then from $\mathrm{Tr}(A) = \mathrm{Tr}(\phi(A)) = \lambda\,\mathrm{Tr}(A)$ we deduce $\mathrm{Tr}(A) = 0$.

In the finite dimensional case $\dim \mathcal{H} < \infty$ all spaces $\mathcal{J}_p$ are equal and of course $\sigma_p(\phi)$ is independent of p. Since all norms $||\ ||_p$ are equivalent, by its definition $r_p(\phi)$ is also independent of p, as it should be.

To formulate the Perron-Frobenius theorem in the present context, we make the following definition, which is the just an adaption of the usual definition. Since any positive map ϕ in $\mathcal{J}_p$ is bounded $\exp t\phi$ is a well defined positive map in $\mathcal{J}_p$ for all $t > 0$. Also its inverse $\exp -t\phi$ is a well defined bounded map in $\mathcal{J}_p$, not necessarily positive .

Definition 3.1. *A positive map ϕ in $\mathcal{J}_p$ $(1 \leq p \leq \infty)$ is positivity improving if $\phi(A) > 0$ for any $A \geq 0$, $A \neq 0$. ϕ is ergodic if for any $A \geq 0, A \neq 0$ there is $t_A > 0$ with $(\exp t_A\phi)(A) > 0$.*

If ϕ is ergodic then $(\exp t\phi)(A) \geq (\exp t_A\phi)(A) > 0$ for all $t \geq t_A$. Obviously ϕ is ergodic if it is positivity improving.

A simple necessary criterion for ergodicity is given by

Lemma 3.1. *If ϕ is ergodic then $Ker\phi|_{C_p} = 0$.*

Proof. Assume that there is $0 \leq A \neq 0$ with $\phi(A) = 0$, such that $(\exp t\phi)(A) = A$ for all t. If A is not positive definite, then we have a contradiction, so let $A > 0$. We claim there is $0 \leq A' \neq 0$ which is not positive definite with $A' \leq A$. In fact, we may take A' to be a one-dimensional orthogonal projection onto an eigenvector of A times the corresponding eigenvalue, which we choose not to be zero. This gives $0 \leq \phi(A') \leq \phi(A) = 0$ and we are back to the first situation. □

Some remarks concerning the finite dimensional case are in order.

First we will show that when $\dim \mathcal{H} < \infty$ the present definition of ergodicity of ϕ is equivalent to the condition $(1+\phi)^{\dim \mathcal{H}-1}(A) > 0$ for any $0 \leq A \neq 0$, a criterion used in [7]. Obviously this last condition implies ergodicity since $\exp ta \geq 1/(n!)^2 \min(1,t)(1+a)^n$ for all n and all $t, a \geq 0$. As for the converse assume there is $0 \leq A \neq 0$ such that $(1+\phi)^{\dim \mathcal{H}-1}(A)$ is not positive definite. So there is an orthogonal projection $P \neq 0, \mathbb{I}$ and $\lambda > 0$ such that $\phi(P) \leq \lambda P$ (see [7]). This gives $(\exp t\phi)(P) \leq (\exp t\lambda)P$ for all $t > 0$, so ϕ is not ergodic.

We have the following necessary and simple criterion for ϕ to be ergodic.

Lemma 3.2. *If $\dim \mathcal{H} < \infty$ then $\phi(\mathbb{I}) > 0$ for ergodic ϕ.*

The converse is not true as may be seen by taking ϕ to be given by (9), where the α_i's are taken to be a nontrivial decomposition of unity. Then $\phi(\mathbb{I}) = \mathbb{I}$ whereas $\phi(\alpha_i) = \alpha_i$ for all i, such that ϕ is not ergodic.

Proof. Assume that $\phi(\mathbb{I}) \geq 0$ holds such that there is $\varphi \in \mathcal{H}$ with $\langle \varphi, \phi(\mathbb{I})\varphi\rangle = 0$. Then $\langle \varphi, \phi(A)\varphi\rangle = 0$ holds for all selfadjoint A due to the bound $-||A||\mathbb{I} \leq A \leq ||A||\mathbb{I}$ and therefore $\langle \varphi, \phi^n(A)\varphi\rangle = 0$ for all $n \geq 1$ and all $A = A^\star$. This in turn implies $\langle \varphi, (\exp t\phi)(\phi(A))\varphi\rangle = 0$ for all $t > 0$ and all $A = A^\star$. Since $\phi \neq 0$ by definition, there is $A \geq 0$ with $0 \leq \phi(A) \neq 0$ (see the decomposition (2)), so ϕ is not ergodic. □

Returning to the general case it is clear that ϕ_α given by (9) is not ergodic if the α_i's satisfy condition (11). Also in the context $p = 1$ and with a continuous, one parameter family of unitary operators $U(t)$ defining a quantum mechanics on $\mathcal{H}$ in terms of its infinitesimal generator H, then $\phi_{U(t)}$ (which describes the time evolution in the Heisenberg picture) for fixed t is not ergodic. If the completely positive map ϕ in $\mathcal{J}_p$ is given by (13) in terms of a unitary representation of a compact group, which contains at least one irreducible representation with finite multiplicity, then by Schur's Lemma ϕ is easily seen to be ergodic (and even positivity improving) if and only if the representation is irreducible. Then also $\mathcal{H}$ is of course finite dimensional and the map ϕ is a conditional expectation. The following two lemmas are almost obvious.

Lemma 3.3. *A completely positive map ϕ in $\mathcal{J}_1$ represented as $\phi_{\underline{\alpha}}$ is positivity improving if and only for any $0 \neq \varphi \in \mathcal{H}$ the closed linear hull of the set of vectors $\{\alpha_i^\star\varphi\}$ is all of $\mathcal{H}$.*

Proof. It suffices to consider $\phi(P)$, where P is any one-dimensional orthogonal projection (compare the proof of Lemma 3.1). Let $0 \neq \varphi_0 \in P\mathcal{H}$ be a unit vector, so we have to consider

$$\langle \varphi, \phi(P)\varphi\rangle = \sum_{i\in\mathbb{N}} |\langle \varphi, \alpha_i\varphi_0\rangle|^2 \tag{17}$$

and the claim follows. □

This proof shows that in the infinite dimensional case no $\phi_{\underline{\alpha}}$ with only finitely many non-vanishing α_i's can be positivity improving. In fact, for given P choose $0 \neq \varphi$ to be orthogonal to all $\alpha_i\varphi_0$. Then (17) vanishes and therefore $\phi(P)$ is not positive definite.

Let $\mathcal{A}(\underline{\alpha})$ be the algebra (not *-algebra) generated by the $\alpha_i^\star$'s.

Lemma 3.4. *A completely positive map ϕ in $\mathcal{J}_1$ represented as $\phi_{\underline{\alpha}}$ is ergodic if and only if every non-zero vector φ in $\mathcal{H}$ is cyclic for $\mathcal{A}(\underline{\alpha})$, i.e. the strong closure of the linear space $\mathcal{A}(\underline{\alpha})\varphi$ is all of $\mathcal{H}$.*

These two results apply to all Kraus representations $\phi_{\underline{\alpha}}$ of ϕ.

Proof. We use the same notation as in the proof of the previous lemma. In addition write $\alpha_{\underline{i}} = \alpha_{i_1}\alpha_{i2}..\alpha_{i_n}$ for $\underline{i} = (i_1, i_2, ..i_n)$ and set $|\underline{i}| = n$. Then

$$\langle\varphi, (\exp t\phi)(P)\varphi\rangle = \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{\underline{i}:|\underline{i}|=n}|\langle\varphi, \alpha_{\underline{i}}\varphi_0\rangle|^2$$

and again the claim follows. □

The next theorem is part of the Perron-Frobenius theorem in the present context. It is part because it assumes that $r_p(\phi)$ is an eigenvalue, a standard assumption one has to make in the infinite dimensional context if no other information is available.

Theorem 3.1. *Let the positive map ϕ in $\mathcal{J}_p$ $(1 \le p \le \infty)$ be ergodic and assume that $r_p(\phi) = ||\phi||_p$. If $r_p(\phi)$ is an eigenvalue of ϕ, then this eigenvalue is simple and the eigenvector A may be chosen to be positive definite.*

Remark 3.1. *According to usual terminology such a vector is called a Perron- Frobenius vector. When $\dim\mathcal{H} < \infty$ then $r_p(\phi)$ is independent of p and is always an eigenvalue. This case has been covered by Evans and Hoegh-Krohn* [7] *with an application to quantum Monte-Carlo processes and by Groh* [14]. *At present we do not know whether in the infinite dimensional case the condition $r_p(\phi) = ||\phi||_p$ can be dropped.*

For the proof of the theorem we need the following simple

Lemma 3.5. *For any $B > 0$ and $0 \ne B' \ge 0$ in $\mathcal{J}_p$, $1 \le p \le \infty$ the following strict inequality*

$$||B - B'||_p < ||B + B'||_p \tag{18}$$

holds.

The condition $B > 0$ cannot be weakened to $B \ge 0$ as the choice $B = P \ne 0$ and $B' = Q \ne 0$ for two finite dimensional orthogonal projections with $PQ = 0$ shows.

Proof. The estimate (18) for the case $p = \infty$ is trivial, so let $p < \infty$. Since $B - B' \in \mathcal{J}_p$ is self-adjoint, there is a complete set orthonormal eigenvectors φ_n with real eigenvalues λ_n of $B - B'$. With $\lambda_n = \langle\varphi_n, (B - B')\varphi_n\rangle$ we obtain

$$||B - B'||_p^p = \sum_n |\lambda_n|^p = \sum_n |\langle\varphi_n, B\varphi_n\rangle - \langle\varphi_n, B'\varphi_n\rangle|^p. \tag{19}$$

Observe that

$$\begin{aligned} |\langle\varphi_n, B\varphi_n\rangle - \langle\varphi_n, B'\varphi_n\rangle|^p &\le \max(\langle\varphi_n, B\varphi_n\rangle^p, \langle\varphi_n, B'\varphi_n\rangle^p) \\ &\le (\langle\varphi_n, B\varphi_n\rangle + \langle\varphi_n, B'\varphi_n\rangle)^p \\ &= \langle\varphi_n, (B + B')\varphi_n\rangle^p \\ &\le \langle\varphi_n, (B + B')^p\varphi_n\rangle \end{aligned} \tag{20}$$

holds, since $B > 0$ and $B' \ge 0$. In the last estimate in (20) we used the estimate (7).

Furthermore $\langle\varphi_n, B\varphi_n\rangle > 0$ holds for all n. Also $\langle\varphi_n, B'\varphi_n\rangle > 0$ for at least one n. Otherwise we would also have $\langle\varphi_n, B'\varphi_{n'}\rangle = 0$ for all n, n' by Schwarz inequality and hence $B' = 0$, contradicting the assumption. Thus the first inequality in (20) is actually strict for at least one n. Inserting (20) into (19) and using this last observation proves (18). □

$Proof\ of\ the\ Theorem$: Let $A \ne 0$ be any eigenvector for ϕ with eigenvalue $r_p(\phi)$. By an observation made above, we may assume that A is self-adjoint. Observe the following estimate, valid for any $t > 0$.

$$\begin{aligned} \exp tr_p(\phi)||A||_p &= ||\exp tr_p(\phi)A||_p = ||(\exp t\phi)(A)||_p \\ &= ||\,|(\exp t\phi)(A)|\,||_p \le ||(\exp t\phi)(|A|)||_p \\ &\le \exp t||\phi||_p||\,|A|\,||_p = \exp t||\phi||_p||A||_p. \end{aligned} \tag{21}$$

Here we have used the estimate (6) in combination with (3) for the map $\exp t\phi$.

Since by assumption $r_p(\phi) = ||\phi||_p$, all inequalities in (21) actually have to be equalities. In particular for all $t > 0$

$$
\begin{aligned}
||(\exp t\phi)(A)||_p &= ||(\exp t\phi)(A_+) - (\exp t\phi)(A_-)||_p \\
&= ||(\exp t\phi)(|A|)||_p = ||(\exp t\phi)(A_+) + (\exp t\phi)(A_-)||_p.
\end{aligned} \tag{22}
$$

If $A_+ = 0$ or $A_- = 0$ there is nothing to prove so assume that $A_\pm \neq 0$. By the ergodicity of ϕ we have $(\exp t\phi)(A_\pm) > 0$ for all large $t > 0$. But then (22) contradicts (18). So we must have either $A_+ = 0$ or $A_- = 0$ and by replacing A by $-A$ if necessary we may without restriction assume that $A = |A| = A_+$. Again by ergodicity we have $\exp tr_p(\phi)A = (\exp t\phi)(A) > 0$ for all large t such that $A > 0$. The proof of non-degeneracy is now easy. Assume there are two linearly independent eigenvectors $A_1 > 0$ and $A_2 > 0$, which we may normalize to $||A_1||_p = ||A_2||_p = 1$. Then $0 \neq A_1 - A_2$ is also an eigenvector and by our previous discussion we must have either $A_1 > A_2$ or $A_2 > A_1$. This would imply that $||A_1||_p > ||A_2||_p$ or $||A_2||_p > ||A_1||_p$ respectively, again an easy consequence of (6) and its proof. This is a contradiction, so we must have $A_1 = A_2$, thus concluding the proof of the theorem.

4. Existence of eigenvalues

In this section we want in particular to establish conditions for positive maps ϕ in $\mathcal{J}_p$, which are sufficient to show that $r_p(\phi)$ is an eigenvalue. We start with a preparation. For any $A \in \mathcal{B}(\mathcal{H})$ we write $A = U_A|A|$ with unique U_A for its polar decomposition. More precisely U_A is isometric on $\overline{\operatorname{Ran}|A|} = (KerA)^\perp$, i.e. $U_A^\star U_A\varphi = \varphi$ for $\varphi \in \overline{\operatorname{Ran}|A|} = (KerA)^\perp$, and $U_A\varphi = 0$ for $\varphi \in (\operatorname{Ran}|A|)^\perp = KerA$.

The next two lemmas replace the Kadison-Schwarz inequality (see e.g. the second reference in [18] and [34, 4, 24] for other early references) used in the context of $C^\star$-algebras in [7, 14] and in the context of von Neumann algebras in [2]. Since the spaces $\mathcal{J}_p$, $p < \infty$ are not algebras these two lemmas will provide the appropriate substitute.

Lemma 4.1. *Let the map ϕ in $\mathcal{J}_p$, $1 \leq p \leq \infty$ be 2-positive. Then for all $\varphi, \varphi' \in \mathcal{H}$ and all $A \in \mathcal{J}_p$*

$$
|\langle \varphi', \phi(A)\varphi\rangle|^2 \leq \langle \varphi, \phi(|A|)\varphi\rangle\langle\varphi', \phi\circ\phi_{U_A}(|A|)\varphi'\rangle. \tag{23}
$$

Also if not all terms in (23) are vanishing, then there is equality if and only if both relations

$$
\begin{pmatrix} \phi(|A|) & \phi(A^\star) \\ \phi(A) & \phi\circ\phi_{U_A}(|A|) \end{pmatrix}
\begin{pmatrix} \langle\varphi', \phi\circ\phi_{U_A}(|A|)\varphi'\rangle & \varphi \\ -\langle\varphi', \phi(A)\varphi\rangle & \varphi' \end{pmatrix} = 0 \tag{24}
$$

and

$$
\begin{pmatrix} \phi(|A|) & \phi(A^\star) \\ \phi(A) & \phi\circ\phi_{U_A}(|A|) \end{pmatrix}
\begin{pmatrix} & \langle\varphi, \phi(A)^\star\varphi'\rangle\varphi \\ - & \langle\varphi, \phi(|A|)\varphi\rangle\varphi' \end{pmatrix} = 0 \tag{25}
$$

hold.

For $A = A^\star$ estimate (23) follows from (3), since then $U_A^\star$ commutes with A such that $U_A|A|U_A^\star = AU_A^\star = U_A{}^\star A = U_A^\star U_A|A| = |A|$.

Proof. Here and in the proof of the next lemma we will mimic and extend Exercise 3.4 in [26], page 39 in the present context. Set $A_1 = |A|^{1/2}$, $A_2 = |A|^{1/2}U_A^\star$ such that $A_1^2 = |A|$, $A_2^\star A_1 = A$, $A_2^\star A_2 = U_A|A|U_A^\star = \phi_{U_A}(|A|)$ are all in $\mathcal{J}_p$. Consider

$$
\begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix}^\star \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} |A| & A^\star \\ A & \phi_{U_A}(|A|) \end{pmatrix} \geq 0 \tag{26}
$$

in $\mathcal{H} \oplus \mathcal{H} \cong \mathcal{H} \otimes \mathbb{C}^2$, which defines an element in $\mathcal{C}_p(\mathcal{H} \oplus \mathcal{H})$. By assumption

$$
\phi_2 \begin{pmatrix} |A| & A^\star \\ A & \phi_{U_A}(|A|) \end{pmatrix} = \begin{pmatrix} \phi(|A|) & \phi(A^\star) \\ \phi(A) & \phi\circ\phi_{U_A}(|A|) \end{pmatrix} \geq 0. \tag{27}
$$

Consider the following linear transformation T on $\mathbb{C}^2$ (equipped with the natural scalar product) given as the 2×2 matrix

$$T = \begin{pmatrix} \langle\varphi, \phi(|A|)\varphi\rangle & \langle\varphi, \phi(A)^\star\varphi'\rangle \\ \langle\varphi', \phi(A)\varphi\rangle & \langle\varphi', \phi\circ\phi_{U_A}(|A|)\varphi'\rangle \end{pmatrix} \tag{28}$$

such that for all $a_1, a_2 \in \mathbb{C}$

$$\langle\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, T\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\rangle = \langle\begin{pmatrix} a_1\,\varphi \\ a_2\,\varphi' \end{pmatrix}, \begin{pmatrix} \phi(|A|) & \phi(A^\star) \\ \phi(A) & \phi\circ\phi_{U_A}(|A|) \end{pmatrix}\begin{pmatrix} a_1\,\varphi \\ a_2\,\varphi' \end{pmatrix}\rangle \geq 0 \tag{29}$$

with the obvious notation for the scalar product in $\mathcal{H}\oplus\mathcal{H}$. A linear transformation in $\mathbb{C}^2$ is ≥ 0 if and only if its trace and its determinant are both ≥ 0. Hence the bound (23) follows. This discussion also easily gives the last claim in Lemma 4.1. Indeed, if $T \neq 0$, its determinant is equal to zero if and only if there is a non-zero eigenvector of T with eigenvalue zero and this is the case if and only if equality in (23) holds and then such an eigenvector of T is given either as

$$\begin{pmatrix} \langle\varphi', \phi\circ\phi_{U_A}(|A|)\varphi'\rangle \\ -\langle\varphi', \phi(A)\varphi\rangle \end{pmatrix}$$

or as

$$\begin{pmatrix} \langle\varphi, \phi(A)^\star\varphi'\rangle \\ -\langle\varphi, \phi(|A|)\varphi\rangle \end{pmatrix}.$$

More precisely, if all matrix elements of T are non-vanishing then these two eigenvectors have non-zero entries and are proportional. If $T \neq 0$ with $\det T = 0$ and if one diagonal element is vanishing, such that also the off diagonal elements are vanishing, then exactly one of these vectors is the null vector. In view of (29) this concludes the proof of Lemma 4.1. □

Lemma 4.2. *Let the map ϕ in $\mathcal{J}_p,\ 1 \leq p \leq \infty$ be 2-positive. Then the bound*

$$||\phi(A)||_p \leq ||\phi(|A|)||_p^{1/2}\ \ ||\phi\circ\phi_{U_A}(|A|)||_p^{1/2} \tag{30}$$

holds for all $A \in \mathcal{J}_p$. In particular $A \in Ker\phi$ if $|A| \in Ker\phi$ or $|A| \in Ker\phi\circ\phi_{U_A}$.

Proof. Choosing $\varphi' = U_{\phi(A)}\varphi$ in (23) we have the estimate

$$\langle\varphi, |\phi(A)|\varphi\rangle \leq \langle\varphi, \phi(|A|)\varphi\rangle^{1/2}\langle U_{\phi(A)}\varphi, \phi\circ\phi_{U_A}(|A|)U_{\phi(A)}\varphi\rangle^{1/2} \tag{31}$$

for all $\varphi \in \mathcal{H}$. Thus (30) follows trivially from (31) in case $p = \infty$. So let $1 \leq p \leq \infty$. Take φ_n to be an orthonormal basis in $\mathcal{H}$ diagonalizing $|\phi(A)|$ with eigenvalues $\mu_n \geq 0$. Then we obtain

$$\begin{aligned} ||\phi(A)||_p^p &= \sum_n \mu_n^p = \sum_n \langle\varphi_n, |\phi(A)|\varphi_n\rangle^p \\ &\leq \sum_n \langle\varphi_n, \phi(|A|)\varphi_n\rangle^{p/2}\langle U_{\phi(A)}\varphi_n, \phi\circ\phi_{U_A}(|A|)U_{\phi(A)}\varphi_n\rangle^{p/2} \\ &\leq \left(\sum_n \langle\varphi_n, \phi(|A|)^p\varphi_n\rangle\right)^{1/2} \cdot \\ &\quad \left(\sum_n \langle U_{\phi(A)}\varphi_n, (\phi\circ\phi_{U_A}(|A|))^p U_{\phi(A)}\varphi_n\rangle\right)^{1/2} \\ &\leq ||\phi(|A|)||_p^{p/2}\ \ ||\phi\circ\phi_{U_A}(|A|)||_p^{p/2}. \end{aligned} \tag{32}$$

Here we have used (31) for the first inequality. The second inequality is a consequence of Schwarz inequality and (7). In what follows this will become important, since we will encounter the situation when all inequalities are actually equalities. Also we have used the fact that with φ_n being an orthonormal basis also $U_{\phi(A)}\varphi_n$ is a possibly incomplete set of orthonormal vectors and which is responsible for the last inequality. □

$\lambda \in \mathbb{C}$ is called a peripheral eigenvalue of the positive map ϕ in $\mathcal{J}_p$ if λ and hence also $\overline{\lambda}$ is an eigenvalue with $|\lambda| = r_p(\phi)$.

Theorem 4.1. *Let the map ϕ in $\mathcal{J}_p$, $1 \le p < \infty$ be 2-positive with $r_p(\phi) = ||\phi||_p$. Assume that $\lambda \in \mathbb{C}$ is a peripheral eigenvalue with eigenvector A. Then $r_p(\phi)$ is also an eigenvalue with eigenvector $|A|$. In particular, if ϕ is 2-positive and compact with $r_p(\phi) = ||\phi||_p$, then $r_p(\phi)$ is an eigenvalue.*

Here and in contrast to the previous discussions the case $p = \infty$ is not covered by the present discussion. Since $\mathcal{J}_{p=\infty}$ is a $C^\star$-algebra, this case, however, is covered by the discussion in [36, 14]. Then and similarly in the context of von Neumann algebras one can actually prove more. In fact, if the spectrum is rescaled via $\lambda \to \lambda/r_p(\phi)$, then the peripheral eigenvalues form a discrete sub-group of the unit circle group, see [2] and the first reference in [14].

Proof. We want to prove that $|A| \neq 0$ is also an eigenvector with eigenvalue $|\lambda| = r_p(\phi)$, i.e. $\phi(|A|) = r_p(\phi)|A|$. Since $|\phi(A)| = |\lambda A| = r_p(\phi)|A|$ (such that also $U_{\phi(A)} = (\lambda/|\lambda|)U_A$ and hence $\phi_{U_{\phi(A)}} = \phi_{U_A}$ by the uniqueness of U_A) it suffices to show that

$$\phi(|A|)\varphi_n = \mu_n \varphi_n \tag{33}$$

holds for all n, where the φ_n are as in the previous lemma. Also by the previous lemma we have

$$\begin{aligned} r_p(\phi)||A||_p &= ||\lambda A||_p = ||\phi(A)||_p = ||\,|\phi(A)|\,||_p \\ &\le ||\phi(|A|)||_p^{1/2}||\phi \circ \phi_{U_A}(|A|)||_p^{1/2} \\ &\le ||\phi||_p \ \ ||A||_p. \end{aligned} \tag{34}$$

Here we have used the fact that $||\phi_{U_A}||_p \le 1$. By the assumption $r_p(\phi) = ||\phi||_p$ we must have equality. In particular this implies that

$$||\,|\phi(A)|\,||_p = ||\phi(|A|)||_p^{1/2}||\phi \circ \phi_{U_A}(|A|)||_p^{1/2}. \tag{35}$$

Inspection of the proof of Lemma 4.2 shows that we must have equality in (31) when $\varphi = \varphi_n$ for all n. The relations (24) and (25) in Lemma 4.1 then imply that we must have

$$\langle \varphi_n', \phi \circ \phi_{U_A}(|A|)\varphi_n'\rangle \phi(|A|)\varphi_n = \langle \varphi_n', \phi(A)\varphi_n\rangle \phi(A)^\star \varphi_n' \tag{36}$$

and

$$\langle \varphi_n, \phi(|A|)\varphi_n\rangle \phi \circ \phi_{U_A}(|A|)\varphi_n' = \langle \varphi_n, \phi(A)^\star \varphi_n'\rangle \phi(A)\varphi_n \tag{37}$$

for all n with $\varphi_n' = U_{\phi(A)}\varphi_n$. Assume n to be such that $\mu_n > 0$. Since $\varphi_n \in \mathrm{Ran}\,|\phi(A)|$ we have $U^\star_{\phi(A)}\varphi_n' = \varphi_n$ and $\phi(A)^\star\varphi_n' = |\phi(A)|\varphi_n = \mu_n\varphi_n$ for all such n. Similarly $\phi(A)\varphi_n = U_{\phi(A)}|\phi(A)|\varphi_n = \mu_n\varphi_n'$. Hence we may rewrite (36) and (37) as

$$\langle \varphi_n', \phi \circ \phi_{U_A}(|A|)\varphi_n'\rangle \phi(|A|)\varphi_n = \mu_n^2 \varphi_n \tag{38}$$

and

$$\langle \varphi_n, \phi(|A|)\varphi_n\rangle \phi \circ \phi_{U_A}(|A|)\varphi_n' = \mu_n^2 \varphi_n' \tag{39}$$

whenever $\mu_n > 0$.

Furthermore we recall that we made use of Schwarz inequality in the estimate (32). For this to be an equality we claim that we actually must have

$$\langle \varphi_n, \phi(|A|)\varphi_n\rangle = \langle \varphi_n', \phi \circ \phi_{U_A}(|A|)\varphi_n'\rangle \tag{40}$$

for all n. Indeed, the Schwarz inequality in this context is an equality if and only if both sides of (40) are proportional with a common proportionality factor for all n. By the equality (35) this proportionality factor has to be equal to 1, thus proving (40).

If $\mu_n = 0$ then $\varphi_n' = 0$ and hence by (40) $\phi(A)\varphi_n = 0$. Thus we have established (33) in the case when $\mu_n = 0$. If $\mu_n > 0$ then the left hand sides of (38) and (39) are both non-vanishing since then $\varphi_n' \neq 0$. Therefore φ_n is an eigenvector of $\phi(|A|)$ with eigenvalue

$$\begin{aligned} \langle \varphi_n, \phi(|A|)\varphi_n\rangle &= \frac{\mu_n^2}{\langle \varphi_n', \phi \circ \phi_{U_A}(|A|)\varphi_n'\rangle} \\ &= \frac{\mu_n^2}{\langle \varphi_n, \phi(|A|)\varphi_n\rangle}. \end{aligned}$$

This gives (33) in the case when $\mu_n > 0$, thus concluding the proof of the theorem, since for compact operators the spectrum is pure point. $\square$

The use of (7) in the second inequality in (32) does not give any additional mileage when $p \neq 1$ since the φ_n are eigenvectors of $\phi(|A|)$ and the φ'_n eigenvectors of $\phi \circ \phi_{U_A}$ and hence (7) gives an equality for the case at hand.

The strategy of the proof was motivated by the following observation in the classical context. With the notation introduced after (3) assume that $S\underline{z} = \lambda\underline{z}$. This gives $|\lambda|\,|\underline{z}| = |\lambda\underline{z}| = |S\underline{z}| \leq S|\underline{z}|$. Taking its usual Hilbert space norm in $\mathbb{C}^n$ we obtain $|\lambda|\,||\,|\underline{z}|\,|| \leq ||S|\underline{z}|\,|| \leq ||S||\,||\,|\underline{z}|\,||$. So if $|\lambda| = ||S||$ we must have $S|\underline{z}| = ||S||\,|\underline{z}|$.

The next result extends Theorem 3.1.

Theorem 4.2. *Let the map ϕ in $\mathcal{J}_p,\ 1 \leq p < \infty$ be 2-positive and ergodic with $r_p(\phi) = ||\phi||_p$. Assume that $\lambda \in \mathbb{C}$ is a peripheral eigenvalue. Then λ is a simple eigenvalue and the corresponding eigenvector A is a normal operator.*

Proof. Assume there are two eigenvectors A_1 and A_2 in $\mathcal{J}_p$ with eigenvalue λ, normalized to $||A_1||_p = ||\,|A_1|\,||_p = ||A_2||_p = ||\,|A_2|\,||_p = 1$. By Theorem 3.1, its proof and by Theorem 4.1 we have that $0 < |A_1| = |A_2|$ is an eigenvector with eigenvalue $r_p(\phi)$. In particular both U_{A_1} and U_{A_2} are unitaries. Since $A_1 - \exp i\tau\ A_2$ is also an eigenvector for the same eigenvalue for all $0 \leq \tau \leq 2\pi$ by the same argument we must have

$$|A_1 - \exp i\tau\ A_2| = ||A_1 - \exp i\tau\ A_2||_p |A_1|. \tag{41}$$

The function $\tau \mapsto ||A_1 - \exp i\tau\ A_2||_p$ is easily seen to be continuous. Hence there is τ_0 such that

$$||A_1 - \exp i\tau_0\ A_2||_p = \min_{0 \leq \tau \leq 2\pi} ||A_1 - \exp i\tau\ A_2||_p. \tag{42}$$

Take the square of (41) and use (42) to obtain

$$|A_1 - \exp i\tau_0\ A_2|^2 \leq |A_1 - \exp i\tau\ A_2|^2.$$

Due to the relation $\overline{\mathrm{Ran}\ |A_1|} = \mathcal{H}$ this gives

$$||(U_{A_1} - \exp i\tau_0\ U_{A_2})\varphi||^2 \leq ||(U_{A_1} - \exp i\tau\ U_{A_2})\varphi||^2$$

which written out in turn gives

$$\Re\left(e^{i\tau}\langle U_{A_1}\varphi, U_{A_2}\varphi\rangle\right) \leq \Re\left(e^{i\tau_0}\langle U_{A_1}\varphi, U_{A_2}\varphi\rangle\right) \tag{43}$$

for all $0 \leq \tau \leq 2\pi$ and all $\varphi \in \mathcal{H}$. We distinguish two possible cases. First if $A_1 - \exp i\tau_0\ A_2 = 0$ and hence $U_{A_1} - \exp i\tau_0\ U_{A_2} = 0$ there is nothing to prove. Otherwise $U = U_{A_1}^{-1} \exp i\tau_0\ U_{A_2}$ is a unitary operator $\neq \mathbb{I}$ and hence by the spectral theorem for unitary operators there is $\varphi \in \mathcal{H}$ such that $\Re(\langle\varphi, U\varphi\rangle) < |\langle\varphi, U\varphi\rangle|$. Choose τ such that $\exp i(\tau - \tau_0)\langle\varphi, U\varphi\rangle = |\langle\varphi, U\varphi\rangle|$. This contradicts (43), thus concluding the proof of the first part of the theorem. As for the last part U_A is unitary, as already noted. Relation (39) shows that $\phi_{U_A}(|A|)$ is also an eigenvector of ϕ with eigenvalue $r_p(\phi)$. Hence $\phi_{U_A}(|A|) = |A|$ by uniqueness and normalization. So U_A commutes with $|A|$ and therefore A is normal with $U_A^\star = U_{A^\star}$ and $|A^\star| = |A|$, where we recall that $A^\star$ is an eigenvector with eigenvalue $\overline{\lambda}$. $\square$

The next result is of relevance in quantum physics. We recall that for completely positive, trace preserving maps ϕ in $\mathcal{J}_1$ one has $r_1(\phi) = ||\phi||_1 = 1$. So we immediately obtain the following existence result for Perron-Frobenius vectors.

Corollary 4.1. *Let the map ϕ in $\mathcal{J}_1$ be completely positive, trace preserving, compact and ergodic. Then there is a unique density matrix $\rho > 0$ which is left invariant under ϕ.*

A converse of this result is given by Theorem 5.1 below.

The following result is of the type needed in the standard context of Monte Carlo simulations in lattice models of statistical mechanics. It states that, starting from any non-negative initial configuration, by iterations of an ergodic up-dating procedure (heat bath method) one eventually arrives at the desired equilibrium configuration which typically is given by a Gibbs distribution and which by construction of the up-dating is a Perron-Frobenius vector for the up-dating procedure.

Corollary 4.2. *Let the map ϕ in $\mathcal{J}_2$ be 2-positive, selfadjoint, compact and ergodic with Perron-Frobenius vector $A > 0$ normalized to $||A||_2 = 1$. Then for any $B \in \mathcal{J}_2$*

$$s - \lim_{t\to\infty} \exp(t(\phi - ||\phi||_2))(B) = \langle A, B\rangle_2 A. \tag{44}$$

If $0 \leq B \neq 0$ then the right hand side of (44) is non-vanishing. It would be interesting to find an analogous formulation when $p \neq 2$.

Proof. We use familiar arguments. Let the A_j form an orthonormal basis of eigenvectors for ϕ arranged such that $A_1 = A$ and $\sigma_{j+1} \leq \sigma_j$. Here the σ_j's are the eigenvalues, i.e. $\phi(A_j) = \sigma_j A_j$. Since $\sigma_1 = ||\phi||_2$ is a simple eigenvalue, $\sigma_1 > \sigma_2$. By Plancherel's theorem we have $B = \sum_j b_j A_j$ with $b_j = \langle A_j, B\rangle_2$ and $\sum_j b_j^2 = ||B||_2^2$. Obviously

$$\exp(t(\phi - ||\phi||_2))(B) = \sum_j b_j e^{t(\sigma_j - \sigma_1)} A_j$$

holds such that

$$||(\exp(t(\phi - ||\phi||_2)))(B) - \langle A, B\rangle_2 A||_2 \leq e^{-t(\sigma_1-\sigma_2)}||B||_2 \tag{45}$$

and the claim follows. □

By a slight modification of the proof using the spectral representation of ϕ one may replace the compactness condition by the condition that $||\phi||_2$ is an isolated eigenvalue, i.e. $||\phi||_2$ is separated by a distance m^2 (the "mass gap" in physical language) from the remainder of the spectrum of ϕ, which otherwise may be arbitrary. Then the estimate (45) is still valid with $\sigma_1 - \sigma_2$ being replaced by m^2.

5. Examples

In this section we will provide non-trivial examples when $\dim \mathcal{H} = \infty$. The issue is to show that all 4 conditions in Corollary 4.1 can be fulfilled simultaneously. In the finite dimensional case the compactness criterion is automatically satisfied and then it is easy to construct non-trivial examples. In general the existence of completely positive, trace preserving, compact maps in $\mathcal{J}_1$ is clarified by

Lemma 5.1. *Let $\underline{\alpha} = \{\alpha_i\}_{i\in\mathbb{N}}$ be such that $\sum_{i\in\mathbb{N}} \alpha_i^\star \alpha_i = \mathbb{I}$ and all $\alpha_i \in \mathcal{B}(\mathcal{H})$ are compact. Then the map $\phi_{\underline{\alpha}}$ in $\mathcal{J}_1$ is compact.*

Proof. We recall that $\phi_{\underline{\alpha}}$ is the norm limit of $\phi_{\underline{\alpha}_N}$ as $N \to \infty$. Since the space of compact operators is closed w.r.t. the norm topology it suffices to prove that all $\phi_{\underline{\alpha}_N}$ are compact. By assumption to each i there is a sequence $\alpha_{i,k}$ of finite rank operators in $\mathcal{BH})$ such that

$$\lim_{k\to\infty} ||\alpha_i - \alpha_{i,k}|| = \lim_{k\to\infty} ||\alpha_i^\star - \alpha_{i,k}^\star|| = 0.$$

Since $\alpha_{i,k}^\star$ is of finite rank $\phi_{\underline{\alpha}_{N,k}}$ with $\underline{\alpha}_{N,k} = (\alpha_{1,k}, ..., \alpha_{N,k})$ is also of finite rank and in particular compact. Using the a priori estimate $||ABC||_1 \leq ||A||\,||B||_1\,||C||$ and a standard $\epsilon/3$ argument gives

$$\lim_{k\to\infty} ||\phi_{\underline{\alpha}_N} - \phi_{\underline{\alpha}_{N,k}}||_1 = 0.$$

□

We are now prepared to provide non-trivial examples, which resulted from a discussion with D. Buchholz. Let $\psi_i,\ i \in \mathbb{N}$ be any orthonormal basis in $\mathcal{H}$ and $c_{ik} > 0$ any set of numbers which satisfy the condition $\sum_i c_{ik}^2 = 1$ for all k. In the Dirac notation we define the following family $\underline{\alpha} = \{\alpha_{ik}\}_{1\leq i,k<\infty}$ of rank 1 operators

$$\alpha_{ik} = c_{ik}|\psi_i\rangle\langle\psi_k|,$$

such that

$$\alpha_{ik}^\star = c_{ik}|\psi_k\rangle\langle\psi_i|,$$

In other words α_{ik} is the map $\varphi \mapsto c_{ik}\langle\psi_k,\varphi\rangle\psi_i$. We set

$$\phi_{\underline{\alpha}}(A)=\sum_{i,k}\alpha_{ik}A\alpha_{ik}^\star$$

By assumption we have

$$\sum_{i,k}\alpha_{ik}^\star\alpha_{ik}=\sum_{i,k}c_{ik}^2\;|\psi_k\rangle\langle\psi_k|=\sum_k|\psi_k\rangle\langle\psi_k|=\mathbb{I},$$

so $\phi_{\underline{\alpha}}$ is completely positive, compact and trace preserving. We claim that it is also positivity improving by Lemma 3.3. Indeed, given $0\neq\varphi\in\mathcal{H}$, assume there is $0\neq\varphi_0\in\mathcal{H}$ such that

$$0=\langle\varphi_0,\alpha_{ik}^\star\varphi\rangle=c_{ik}\langle\varphi_0,\psi_k\rangle\langle\psi_i,\varphi\rangle$$

holds for all i and k. Since by assumption $c_{ik}>0$ this contradicts the fact that the ψ_i form an orthonormal basis. So this $\phi_{\underline{\alpha}}$ satisfies all conditions of Corollary 4.1. The resulting Perron-Frobenius vector is given as the density matrix

$$\rho=\sum_i\rho_i|\psi_i\rangle\langle\psi_i|,\;\rho_i>0,\sum_i\rho_i=1, \tag{46}$$

where the ρ_i form the components of the Perron-Frobenius vector with eigenvalue 1 for the matrix c_{ik}^2, i.e. $\sum_k c_{ik}^2\rho_k=\rho_i$. As a special case consider the choice $c_{ik}=c_i>0$ for all k with $\sum_i c_i^2=1$. Then $\rho_i=c_i^2$.

This discussion also gives the following converse to Corollary 4.1, which in its standard formulation forms one of the starting points for Monte Carlo simulations in lattice models of statistical mechanics and (euclidean) quantum field theory including lattice gauge theories.

Theorem 5.1. *Given an arbitrary density matrix $\rho>0$, there is a completely positive, trace preserving, compact and ergodic map ϕ in $\mathcal{J}_1$ for which ρ is the resulting Perron-Frobenius vector. In addition ϕ may be chosen to be idempotent (i.e. ϕ satisfies the characteristic equation $\phi\circ(1-\phi)=0$) and $\sigma_1(\phi)=\{0,1\}$.*

Proof. Take the ψ_i's to form an orthonormal basis in which ρ is diagonal, i.e. such that ρ takes the form (46) and choose $\phi=\phi_{\underline{\alpha}}$ with $\underline{\alpha}$ as above with $c_{ik}=\rho_i^{1/2}$. Then we have $\phi(A)=\mathrm{Tr}\,A\cdot\rho$ for all $A\in\mathcal{J}1$, such that ϕ is idempotent. Also the kernel of ϕ consists of all elements which may be written in the form $A-\mathrm{Tr}\,A\cdot\rho$. Moreover let $\lambda\neq0,1$. Then for given A' the equation $(\lambda-\phi)(A)=A'$ has a solution in A of the form

$$A=\frac{1}{\lambda(1-\lambda)}\,\mathrm{Tr}\,A'\cdot\rho+\frac{1}{\lambda}A'.$$

□

As in the corresponding (finite dimensional) standard context ϕ with prescribed Perron-Frobenius vector is of course not unique. We provide another example which in spirit is much closer to the usual "local" up-grading procedure in Monte Carlo simulations. Set $p_{\nu,i}=\rho_{\nu+i}/(\rho_{i+1}+\rho_i+\rho_{i-1})$ for $\nu=0,\pm1$ and with the convention $\rho_0=0$. Then $\sum_{\nu=0,\pm1}p_{\nu,i}=1$ for all i and $p_{\nu,i}>0$ for all ν and i unless $\nu=-1$ and $i=1$. Set

$$T_{\nu,i}=p_{\nu,i}^{1/2}\;|\psi_{\nu+i}\rangle\langle\psi_i|$$

and define ϕ_T as

$$\phi_T(A)=\sum_{\nu,i}T_{\nu,i}AT_{\nu,i}^\star.$$

An easy calculation shows that ϕ_T is completely positive, trace preserving, compact and ergodic.

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