Renormalization group flow with unstable particles

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The renormalization group flow of an integrable two dimensional quantum field theory which contains unstable particles is investigated. The analysis is carried out for the Virasoro central charge and the conformal dimensions as a function of the renormalization group flow parameter. This allows to identify the corresponding conformal field theories together with their operator content when the unstable particles vanish from the particle spectrum. The specific model considered is the $SU(3)_2$-homogeneous Sine-Gordon model.

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The study of two-dimensional quantum field theories (2D-QFT) has turned out to be a fruitful venture since almost three decades. In particular when exploiting integrability many non-perturbative methods have been developed over the years. Besides the challenge to understand the underlying mathematical structures and the intriguing physical applications in two dimensions itself, e.g. to describe measurable quantities of carbon nanotubes [1], the ultimate goal is to extrapolate ones findings to higher dimensions. In particular for the celebrated c-theorem of Zamolodchikov [2], which originally describes the renormalization group trajectory of a function which at the renormalization group fixed point corresponds to the Virasoro central charge, various counterparts have been developed in higher dimensions [3].

Fairly recently a class of massive integrable quantum field theories, the homogeneous Sine-Gordon models (HSG) [4], has been proposed introducing the feature of possessing unstable particles inside its particle spectrum, which hitherto have not been considered in the context of two-dimensional massive quantum field theories. In general the HSG models are associated to integrable perturbations of G-parafermions of level k [5], i.e. WZNW-coset theories of the form $G_k/U(1)^\ell$ with ℓ being the rank of a compact Lie group G. As free parameters the model contains ℓ different mass scales and $\ell-1$ different scales for the resonance parameter σ which enter the Breit-Wigner formula [6]

$$M_{\tilde{c}}{}^2 - \frac{\Gamma_{\tilde{c}}{}^2}{4} = M_a{}^2 + M_b{}^2 + 2M_a M_b \cosh\sigma\cos\bar{\sigma} \qquad (1)$$

$$M_{\tilde{c}}\Gamma_{\tilde{c}} = 2M_a M_b \sinh|\sigma|\sin\bar{\sigma}\ , \qquad (2)$$

therefore characterizing the energy scale of the unstable particles of type $\tilde{c}$ with mass $M_{\tilde{c}}$ and decay width $\Gamma_{\tilde{c}}$. When the scattering of the stable particles of type a and b with masses M_a and M_b is described by an S-matrix $S_{ab}(\theta)$ as function of the rapidity θ, its resonance pole is situated at $\theta_R = \sigma - i\bar{\sigma}$. Eliminating the decay width from the Breit-Wigner formula we can express the mass of the unstable particles $M_{\tilde{c}}$ in the model as a function of the masses of the stable particles m_a, m_b and the resonance parameter σ

$$M_{\tilde{c}}^2 = (m_a^2 + m_b^2)\left(1+\sqrt{1+M_r\sinh^2\sigma}\right). \qquad (3)$$

It is convenient to introduce a reduced mass like parameter $M_r = 4\ m_a^2 m_b^2/(m_a^2+m_b^2)^2$ which is one when the two mass scales m_a, m_b coincide.

In [7] the relative mass scales between the unstable and stable particles and the stable particles themselves were investigated by computing the finite size scaling function from the thermodynamic Bethe ansatz (TBA). A consistent physical picture was obtained for the overall identification of the flow between different coset models. It remained, however, an open question how to identify the operator content. In general this question is left unanswered in the context of the TBA. For theories with certain properties, it is sometimes possible to determine at least the dimension of the perturbing operators by investigating periodicities in the so-called Y-systems [8]. Resorting to a different method, namely by appealing to sum rules which are expressible in terms of correlation functions, the major part of the operator content was successfully identified for some of the HSG models [9]. The purpose of this manuscript is on one hand to confirm and refine the TBA results by the latter method, i.e. by investigating the renormalization group flow described by the Zamolodchikov c-function [2]. We will precisely study the onset of the mass scale of the unstable particles and investigate how a particular coset flows to another one. On the other hand, we study in addition the flow of the operator content of one conformal field theory to another one by exploiting the flow provided by the Δ-sum rule of Defino, Simonetti and Cardy [10].

Denoting by r the radial distance and by $t=\ln r^2$ the renormalization group parameter, the functions $c(t)$ and $\Delta(t)$ were defined in [2] and [10], respectively, obeying the differential equations

$$\frac{dc(t)}{dt} = -\frac{3}{4}\,e^t\,\langle\Theta(t)\Theta(0)\rangle \qquad (4)$$

$$\frac{d\Delta(t)}{dt} = \frac{1}{4}\,e^t\,\langle\Theta(t)\mathcal{O}(0)\rangle\ . \qquad (5)$$

The r.h.s. of these equations involve the two-point correlation functions of the trace of the energy-momentum tensor Θ and an operator $\mathcal{O}$, which is a primary field in the sense of [11]. In general these equations are integrated from $t=-\infty$ to $t=\infty$ and one consequently compares the difference between the ultraviolet and the infrared fixed points. In order to exhibit the quantitative onset of the mass scale of the unstable particles we integrate these equations instead from some finite value t_0 to infinity. Restricting our attention to purely massive theories we use the fact that for those theories the infrared central charges are zero, such that

$$c(r_0)=\frac{3}{2}\int_{r_0}^{\infty} dr\, r^3 \ \langle \Theta(r)\Theta(0)\rangle \ . \tag{6}$$

Instead of the integral representation (6), the c-function is equivalently expressible in terms of a sum of correlators involving also other components of the energy momentum tensor [2]. In deriving (4) these terms have been eliminate by means of the conservation law of the energy momentum tensor. We find (6) most convenient. The flow of $c(r_0)$ will surpass various steps: Starting with $r_0=0$ the theory will leave its ultraviolet fixed point and at a certain definite value, say $r_0=r_u$, the unstable particle will become massive such that $c(r_0>r_u)$ can be associated to a different conformal field theory. It appears natural to identify the mass $M_{\tilde{c}}$ as the point at which $c(r_0)$ is half the difference between the two coset values of c. As a consequence of the Breit-Wigner formula we may relate the masses of the unstable particles at different values of the resonance parameter σ, σ' and expect $M_{\tilde{c}}(r_u,\sigma)=M_{\tilde{c}}(r_u',\sigma')$. We will employ the latter equality evaluated in the form (3) not only as a consistency requirement, but also as a confirmation of the fact that the renormalization group flow is indeed achieved by $m\to r_0\, m$. Increasing r_0 further, the energy scale of the stable particles will eventually be reached at, say at $r_0=r_a,r_b,\ldots,r_n$. Depending on the relative mass scales between the stable particles these points may coincide. Finally the flow will reach its infrared fixed point $c(r_0=r_{ir})=0$.

Likewise we can integrate equation (5)

$$\Delta(r_0)=-\frac{1}{2\,\langle \mathcal{O}(0)\rangle}\int_{r_0}^{\infty} dr\, r \ \langle \Theta(r)\mathcal{O}(0)\rangle \ , \tag{7}$$

which allows to keep track of the manner the operator content of the various conformal field theories are mapped into each other. We used that all conformal dimensions vanish in the infrared limit. Fortunately, we have $\langle\Theta(r)\mathcal{O}(0)\rangle\sim\langle\mathcal{O}(0)\rangle$ in many applications such that the vacuum expectation value $\langle\mathcal{O}(0)\rangle$ cancels often. One should note, however, that (7) is only applicable to those operators for which its two-point correlator with the trace of the energy momentum tensor is non-vanishing, such that one may not be in the position to investigate the flow of the entire operator content by means of (7).

In order to evaluate (6) and (7) we have to compute the two-point correlation functions in some way. In 2D-QFT this is probably most efficiently achieved, by expanding them in terms of n-particle form factors, i.e. the matrix elements of some local operator $\mathcal{O}(\vec{x})$ located at the origin between a multiparticle in-state and the vacuum denoted by $\langle 0|\mathcal{O}(0)|V_{\mu_1}(\theta_1)V_{\mu_2}(\theta_2)\ldots V_{\mu_n}(\theta_n)\rangle_{\text{in}}=: F_n^{\mathcal{O}|\mu_1\ldots\mu_n}(\theta_1,\ldots,\theta_n)$. Here the $V_\mu(\theta)$ are some vertex operators representing a particle of species μ. Abbreviating the sum of the on-shell energies as $E=\sum_{i=1}^n m_{\mu_i}\cosh\theta_i$, one may write

$$\langle \mathcal{O}(r)\mathcal{O}'(0)\rangle=\sum_{n=1}^{\infty}\sum_{\mu_1\ldots\mu_n}\int_{-\infty}^{\infty}\frac{d\theta_1\ldots d\theta_n}{n!(2\pi)^n}e^{-r\,E} \tag{8}$$
$$\times\ F_n^{\mathcal{O}|\mu_1\ldots\mu_n}(\theta_1,\ldots,\theta_n)\left(F_n^{\mathcal{O}'|\mu_1\ldots\mu_n}(\theta_1,\ldots,\theta_n)\right)^* .$$

Using this expansion we replace the correlation functions in the expression of the c-function $c(r_0)$ and the scaled conformal dimension $\Delta(r_0)$ and perform the r integrations thereafter. Thus we obtain

$$c(r_0)=3\sum_{n=1}^{\infty}\sum_{\mu_1\ldots\mu_n}\int_{-\infty}^{\infty}\frac{d\theta_1\ldots d\theta_n}{n!(2\pi)^n}e^{-r\,E} \tag{9}$$
$$\times\left|F_n^{\Theta|\mu_1\ldots\mu_n}(\theta_1,\ldots,\theta_n)\right|^2\frac{(6+6r_0E+3r_0^2E^2+r_0^3E^3)}{2E^4}$$

and

$$\Delta(r_0)=-\sum_{n=1}^{\infty}\sum_{\mu_1\ldots\mu_n}\int_{-\infty}^{\infty}\frac{d\theta_1\ldots d\theta_n}{n!(2\pi)^n}\frac{(1+r_0E)e^{-r\,E}}{2E^2} \tag{10}$$
$$\times F_n^{\Theta|\mu_1\ldots\mu_n}(\theta_1,\ldots,\theta_n)\left(F_n^{\mathcal{O}|\mu_1\ldots\mu_n}(\theta_1,\ldots,\theta_n)\right)^* .$$

We will now analyze (9), (10) and (3) for the $SU(3)_2$-HSG model. This model contains only two self-conjugate solitons which we denote by "+", "−" and one unstable particle, which call $\tilde{u}$. The corresponding scattering matrix was found [13] to be $S_{\pm\pm}=-1$, $S_{\pm\mp}(\theta)=\pm\tanh\left(\theta\pm\sigma-i\pi/2\right)/2$, which means the resonance pole is situated at $\theta_R=\mp\sigma-i\pi/2$. Stable bound states may not be formed.

A huge class of form factors corresponding to various operators related to this model were constructed in [12,9]. Labelling an operator by four quantum numbers μ,ν,τ,τ' the general n-particle solution reads

$$F_{2s+\tau,2t+\tau'}^{\mathcal{O}^{\mu,\nu}_{\tau,\tau'}|M^+M^-}(\theta_1,\ldots,\theta_n)=H_{2s+\tau,2t+\tau'}^{\mathcal{O}^{\mu,\nu}_{\tau,\tau'}|M^+M^-}\det\mathcal{A}^{\mu,\nu}_{2s+\tau,2t+\tau'}$$
$$\left(\sigma^+_{2s+\tau}\right)^{s-t+\frac{\tau-1-\nu}{2}}\left(\sigma^-_{2t+\tau'}\right)^{\frac{1+\tau-\tau'-\mu}{2}-t}\prod_{i<j}\hat{F}^{\mu_i\mu_j}(\theta_{ij})\,. \tag{11}$$

We used here a particular ordering by starting with $2s+\tau$ particles of the type $\mu=+$ followed by $2s+\tau'$ particles of the type $\mu=-$, collected in the sets $M^\pm=\{\pm,\ldots,\pm\}$.

Once these expressions are known, all other form factors related to it by permutations of the particles may be constructed trivially by exploiting Watson's equations [14], see [12,9] for details concerning the HSG-models. The function $\hat{F}^{\mu_i \mu_j}$ for all combinations of the μ's are

$$\hat{F}^{\pm\pm}(\theta) = -i/2 \tanh \frac{\theta}{2} \exp(\mp\theta/2) \tag{12}$$

$$\hat{F}^{\pm\mp}(\theta) = 2^{\frac{1}{4}} e^{\frac{i\pi(1\mp1)\pm\theta}{4} - \frac{G}{\pi} - \int_0^\infty \frac{dt}{t} \frac{\sin^2\left((i\pi-\theta\mp\sigma)\frac{t}{2\pi}\right)}{\sinh t \cosh t/2}}, \tag{13}$$

with $G = 0.91597\ldots$ being the Catalan constant. The $(t+s)\times(t+s)$-matrix

$$\left(\mathcal{A}^{\mu,\nu}_{2s+\tau,2t+\tau'}\right)_{ij} = \begin{cases} \sigma^{+}_{2(j-i)+\mu}, & 1 \leq i \leq t \\ \hat{\sigma}^{-}_{2(j-i)+2t+\nu}, & t < i \leq s+t \end{cases} \tag{14}$$

has as its entries elementary symmetric polynomials (see e.g. [15] for properties) depending on different sets of variables. We use the notation $\sigma^{\pm}$ when they depend on the variable $x = \exp\theta$ associated to the sets $M^{\pm}$ and $\hat{\sigma}$ to indicate that all variables are multiplied by a factor $ie^{-\sigma}$. The overall constant was computed to

$$H^{\mathcal{O}^{\mu,\nu}_{\tau,\tau'}|M^+M^-}_{2s+\tau,2t+\tau'} = i^{s(2\tau+\tau'+\nu+2)} 2^{s(2s-2t-\tau'-1+2\tau)} \times e^{s\sigma(2t+\tau')/2} H^{\mathcal{O}^{\mu,\nu}_{\tau,\tau'}}_{\tau,2t+\tau'}, \tag{15}$$

where the value of $H^{\mathcal{O}^{\mu,\nu}_{\tau,\tau'}}_{\tau,2t+\tau'}$ is fixed by the lowest non-vanishing form factor. In particular we need

$$F^{\Theta}_{2s,2t} = \sigma_1(x_1,\ldots,x_n)\sigma_1(x_1^{-1},\ldots,x_n^{-1}) F^{\mathcal{O}^{1,1}_{2,2}}_{2s,2t}. \tag{16}$$

Having assembled all the ingredients we can evaluate the expressions (9) and (10). We carry out the integrals by means of a Monte Carlo computation. For $c(r_0)$ we take contributions up to the 4-particle form factor into account and display our results in figure 1.

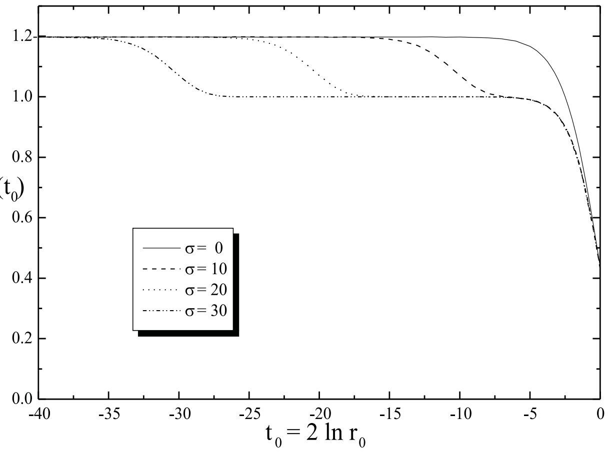

Figure 1: Renormalization group flow for the Virasoro central charge $c(r_0)$ for various values of the resonance parameter σ.

Following the renormalization group flow from the ultraviolet to the infrared, figure 1 illustrates the flow from the $SU(3)_2/U(1)^2$- to the $SU(2)_2/U(1)\otimes SU(2)_2/U(1)$-coset when the unstable particles become massive. This confirms qualitatively the previous observation from the TBA analysis [7]. Here we also want to compare the value of the unstable mass at different points of the resonance parameter σ and t_0. Since the flow between the two coset is smooth and takes places over some range of t_0, we have to select one particular point t_u. As already indicated in general, it appears natural to identify $M_{\tilde{u}}$ as the point at which $c(t_0)$ is half the difference between the two coset values of c. It is clear from figure 1, that since the overall shape of the curves between two values of c is identical for different values of σ, any other value in the interval would lead to the same results in comparative considerations. Taking now the mass scales of the stable particle to be the same, i.e. $m_+ = m_- = m$, we compute the mass of the unstable particle in this manner (3) and find

$$M_{\tilde{u}}(t_u = -30.8, \sigma = 30) = M_{\tilde{u}}(t_u = -20.8, \sigma = 20) = M_{\tilde{u}}(t_u = -10.8, \sigma = 10) = 0.67m. \tag{17}$$

Equations (17) confirm our general assertions outlined above.

For the evaluation of the scaled conformal dimension (10) we proceed similarly. For the solutions corresponding to the operators $\mathcal{O}^{0,0}_{0,0}$, $\mathcal{O}^{0,1}_{0,2}$ and $\mathcal{O}^{1,0}_{2,0}$ we take up to the 6-particle form factors into account. For the former two operators our results are presented in figure 2 and 3.

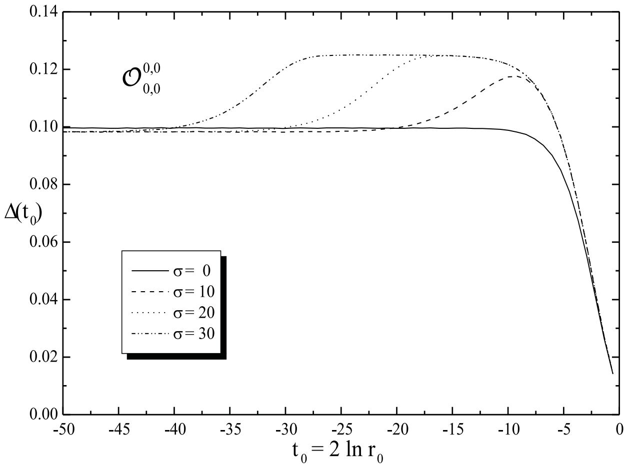

Figure 2: Renormalization group flow for the conformal dimension $\Delta(r_0)$ of the operator $\mathcal{O}^{0,0}_{0,0}$ for various values of the resonance parameter σ.

We observe that the conformal dimension of the operator $\mathcal{O}^{0,0}_{0,0}$ flows to the value 1/8, which is twice the conformal dimension of the disorder operator μ in the Ising model. The factor 2 is expected from the mentioned coset structure, i.e. we find two copies of $SU(2)_2/U(1)$. The nature of the operator is also anticipated, since by construction $F_n^{\mathcal{O}^{0,0}_{0,0}|M^+M^-}$ of the $SU(3)_2$-HSG model coincides precisely with F_n^{μ} of the thermally perturbed Ising model when one of the sets $M^{\pm}$ is empty. It is also clear that we could alternatively obtain (17) from the analysis of $\Delta(r_0)$.

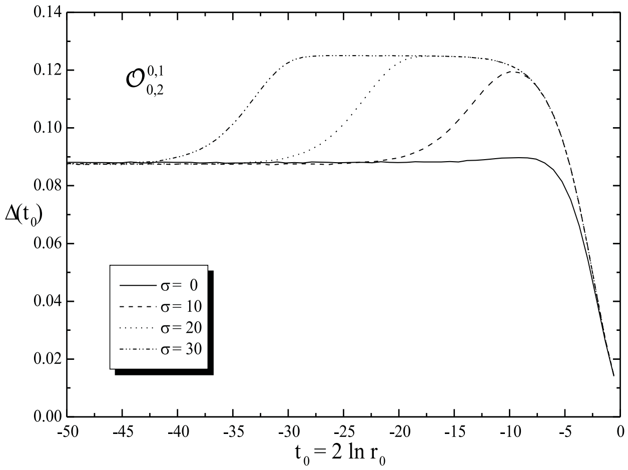

Figure 3: Renormalization group flow for the conformal dimension $\Delta(r_0)$ of the operator $\mathcal{O}^{0,1}_{0,2}$ for various values of the resonance parameter σ.

Despite the fact that the explicit expressions for the form factors of $\mathcal{O}^{0,1}_{0,2}$ and $\mathcal{O}^{1,0}_{2,0}$ differ the values of $\Delta(r_0)$ are hardly distinguishable and we therefore omit the plots for the latter case. We also note the previously observed fact [9], that the higher particle contributions for the latter operators are more important than for $\mathcal{O}^{0,0}_{0,0}$, which explains the fact that the starting point at ultraviolet fixed point is not quite 0.1. The operators also flow to the value 1/8, such that the degeneracy of the $SU(3)_2$-HSG model disappears surjectively when the unstable particles become massive.

In comparison with other methods it would be extremely desirable to elaborate on the precise relationship between $c(r_0)$ and the finite size scaling function of the thermodynamic Bethe ansatz. Also the relation to the intriguing proposal in [16] of a renormalization group flow between Virasoro characters remains unclarified. The analogue of $\Delta(r_0)$ still needs to be identified in the TBA as well as in the context of [16]. In addition one may pose the question whether there exist higher dimensional counterparts of the function $\Delta(r_0)$ in analogy to the results obtained in [3] for $c(r_0)$. Concerning the specific status of the HSG-models it remains a challenge to extend the results to other Lie groups [17].

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