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The problem of integrable discretization: Hamiltonian approach

A skeleton of the book

Preface

This book is devoted to a discipline which is growing quickly on the frontier between the Numerical Analysis and the Theory of Integrable Dynamical Systems. It deals with the following general problem: *how to discretize one or several of independent variables in a given integrable system, maintaining the integrability property?*

The Theory of Integrable Systems, or Theory of Solitons was born thirty three years ago, in the celebrated paper Gardner, Greene, Kruskal and Miura (1967), and has grown since then so tremendously, that it is difficult to keep an overview even of the whole variety of different notions and contexts of integrability, not to say about the concrete results. (Recall that recently there appeared a thick book under the title "What is integrability?" intended to provide an overview of the different definitions and aspects of integrablity, see Zakharov (1991).) The theory is now mature, and there exist several monographs representing the viewpoints of various researchers and research groups. Certainly, anyone working in this area has his own preferred books. My personal favorites are Newell (1985), Faddeev and Takhtajan (1987), Dickey (1991), but this choice is probably more exotic than canonical, and other people might prefere other books, like Calogero and Degasperis (1982) or Ablowitz and Clarkson (1991). The title of the present work is intentionally made similar to the title of the book Faddeev and Takhtajan (1987) which reflected the so called Leningrad view on the Soliton Theory, i.e. the view developed by the Leningrad school around L.D. Faddeev. Nowadays representatives of this school are scattered around the world, so it can hardly be considered as something material. Unfortunately, although coming from Leningrad (now St.Petersburg), I did not formally belong to this school. Nevertheless, I hope that its spirit is definitely reflected in the present book. Anyway, the symbol $\{L \overset{\otimes}{,} L\}$ and the notion of the r–matrix, which are, so to say, trademarks of the Leningrad school, are heavily used in our presentation. More seriously, the Hamiltonian formalism is one of our main working tools here.

The importance of the problem of integrable discretization is obvious for anyone who agrees with the following wisdoms (or, rather, trivial common places):

- Differential equations form an extremely useful and important instrument in the sciences. They constitute the heart (or, if it will be allowed to say so, one of the hearts) of Mathematics and Mathematical Physics.
- In order to extract quantitative information from the models described by differential equations, it is often necessary to solve them numerically with the help of various discretization methods.
- By the investigations of the long–term dynamics, i.e. the features of dynamical systems on very long (potentially infinite) time intervals, their qualitative features become of first rank importance.
- It is therefore crucial to assure that the discretized models exhibit the same qualitative features of the dynamics as their continuous counterparts.

To assure the coincidence of the qualitative properties of discretized models with that of the continuous ones becomes one of the central ideas of the modern Numerical Analysis, which therefore gets into a close interplay with different aspects of the Dynamical Systems theory. One of the most advanced examples of this approach is the theory and practice of Symplectic Integration, which recently achieved the stage of the first monograph, see Sanz-Serna and Calvo (1994).

The problem of integrable discretization constitutes another aspect of the general line of thinking described above. It arose in the course of development of the Solitons Theory, and various approaches to this problem began to be discussed in the soliton literature sporadically starting from the mid–70–es. The following ones should be mentioned:

1. Any integrable system possesses a zero curvature representation, i.e. a representation as a compatibility condition of two auxiliary linear problems. Realization of this led naturally to a proposition to discretize one or both of these linear problems, see Ablowitz and Ladik (1976, 1977). This, however, can be done in a great variety of ways. To reduce the arbitraryness, some additional structures have to be invoked. One attempt to fix a discretization of the zero curvature representation cristallized with the further development of the Hamiltonian approach. Namely, Faddeev and Takhtajan (1987) formulated the following requirement to be fulfilled by the transition from models with one continuous space variable to lattice models: the r–matrix should be preserved, the linear Poisson bracket being replaced by a quadratic one. The last two chapters of Faddeev and Takhtajan (1987) show the productivity of this heuristic approach, nevertheless it can hardly be considered as a firm prescription.

2. One of the most intriguing and universal approaches to integrable systems is the Hirota's bilinear method, cf. Hirota (1973a,b, 1974, 1975, 1977a,b,c, 1978, 1979). It seems to be able to produce discrete versions of the majority of the soliton equations, but it still remains somewhat mysterious, and the mechanism behind is yet to be fully understood. One successful way to do it was proposed by Date, Jimbo and Miwa (1982–3), who also derived a large number of integrable discretizations. One of the most interesting products of this approach is the so called Hirota–Miwa equation, cf. Hirota (1981) which is sometimes claimed to contain "everything". Indeed, the majority if not all soliton equations (continuous and discrete) are particular or limiting cases of this single equation, cf. Zabrodin (1997).

3. A fruitful method is based on the "direct linearization", see Nijhoff, Capel and Wiersma (1984), Quispel *et al.* (1984), Capel, Wiersma and Nijhoff (1986), Wiersma and Capel (1987-8), Nijhoff *et al.* (1992), Nijhoff and Capel (1995). Its basic idea is to derive integrable nonlinear differential and difference equations which are satisfied by the solutions of certain linear integral equations. A large variety of continuous and discrete soliton equations has been obtained in this way. Actually, this approach is closely related to the approaches of Date, Jimbo and Miwa (1982–3), as well as of Bogdanov and Konopelchenko (1995, 1998, 1999).

4. Differential equations describing various geometric problems (surfaces of the constant Gaussian or mean curvature, motion of a curve in a space, etc.) turn out to be integrable, see, e.g., Sym (1985), Bobenko (1991, 1994), Cieslinski, Goldstein and Sym (1995). Correspondingly, a discretization of geometric notions naturally leads to discrete integrable equations. The area of discrete integrable geometry became flourishing in the recent years, see Bobenko and Pinkall (1996a,b, 1999), Pedit and Wu (1995), Doliwa and Santini (1995, 1997, 1999), Cieslinski, Doliwa and Santini (1997), Doliwa (1997c, 1999a,b), Doliwa, Manakov and Santini (1998), Konopelchenko and Schief (1998, 1999), Bobenko (1999a,b), Cieslinski (1999), Akhmetshin, Krichever and Volvovsky (1999), Hertrich-Jeromin, Hoffmann and Pinkall (1999), Bobenko and Schief (1999), Doliwa, Santini and Mañas (2000), Hoffmann (2000a,b).

5. Considering stationary and restricted flows of soliton hierarchies, closely related to the "nonlinearization" of spectral problems, often leads to interesting discrete equations, cf. Quispel, Roberts and Thompson (1989), Ragnisco (1991, 1992), Ragnisco and Rauch-Wojciechowski (1996).

6. Approach based on the variational principle (discrete Lagrangian equations), combined with matrix factorizations, was pushed forward by Veselov (1988, 1991a), Moser and Veselov (1991). A set of examples with similar properties, which also belong to the most beautiful ones, but were derived by guess, without any systematical approach, was given by Suris (1990a,b, 1994b,c,d). Although this approach is heavily based on guesswork, and its applications were up to now rather restricted (see Bobenko and Suris (1999a), Suris (2000a), Penskoi (2000) for recent developments), historically it was the work of Veselov that consolidated the subject of integrable discretizations to a separate branch of the theory of integrable systems.

7. Finally, we mention the method of integrable discretization based on the notion of Bäcklund transformations for r–matrix hierarchies. This approach was pushed forward in the recent papers Suris (1995a, 1996a,b,c, 1997a,b,e,f, 1999a,b,c). Its clear advantage is its universality: it can be, in principle, applied to any system admitting an r–matrix interpretation, which is the common feature of the great majority of the known integrable systems.

All these methods produced a large collection of discrete integrable systems. Mysteriously enough, one and the same equations of this collection appear often in different approaches, which indicates on some general common constructions hidden behind all of them. It would be very important to bring some system into this seemingly unordered collection. This is one of the aims of the present book. It is based on the last of the above mentioned approaches, but includes also results first found along different routes. Several words should be said about the things contained here and those *not* contained.

The genus of this book may be described as a rare mixture of an expository monograph and a handbook. It is intended to be self–contained to a large extent, and exhaustive within its scope. This scope may be formulated as "integrable discretizations of integrable finite–dimensional dynamical systems". So, we deal only with ordinary differential equations, and say *almost nothing* about integrable discretizations of integrable partial differential equations. Further, we *do not* study discretizations of non–autonomous systems like Painlevé equations, the latter topic being a very popular separate branch of the integrable science nowadays. We *do* treat finite dimensional truncations of the so called lattice, or differential–difference systems like Toda lattice. Moreover, their treatment constitutes the largest part of the book. The results can be transferred to infinite lattices without any difficulties. We believe that within this scope almost all known interesting discretizations appear in our text.

Our main concern is the *derivation* of discrete integrable systems. We say practically nothing about *solving* them: we do not discuss such methods of obtaining solutions as inverse scattering method with its numerous variants or algebro–geometric methods: it would require writing another book of a comparable volume (only hints on the solution by means of matrix factorizations are given). However, we always have in mind one of the motivations of integrable discretization, namely a possibility of applying these discretizations for *actual* numerical computations. Therefore we always tried to design difference equations which are easy to implement for such computations. In particular, the *locality* of arising difference schemes was a subject of our special concern. Further, we do not say anything about the *physics* or *applications* of these equations; this book belongs to "pure" mathematics and tries to further uncover mathematical structures behind integrable equations.

The structure of the book is simple. It consists of three parts. Part I "General theory" consists of two Chapters. In the first one we review the important notions and results of the Hamiltonian theory on general Poisson manifolds. The second Chapter contains a compact exposition of the r–matrix theory and a formulation of the general recipe for integrable discretization.

The rest of the book, consisting of Part II "Lattice systems" and Part III "Systems of classical mechanics", contains a detailed account of various classes of integrable finite–dimensional dynamical systems. Each Chapter is devoted to one class consisting of systems of the same genealogy. It starts with an Introduction, where the most important equations studied in this Chapter and their discretizations are collected. This is intended for the reader using the book in the "handbook mode", to save browsing through the main text. The Introductions contain also short overviews of the content of each Chapter. The main text of each Chapter includes elaborating on the Hamiltonian and r–matrix interpretation of the systems under consideration, and their discretization along the lines of our general recipe. Each Chapter ends with Bibliographical remarks, which are intended to allow the reader to trace the origin of the results presented in the text, and to figure out suitable sources for further reading. The Bibliography, though being very voluminous, cannot pretend to be exhaustive. It consists mainly of the literature we explicitly refer to.

Writing this book consisted of two periods, during both of which the author was financially suported by DFG (the Deutsche Forschungsgemeinschaft), which is gratefully acknowledged. During the first period (Bremen, 1996–1997) I received the Habilitationsstipendium of DFG. At this point I want to express my deep and cordial gratitude to H.-O. Peitgen. Not only did he provide me with excellent working conditions in his Institute CeVis at the University of Bremen, but also his personal assistance in all sorts of mathematical and non–mathematical problems cannot be overestimated. The result of the Bremen period was the first version of this text defended as Habilitation at the Mathematics Department there in 1999. During the second period (Berlin, 1998-2000) I participated in the SFB (Sonderforshungsbereich) 288 "Differential geometry and quantum physics", also financed by DFG. I thank U. Pinkall, the Spokesman of SFB, for his patience during the seemingly endless process of writing the final version of this book.

Some results of the book were reported during 1996–2000 at the Workshop on Discrete Geometry and Condensed Matter Physics at the Erwin Schrödinger Institute (Vienna, 1996), at the International Meetings on Symmetry and Integrability of Difference Equations (SIDE II in Canterbury, 1996, SIDE III in Sabaudia near Rome, 1998, and SIDE IV in Tokyo, 2000), at the Colloquium on Discrete Systems in the framework of the Integrable Systems Semester at the Institute Henri Poincaré (Paris, 1996), at the Workshop on Nonlinear Systems, Solitons and Geometry (Oberwolfach, 1998), at the International Workshop on Nonlinear Evolutionary Equations and Dynamical Systems (NEEDS'99 in Kolymbari, Crete), at the International Conference on New Applications of Multisymplectic Field Theory (Salamanka, 1999), at the Workshop on Mathematical Methods of Regular Dynamics dedicated to the 150th anniversary of Sophie Kovalevskaya (Leeds, 2000), and at the Simposium dedicated to the 150th anniversary of Sofia Kovalevskaya (St. Petersburg, 2000). I thank the organizers of these events for inviting me there and for the possibility to speak on this topic. I am grateful to all collegues with whom I discussed the relevant problems, for their interest and constructive criticisms. Finally, I am thankful to my co-authors O. Ragnisco and A. Bobenko for the fruitful collaboration in the works which constitute the corresponding parts of this book.

Part I

General theory

Chapter 1

Hamiltonian mechanics

Contents

1.1 Introduction: The problem of integrable discretization

Let us start with a precise formulation of the problem of integrable discretization. The notions used in this formulation are the fundamental notions of Hamiltonian mechanics. The main definitions and results thereof are collected in the subsequent Sections of this Chapter in the form convenient for further presentation.

Let $\mathcal{P}$ be a Poisson manifold with a Poisson bracket $\{\cdot,\cdot\}$. Let H be a Hamilton function of a completely integrable Hamiltonian flow on $\mathcal{P}$:

$$\dot{x} = f(x) = \{H, x\}\ ; \tag{1.1.1}$$

so, it is supposed that this flow possesses many enough functionally independent integrals $I_k(x)$ in involution.

The ***problem of integrable discretization*** consists of finding a family of diffeomorphisms $\mathcal{P} \mapsto \mathcal{P}$,

$$\widetilde{x} = \Phi(x; h)\ , \tag{1.1.2}$$

depending smoothly on a small parameter $h > 0$, and satisfying the following requirements:

1. The maps (1.1.2) *approximate* the flow (1.1.1) in the following sense:
$$\Phi(x; h) = x + hf(x) + O(h^2)\ . \tag{1.1.3}$$
(Of course, one might require also a higher order of approximation). In all our considerations and formulas we pay a special attention to a simple and transparent control of the continuous limit $h \to 0$.

2. The map (1.1.2) is *Poisson* with respect to the bracket $\{\cdot,\cdot\}$ on $\mathcal{P}$ or with respect to some its deformation $\{\cdot,\cdot\}_h$ such that $\{\cdot,\cdot\}_h = \{\cdot,\cdot\} + O(h)$.

3. The map (1.1.2) is *integrable*, i.e. possesses the necessary number of independent integrals in involution $I_k(x; h)$ approximating the integrals of the original system: $I_k(x; h) = I_k(x) + O(h)$.

Remark. Usually in applications the maps (1.1.2) will be described by *implicit equations of motion* of the type

$$\widetilde{x} - x = h\Psi(x, \widetilde{x}; h)\ . \tag{1.1.4}$$

The implicit function theorem assures in general the (local) solvability of such equations, and the approximation requirement above is satisfied, if

$$\Psi(x, x; 0) = f(x)\ . \tag{1.1.5}$$

Already the above formulations indicate that we adopt the viewpoint according to which the notion lying at the basis of Hamiltonian mechanics is that of the *Poisson brackets*. We discuss some general properties of Poisson manifolds and Hamiltonian flows in Sect. 1.2. A particular class of Poisson manifolds enjoying an extraordinary importance constitute *symplectic manifolds*, see Sect. 1.3.

Several further Sections are devoted to producing new Poisson structure out of known ones. The simplest possible situation appears when a given Poisson bracket can be restricted to a *Poisson submanifold*; the relevant notions are introduced in Sect. 1.4. A particular situation, when this cannot be done, but certain *reduction* procedure is applicable, leads to the so called *Dirac bracket*; see Sect. 1.5. Further, in Sect. 1.6 we discuss some general aspects of the *Poisson reduction* related to the notion of *symmetry group*.

The key notion for this book is the concept of *complete integrability* which is based on the so called *Liouville–Arnold theorem* and its various generalizations. This is the subject of Sect. 1.7. An alternative route to integrability is through the so called *bi–Hamiltonian structure* discussed in Sect. 1.8.

One of the most important and natural ways for Hamiltonian systems to appear in apllications is through the variational (Lagrangian) approach. Several aspects thereof, including Lagrangian mechanics on Lie groups and Lagrangian reduction, are worked out in Sect. 1.9–1.13.

We close the Chapter with two appendices containing some notations and background information on smooth manifolds and the Lie groups theory, respectively.

1.14 Bibliographical remarks

1.2–1.6. There exist several comprehensive textbooks covering all the material reviewed in these Sections. Our presentation is based mainly on Abraham and Marsden (1978), Arnold (1989), Marsden and Ratiu (1994). These excellent books treat also the historical aspects of the development of mathematical methods in classical mechanics, and provide a rich bibliography, so we do not feel to have to give here an additional account of this theme, and simply urge the reader to consult the above textbooks.

1.7. The modern version of the integrability concept, known under the name of the Liouville–Arnold integrability, appeared in the first editions of Arnold (1989). The noncommutative integrability concept as treated in Theorem 1.25, is due to Nekhoroshev (1972). There exists also another version of the noncommutative integrability, due to Mishchenko and Fomenko (1987), with stronger hypothesis about the integrals F_k. Namely, it is supposed that they span a Lie algebra g with respect to the Poisson bracket (i.e. $\{F_i, F_j\}$ is a linear combination of F_k's), and $\dim(g) = N + s$, $\mathrm{ind}(g) = N - s$. Then, as in Theorem 1.25, a compact connected component of the common level set of F_k's is diffeomorphic to a torus $\mathbb{T}^{N-s}$.

1.8. The notion of a bi–Hamiltonian system was discovered by Magri (1978). A closely related notion of a recursion operator was introduced a bit earlier by Olver (1977), and then studied by (Fuchssteiner 1979). Other important early papers include Gelfand and Dorfman (1979,1980), Fokas and Fuchssteiner (1980), Fuchssteiner and Fokas (1981). Since then the theory of bi–Hamiltonian became a mature and rich research area which is partly reflected in the monographs Olver (1986), Dorfman (1993), Blaszak (1998). A review of the further developments achieved by the Magri's school can be found in Magri (1995, 1997), Magri, Falqui and Pedroni (2000). Recently deep results on the geometry of bi–Hamiltonian manifolds were found in Gelfand, Zakharevich (1999), Zakharevich (2000).

1.9–1.10. The continuous time Lagrangian mechanics is a classical subject, see, e.g. Abraham and Marsden (1978), Arnold (1989), Marsden and Ratiu (1994). The discrete time Lagrangian mechanics on manifolds was pushed forward by Veselov (1988), see also Moser and Veselov (1991), Marsden and Wendlandt (1997), Wendlandt and Marsden (1997). A generalization to partial differential equations and their variational discretization is given in Marsden, Patrick and Shkoller (1998).

1.11. Specific features of the Lagrangian mechanics on Lie groups in the continuous time case are elaborated in the above mentioned textbooks Abraham and Marsden (1978), Arnold (1989), Marsden and Ratiu (1994), see also Cushman and Bates (1997), who call the formulas (1.11.11), (1.11.12) for the Poisson bracket on the trivialized cotangent bundle of a Lie group "perhaps the basic formula of Hamiltonian mechanics" (p. 405). The case of the discrete time was treated in detail by Bobenko and Suris (1999).

1.12–1.13. For historical remarks on the Lie–Poisson bracket and the Euler–Poincaré equations, which were known already to Lie and Poincaré, but rediscovered independently in the 60-es by Beresin, Kirillov, Souriau, and Kostant, see, e.g., Marsden and Ratiu (1994). The Hamiltonian version of the semi–direct product reduction was developed in the 80-es, see, e.g., Marsden, Ratiu, and Weinstein (1984a,b). The Lagrangian version was, strangely enough, formulated only recently, see Holm, Marsden, and Ratiu (1998), Cendra, Holm, Marsden, and Ratiu (1998). The discrete time conterparts of these constructions were elaborated by Bobenko and Suris (1999a,b), see also Marsden, Pekarsky, and Shkoller (1999).

Chapter 2

R–matrix hierarchies

Contents

2.1 Introduction

Almost all (perhaps, all) known integrable systems possess ***Lax representations***. In the situation of systems described by ordinary differential equations, a Lax representation for a given system means that there exist two maps $L : \mathcal{P} \mapsto \boldsymbol{g}$ and $B : \mathcal{P} \mapsto \boldsymbol{g}$ from the system's phase space $\mathcal{P}$ into some Lie algebra $\boldsymbol{g}$ such that the equations of motion are equivalent to

$$\dot{L} = [L, B] \ . \tag{2.1.1}$$

(In fact, as a rule, there exist several different Lax representations, with different algebras $\boldsymbol{g}$; numerous examples of this circumstance will be given in this book). The matrix L, or, better, the map $L : \mathcal{P} \mapsto \boldsymbol{g}$ is called the ***Lax matrix***, while the matrix B is called the ***auxiliary matrix*** of the Lax representation. The pair (L, B) is called the ***Lax pair*** (and sometimes one uses, somewhat loosely, this term for the equation (2.1.1) itself). Finding a Lax representation for a given system usually implies its integrability, due to the fact that Ad–invariant functions on the Lie algebra $\boldsymbol{g}$ are integrals of motion of the systems of the type (2.1.1), and therefore the values of such functions composed with the map L deliver functions on $\mathcal{P}$ serving as integrals of motion of the original system. Of course, in the Hamiltonian context, there remains something to be done in order to establish the complete integrability, namely, to show that the number of functionally independent integrals thus found is large enough, and that they are in involution. There exists an approach which incorporates an involutivity property in the very construction of Lax equations, namely the ***r–matrix approach***.

It is based on the following important observation (which stems, let us stress it once more, from considering a large number of examples): usually the auxiliary matrix B in (2.1.1) may be presented as

$$B = \mathrm{R}(f(L)) \ , \tag{2.1.2}$$

where $\mathrm{R} : \boldsymbol{g} \mapsto \boldsymbol{g}$ is a linear operator, and $f : \boldsymbol{g} \mapsto \boldsymbol{g}$ is an Ad–covariant function:

$$\mathrm{Ad}\, g \cdot f(L) = f(\mathrm{Ad}\, g \cdot L) \quad \forall g \in \boldsymbol{G} \ ,$$

where $\boldsymbol{G}$ is a Lie group with the Lie algebra $\boldsymbol{g}$. (Notice that gradients $f = \nabla F$ of Ad–*invariant* functions $F : \boldsymbol{g} \mapsto \mathbb{C}$ are the basic examples of Ad–*covariant* functions $f : \boldsymbol{g} \mapsto \boldsymbol{g}$). This observation is related to the fact that integrable systems appear not separately, but are organized in ***hierarchies***. Namely, to every covariant function $f(L)$ there corresponds a Lax equation of the form (2.1.1) with the matrix B given by (2.1.2). We shall call R an ***R–operator governing the corresponding hierarchy***. In some examples the operator R depends on $L = L(P)$, or, what is the same, on the point $P \in \mathcal{P}$ (and then it is called *dynamical*), but in the majority of cases it does not. We will be dealing here almost exclusively with this latter possibility, namely that of *constant R*–operators.

A remarkable feature of equations (2.1.1) is that they often can be included into an abstract framework of Hamiltonian equations on $\boldsymbol{g}$. More precisely, there can be defined such Poisson structures on $\boldsymbol{g}$ that the corresponding Hamiltonian equations have the form (2.1.1), and the map $L : \mathcal{P} \mapsto \boldsymbol{g}$ is Poisson. There exist several variants of the so called ***r–matrix Poisson brackets*** on $\boldsymbol{g}$.

The *linear r–matrix bracket* is studied in Sect. 2.3, 2.4. Actually, its most natural interpretation is as a certain *Lie–Poisson* bracket on $\boldsymbol{g}^*$, where the dual space $\boldsymbol{g}^*$ is identified with $\boldsymbol{g}$ by means of an invariant scalar product. Therefore, we found it useful to give in Sect. 2.2 an account of the Lie–Poisson brackets, independent of the reduction procedure discussed in Sect. 1.12.

The definition and the properties of the *quadratic r–matrix bracket* are given in Sect. 2.5. A convenient framework for these brackets constitute *associative algebras*, so that somewhat more structure than just a Lie commutator is required. Further, Sect. 2.6 contains an account of the quadratic brackets in an important case when the underlying algebra is a direct product $\boldsymbol{g}^{\otimes m}$ of several copies of $\boldsymbol{g}$. The corresponding Lax equations are of the form

$$\dot{L}_j = L_j B_{j-1} - B_j L_j \ , \tag{2.1.3}$$

where the auxiliary matrices B_j are expressed with the help of the R–operator governing the corresponding hierarchy as

$$B_j = \mathrm{R}(f(T_j)) \ , \qquad T_j = L_j \cdot \ldots \cdot L_1 \cdot L_m \cdot \ldots \cdot L_{j+1} \ . \tag{2.1.4}$$

One meets often the equations of the type (2.1.3), called sometimes also ***Lax triads***, in the theory of integrable lattice systems.

One wide class of R–operators plays an especially important role in the theory of integrable systems, namely operators

$$\mathrm{R} = \pi_+ - \pi_- \tag{2.1.5}$$

coming from the splitting $\boldsymbol{g} = \boldsymbol{g}_+ \oplus \boldsymbol{g}_-$ of the Lie algebra $\boldsymbol{g}$ into a direct sum of its two subspaces which are also Lie subalgebras. As we shall see in Sect. 2.7, in this case the Lax equations (2.1.1), (2.1.3) may be formally solved in terms of the factorization problem in the Lie group $\boldsymbol{G}$ related to the multiplicative decomposition $\boldsymbol{G} = \boldsymbol{G}_+ \cdot \boldsymbol{G}_-$, where $\boldsymbol{G}_\pm$ are the subgroups of $\boldsymbol{G}$ with the Lie algebras $\boldsymbol{g}_\pm$.

One can find also *difference equations* whose solutions are found in terms of the factorization problem. As it turns out, these difference equations describe nothing else than iterations of the ***Bäcklund transformations*** having very old and classical origin (Sect. 2.8). This plays the role of the cornerstone in our recipe for integrable discretizations, which is formulated in Sect. 2.9 and is a culmination of the first, "theoretic" part of this book.

2.7 R–operators from splitting $\boldsymbol{g} = \boldsymbol{g}_+ \oplus \boldsymbol{g}_-$

We have seen in the previous Sections that different versions of the r–matrix construction allows to give a Hamiltonian or even a bi–Hamiltonian interpretation to Lax equations of the form

$$\dot{L} = \frac{1}{2}\Big[L, \mathrm{R}(f(L))\Big] , \tag{2.7.1}$$

where linear operator $\mathrm{R} : \boldsymbol{g} \mapsto \boldsymbol{g}$ satisfies the modified Yang–Baxter equation. A very important class of such R–operators is given by the following construction. Let $\boldsymbol{g}$ be a Lie algebra of some associative algebra, equipped with a non–degenerate bi–invariant scalar product. Let $\boldsymbol{g}$, as a linear space, be a direct sum of its two subalgebras:

$$\boldsymbol{g} = \boldsymbol{g}_+ \oplus \boldsymbol{g}_- , \tag{2.7.2}$$

and let π_+, π_- denote the projections from $\boldsymbol{g}$ to the corresponding subspaces:

$$L = \pi_+(L) + \pi_-(L) , \quad \pi_\pm(L) \in \boldsymbol{g}_\pm , \quad \forall L \in \boldsymbol{g} . \tag{2.7.3}$$

Then it is easy to check that the operator

$$\mathrm{R} = \pi_+ - \pi_- \tag{2.7.4}$$

satisfies the modified Yang–Baxter equation (2.3.6) with $\alpha = 1$, which allows to define a linear bracket (2.3.2) on $\boldsymbol{g}$, and, under additional conditions, a set of its quadratizations (2.5.2). Notice that all operators considered as Examples in Sect. 2.3 are of the sort (2.7.4).

The corresponding Hamiltonian equations on $\boldsymbol{g}$ with Ad–invariant Hamilton functions have the form (2.7.1), which in the present case may be recast in two equivalent forms:

$$\dot{L} = \Big[L, \pi_+(f(L))\Big] = -\Big[L, \pi_-(f(L))\Big] . \tag{2.7.5}$$

Such a general setting is enough to assure that the equation (2.7.5) posesses several remarkable properties. First of all, different flows of the type (2.7.5) commute. Second, they admit an explicit solution in terms of a factorization problem in a Lie group. Both these properties have a purely kinematical nature and do not depend on the Hamiltonian theory (though the latter provides a deeper insight into the situation). For completeness, we give here kinematic proofs of these results.

Theorem 2.23 *Let $H_1, H_2 \in \mathcal{F}(\boldsymbol{g})$ be two* Ad*–invariant functions, so that the functions $f_1 = \nabla H_1, f_2 = \nabla H_2 : \boldsymbol{g} \mapsto \boldsymbol{g}$ are* Ad*–covariant. Then the two flows on $\boldsymbol{g}$ generated by the differential equations*

$$L_t = \Big[L, \pi_+(f_1(L))\Big] \quad \textit{and} \quad L_s = \Big[L, \pi_+(f_2(L))\Big] \tag{2.7.6}$$

commute.

Proof. ... ■

So, we proved the commutativity of different flows of the type (2.7.5) without any reference to the Hamiltonian formalism. However, the linear (say) r–matrix construction would also imply this statement, because such flows are, as we know, generated by Ad–invariant Hamilton functions, which are in involution.

We turn now to the explicit solution of the flows (2.7.5). Let $\boldsymbol{G}$ be a Lie group with the Lie algebra $\boldsymbol{g}$, and let $\boldsymbol{G}_+$ and $\boldsymbol{G}_-$ be its two subgroups having $\boldsymbol{g}_+$ and $\boldsymbol{g}_-$, respectively, as Lie algebras. Then in a certain component Ω of the group unit I following factorization is uniquely defined:

$$g = \Pi_+(g)\,\Pi_-(g)\;, \quad \Pi_\pm(g) \in \boldsymbol{G}_\pm\;, \quad \forall g \in \Omega \subset \boldsymbol{G}\;. \tag{2.7.7}$$

In what follows we suppose, for the sake of notational simplicity, that $\boldsymbol{G}$ is a matrix group, and write the adjoint action of the group on $\boldsymbol{g}$ as a conjugation by the corresponding matrices. [In this context we write $\Pi_\pm^{-1}(g)$ for $(\Pi_\pm(g))^{-1}$]. Correspondingly, we shall call Ad–covariant functions $f : \boldsymbol{g} \mapsto \boldsymbol{g}$ also "conjugation covariant". This notation has an additional advantage of being applicable also to functions $F : \boldsymbol{g} \mapsto \boldsymbol{G}$. Namely, we shall call such a function conjugation covariant, if $F(\mathrm{Ad}\,g \cdot L) = gF(L)g^{-1}$.

Theorem 2.24 *Let $f : \boldsymbol{g} \mapsto \boldsymbol{g}$ be a conjugation covariant function. Then the solution of the differential equation* (2.7.5) *with the initial condition $L(0) = L_0$ is given, at least for t small enough, by*

$$L(t) = \Pi_+^{-1}\Big(e^{tf(L_0)}\Big)\cdot L_0 \cdot \Pi_+\Big(e^{tf(L_0)}\Big) = \Pi_-\Big(e^{tf(L_0)}\Big)\cdot L_0 \cdot \Pi_-^{-1}\Big(e^{tf(L_0)}\Big)\;. \tag{2.7.8}$$

Proof. Denote

$$\mathcal{L}(t) = \Pi_+\Big(e^{tf(L_0)}\Big)\,, \qquad \mathcal{R}(t) = \Pi_-\Big(e^{tf(L_0)}\Big)\,,$$

so that

$$e^{tf(L_0)} = \mathcal{L}(t)\,\mathcal{R}(t)\;, \qquad \mathcal{L}(t) \in \boldsymbol{G}_+\;, \quad \mathcal{R}(t) \in \boldsymbol{G}_-\;. \tag{2.7.9}$$

Now set

$$L(t) = \mathcal{L}^{-1}(t)\,L_0\,\mathcal{L}(t) = \mathcal{R}(t)\,L_0\,\mathcal{R}^{-1}(t)\;. \tag{2.7.10}$$

(These two expressions for $L(t)$ are equal due to Ad–covariance of $f(L)$). We now check by direct calculation that this $L(t)$ satisfies the differential equation (2.7.5). The Theorem will follow by the uniqueness of solution. We see immediately that $L(t)$ satisfies the following Lax type equation:

$$\dot{L} = [L, \mathcal{L}^{-1}\dot{\mathcal{L}}] = -[L, \dot{\mathcal{R}}\,\mathcal{R}^{-1}]\;,$$

and it remains to show that

$$\mathcal{L}^{-1}\dot{\mathcal{L}} = \pi_+(f(L))\;, \qquad \dot{\mathcal{R}}\,\mathcal{R}^{-1} = \pi_-(f(L))\;.$$

Since, obviously,

$$\mathcal{L}^{-1}\dot{\mathcal{L}} \in \boldsymbol{g}_+\;, \qquad \dot{\mathcal{R}}\,\mathcal{R}^{-1} \in \boldsymbol{g}_-\;,$$

we need to demonstrate only that

$$\mathcal{L}^{-1}\dot{\mathcal{L}} + \dot{\mathcal{R}}\,\mathcal{R}^{-1} = f(L)\;. \tag{2.7.11}$$

To do this, we differentiate (2.7.9) and derive, using Ad–covariance of f and the definition (2.7.10):

$$\dot{\mathcal{L}}\,\mathcal{R} + \mathcal{L}\,\dot{\mathcal{R}} = e^{tf(L_0)}f(L_0) = \mathcal{L}\,\mathcal{R}f(L_0) = \mathcal{L}\,f(L)\,\mathcal{R}\;.$$

This is equivalent to (2.7.11). ■

Remark. Although we have seen that that the equation (2.7.5) admits a Hamiltonian (often even a bi–Hamiltonian) interpretation, these structures are not needed to formulate or to prove the above Theorem. In this sense it is a purely kinematic result. However, the Hamiltonian property provides us with some useful additional information. In particular, if the set of Lax matrices $L(\mathcal{P})$ for the system at hand forms a Poisson submanifold for the linear bracket (2.3.2) or/and for some of its quadratizations (2.5.2), then this manifold is invariant under the flow (2.7.5). The functions on $\mathcal{P}$ of the form $F \circ T$, where F are conjugation invariants of $\boldsymbol{g}$, are integrals of motion of the corresponding system and in involution with respect to $\{\cdot, \cdot\}_{\mathcal{P}}$.

We close this Section by giving an analogous result for the Lax equations generated by the operator (2.7.4) on the direct product $\mathbf{g} = \bigotimes_{j=1}^{m} \boldsymbol{g}$. Its proof is kinematic and absolutely parallel to the case of the "small" algebra $\boldsymbol{g}$.

Theorem 2.25 *For a conjugation covariant function $f : \boldsymbol{g} \mapsto \boldsymbol{g}$ consider the following system of differential equations on $\mathbf{g}$:*

$$\begin{aligned} \dot{L}_j &= L_j \cdot \pi_+\Big(f(T_{j-1})\Big) - \pi_+\Big(f(T_j)\Big) \cdot L_j \\ &= -L_j \cdot \pi_-\Big(f(T_{j-1})\Big) + \pi_-\Big(f(T_j)\Big) \cdot L_j \,, \end{aligned} \tag{2.7.12}$$

where $T_j = L_j \cdot \ldots \cdot L_1 \cdot L_m \cdot \ldots \cdot L_{j+1}$. Its solution with the initial value $\boldsymbol{L}(0)$ is given, at least for t small enough, by the formula

$$\begin{aligned} L_j(t) &= \Pi_+^{-1}\Big(e^{tf(T_j(0))}\Big) \cdot L_j(0) \cdot \Pi_+\Big(e^{tf(T_{j-1}(0))}\Big) \\ &= \Pi_-\Big(e^{tf(T_j(0))}\Big) \cdot L_j(0) \cdot \Pi_-^{-1}\Big(e^{tf(T_{j-1}(0))}\Big) \,. \end{aligned} \tag{2.7.13}$$

2.8 Bäcklund transformations

Theorem 2.24 paves another way to the kinematic demonstration of the commutativity of the different flows (2.7.5). Towards this aim, one needs additionally the following fundamental definition.

Definition 2.26 *Consider the hierarchy of flows* (2.7.5) *on $\boldsymbol{g}$. For an arbitrary conjugation covariant function $F : \boldsymbol{g} \mapsto \boldsymbol{G}$ define the* ***Bäcklund transformation*** $\mathrm{BT}_F : \boldsymbol{g} \mapsto \boldsymbol{g}$ *of this hierarchy as*

$$\widetilde{L} = \mathrm{BT}_F(L) = \Pi_+^{-1}\left(F(L)\right) \cdot L \cdot \Pi_+\left(F(L)\right) = \Pi_-\left(F(L)\right) \cdot L \cdot \Pi_-^{-1}\left(F(L)\right) \,. \tag{2.8.1}$$

One has, of course, to convince himself that two expressions in (2.8.1) are indeed equal to one another. This is equivalent to

$$\mathrm{Ad}\, F(L) \cdot L = F(L) \cdot L \cdot F(L)^{-1} = L \,, \tag{2.8.2}$$

and follows from the conjugation covariance for F.

One of the most important properties of Bäcklund transformations, implying also other ones, is contained in the following theorem.

Theorem 2.27 *For two arbitrary conjugation covariant functions $F_1, F_2 : \boldsymbol{g} \mapsto \boldsymbol{G}$*

$$\mathrm{BT}_{F_2} \circ \mathrm{BT}_{F_1} = \mathrm{BT}_{F_2 F_1} \,, \tag{2.8.3}$$

so that the Bäcklund transformations BT_{F_1}, BT_{F_2} commute.

Proof. Denote:

$$L_1 = \mathrm{BT}_{F_1}(L) \,, \qquad \widetilde{L} = \mathrm{BT}_{F_2} \circ \mathrm{BT}_{F_1}(L) = \mathrm{BT}_{F_2}(L_1) \,.$$

So, by definition we have:

$$L_1 = \mathcal{L}_1^{-1} L \mathcal{L}_1 = \mathcal{R}_1 L \mathcal{R}_1^{-1} \,, \qquad \widetilde{L} = \mathcal{L}_2^{-1} L_1 \mathcal{L}_2 = \mathcal{R}_2 L_1 \mathcal{R}_2^{-1} \,, \tag{2.8.4}$$

where the matrices $\mathcal{L}_i \in \boldsymbol{G}_+$, $\mathcal{R}_i \in \boldsymbol{G}_-$ $(i = 1, 2)$ come from the following factorizations:

$$F_1(L) = \mathcal{L}_1 \mathcal{R}_1 \,, \qquad F_2(L_1) = \mathcal{L}_2 \mathcal{R}_2 \,.$$

From (2.8.4) we have:

$$\widetilde{L} = \mathcal{L}^{-1} L \mathcal{L} = \mathcal{R} L \mathcal{R}^{-1} \,,$$

where

$$\mathcal{L} = \mathcal{L}_1 \mathcal{L}_2 \in \boldsymbol{G}_+ \,, \qquad \mathcal{R} = \mathcal{R}_2 \mathcal{R}_1 \in \boldsymbol{G}_- \,, \tag{2.8.5}$$

and it remains to prove that

$$\mathcal{L} = \Pi_+\Big(F_2(L)F_1(L)\Big)\,, \qquad \mathcal{R} = \Pi_-\Big(F_2(L)F_1(L)\Big)\,.$$

In view of (2.8.5) it is enough to demonstrate that

$$\mathcal{L}\mathcal{R} = F_2(L)F_1(L)\,.$$

To show this equality, we notice:

$$\mathcal{L}\mathcal{R} = \mathcal{L}_1F_2(L_1)\mathcal{R}_1 = \mathcal{L}_1F_2(\mathcal{L}_1^{-1}L\mathcal{L}_1)\mathcal{R}_1 = F_2(L)\mathcal{L}_1\mathcal{R}_1 = F_2(L)F_1(L)\,,$$

and (2.8.3) is proved.

To demonstrate the last statement of the Theorem, we prove that for two arbitrary conjugation covariant functions $F_1, F_2 : \boldsymbol{g} \mapsto \boldsymbol{G}$ one has:

$$F_2(L)F_1(L) = F_1(L)F_2(L)\,. \tag{2.8.6}$$

Indeed, due to (2.8.2), we have:

$$F_2(L)F_1(L)F_2(L)^{-1} = F_1(\mathrm{Ad}\,F_2(L)\cdot L) = F_1(L)\,.$$

The Theorem is proved. ■

Corollary. *An arbitrary Bäcklund transformation commutes with an arbitrary flow of the hierarchy* (2.7.5)*. In other words, any Bäcklund transformation maps solutions of* (2.7.5) *onto solutions.*

Indeed, according to Theorem 2.24 any flow governed by a differential equation (2.7.5) consists of Bäcklund transformations BT_F with $F(L) = e^{tf(L)}$. As a further consequence, we get a new proof of the fact that any two flows of the type (2.7.5) commute. As another important consequence of Theorem 2.27 we have the following statement.

Theorem 2.28 *Let $F : \boldsymbol{g} \mapsto \boldsymbol{G}$ be a conjugation covariant function. Consider the formula* (2.8.1) *for the Bäcklund transformation* BT_F *as the difference equation*

$$\widetilde{L} = \Pi_+^{-1}(F(L))\cdot L\cdot \Pi_+(F(L)) = \Pi_-(F(L))\cdot L\cdot \Pi_-^{-1}(F(L)) \tag{2.8.7}$$

for $L = L(n)$, $\widetilde{L} = L(n+1)$, with the initial condition $L(0) = L_0$. Then the solution of this difference equation is given by

$$L(n) = \Pi_+^{-1}(F^n(L_0))\cdot L_0\cdot \Pi_+(F^n(L_0)) = \Pi_-(F^n(L_0))\cdot L_0\cdot \Pi_-^{-1}(F^n(L_0))\,. \tag{2.8.8}$$

Proof. By induction from Theorem 2.27,

$$\mathrm{BT}_{F_n}\circ\ldots\circ\mathrm{BT}_{F_2}\circ\mathrm{BT}_{F_1} = \mathrm{BT}_{F_n\ldots F_2F_1}\,.$$

In particular, for $F_1 = F_2 = \ldots = F_n = F$ we obtain:

$$(\mathrm{BT}_F)^n = \mathrm{BT}_{F^n}\,,$$

which is the statement of the Theorem. ■

This Theorem gives a discrete–time counterpart of Theorem 2.24. It is purely kinematic, as well. However, the following additional remarks are in order. Comparing the formulas (2.8.8), (2.7.8), we see that the map (2.8.7) is interpolated by the flow (2.7.5) with the time step h, if

$$e^{hf(L)} = F(L) \quad \Leftrightarrow \quad f(L) = h^{-1}\log(F(L))\,.$$

This interpolating flow is Hamiltonian with respect to the linear bracket (2.3.2) and for its quadratizations (2.5.2), with the Hamilton function $H_1(L)$ or $H_2(L)$, respectively, where

$$\nabla H_1(L) = h^{-1}\log(F(L)) \quad \text{and} \quad dH_2(L) = h^{-1}\log(F(L))$$

(the existence of such H_1, H_2 is obvious, if $F(L)$ is an analytic function on matrices, which is usually the case in applications).

In particular, if the set of Lax matrices $L(\mathcal{P})$ for the system at hand forms a Poisson submanifold for the linear bracket (2.3.2) or/and for some of its quadratizations (2.5.2), then the manifold $L(\mathcal{P})$ is automatically invariant under the map (2.8.7), and, moreover, this map is Poisson on $L(\mathcal{P})$ with respect to the corresponding bracket.

We close this Section by formulating the analogs of Definition 2.26 and Theorem 2.28 for the case of direct product algebras $\mathbf{g} = \boldsymbol{g}^{\otimes m}$.

Definition 2.29 *Consider the hierarchy of flows* (2.7.12) *on* $\mathbf{g}$. *For an arbitrary conjugation covariant function* $F : \boldsymbol{g} \mapsto \boldsymbol{G}$ *define the **Bäcklund transformation*** $\mathrm{BT}_F : \mathbf{g} \mapsto \mathbf{g}$ *of this hierarchy as*

$$\begin{aligned} \widetilde{L}_j &= \Pi_+^{-1}\left(F(T_j)\right) \cdot L_j \cdot \Pi_+\left(F(T_{j-1})\right) \\ &= \Pi_-\left(F(T_j)\right) \cdot L_j \cdot \Pi_-^{-1}\left(F(T_{j-1})\right) . \end{aligned} \tag{2.8.9}$$

Theorem 2.30 *For a conjugation covariant function* $F : \boldsymbol{g} \mapsto \boldsymbol{G}$, *consider the formulas* (2.8.9) *for the Bäcklund transformation as a system of difference equations on* $\mathbf{g}$, *for* $\boldsymbol{L} = \boldsymbol{L}(n)$, $\widetilde{\boldsymbol{L}} = \boldsymbol{L}(n+1)$. *Its solution with the initial value* $\boldsymbol{L}(0)$ *is given by the formula*

$$\begin{aligned} L_j(n) &= \Pi_+^{-1}\Big(F^n(T_j(0))\Big) \cdot L_j(0) \cdot \Pi_+\Big(F^n(T_{j-1}(0))\Big) \\ &= \Pi_-\Big(F^n(T_j(0))\Big) \cdot L_j(0) \cdot \Pi_-^{-1}\Big(F^n(T_{j-1}(0))\Big) . \end{aligned} \tag{2.8.10}$$

All properties of Bäcklund transformation mentioned above, hold also in the present situation. In particular, what concerns a Hamiltonian interpretation, one can extract from Theorem 2.22 a large family of quadratic Poisson brackets on $\mathbf{g}$ making possible such an interpretation for the equations (2.7.12). The difference equations (2.8.9) give then maps which are Poisson with respect to these brackets.

2.9 Recipe for integrable discretization

The results of the previous Section suggest the following prescription, which might be roughly formulated as *"discretization is a Bäcklund transformation close to identity"*.

Recipe. *Suppose you are looking for an integrable discretization of an integrable system on* $\Big(\mathcal{P}, \{\cdot,\cdot\}\Big)$ *allowing a Lax representation of the form*

$$\dot{L} = \Big[L, \pi_+(f(L))\Big] = -\Big[L, \pi_-(f(L))\Big] , \tag{2.9.1}$$

where $L : \mathcal{P} \mapsto \boldsymbol{g}$ *is the Lax matrix living in the Lie algebra* $\boldsymbol{g}$; $f : \boldsymbol{g} \mapsto \boldsymbol{g}$ *is a conjugation covariant function; and* $\pi_\pm : \boldsymbol{g} \mapsto \boldsymbol{g}_\pm$ *are projections to two complementary subalgebras of* $\boldsymbol{g}$ *such that* $\boldsymbol{g} = \boldsymbol{g}_+ \oplus \boldsymbol{g}_-$. *Then as a solution of your task you may take the difference equation*

$$\widetilde{L} = \Pi_+^{-1}(F(L)) \cdot L \cdot \Pi_+(F(L)) = \Pi_-(F(L)) \cdot L \cdot \Pi_-^{-1}(F(L)) \tag{2.9.2}$$

with the same Lax matrix L *and some conjugation covariant function* $F : \boldsymbol{g} \mapsto \boldsymbol{G}$ *such that*

$$F(L) = I + hf(L) + o(h) .$$

Here $\boldsymbol{G}$ *is the Lie group with the Lie algebra* $\boldsymbol{g}$; $\boldsymbol{G}_\pm$ *are its two subgroups with the Lie algebras* $\boldsymbol{g}_\pm$, *respectively; and* $\Pi_\pm : \boldsymbol{G} \mapsto \boldsymbol{G}_\pm$ *map* $g \in \boldsymbol{G}$ *to the unique solutions of the following factorization problem:*

$$g = \Pi_+(g)\Pi_-(g) , \quad \Pi_\pm(g) \in \boldsymbol{G}_\pm .$$

Analogously, if your system has a Lax representation

$$\dot{L}_j = L_j \cdot \pi_+\Big(f(T_{j-1})\Big) - \pi_+\Big(f(T_j)\Big) \cdot L_j = -L_j \cdot \pi_-\Big(f(T_{j-1})\Big) + \pi_-\Big(f(T_j)\Big) \cdot L_j \tag{2.9.3}$$

in the algebra $\mathbf{g} = \bigotimes_{j=1}^{m} \boldsymbol{g}$, *with the Lax matrix* $\boldsymbol{L} = (L_1, \ldots, L_m) : \mathcal{P} \mapsto \mathbf{g}$, *then you may take as a solution of your task the difference Lax equation*

$$\widetilde{L}_j \;=\; \Pi_+^{-1}\left(F(T_j)\right) \cdot L_j \cdot \Pi_+\left(F(T_{j-1})\right) \;=\; \Pi_-\left(F(T_j)\right) \cdot L_j \cdot \Pi_-^{-1}\left(F(T_{j-1})\right)\;, \tag{2.9.4}$$

with F *as above.*

Of course, this prescription makes sense only if the factors $\Pi_\pm(F(L))$ [resp. $\Pi_\pm(F(T_j))$] admit more or less explicit expressions, allowing to write down the corresponding difference equations in a more or less closed form. The choice of $F(L)$ is a transcendent problem, which however turns out to be solvable for many of (hopefully, for the majority of or even for all) the known integrable systems. Miraculously, the simplest possible choice $F(L) = I + hf(L)$ works perfectly well for a vast set of examples (when it makes sense, i.e. when $I + hf(L) \in \boldsymbol{G}$).

Let us stress once more the advantages of this approach to the problem of integrable discretization.

- Although we formulate our recipe only for the hierarchies governed by the operators of the form (2.7.4), it is in fact more universal. Almost without changes it may be reformulated for hierarchies governed by any R–operators satisfying the modified Yang–Baxter equation. Hence, it is rather *universal.*

- The discretizations obtained in this way share the Lax matrix and the integrals of motion with their underlying continuous time systems.

- Suppose that the hierarchy of continuous time systems (2.9.1) [resp. (2.9.3)] is Hamiltonian with respect to some Poisson bracket. Then our discretizations have the Poisson property with respect to the same bracket. In particular, if the Lax matrices L [resp. $\boldsymbol{L}$] form a Poisson submanifold for some (linear or quadratic) r–matrix bracket, then this submanifold is left invariant by the corresponding Poisson map (2.9.2) [resp. (2.9.4)].

- The initial value problem for our discrete time equations can be solved in terms of the same factorization in a Lie group as the initial value problem for the continuous time system.

- Interpolating Hamiltonians also belong to the set of granted by–products of this approach.

The following property is not guaranteed by a general theory, but it is confirmed in the following Chapters by a large amount of concrete examples.

- Each time when the continuous time system admits a variational (Lagrangian) formulation, the same holds true for its discretizations obtained by the above approach.

Let us stress that, although our recipe solves the problem of integrable discretization *in principle*, each concrete systeme requires for a separate investigation in order to choose the function $F(L)$ properly and to put the resulting equations of motion in an acceptable form *in local coordinates on* $\mathcal{P}$. This last part is, in principle, non–formalizable. A good guess is often necessary to this end.

2.10 Bibliographical remarks

2.3. The concept of the classical r–matrix, and the formulas like (2.3.15), (2.3.18), including the very notation $\{L \overset{\otimes}{,} L\}$, appeared in the work of Sklyanin (1979, 1980) as a technical device allowing to make manageable difficult calculations of Poisson brackets of various scattering data in integrable models like the Landau–Lifshits and the nonlinear Schrödinger equations. Actually, it was a by–product of the development of the quantum version of the inverse scattering method by the Leningrad school around L.D.Faddeev, see Takhtajan and Faddeev (1979) for an early report on this development.

A Lie–algebraic meaning of the Sklyanin's formulas was clarified and the notion of a linear r–matrix bracket has cristallized in the paper by Semenov-Tian-Shansky (1983). The Theorems 2.4 and 2.6 in this generality also belong to him. Recently, a somewhat different formalization of the notion of R–operator was proposed in Kupershmidt (1999).

Loop algebras were introduced into this context by Reyman and Semenov-Tian-Shansky (1979, 1981), Reyman (1980), and by Adler and van Moerbeke (1980). Further important contributions include Drinfeld and Sokolov (1984), Segal and Wilson (1985).

More historical remarks in this connection can be found in Reyman and Semenov-Tian-Shansky (1994). The monograph Faddeev and Takhtajan (1987) is devoted to a consequent presentation the whole body of the soliton theory from the r–matrix ("Leningrad") viewpoint.

2.4. The constructions of this Section were introduced in Suris (1999d). Related constructions are contained in Reyman, Semenov-Tian-Shansky (1987) and in Fordy, Reyman and Semenov-Tian-Shansky (1989).

2.5. The simplest possible quadratic bracket, known as the Sklyanin bracket, (corresponding to $\mathrm{A}_1 = \mathrm{A}_2 = \mathrm{R}$, $\mathrm{S} = 0$ in our notations), also appeared in the work of Sklyanin (1979, 1980, 1982, 1983). Some related work was: Gelfand and Dickey (1976, 1977), Adler (1979), Gelfand and Dorfman (1979, 1980, 1981, 1982). A formalisation of these construction was achieved in Drinfeld (1983) and Semenov-Tian-Shansky (1983). More precisely, in the work by Drinfeld a version of the Sklyanin bracket applicable to Lie groups rather then to associative algebras was considered, and its remarkable property was underlined: it defines a Lie–Poisson group, i.e. the group multiplication is a Poisson map. A quantization of this notion undertaken in Drinfeld (1985, 1986) led to the notion of a quantum group, which enjoyed an explosion of interest in the following years. Also the Lie–Poisson groups themselves were a subject of deep studies, e.g. Semenov-Tian-Shansky (1985, 1986, 1987, 1992), Lu and Weinstein (1990), Alekseev and Malkin (1994). In particular, in Semenov-Tian-Shansky (1985, 1986) appeared, under the name of a "dual bracket on the Poisson–Lie group", an interesting example of the quadratic bracket with $\mathrm{A}_1 = -\mathrm{A}_2 = \mathrm{R}$, $\mathrm{S} = I - \mathrm{R}$, $\mathrm{S}^* = I + \mathrm{R}$.

Another line of development was connected with generalizations of the Sklyanin bracket on associative algebras. A quadratic bracket with $\mathrm{A}_1 = \mathrm{A}_2$, $\mathrm{S} = \mathrm{S}^*$ was introduced independently in Li and Parmentier (1989) and in Oevel and Ragnisco (1989). The most general bracket (2.5.2), with the properties given in Theorems 2.13, 2.14, and Proposition 2.15, was also discovered several times independently: Freidel and Maillet (1991), Parmentier (1991), Suris (1993b). The main motivation of Freidel and Maillet (1991) was the problem of quantization, the work Parmentier (1991) contains a description of symplectic leaves of the general quadratic bracket à la Semenov-Tian-Shansky (1987), while the main interest in Suris (1993b) was to find an r–matrix interpretation for some concrete systems, like the Toda and the relativistic Toda lattices. For further developments, see Semenov-Tian-Shansky (1992a,b), Semenov-Tian-Shansky and Sevostyanov (1996), Oevel (1996).

2.6. The most general quadratic brackets on $\boldsymbol{g} \otimes \boldsymbol{g}$ were studied in Semenov-Tian-Shansky (1992a,b), where also an important observation was made that the sufficient condition for the Poisson property of $\mathrm{PB}_2(\mathrm{A}_1, \mathrm{A}_2, \mathrm{S})$ is equivalent to the modified Yang–Baxter equation for the operator $\begin{pmatrix} \mathrm{A}_1 & \mathrm{S} \\ \mathrm{S}^* & \mathrm{A}_2 \end{pmatrix}$ on $\boldsymbol{g} \otimes \boldsymbol{g}$ (see Remark 2 from subsect. 2.6.1).

Strangely enough, the most general (fully nonlocal) quadratic brackets on $\boldsymbol{g}^{\otimes m}$ with $m > 2$ remained unconsidered until the paper Suris (1997d), presumably, due to the lack of interesting examples and applications. Theorem 2.22 in this generality was found also in Suris (1997d), and applied there to the Bogoyavlensky lattices (several further important applications of Theorem 2.22 to lattice systems will be given in the subsequent Chapters, where its full generality will be required). This result may be considered as a final link in the chain of generalizations: Semenov-Tian-Shansky (1983), Semenov-Tian-Shansky (1987), Li and Parmentier (1989), Li and Parmentier (1991), Deift and Li (1991), Li (1992), Nijhoff *et al.* (1992), Semenov-Tian-Shansky (1992b), Semenov-Tian-Shansky and Sevostyanov (1996). However, the operators $\mathbf{A}_1, \mathbf{A}_2, \mathbf{S}, \mathbf{S}^*$ for the nonlocal quadratic Poisson structures discussed in these papers always had only few nonvanishing "operator entries", namely $(\mathbf{A}_1)_{jj}$, $(\mathbf{A}_2)_{jj}$, $(\mathbf{S})_{j+1,j}$, and $(\mathbf{S}^*)_{j,j+1}$.

Another example of nonlocal brackets on direct products (however in the Lie groups rather than in the associative algebras framework) is delivered by the so called "polyubles" introduced by Fock and Rosly (1993, 1998). These are Poisson–Lie groups, hence $\mathbf{A}_1 = \mathbf{A}_2 = \mathbf{R}$, $\mathbf{S} = 0$, but, as opposed to the Sklyanin bracket, the operator $\mathbf{R}$ is nonlocal: $(\mathbf{R})_{jj} = \mathrm{R}$, $(\mathbf{R})_{ij} = \mathrm{R} + I$ for $i < j$, and $(\mathbf{R})_{ij} = \mathrm{R} - I$ for $i > j$. These Poisson–Lie groups turn out to play an important role in the theory of Poisson structures on graph connections and on moduli spaces of flat connections on Riemann surfaces.

2.7. The situation encoded in the formula $\boldsymbol{g} = \boldsymbol{g}_+ \oplus \boldsymbol{g}_-$ is the basic for the so called Adler–Kostant–Symes scheme. It were the papers Adler (1979), Kostant (1979), Symes (1980, 1982) where Theorem 2.24 appeared and was applied to several integrable systems, including the Toda lattice and its generalizations. For a more geometrical proof of Theorem 2.24, more on its its history, and more references, the reader should consult Reyman and Semenov-Tian-Shansky (1994).

2.8. Usually, one understands under a Bäcklund transformation for a given system of differential equations a map which brings solutions into solutions. In our framework of integrable evolutionary dynamical systems this means that the map has to commute with the phase flow. For general bibliographic comments on Bäcklund transformations, see below (cooments to Appendix A). Theorem 2.27 delivers a general construction of Bäcklund transformations in the context of the Adler–Kostant–Symes scheme. It is, probably, known to the specialists, but we find it difficult to give a precise reference. Its consequence Theorem 2.28 goes back to the work Symes (1980,1982), where the relation between the Toda lattice and the QR algorithm was discovered. However, it is also difficult to give a reference for Theorem 2.28 in the form given here which would be earlier than Suris (1991). Different approach to Bäcklund transformations in situations close to the one considered here, are being developed in Kuznetsov and Sklyanin (1998), Sklyanin (1999a,b), and in Hone, Kuznetsov and Ragnisco (1999, 2000). This approach also uses Lax representations, but does not refer at all to the factorization theorem. Instead, an emphasis is being made on the novel property of spectrality, discovered in Kuznetsov and Sklyanin (1998) and still escaping a systematical explanation. The paper Kuznetsov and Vanhaecke (2000) pushed forward an approach based on the methods of algebraic geometry.

2.9. The recipe for integrable discretization, as given here, was clearly formulated for the first time in Suris (1995, 1996a), and was used in the subsequent publications by the author. A viewpoint according to which Bäcklund transformations lie in the basis of discretization is now well established also in the discrete geometry, cf. Bobenko (1999) or Doliwa, Santini, and Mañas (2000).

Appendix A. Bäcklund transformations, as transformations of partial differential equations bringing solutions into solutions, were invented in the differential–geometric context by Bäcklund (1876, 1880, 1882). Independently, Darboux (1882) came to similar transformations in the context of second order ordinary differential equations. The literature devoted to these Bäcklund–Darboux transformations is nowadays enormous, and it is impossible to give here a complete review. Two monographs should be mentioned: Rogers and Shadwick (1982) is devoted to somewhat more applied aspects of Bäcklund transformations in the framework of a geometric theory of partial differential equations; Matveev and Salle (1991) follow and further develop the algebraic part of the approach by Darboux. They derive nice determinant representations for iterated Darboux transformations for numerous integrable systems, which allows them to find closed formulas for multi–soliton and more complicated solutions.

The important work of Flaschka and McLaughlin (1976) was the first one to treat spectral and Hamiltonian properties of Bäcklund transformations in the case of KdV–like systems under rapidly decaying boundary conditions. They pointed out that these transformations are, generally speaking, not isospectral. An account of this property from the viewpoint of the theory of linear operators was given by Deift (1978). An algebro–geometric account of the spectral properties of Bäcklund transformations for KdV in the periodic (or, better, finite–gap) situation was given by Ehlers and Knoerrer (1982), see a simplified exposition in Gesztesy and Holden (1999). Hamiltonian properties of Bäcklund transformations for the Toda lattice, sine–Gordon and nolinear Schrödinger equations were elaborated in Bobenko (1982). Generalizations related to energy–dependent Schrödenger operators and to Kac–Moody algebras are considered in Fordy, Shabat and Veselov (1995). An exposition of the approach to Bäcklund transformations relating them to factorizations in loop groups is given in Terng and Uhlenbeck (2000).

Part II

Lattice systems

Chapter 3

Toda lattice

Contents

3.1 Introduction

The second part of our book is devoted to the exposition of concrete examples, belonging to the big area of ***lattice, or differential–difference, systems***. The most famous integrable lattice system is undoubtedly the ***Toda lattice***.

The celebrated Newtonian equations of motion of the Toda lattice read:

$$\ddot{x}_k = e^{x_{k+1}-x_k} - e^{x_k - x_{k-1}} \, . \tag{3.1.1}$$

This system (like all lattice systems) may be considered on an infinite lattice ($k \in \mathbb{Z}$); in this case the suitable boundary conditions are those of decay on the both infinities, $x_k, \dot{x}_k \to 0$ by $k \to \pm\infty$. We shall, however, *always* restrict ourselves to finite–dimensional versions ($1 \le k \le N$) of lattice systems, subject to one of the two types of boundary conditions: open–end or periodic. For the equations (3.1.1) the open–end boundary conditions read: $x_0 = \infty$, $x_{N+1} = -\infty$, while the periodic boundary conditions mean that all subscripts are taken (mod N), so that $x_0 \equiv x_N$, $x_{N+1} \equiv x_1$.

If one introduces the variables (a, b) according to

$$b_k = \dot{x}_k \; , \qquad a_k = e^{x_{k+1}-x_k} \; , \tag{3.1.2}$$

then their time evolution is governed by the following equations of motion:

$$\dot{b}_k = a_k - a_{k-1} \; , \qquad \dot{a}_k = a_k(b_{k+1} - b_k) \; , \qquad 1 \le k \le N \; , \tag{3.1.3}$$

with one of two types of boundary conditions: open–end ($a_0 = a_N = 0$), or periodic (all subscripts are taken (mod N), so that $a_0 \equiv a_N$, $b_{N+1} \equiv b_1$). This innocently looking change of variables has however very far going consequences. The flow (3.1.3), denoted hereafter TL, turns out to have much richer structure than the original Newtonian equations of motion (3.1.1).

The flow TL is multi–Hamiltonian, i.e. allows several Hamiltonian formulations with respect to different but compatible Poisson brackets. *Three* of these brackets turn out to be local in the coordinates (a, b), they are discussed in the Sect. 3.2.

The flow TL allows a Lax representation (in fact, even several different but closely related ones). Two of them are discussed in Sect. 3.4 and 3.8. All these Lax representations serve as classical examples of an r–matrix structure. See Sect. 3.5, 3.6.

As every multi–Hamiltonian and completely integrable system, the flow TL belongs to a whole integrable hierarchy. Each flow of this hierarchy (being also multi–Hamiltonian and completely integrable) may be solved by means of matrix factorization, which is a finite–dimensional analog of the inverse scattering method. This is the ground for TL to deliver a quite transparent model for our approach to the problem of integrable discretization, which is elaborated in the Sect. 3.7. The resulting discretization shares invariant Poisson structures and integrals of motion with TL. In short, it belongs to the hierarchy attached to TL.

However, it displays a typical phenomenon: the equations of motion governing this discretization are non–local. We discuss, therefore, in Sect. 3.9 a general device allowing to circumvent this drawback, namely *localizing changes of variables*. The first application of this device to the discretization of TL is presented in Sect. 3.10. The resulting local discretization reads:

$$\widetilde{\mathbf{b}}_k + h\widetilde{\mathbf{a}}_{k-1} = \mathbf{b}_k + h\mathbf{a}_k \; , \qquad \widetilde{\mathbf{a}}_k(1 + h\widetilde{\mathbf{b}}_k) = \mathbf{a}_k(1 + h\mathbf{b}_{k+1}) \; . \tag{3.1.4}$$

A universal character of our constructions is illustrated further in Sect. 3.11–3.14, where they are applied to the second and to the third flows of the Toda hierarchy.

Actually, the localizing changes of variables turn out to be close relatives, or, better, particular instances of the so called *Miura transformations*. Apart from a novel observation concerning their role in the discretization problem, they are well studied and widely used in the soliton theory as a mean for producing new integrable hierarchies out of the known ones, – a procedure called *modification*. Normally, each modification is connected with loosing one local Hamiltonian structure. Correspondingly, the tri–Hamiltonian flow TL allows a tower of *three* consecutive modifications. We write down here the corresponding equations of motion in notations slightly different from those used in the main text.

The first modification of the Toda lattice with the modification parameter ϵ, or MTL(ϵ), is the following system:

$$\dot{b}_k = (1 + \epsilon b_k)(a_k - a_{k-1}) \; , \qquad \dot{a}_k = a_k(b_{k+1} - b_k) \; . \tag{3.1.5}$$

This bi–Hamiltonian system is introduced in Sect. 3.15. Actually, the discretization (3.1.4) belongs to the hierarchy of MTL(h). This means that (3.1.4) shares with MTL(h) invariant Poisson structures and integrals of motion. This is an example of discretizations deforming both the Poisson structure and the integrals of motion. (Let us stress once more that in the present case these deformations are due to the Miura transformation playing the role of a localizing change of variables).

We discretize MTL(ϵ) in Sect. 3.16, obtaining a discrete time system from the MTL(ϵ) hierarchy described by nonlocal equations of motion. Again, a localizing change of variables can be found, resulting in the following local discretization, see Sect. 3.17:

$$(1+\epsilon\widetilde{\mathbf{b}}_k)(1+h\epsilon\widetilde{\mathbf{a}}_{k-1})=(1+\epsilon\mathbf{b}_k)(1+h\epsilon\mathbf{a}_k)\;,\qquad \widetilde{\mathbf{a}}_k(1+h\widetilde{\mathbf{b}}_k)=\mathbf{a}_k(1+h\mathbf{b}_{k+1})\;. \tag{3.1.6}$$

Actually, this discretization belongs to the hierarchy of M^{2}TL(ϵ,h), the double modified Toda lattice with the modification parameters ϵ, h.

The general M^{2}TL(ϵ,δ) with the modification parameters ϵ, δ is the following system:

$$\dot{b}_k=(1+\epsilon b_k)(1+\delta b_k)(a_k-a_{k-1})\;,\qquad \dot{a}_k=a_k(1+\epsilon\delta a_k)(b_{k+1}-b_k)\;; \tag{3.1.7}$$

This system, possessing one local Hamiltonian structure, is introduced in Sect. 3.18 and discretized in Sect. 3.19. Again, the discretization living in the M^{2}TL(ϵ,δ) hierarchy is nonlocal, and in order to find a local discretization one has to apply a localizing change of variables, bringing the corresponding discrete time equations of motion into the M^{3}TL(ϵ,δ,h) hierarchy (with the modification parameters ϵ, δ, h):

$$\left\{\begin{array}{l} \dfrac{1+\epsilon\widetilde{\mathbf{b}}_k}{1+\delta\widetilde{\mathbf{b}}_k}\cdot\dfrac{1+h\epsilon\widetilde{\mathbf{a}}_{k-1}}{1+h\delta\widetilde{\mathbf{a}}_{k-1}}=\dfrac{1+\epsilon\mathbf{b}_k}{1+\delta\mathbf{b}_k}\cdot\dfrac{1+h\epsilon\mathbf{a}_k}{1+h\delta\mathbf{a}_k}\;, \\[4mm] \dfrac{\widetilde{\mathbf{a}}_k}{1+\epsilon\delta\widetilde{\mathbf{a}}_k}(1+h\widetilde{\mathbf{b}}_k)=\dfrac{\mathbf{a}_k}{1+\epsilon\delta\mathbf{a}_k}(1+h\mathbf{b}_{k+1})\;. \end{array}\right. \tag{3.1.8}$$

The general M^{3}TL(ϵ,δ,γ), the triple modified Toda lattice with the modification parameters ϵ, δ, γ, is the following system:

$$\left\{\begin{array}{l} \dot{b}_k=(1+\epsilon b_k)(1+\delta b_k)(1+\gamma b_k)\left(\dfrac{a_k}{1-\epsilon\delta\gamma b_k a_k}-\dfrac{a_{k-1}}{1-\epsilon\delta\gamma b_k a_{k-1}}\right)\;, \\[4mm] \dot{a}_k=a_k(1+\epsilon\delta a_k)(1+\epsilon\gamma a_k)(1+\delta\gamma a_k)\left(\dfrac{b_{k+1}}{1-\epsilon\delta\gamma b_{k+1}a_k}-\dfrac{b_k}{1-\epsilon\delta\gamma b_k a_k}\right)\;. \end{array}\right. \tag{3.1.9}$$

This system is discussed in Sect. 3.21. No local Hamiltonian formulation is known for it. The lack of Hamiltonian structure for (3.1.9) does not allow us to formulate any statements about its complete integrability in the Liouville–Arnold sense. Nevertheless it is in some sense integrable: it has many integrals of motion and belongs to a hierarchy of commuting flows; the corresponding objects may be found by pulling back analogous objects for M^{2}TL(ϵ,δ) under the Miura transformations.

In the tower of successive modifications each flow is a regular one–parameter deformation of the previous one. Very remarkable is the symmetry of each flow with respect to its parameters. Each one of these flows possesses many integrals of motion and belongs to a hierarchy of commuting flows. The number of local invariant Poisson structures diminishes by one with each further modification. All these statements seem to hold in much more general situations. It is also very interesting to observe that the Miura transformations play also the role of localizing changes of variables, so that finding local discretizations inevitably leads to climbing upstairs in the modifications tower.

Returning to the Newtonian equations of motion (3.1.1), we must say that, due to the multi–Hamiltonian structure, the system (3.1.3) describes not only these ones, but also several apparently different integrable Newtonian systems living their own life in the mathematical physics – a fact overlooked by many researchers in this area. All these systems allow beautiful discretizations in form of “discrete time Newtonian equations”. These results constitute the contents of Chapter 5.

3.22 Bibliographical remarks

3.1. An integrable lattice with an exponential interaction of nearest neighbors was discovered by Toda (1967) and since then became one of the most popular and important integrable models in general. The bibliography of the Toda lattice is nowadays too vast to be reviewed here. The best general reference remains the monograph Toda (1989), where also the story of the discovery of this system is given from the first hand. The change of variables $(x, \dot{x}) \mapsto (a, b)$ was introduced in Flaschka (1974), Manakov (1974).

A two–dimensional generalization of the Toda lattice,

$$(u_k)_{xy} = e^{u_{k+1}-u_k} - e^{u_k - u_{k-1}} ,$$

or the *Toda field model*, was introduced in the soliton theory by Mikhailov (1979). It was interpreted as a simplest representative of the huge two–dimensional Toda hierarchy by Ueno and Takasaki (1984). Later it was realized that the equivalent equation

$$(\log\, v_k)_{xy} = v_{k+1} - 2v_k + v_{k-1} ,$$

which is obtained by setting $v_k = \exp(u_{k+1} - u_k)$, appeared already in Darboux (1888), as the equation satisfied by the so called *Laplace invariants* of the chain of *Laplace transformations* of a given second order hyperbolic partial differential equation.

3.2. The tri–Hamiltonian structure of the Toda lattice belongs to the most classical results. The linear bracket was known from the very beginning of the story, since it is a simple consequence of the Flaschka–Manakov change of variables. The quadratic one appeared in Adler (1979). The cubic bracket is due to Kupershmidt (1985). This latter reference contains also a systematic account of the Hamiltonian formalism for lattice systems. However do not use here this formalism, partly because we tend to consider only finite dimensional reductions of such systems which remain in the applicability domain of the usual Hamiltonian formalism, and partly because we replace it with the r–matrix approach. See also Damianou (1990, 1994) for a construction of higher (nonlocal) brackets with the help of the notion of master symmetry. Further relevant references include Das and Okubo (1989), Fernandes (1993). The geometry of the bi–Hamiltonian manifold of the Lax matrices of the Toda lattice is studied in Gelfand and Zakharevich (1999).

3.4. The Lax representation of the Toda lattice is due to Flaschka (1974) and Manakov (1974). An important early account of the Toda lattice based on the Lax representation is due to Moser (1975). For the relations between the flows of the Toda type and different matrix factorizations see Deift *et al.* (1989), Deift and Li (1989).

3.5. The linear r–matrix structure for the Toda lattice is among the cornerstones for the whole Hamiltonian approach to the theory of solitons. For the detailed account and for the references the reader should consult Reyman and Semenov-Tian-Shansky (1994). Among the most important early papers are Kostant (1979), Adler and van Moerbeke (1980), Symes (1980). Our first proof here follows that of the Reyman and Semenov-Tian-Shansky, our second proof follows the one of Belavin and Drinfeld (1982).

3.6. Theorem 3.7 is, surprisingly, a much more recent achievement. A more simple QR version appeared in Oevel and Ragnisco (1989). An LU version given here requires the most general quadratic r–matrix structure, and appeared only in Suris (1993b). See also Morosi and Pizzocchero (1996a).

3.7. As we know, the discretization is closely related to (or, better, essentially coincides with) Bäcklund–Darboux transformations. The Darboux transformation for the Toda lattice, which essentially coincides with the map given in Theorem 3.10, was given for the first time in Matveev (1979), Matveev and Salle (1979). For the reader's convenience we reproduce here their result. The flow TL is a compatibility condition of the pair of linear problems

$$a_k f_{k+1} + b_k f_k + f_{k-1} = \lambda f_k , \qquad \dot{f}_k = b_k f_k + f_{k-1} .$$

Let ψ_k be an arbitrary solution of the linear equation

$$a_k \psi_{k+1} + b_k \psi_k + \psi_{k-1} = \lambda_0 \psi_k$$

with some fixed λ_0. The Darboux transformation $f \mapsto \phi$ is defined by the formula

$$\phi_k = f_k - \sigma_k f_{k+1} , \quad \text{where} \quad \sigma_k = \frac{\psi_k}{\psi_{k+1}} .$$

Then the sequence ϕ_k satisfies two linear problems analogous to those for f_k but with (a_k, b_k) replaced by $(\widetilde{a}_k, \widetilde{b}_k)$, where

$$\widetilde{b}_k = b_k + \sigma_{k-1} - \sigma_k , \qquad \widetilde{a}_k = \frac{\sigma_k}{\sigma_{k+1}}\, a_{k+1} .$$

Matveev and Salle (1979) also generalized these formulas for iterated Darboux transformations. Let us show the relation of this form of the Darboux transformation to matrix factorizations. Indeed, if we define (infinite) matrices

$$T=\sum a_kE_{k,k+1}+\sum b_kE_{kk}+\sum E_{k+1,k}$$

and

$$A=I-\sum \sigma_kE_{k,k+1}\ ,\qquad B=\sum \tau_kE_{kk}+\sum E_{k+1,k}\ .$$

then the above equations for $(\widetilde{a}_k,\widetilde{b}_k)$ are equivalent to the discrete time Lax equation $\widetilde{T}A=AT$, while the linear equation for ψ_k, rewritten in terms of σ_k as

$$a_k+b_k\sigma_k+\sigma_k\sigma_{k-1}=\lambda_0\sigma_k\ ,$$

is exactly the recurrent relation for the entries of the matrix S defined from the factorization $T-\lambda_0I=BA$.

A discretization of the flow TL equivalent to our one was studied independently also by Gibbons and Kupershmidt (1992). However, the relation to the factorization problem lacked also in this paper. Hence, the nature of this discretization as a member of the Toda hierarchy was not understood properly. A complete account was given in Suris (1995).

The work Rutishauser (1954) contains not only the definition of the *qd* algorithm from the viewpoint of the numerical analist, but also the equations of motion of the Toda lattice (under the name of a "continuous analogue of the *qd* algorithm")! The relation of the *qd* algorithm to integrable systems might have important implications for the numerical analysis, cf. Deift *et al.* (1991), Nagai and Satsuma (1995).

3.9. The original Miura transformation for the Korteweg–de Vries equation appeared in Miura (1968), and played an extremely important role in the development of the soliton theory, see further references in remarks to Appendix A. In the context of lattice systems Miura–like transformations were discussed in Kupershmidt (1985), Volkov (1988), Yamilov (1994). The observation that they play also a role of localizing changes of variables for integrable discretizations is due to Suris (1999c).

3.10. The discretization (3.10.2) appeared, along with the Lax representation, in Hirota, Tsujimoto, and Imai (1993), Hirota and Tsujimoto (1995). However, its place in the (modified) Toda hierarchy was not recognized there, and its Hamiltonian structure was not discussed. On the other hand, the Miura transformation (3.10.1), along with the equations (3.10.12), were discussed in Kupershmidt (1983, 1985), but without any relation to the time discretization. Both themes were unified in Suris (1999c). The relation of (3.10.2) to the *qd* algorithm was pointed out by Sogo (1993). The multi–Hamiltonian structure of the *qd* algorithm was elaborated in Suris (1995, 1999c).

3.11–3.14. The results on the discretization of the second and of the third flows of the Toda hierarchy, in particular, the local equations of motion (3.12.2) and (3.14.2), (3.14.3), are new.

3.15–3.17. The definition of the modified Toda lattice is due to Kupershmidt (1983,1985), as well as its bi–Hamiltonian structure, the Miura transformations to the Toda lattice, and their Poisson properties. The interpretation of the invariant brackets (3.15.3) in terms of the r–matrix structure on $\boldsymbol{g}\otimes\boldsymbol{g}$ is a new result. The time discretization of MTL and its local form are also new. However, the particular case $h=\epsilon$ of the equations (3.17.3) appeared in Hirota and Tsujimoto (1995) as a "discrete time Toda lattice of type II".

3.18–3.21. The definition of the double modified Toda lattice is again due to Kupershmidt (1983,1985), and the time discretization is a new result, as well as its local form, of course. The corresponding localizing change of variables, as well as all other *rational* Miura transformations of this Chapter were not known previously, to the best of the author's knowledge. The triple modification of the Toda lattice also seems not to appear previously in the literature.

Appendix A. The complete integrabilty of KdV with respect to the first Hamiltonian structure (A.2) was established in Zakharov and Faddeev (1971) and Gardner (1971). This may be considered as a begin of the Hamiltonian approach to the soliton theory. The second Hamiltonian structure is due to Magri (1978), Gelfand and Dickey (1978), Adler (1979). The Miura transformation with a parameter, known also as the Gardner transformation, was introduced in Miura, Gardner, and Kruskal (1968), and was put in a more broad kontext in Kupershmidt (1981). General results on the Hamiltonian properties of Miura maps and their connection with factorization of Lax operators were discussed, e.g., in: Kupershmidt and Wilson (1980), Fordy and Gibbons (1980b, 1981), Kupershmidt

(1982a,b, 1995), Antonowicz and Fordy (1989), Fordy, Shabat and Veselov (1995). In particular, Antonowicz and Fordy found systems with $m+1$ Hamiltonian formulations which admit towers of m consequtive modifications via Miura maps. An r–matrix interpretation of these Miura maps was found by Marshall (1990, 1994).

A sort of a rational explanation of some of the "magic" properties of Miura transformations was proposed in Wilson (1989).

Deformations of integrable systems coming from parameter–dependent families of Miura transformations were studied in Kupershmidt (1983), Fordy (1983).

Chapter 4

Volterra lattice

Contents

4.1 Introduction

We next turn our attention to the second most famous integrable lattice system, known under several different names such as Volterra lattice, discrete KdV equation, Langmuir lattice, Kac–van Moerbecke lattice, and so on. We shall adopt the name ***Volterra lattice*** and the abbreviation VL for this system.

The equations of motion of VL read:

$$\dot{a}_k = a_k(a_{k+1} - a_{k-1})\ . \tag{4.1.1}$$

As usual, two types of boundary conditions will be of a major interest for us here: open–end ($a_0 = a_N = 0$, so that actually the lattice consists of $N-1$ sites, or particles), and periodic (all indices are taken (mod N), so that $a_0 \equiv a_N$, $a_{N+1} \equiv a_1$).

This system is related to the Toda lattice in two different ways. First of all, (4.1.1) can be seen as the $b_k = 0$ reduction of the second flow TL2 of the Toda hierarchy (3.11.1). It should be remarked that the first flow TL, the third flow TL3, and in general all flows of the Toda hierarchy with odd numbers do not admit this reduction. On the contrary, all flows of the Toda hierarchy with even numbers do allow this reduction, and this gives rise to the ***Volterra hierarchy***.

Only one of the three local Poisson brackets of the Toda hierarchy survives this reduction, producing an invariant Poisson bracket for VL. Nevertheless, for VL there appeares a new local invariant Poisson bracket, turning it into a bi–Hamiltonian system. This is discussed in Sect. 4.2.

As for the Lax representation of VL, it follows without any problems from the Lax representation of the second Toda flow TL2, see Sect. 4.3. The r–matrix structure behind it appears to be a Dirac reduction of the quadratic r–matrix bracket relevant for the Toda hierarchy, see Sect. 4.4.

We can apply to VL our recipe of integrable discretization. This is performed in Sect. 4.5. As usual, this leads to nonlocal equations. The localizing change of variables can be found, see Sect. 4.6. The resulting local discretization of the flow (4.1.1) is:

$$\widetilde{\mathbf{a}}_k(1 + h\widetilde{\mathbf{a}}_{k-1}) = \mathbf{a}_k(1 + h\mathbf{a}_{k+1})\ . \tag{4.1.2}$$

The same procedure of integrable discretization can be applied to higher flows of the Volterra hierarchy; for the second flow this is done in Sect. 4.8–4.9.

The localizing change of variables for the time discertization of VL may be viewed as a Miura transformation, allowing to *modify* VL. The ***modified Volterra lattice*** with the modification parameter ϵ, or MVL(ϵ), reads:

$$\dot{a}_k = a_k(1 + \epsilon a_k)(a_{k+1} - a_{k-1})\ . \tag{4.1.3}$$

Actually the discretization (4.1.2) belongs to the hierarchy of MVL(h). This means that the map (4.1.2) shares the integrals and the invariant Poisson structure with MVL(h).

We introduce MVL(ϵ) and its Hamiltonian formulation in Sect. 4.11, and discretize it in Sect. 4.12. Again, only nonlocal equations of motion are obtained upon applying our recipe. Fortunately, also for this discretization a localizing change of variables can be found, see Sect. 4.13. The corresponding local discretization of (4.1.3) reads:

$$\frac{\widetilde{\mathbf{a}}_k}{1 + \epsilon\widetilde{\mathbf{a}}_k}\,(1 + h\widetilde{\mathbf{a}}_{k-1}) = \frac{\mathbf{a}_k}{1 + \epsilon\mathbf{a}_k}\,(1 + h\mathbf{a}_{k+1})\ . \tag{4.1.4}$$

Another form of MVL and its discretizations are sometimes more convenient. The system MVL(ϵ) is equivalent to

$$\dot{c}_k = (c_k^2 - \nu^2)(c_{k+1} - c_{k-1})\ , \tag{4.1.5}$$

and a local discretization of this system reads:

$$\frac{\widetilde{\mathbf{c}}_k - \nu}{\widetilde{\mathbf{c}}_k + \nu} \cdot \frac{1 + h\nu\widetilde{\mathbf{c}}_{k-1}}{1 - h\nu\widetilde{\mathbf{c}}_{k-1}} = \frac{\mathbf{c}_k - \nu}{\mathbf{c}_k + \nu} \cdot \frac{1 + h\nu\mathbf{c}_{k+1}}{1 - h\nu\mathbf{c}_{k-1}}\ , \tag{4.1.6}$$

This form of MVL and dMVL is discussed in Sect. 4.14, 4.15.

Considering the localizing change of variables for the discretization of MVL(ϵ) as a new Miura transformation allows to define a further modication of MVL(ϵ), which we denote by M^{2}VL(ϵ, δ). The equations of motion of $\mathrm{M}^2\mathrm{VL}(\epsilon,\delta)$, the ***double modified Volterra lattice*** with the modification parameters ϵ, δ read:

$$\dot{a}_k = a_k(1+\epsilon a_k)(1+\delta a_k)\left(\frac{a_{k+1}}{1-\epsilon\delta a_{k+1}a_k} - \frac{a_{k-1}}{1-\epsilon\delta a_k a_{k-1}}\right) . \tag{4.1.7}$$

(Actually, the map (4.1.4) belongs to the hierarchy of $\mathrm{M}^2\mathrm{VL}(\epsilon,h)$). This system is discussed in Sect. 4.16; we do not know any local Hamiltonian structure for it. However, it belongs to a hierarchy of commuting flows and possesses a large set of integrals of motion. Actually, the tower of successive modifications of VL consists of two levels (in accordance with the bi–Hamiltonian nature of VL), so that the second modification is the last one (at least, as long as one restricts himself with local equations of motion).

Returning to the Volterra lattice VL itself, one finds that there exists also another relation to the Toda lattice TL of a completely different nature. To describe it, we consider VL as a sort of the modified TL, related to TL by means of a Miura transformation. As usual, the origin of this Miura transformation lies in factorization of the Lax matrix for TL. On this way VL appears in the *two–field form* (Sect. 4.17):

$$\dot{u}_k = u_k(v_k - v_{k-1}) , \qquad \dot{v}_k = v_k(u_{k+1} - u_k) . \tag{4.1.8}$$

To see the equivalence of the systems (4.1.1), (4.1.8), it is enough to perform the re–naming of variables

$$u_k \to a_{2k-1} , \qquad v_k \to a_{2k} . \tag{4.1.9}$$

Notice, however, that if we let the subscript k in (4.1.8) vary in the interval $1 \le k \le N$, then the corresponding periodic VL consists of an *even* number $2N$ of particles u_k, v_k $(1 \le k \le N)$, and the open–end version consists of an *odd* number $2N-1$ of particles u_k $(1 \le k \le N)$, v_k $(1 \le k \le N-1)$.

The natural Lax representation for VL in the form (4.1.8) lives in $\boldsymbol{g}\otimes\boldsymbol{g}$, and it delivers the simplest example for constructions on direct products. It is discussed in Sect. 4.18. The Poisson geometry of the matrix factorization delivering an r–matrix interpretation of the Lax equations in $\boldsymbol{g}\otimes\boldsymbol{g}$, is the subject of Sect. 4.19.

A somewhat astonishing fact is the coincidence of time discretizations of VL based on two different Lax representations. This is established in Sect. 4.20. The corresponding localizing change of variables is studied in Sect. 4.21, leading to the discrete time equations of motion:

$$\widetilde{\mathbf{u}}_k(1+h\widetilde{\mathbf{v}}_{k-1}) = \mathbf{u}_k(1+h\mathbf{v}_k) , \qquad \widetilde{\mathbf{v}}_k(1+h\widetilde{\mathbf{u}}_k) = \mathbf{v}_k(1+h\mathbf{u}_{k+1}) . \tag{4.1.10}$$

Actually, the whole Volterra hierarchy, with all its continuous and discrete time flows, can be equally well studied on the base of the $\boldsymbol{g}\otimes\boldsymbol{g}$ Lax representation, the results being always coherent with those obtained on the base of the more usual one, living in $\boldsymbol{g}$. This is once more illustrated by the example of the second flow VL2 and its discretization dVL2 in Sect. 4.22.

Finally, a two–field reformulation of $\mathrm{MVL}(\epsilon)$ is also of interest:

$$\dot{u}_k = u_k(1+\epsilon u_k)(v_k - v_{k-1}) , \qquad \dot{v}_k = v_k(1+\epsilon v_k)(u_{k+1} - u_k) . \tag{4.1.11}$$

(The re-naming (4.1.9) brings this system into the form (4.1.3)). In slightly different notations, the system (4.1.11) is considered in Sect. 4.23, while its local integrable discretization,

$$\frac{\widetilde{\mathbf{u}}_k}{1+\epsilon\widetilde{\mathbf{u}}_k}(1+h\widetilde{\mathbf{v}}_{k-1}) = \frac{\mathbf{u}_k}{1+\epsilon\mathbf{u}_k}(1+h\mathbf{v}_k) , \qquad \frac{\widetilde{\mathbf{v}}_k}{1+\epsilon\widetilde{\mathbf{v}}_k}(1+h\widetilde{\mathbf{u}}_k) = \frac{\mathbf{v}_k}{1+\epsilon\mathbf{v}_k}(1+h\mathbf{u}_{k+1}) , \tag{4.1.12}$$

is derived in Sect. 4.24–4.25.

4.26 Bibliographical remarks

4.1. The integrability of the Volterra lattice was established in Manakov (1974), Kac and van Moerbeke (1975), and an N–soliton solution was found by Hirota and Satsuma (1976). Since then it became one of the most important integrable differential–difference equations. Among its more recent applications the one to the Liouville field theory should be mentioned Faddeev and Takhtadjan (1986), Volkov (1992), Faddeev and Volkov (1994). See also recent expositions in Bulla *et al.* (1998), Teschl (2000). Generalizations of the finite Volterra lattice, related to simple Lie algebras, were discovered by Bogoyavlensky (1988a,b).

4.2. The bi–Hamiltonian structure of the Volterra hierarchy was found independently by Kupershmidt (1985) and Faddeev and Takhtadjan (1986). The r–matrix formulation of the cubic bracket $\{\cdot,\cdot\}_3$ in terms of a 2×2 Lax representation was given by Volkov (1992). Correspondingly, in the physical literature this cubic bracket is known as the FTV bracket (after Faddeev–Takhtajan–Volkov).

One can consider the Volterra hierarchy as a spatial discretization of the Korteweg–de Vries hierarchy. A precise treatment of this relation, including the relations between the corresponding Hamiltonian formulations, is given in Zeng and Rauch–Wojciechowski (1995) and in Morosi and Pizzochero (1996c).

4.3. The Lax representation in $\boldsymbol{g}$ was found already in Manakov (1974) and in Kac and van Moerbeke (1975), and was used as a basis for applying the inverse scattering method to VL. A 2×2 version of these results appeared in Ablowitz and Ladik (1975, 1976a).

4.5. The r–matrix interpretation of the Lax representation (4.3.3), involving the Dirac reduction of a quadratic bracket on $\boldsymbol{g}$, seems to be given for the first time in Suris (1994a).

4.6, 4.7. A time discretization of the Volterra lattice (4.7.2) was first found in Tsujimoto, Hirota and Oishi (1993), where it was derived as a reduction of the lattice KP equation in the bilinear (Hirota's) representation. See also Hirota, Tsujimoto and Imai (1993), Hirota and Tsujimoto (1995). A discrete time Lax representation was also found in these papers. However, the Poisson properties of this discretization and the Miura relation to the original Volterra hierarchy were not discussed there. Another attempt to discretize VL was undertaken in Kupershmidt (1996). His approach is very similar to our one, and leads to the same nonlocal equations of motion (4.5.2), however it is based on an *ad hoc* ansatz for the discrete Lax pair rather than on a systematic factorization procedure (the author himself calles his ansatz "bizarre"). As a consequence, the Hamiltonian interpretation (and local equations of motion) were missed in this work. Our present construction explains all previous results and adds much new information. It was introduced in Suris (1996b, 1999c). A nice observation that in the periodic case the regular solution for $\widetilde{\mathbf{a}}_k$ is rational in $\mathbf{a}_j$, is due to Narita, Saito, Saitoh and Yoshida (2000).

4.8–4.10. The discretization of the second flow of the Volterra hierarchy (including the corresponding local equations of motion) was performed in Suris (1999c). A different (and highly non-local) integrable difference scheme for the KdV equation was proposed in Taha and Ablowitz (1984) and tested in real computations in Taha and Ablowitz (1988).

4.11–4.14. The modified Volterra lattice (in the two–field version) was introduced under the name "self–dual electrical network" in Hirota (1973); also the multi–soliton solutions were found in this work. Ablowitz and Ladik (1975, 1976a) found a 2×2 Lax formulation and developed the inverse scattering approach to this system. The Miura relations between the modified Volterra lattice and the usual Volterra and the Toda lattices were first observed in Hirota (1975), see also Wadati (1976a,b), Hirota and Satsuma (1976b). Later the modified Volterra lattice appeared also in Yamilov (1983) in the course of classifying integrable lattice equations with nearest neighbors interactions. Other relevant references by the latter author are Shabat and Yamilov (1990) and Yamilov (1994).

The Hamiltonian properties of the Miura relations between the modified Volterra lattice and the usual one were studied in Kupershmidt (1985). The nonlocal Poisson structure of MVL obtained as a pull–back of the bracket $\{\cdot,\cdot\}_2$ for VL, was written down in Tsuchida and Wadati (1998) and in Faybusovich and Gekhtman (1999). The latter reference contains also a discussion of several gauge–equivalent versions of the Lax representation for MVL. Let us mention also the appearance of the bi–Hamiltonian structure from Sect. 4.15 in a seemingly different context of "dressing chains" in Weiss (1986, 1987), Veselov and Shabat (1993). Actually, the dressing chains and the modified Volterra lattice in the $\nu\to 0$ limit belong to the same bi–Hamiltonian hierarchy.

Boundry conditions different from the open–end and the periodic ones, but still assuring the complete integrability of MVL, were studied in Kajinaga and Wadati (1997).

The systematic discretization of MVL is a new result, however the local discrete time equations were derived in Tsujimoto and Hirota (1995) based on the bilinear formalism. The relation of this local discretization to the original MVL hierarchy was missed in that paper.

4.16. The double modification of the Volterra lattice appeared in the work Yamilov (1983) as one of the more complicated members of his exhaustive list of integrable lattice systems with nearest neighbours interactions. The Miura relation to the MVL was given in Yamilov (1994) (and earlier in an unpublished thesis of the same author). For the reader's convenience we reproduce here the Yamilov's form of $\mathrm{M}^2\mathrm{VL}$, since it is seamingly different from our one:

$$\dot{w}_k = (w_k^2 - 1)(\nu^2 - \mu^2 w_k^2)\left(\frac{1}{w_{k+1} + w_k} - \frac{1}{w_k + w_{k-1}}\right) .$$

However, a simple transformation

$$y_k = \frac{w_k + 1}{w_k - 1} ,$$

accompanied by a time rescaling, brings the above system into the form (14.16.1) (with $\epsilon\delta = 1$).

4.17, 4.18. The Miura relation between the Toda lattice and the Volterra lattice is an old observation: already Kac and van Moerbeke (1975) established the fact that VL contains two different TL's and provides a Bäcklund transformation between them. A clear presentation of the origin of this Bäcklund transformation in the matrix factorization (in the Cholesky version thereof) was given in Adler (1981). This latter reference contains also the Lax formulation of the two–field version of VL as a Lax triad (again, in the version corresponding to the Cholesky decomposition). The Lax representation of Theorem 4.28 is due to Kupershmidt (1985).

The nonlocal Poisson brackets of VL resulting from the linear bracket $\{\cdot,\cdot\}_1$ of TL, pulled–back under the Miura maps $M_{1,2}$, was written down in in Tsuchida, Kajinaga and Wadati (1997) and in Faybusovich and Gekhtman (1999).

The *qd* algorithm and the *LU* algorithm are among important tools of the numerical analysis, and were invented by Rutishauser (1954), long before they re-appeared in the theory of integrable systems. Notice also that the equations of motion of the Toda lattice were also proposed in Rutishauser (1954) as a continuous analogue of the *qd* algorithm!

4.19. The quadratic r–matrix structure on $\boldsymbol{g} \otimes \boldsymbol{g}$ was found in Suris (1997d). It serves as the simplest example of a situation where the full strength of Theorem 2.22 is required. A more simple version of the treatment of the Poisson geometry of factorization appeared earlier in Deift and Li (1991). This paper dealt with the Cholesky factorization, in which case the usual Sklyanin bracket works.

4.20–4.25. The results here are mainly a two–field reformulation of the previously obtained ones. The only exception represent the translation of the Miura relation between VL and TL hierarchies and between MVL and MTL hierarchies into localizing variables. The formulas (4.21.9), (4.21.10) were found in Kupershmidt (1985), although without any relation to the discretization problem, and also in Hirota and Tsujimoto (1995), but without discussing their Poisson properties. Both themes were unified in Suris (1999c). The pair of formulas (4.22.14), (4.22.15), as well as (4.25.4), (4.25.5), and (4.25.6), (4.25.7), seem to be new.

Chapter 5

Newtonian equations of the Toda type

Contents

5.1 Introduction

Actually, the integrable system originally discovered by Toda was (3.1.1) – a lattice of particles interacting with nearest neighbours via forces exponentially depending on distances. Only later was it rewritten in the Flaschka variables (a, b), i.e. in the form (3.1.3). However, it turns out that the system (3.1.3) has a much richer structure than (3.1.1). In particular, it is a tri–Hamiltonian system. It turns out that each one of invariant Poisson structures, and some of their linear combinations, allow parametrizations by means of canonically conjugate variables (x, p), which leads to a whole variety of Newtonian equations of motions hidden in (3.1.3). The present Chapter is devoted to elaborating the relevant systems, along with their discretizations.

Let us list the main Newtonian equations and their discrete time counterparts arising from (3.1.3) and (3.7.2), respectively, via different parametrizations of the (a, b) variables by canonically conjugate ones.

First of all, we have two parametrizations of the linear bracket, which are in a sense dual to each other, and lead to the following Newtonian systems (which constitute, as already mentioned, the Toda's discovery):

$$\ddot{x}_k = e^{x_{k+1}-x_k} - e^{x_k-x_{k-1}} , \tag{5.1.1}$$

and

$$\ddot{x}_k = \dot{x}_k \left(x_{k+1} - 2x_k + x_{k-1}\right) . \tag{5.1.2}$$

The corresponding discrete time Newtonian equations of motion are:

$$e^{\widetilde{x}_k - x_k} - e^{x_k - \underset{\sim}{x}_k} = h^2 \left(e^{\underset{\sim}{x}_{k+1} - x_k} - e^{x_k - \widetilde{x}_{k-1}} \right) , \tag{5.1.3}$$

and

$$\frac{\widetilde{x}_k - x_k}{x_k - \underset{\sim}{x}_k} = \frac{1 + h(\underset{\sim}{x}_{k+1} - x_k)}{1 + h(x_k - \widetilde{x}_{k-1})} , \tag{5.1.4}$$

respectively.

Further, a parametrization of the quadratic invariant bracket of TL leads to the following Newtonian equations of motion:

$$\ddot{x}_k = \dot{x}_k \left(e^{x_{k+1} - x_k} - e^{x_k - x_{k-1}} \right) . \tag{5.1.5}$$

Their discrete time conterpart read:

$$\frac{e^{\widetilde{x}_k - x_k} - 1}{e^{x_k - \underset{\sim}{x}_k} - 1} = \frac{1 + h e^{\underset{\sim}{x}_{k+1} - x_k}}{1 + h e^{x_k - \widetilde{x}_{k-1}}} . \tag{5.1.6}$$

One can also parametrize linear combinations $\{\cdot,\cdot\}_1 + \epsilon\{\cdot,\cdot\}_2$ of the linear and the quadratic brackets, which leads to a "hybrid" system, whose right–hand side turns out, curiously enough, to be the corresponding linear combination of the right–hand sides of (5.1.1) and (5.1.5):

$$\ddot{x}_k = (1 + \epsilon \dot{x}_k) \left(e^{x_{k+1} - x_k} - e^{x_k - x_{k-1}} \right) . \tag{5.1.7}$$

For dTL the same parametrization of the Flaschka variables leads to the following discretization of the latter system:

$$\frac{\epsilon \left(e^{\widetilde{x}_k - x_k} - 1 \right) + h}{\epsilon \left(e^{x_k - \underset{\sim}{x}_k} - 1 \right) + h} = \frac{1 + h\epsilon e^{\underset{\sim}{x}_{k+1} - x_k}}{1 + h\epsilon e^{x_k - \widetilde{x}_{k-1}}} . \tag{5.1.8}$$

Obviously, the system (5.1.7) may be considered as a perturbation of the standard Toda lattice (5.1.1), if the parameter ϵ is small. Then in the discretization (5.1.8) there are two competitive small parameters, ϵ and h. A very interesting and important particular case of this discretization arises when $\epsilon = h$. The equations of motion then simplify to:

$$e^{\widetilde{x}_k - 2x_k + \underset{\sim}{x}_k} = \frac{1 + h^2 e^{\underset{\sim}{x}_{k+1} - x_k}}{1 + h^2 e^{x_k - \widetilde{x}_{k-1}}} , \tag{5.1.9}$$

and serve as an alternative discretization for (5.1.1).

Further, a linear combination $\{\cdot,\cdot\}_3 + 4\nu\{\cdot,\cdot\}_2$ of the cubic and the quadratic invariant brackets for TL also can be parametrized by canonically conjugate variables, which leads to the following Newtonian equations:

$$\ddot{x}_k = -(\dot{x}_k^2 - \nu^2)\Big(\coth(x_{k+1} - x_k) - \coth(x_k - x_{k-1}) \Big) , \tag{5.1.10}$$

and we find the following discretization thereof:

$$\begin{aligned} &\frac{\sinh(\widetilde{x}_k - x_k + h\nu)}{\sinh(\widetilde{x}_k - x_k - h\nu)} \cdot \frac{\sinh(x_k - \underset{\sim}{x}_k - h\nu)}{\sinh(x_k - \underset{\sim}{x}_k + h\nu)} \\ &\qquad = \frac{\sinh(\underset{\sim}{x}_{k+1} - x_k + h\nu)}{\sinh(\underset{\sim}{x}_{k+1} - x_k - h\nu)} \cdot \frac{\sinh(x_k - \widetilde{x}_{k-1} - h\nu)}{\sinh(x_k - \widetilde{x}_{k-1} + h\nu)} . \end{aligned} \tag{5.1.11}$$

The $\nu = 0$ particular case of the previous equations corresponds to a canonical parametrization of the pure cubic bracket $\{\cdot,\cdot\}_3$, and delivers the following Newtonian equations:

$$\ddot{x}_k = -\dot{x}_k^2 \left(\coth(x_{k+1} - x_k) - \coth(x_k - x_{k-1}) \right) , \tag{5.1.12}$$

while the discretization simplifies considerably and reads:

$$\coth(\widetilde{x}_k - x_k) - \coth(x_k - \underset{\sim}{x}_k) = \coth(\underset{\sim}{x}_{k+1} - x_k) - \coth(x_k - \widetilde{x}_{k-1}) \ . \tag{5.1.13}$$

On the other hand, rescaling in (5.1.10), (5.1.11) $x_k \mapsto \nu x_k$, $\nu \mapsto \gamma\nu$, and *then* sending $\nu \to 0$ results in the Newtonian equations:

$$\ddot{x}_k = -(\dot{x}_k^2 - \gamma^2)\left(\frac{1}{x_{k+1} - x_k} - \frac{1}{x_k - x_{k-1}}\right) \ , \tag{5.1.14}$$

and their discretization

$$\frac{(\widetilde{x}_k - x_k + h\gamma)}{(\widetilde{x}_k - x_k - h\gamma)} \cdot \frac{(x_k - \underset{\sim}{x}_k - h\gamma)}{(x_k - \underset{\sim}{x}_k + h\gamma)} = \frac{(\underset{\sim}{x}_{k+1} - x_k + h\gamma)}{(\underset{\sim}{x}_{k+1} - x_k - h\gamma)} \cdot \frac{(x_k - \widetilde{x}_{k-1} - h\gamma)}{(x_k - \widetilde{x}_{k-1} + h\gamma)} \ . \tag{5.1.15}$$

The system (5.1.14) is dual to (5.1.12), in particular, it also corresponds to a canonical parametrization of the pure cubic bracket $\{\cdot,\cdot\}_3$.

Finally, rescaling $x_k \mapsto \nu x_k$ and then sending $\nu \to 0$ in the equations (5.1.12), (5.1.13), or, equivalently, sending $\gamma \to 0$ in the equations (5.1.14), (5.1.15), we come to the simplest Newtonian equations, still corresponding to a canonical parametrization of the pure cubic bracket $\{\cdot,\cdot\}_3$:

$$\ddot{x}_k = -\dot{x}_k^2\left(\frac{1}{x_{k+1} - x_k} - \frac{1}{x_k - x_{k-1}}\right) \ , \tag{5.1.16}$$

and to their discretization

$$\frac{1}{\widetilde{x}_k - x_k} - \frac{1}{x_k - \underset{\sim}{x}_k} = \frac{1}{\underset{\sim}{x}_{k+1} - x_k} - \frac{1}{x_k - \widetilde{x}_{k-1}} \ . \tag{5.1.17}$$

The present Chapter is devoted to elaborating all these Newtonian equations of motion, both in continuous and discrete time. For all these systems we shall trace back the relations to TL and dTL, respectively, and provide a Lagrangian and (only for continuous time systems) Hamiltonian formulation. For some of them also nice 2×2 Lax representations will be given.

5.11 Bibliographical remarks

5.1. The exponential Toda lattice (5.1.1) was discovered by Toda (1967), see also Toda (1989). The relation to the dual Toda lattice (5.1.2) was instrumental in this discovery. The exponential form of the modified Toda lattice (5.1.7) first appeared in Chen and Liu (1975).

The system (5.1.10), as well as its degenerations (5.1.12), (5.1.14), (5.1.16), seem to be discovered by Yamilov (1989). He proved also that the list of continuous time Toda–like lattices reproduced in Sect. 5.1 exhausts integrable lattice systems of the type

$$\ddot{x}_k = f(\dot{x}_k)\Big(g(x_{k+1} - x_k) - g(x_k - x_{k-1})\Big) \ .$$

More precisely, he demonstrated that this list exhausts such systems possessing conservation laws of a high enough order (see the original paper for a precise formulation). As we show in the present Chapter, these systems are integrable also in the usual Liouville–Arnold sense, possess many enough conservation laws, Lax representations etc. Yamilov also elaborated on the coupled nonlinear Schrödinger equations connected with the above lattices (see also Shabat and Yamilov (1988, 1990) for a far reaching development of this idea). For the reader's convenience, we cite here the corresponding construiction for the standard exponential Toda lattice

$$\left\{\begin{array}{l} (x_k)_t = p_k \ , \\ (p_k)_t = e^{x_{k+1} - x_k} - e^{x_k - x_{k-1}} \ . \end{array}\right.$$

These motion equations are to be supplemented with the analogous equations for the second flow, corresponding to the integral $\frac{1}{3}\sum p_k^3 + \sum(p_{k+1} + p_k)e^{x_{k+1} - x_k}$, namely:

$$\left\{\begin{array}{l} (x_k)_s = p_k^2 + e^{x_{k+1} - x_k} + e^{x_k - x_{k-1}} \ , \\ (p_k)_s = (p_{k+1} + p_k)e^{x_{k+1} - x_k} - (p_k + p_{k-1})e^{x_k - x_{k-1}} \ . \end{array}\right.$$

Now, denoting $u = e^{x_k}$ and $v = e^{-x_{k-1}}$, one arrives after some manipulations at the system of partial differential equations for (u, v):

$$u_s = u_{tt} + 2u^2 v\ , \qquad -v_s = v_{tt} + 2uv^2\ .$$

Miraculously, this procedure works for *all* equations of the list of integrable Newtonian Toda–like systems.

5.2. The discrete time Toda lattice (5.1.3), in the Lagrangian form (5.2.8), (5.2.9), was found by Toda and Wadati (1975), Wadati and Toda (1975) as a Bäcklund transformation for the Toda lattice. The Lagrangian function appeared as a generating function of the Bäcklund transformation, which demonstrated also the symplectic nature of this transformation. Notice that in the infinite lattice situation this Bäcklund transformation is, generically, no longer isospectral; rather, it adds one soliton to the solution. These results are presented also in Hirota and Satsuma (1978), Toda (1989), Gaudin (1983). See also Kuznetsov and Sklyanin (1998), where a novel property of spectrality of Bäcklund transformation is introduced.

The 2×2 Lax representation of the Toda lattice with the matrices (5.2.18), (5.2.19) is due to Faddeev with co-workers, see Takhtajan and Faddeev (1979), Faddeev and Takhtajan (1987). The 2×2 Lax representation of the discrete time Toda lattice with the matrices (5.2.18), (5.2.20) is essentially due to Gaudin (1983). The relation of the $N \times N$ and the 2×2 Lax representations of the discrete time Toda lattice is discussed in detail in Kuznetsov and Sklyanin (1998).

5.3–5.10. The discrete time Newtonian systems (5.1.6), (5.1.8), (5.1.9), and (5.1.17) were found in Suris (1997b), while (5.1.4) and the discretization of the second Toda flow (5.10.9) were found in Suris (1999a). The systems (5.1.11), (5.1.15) may be extracted from Adler (1998, 2000) as particular cases of the corresponding relativistic systems; their relations to the flow dTL is elaborated here for the first time. All 2×2 Lax representations, both continuous and discrete time, given in this Chapter, are new (except, of course, those for the exponential Toda lattice).

Chapter 6

Relativistic Toda lattice

Contents

6.1 Introduction

The system known nowadays as the *relativistic Toda lattice* (RTL) was invented by S.N.M. Ruijsenaars in the form of the following Newtonian equations of motion:

$$\ddot{x}_k = (1+\alpha\dot{x}_{k+1})(1+\alpha\dot{x}_k)\,\frac{e^{x_{k+1}-x_k}}{1+\alpha^2 e^{x_{k+1}-x_k}} - (1+\alpha\dot{x}_k)(1+\alpha\dot{x}_{k-1})\,\frac{e^{x_k-x_{k-1}}}{1+\alpha^2 e^{x_k-x_{k-1}}}\ . \quad (6.1.1)$$

As usual, this system may be considered on an infinite lattice ($k \in \mathbb{Z}$), or on a finite one ($1 \le k \le N$). In the latter case open–end or periodic boundary conditions may be imposed, they look just as for the usual Toda lattice. Here α is a (small) parameter of the model, having physical meaning of the inverse speed of light. The non-relativistic limit to the usual (exponential) Toda lattice (5.1.1) is achieved by sending $\alpha \to 0$. So, (6.1.1) is a one–parameter perturbation of the usual Toda lattice.

Many properties of the Toda lattice have their relativistic counterparts. In particular, one can introduce the change of variables $(x,\dot{x}) \mapsto (a,b)$ which is similar to the Flaschka's one (3.1.2), and is at the same time its $O(\alpha)$–perturbation:

$$1+\alpha b_k = \frac{1+\alpha\dot{x}_k}{1+\alpha^2 e^{x_{k+1}-x_k}}\ , \qquad a_k = (1+\alpha b_k)\, e^{x_{k+1}-x_k}\ . \quad (6.1.2)$$

The equations satisfied by b_k, a_k read:

$$\dot{b}_k = (1+\alpha b_k)(a_k - a_{k-1})\ , \qquad \dot{a}_k = a_k(b_{k+1} - b_k + \alpha a_{k+1} - \alpha a_{k-1})\ , \quad (6.1.3)$$

which may be considered as a one–parameter perturbation of the flow TL. This system turns out to have much richer structure than the original Newtonian equations. It is this system (6.1.3) we shall be mainly occupied with in this Chapter. We use an abbreviated name $\mathrm{RTL}_+(\alpha)$ for this flow. The explanation of the "+" sign in this abbreviation is as follows. The system (6.1.3) is the simplest (the "first") member of a hierarchy of commuting flows. This hierarchy contains one more flow with equations of motion as simple as those of $\mathrm{RTL}_+(\alpha)$:

$$\dot{b}_k = \frac{a_k}{1+\alpha b_{k+1}} - \frac{a_{k-1}}{1+\alpha b_{k-1}}\ , \qquad \dot{a}_k = a_k\left(\frac{b_{k+1}}{1+\alpha b_{k+1}} - \frac{b_k}{1+\alpha b_k}\right)\ , \quad (6.1.4)$$

which is another one–parameter perturbation of TL. We shall see that it is natural to call this flow the "minus first" flow of the $\mathrm{RTL}(\alpha)$ hierarchy, and to assign an abbreviation $\mathrm{RTL}_-(\alpha)$ to it. Note in passing that the system (6.1.4) arises from Newtonian equations slightly distinct from (6.1.1). More precisely, consider the system

$$\ddot{x}_k = (1-\alpha\dot{x}_{k+1})(1-\alpha\dot{x}_k)\,\frac{e^{x_{k+1}-x_k}}{1+\alpha^2 e^{x_{k+1}-x_k}} - (1-\alpha\dot{x}_k)(1-\alpha\dot{x}_{k-1})\,\frac{e^{x_k-x_{k-1}}}{1+\alpha^2 e^{x_k-x_{k-1}}}\ . \quad (6.1.5)$$

Basically, (6.1.1) and (6.1.5) are sent into one another by changing the sign of α, or, alternatively, by inverting the direction of time, $t \mapsto -t$. Introduce the change of variables $(x,\dot{x}) \mapsto (a,b)$ according to:

$$1+\alpha b_k = \frac{1+\alpha^2 e^{x_k-x_{k-1}}}{1-\alpha\dot{x}_k}\ , \qquad a_k = (1+\alpha b_k)\, e^{x_{k+1}-x_k}\ . \quad (6.1.6)$$

Then the evolution of the (a,b) variables in virtue of (6.1.5) is given by (6.1.4).

Both systems $\mathrm{RTL}_\pm(\alpha)$ may be considered either under open–end boundary conditions ($a_0 = a_N = 0$), or under periodic ones (all the subscripts are taken (mod N), so that $b_{N+1} \equiv b_1$, $b_0 \equiv b_N$, $a_{N+1} \equiv a_1$, $a_0 \equiv a_N$).

These two flows have very rich properties, similar to those of the usual Toda lattice, i.e. of the flow TL. At first glance, the "relativistic" α–deformation is much like the modification. Indeed, the equations (6.1.3) of $\mathrm{RTL}_+(\alpha)$ look similar to (and slightly more complicated than) the equations (3.1.5) of $\mathrm{MTL}(\alpha)$. However, this deformation is of a very different nature. It is not generated by any Miura–like transformation, and is not accompanied by a lost of local invariant Poisson structures. Like the flow TL, its relativistic deformations $\mathrm{RTL}_\pm(\alpha)$ possess three local invariant Poisson brackets, discussed in Sect. 6.5. Different parametrizations of invariant Poisson structures via canonically conjugate variables lead to a series of integrable Newtonian equations, including (6.1.1), (6.1.5). These will constitute the subject of Chapter 8.

As for Lax representations of the flows $\mathrm{RTL}_{\pm}(\alpha)$, there exist two apparently different ones. The first one of them, introduced in Sect. 6.2, has an advantage of being very general. Actually, it is applicable to any system governed by a standard Lax representation, and delivers its "relativistic" deformation. Its r–matrix interpretation may be given in terms of generalized linear r–matrix structures, see Sect. 6.3. Also an interpretation in terms of quadratic r–matrix structures is possible, and leads to amasing conclusions, namely, that the quadratic invariant Poisson brackets for all the relativistic deformations *literally coincide* with those of their nonrelativistic counterparts. These results are given in Sect. 6.4.

Although the first Lax representation of the $\mathrm{RTL}(\alpha)$ hierarchy has several advantageos features, it is not well suited for the purpose of discretization. To this end the second Lax representation is introduced in Sect. 6.6. This Lax representation is of a standard type, and can be also given an r–matrix interpretation in terms of the (standard) linear and the quadratic r–matrix structures on $\boldsymbol{g}$, see Sect. 6.7, 6.8.

(Also 2×2 Lax representations are available, see Sect. 6.13, but they do not play an essential role in our presentation).

The second Lax representations of the both flows $\mathrm{RTL}_{\pm}(\alpha)$ serve as welcome objects for applying our general approach to integrable discretization. This is done for the flow $\mathrm{RTL}_{+}(\alpha)$ in Sect. 6.9, and for the flow $\mathrm{RTL}_{-}(\alpha)$ in Sect. 6.11. As usual, discretizations thus obtained lose the property of being described by local equations of motion. Nevertheless, the general device of localizing changes of variables is applicable and leads to beautiful local discretizations. The local discretization of $\mathrm{RTL}_{+}(\alpha)$ is obtained in Sect. 6.10, and reads:

$$\left\{\begin{array}{l} (1+\alpha\widetilde{\mathbf{b}}_k)(1+h\alpha\widetilde{\mathbf{a}}_{k-1})=(1+\alpha\mathbf{b}_k)(1+h\alpha\mathbf{a}_k)\;, \\[2mm] \widetilde{\mathbf{a}}_k(1+h\widetilde{\mathbf{b}}_k)(1+h\alpha\widetilde{\mathbf{a}}_{k-1})=\mathbf{a}_k(1+h\mathbf{b}_{k+1})(1+h\alpha\mathbf{a}_{k+1})\;, \end{array}\right. \tag{6.1.7}$$

while the local discretization of $\mathrm{RTL}_{-}(\alpha)$ is found in Sect. 6.12:

$$\left\{\begin{array}{l} \widetilde{\mathbf{b}}_k+\dfrac{h\widetilde{\mathbf{a}}_{k-1}}{1+\alpha\widetilde{\mathbf{b}}_{k-1}}=\mathbf{b}_k+\dfrac{h\mathbf{a}_k}{1+\alpha\mathbf{b}_{k+1}}\;, \\[4mm] \widetilde{\mathbf{a}}_k\left(1+\dfrac{h\widetilde{\mathbf{b}}_k}{1+\alpha\widetilde{\mathbf{b}}_k}\right)=\mathbf{a}_k\left(1+\dfrac{h\mathbf{b}_{k+1}}{1+\alpha\mathbf{b}_{k+1}}\right) \end{array}\right. \tag{6.1.8}$$

Actually, the variables $(\mathbf{a},\mathbf{b})$ in these formulas are related to (a,b) by two different Miura transformations, so that the discretizations (6.1.7), (6.1.8) do not belong to the $\mathrm{RTL}(\alpha)$ hierarchy anymore, but rather to two apparently different *modified* $\mathrm{RTL}(\alpha)$ hierarchies. As a matter of fact, these two hierarhies turn out to be not really different. They are related to one another by means of an (invertible) change of variables (which intertwins the above mentioned Miura transformations), so that is sufficient to study one of them.

The first flow of the hierarchy $\mathrm{MRTL}(\alpha;\epsilon)$ corresponding to the "+"–version of the Miura map reads:

$$\left\{\begin{array}{l} \dot{b}_k=(1+\epsilon b_k)(1+\alpha b_k)(a_k-a_{k-1})\;, \\[2mm] \dot{a}_k=a_k(1+\epsilon\alpha a_k)\Big(b_{k+1}+\alpha a_{k+1}+\epsilon\alpha b_{k+1}a_{k+1}-b_k-\alpha a_{k-1}-\epsilon\alpha b_k a_{k-1}\Big). \end{array}\right. \tag{6.1.9}$$

The discretization (6.1.7) belongs to the hierarchy $\mathrm{MRTL}(\alpha;h)$. At first glance, the above flow reminds the double modified Toda flow $\mathrm{M^2TL}(\alpha,\epsilon)$. However, in (6.1.9) the parameters α and ϵ enter not quite symmetrically. More important, in the course of modification one local invariant Poisson structure goes lost, so that (6.1.9) is bi–Hamiltonian, see Sect. 6.14. This is opposed to $\mathrm{M^2TL}(\alpha,\epsilon)$ possessing only one local invariant Poisson structure.

The natural Lax representation for the flow (6.1.9) (and for the whole $\mathrm{MRTL}(\alpha;\epsilon)$ hierarchy) lives in $\boldsymbol{g}\otimes\boldsymbol{g}\otimes\boldsymbol{g}$, see Sect. 6.16. This Lax representation belongs to the class covered by our recipe. Hence we perform the discretization of (6.1.9) in Sect. 6.18, and find the corresponding localizing changes of variables in Sect. 6.19. The local discretization of the system (6.1.9) reads:

$$\left\{\begin{array}{l} \dfrac{1+\epsilon\widetilde{\mathbf{b}}_k}{1+\alpha\widetilde{\mathbf{b}}_k}\cdot\dfrac{1+h\epsilon\widetilde{\mathbf{a}}_{k-1}}{1+h\alpha\widetilde{\mathbf{a}}_{k-1}}=\dfrac{1+\epsilon\mathbf{b}_k}{1+\alpha\mathbf{b}_k}\cdot\dfrac{1+h\epsilon\mathbf{a}_k}{1+h\alpha\mathbf{a}_k}\,, \\[4mm] \dfrac{\widetilde{\mathbf{a}}_k}{1+\epsilon\alpha\widetilde{\mathbf{a}}_k}\cdot\dfrac{(1+h\widetilde{\mathbf{b}}_k)(1+h\alpha\widetilde{\mathbf{a}}_{k-1})}{1-h\epsilon\alpha\widetilde{\mathbf{b}}_k\widetilde{\mathbf{a}}_{k-1}}=\dfrac{\mathbf{a}_k}{1+\epsilon\alpha\mathbf{a}_k}\cdot\dfrac{(1+h\mathbf{b}_{k+1})(1+h\alpha\mathbf{a}_{k+1})}{1-h\epsilon\alpha\mathbf{b}_{k+1}\mathbf{a}_{k+1}}\,. \end{array}\right. \tag{6.1.10}$$

As usual, the localizing change of variables leading to (6.1.10) may be used also in order to further modify the flow (6.1.9). The double modified relativistic Toda lattice $\mathrm{M}^2\mathrm{RTL}_+(\alpha;\epsilon,\delta)$ obtained on this way reads:

$$\dot{b}_k = \frac{(1+\alpha b_k)(1+\epsilon b_k)(1+\delta b_k)(a_k-a_{k-1})}{1-\delta\epsilon\alpha D(a_k,b_k,a_{k-1})}\,, \tag{6.1.11}$$

$$\begin{aligned} \dot{a}_k &= a_k(1+\epsilon\alpha a_k)(1+\delta\alpha a_k)(1+\delta\epsilon a_k)\bigg(\frac{b_{k+1}+\alpha a_{k+1}+\alpha(\epsilon+\delta)b_{k+1}a_{k+1}}{1-\delta\epsilon\alpha D(a_{k+1},b_{k+1},a_k)}- \\ &\quad -\frac{b_k+\alpha a_{k-1}+\alpha(\epsilon+\delta)b_k a_{k-1}}{1-\delta\epsilon\alpha D(a_k,b_k,a_{k-1})}\bigg)\,, \end{aligned} \tag{6.1.12}$$

where

$$D(a_k,b_k,a_{k-1}) = a_k b_k + b_k a_{k-1} + \alpha a_k a_{k-1} + \alpha(\epsilon+\delta)a_k b_k a_{k-1}\,. \tag{6.1.13}$$

Notice that this system is, as usual, symmetric with respect to flipping the modification parameters $\epsilon\leftrightarrow\delta$, and that the parameter α playes an asymmetric role. In the "nonrelativistic limit" $\alpha\to 0$ one recovers the double modified Toda lattice $\mathrm{M}^2\mathrm{TL}(\epsilon,\delta)$. Notice that the system $\mathrm{M}^2\mathrm{RTL}_+(\alpha;\epsilon,\delta)$ resembles the triple modification of the Toda lattice $\mathrm{M}^3\mathrm{TL}_+(\alpha,\epsilon,\delta)$ but has somewhat different properties. In particular, it admits one local invariant Hamiltonian structure. We remark also that the discretization (6.1.10) belongs to the hierarchy $\mathrm{M}^2\mathrm{RTL}(\alpha;\epsilon,h)$.

6.21 Bibliographical remarks

6.1. The relativistic Toda lattice (6.1.1) was introduced by Ruijsenaars (1990). Early references, concerning inverse scattering and finite gap solutions, Lax representations and so on, include: Bruschi and Ragnisco (1988, 1989a,b), Alber (1989). (Notice that Ruijsenaars' work circulated a couple of years as a preprint, and was published later than the papers just mentioned). Lie–algebraic generalizations were introduced in Suris (1990b), see also Kuznetsov and Tsiganov (1993). A recent review where RTL is included in a broad physical context is Kharchev, Mironov, and Zhedanov (1995).

6.2–6.5. Our first Lax representation of the relativistic Toda lattice is a particular case of the relativistic deformations of Lax operaotors introduced in Gibbons and Kupershmidt (1990), see also Kupershmidt (1993). In a form without a small parameter, this Lax representation was present already in Bruschi and Ragnisco (1988, 1989a). The linear r–matrix structure was found in the open–end case without a small parameter in Oevel and Ragnisco (1989), our presentation of the general case in Sect. 6.7 follows Suris (1999d). The quadratic r–matrix structure in Sect. 6.8 was found for the first time in Suris (1999d) (see the discussion of earlier unsuccessful attempts in this direction in Oevel and Ragnisco (1989), Morosi (1992)).

The tri–Hamiltonian structure was elaborated in Bruschi and Ragnisco (1988, 1989a) and in Oevel *et al.* (1990), Zhang *et al.* (1991). The latter two papers are based on the master symmetries approach, see in this respect also Damianou (1994) and Nunes da Costa and Marlet (1997).

6.6–6.8. The second Lax representation in terms of $(L,W)\in\boldsymbol{g}\otimes\boldsymbol{g}$ was invented in Suris (1991), although the monodromy matrices $T_{1,2}$ were introduced already in Ruijsenaars (1990). The linear and the quadratic r–matrix structures for this Lax representation were elaborated in Suris (1990, 1993b). The interpretation of the corresponding Lax matrix as lying on an orbit of a Lie–Poisson group, found in the first of these papers, was further developed in Fock and Marshakov (1997) and generalized for all simple Lie groups in Hoffmann *et al.* (2000).

6.9–6.12. The discretization of the relativistic Toda lattice was first performed in Suris (1996a). This was the first nontrivial application of our general recipe of integrable discretization. The localizing changes of variables were found in Suris (1999c).

6.13. 2×2 Lax representations for RTL appeared for the first time in Suris (1990b).

6.14–6.20. All results on the modified relativistic Toda lattice and its discretization are new and published here for the first time.

Chapter 7

Relativistic Volterra lattice

Contents

7.1 Introduction

The previous Chapter was devoted to the study of the relativistic Toda lattice. We traced several points of the similarity between this system and the usual Toda lattice. However, one such point remained untouched, and we turn our attention to it now. The usual Toda lattice (TL) is Miura related to the two–field Volterra lattice (VL). It turns out that the *same* Miura transformations relate the flows RTL$_\pm(\alpha)$ to certain systems which are one–parameter perturbations of VL, and can therefore be called *relativistic Volterra lattices* and denoted RVL$_\pm(\alpha)$, see Sect. 7.2. An interesting phenomenon here is that each one of the flows RTL$_+(\alpha)$, RTL$_-(\alpha)$, being pulled back under two companion Miura transformations, results in two slightly different systems, so that actually each one of the systems RVL$_+(\alpha)$, RVL$_-(\alpha)$ splits into two versions. We give here those ones of RVL$_\pm(\alpha)$

which are related to the Miura map M_2. The flow $\mathrm{RVL}_+(\alpha)$, i.e. the pull–back of $\mathrm{RTL}_+(\alpha)$ under M_2, reads:

$$\left\{\begin{array}{l} \dot{u}_k = u_k(v_k - v_{k-1} + \alpha u_{k+1}v_k - \alpha u_k v_{k-1})\;, \\[2ex] \dot{v}_k = v_k(u_{k+1} - u_k + \alpha u_{k+1}v_k - \alpha u_k v_{k-1})\;. \end{array}\right. \tag{7.1.1}$$

The flow $\mathrm{RVL}_-(\alpha)$, i.e. the pull–back of $\mathrm{RTL}_-(\alpha)$ under M_2, reads:

$$\left\{\begin{array}{l} \dot{u}_k = u_k\left(\dfrac{v_k}{1+\alpha(u_k+v_k)} - \dfrac{v_{k-1}}{1+\alpha(u_{k-1}+v_{k-1})}\right), \\[3ex] \dot{v}_k = v_k\left(\dfrac{u_{k+1}}{1+\alpha(u_{k+1}+v_{k+1})} - \dfrac{u_k}{1+\alpha(u_k+v_k)}\right). \end{array}\right. \tag{7.1.2}$$

These two systems are the simplest flows of a bi–Hamiltonian hierarchy, i.e. they have two invariant local Poisson structures, discussed in Sect. 7.3, and commute with one another.

There exist several Lax representations of the $\mathrm{RVL}(\alpha)$ hierarchy. The first one of them (Sect. 7.5) appears as an application of the general relativistic ansatz. An r–matrix interpretation of this Lax representation is given in Sect. 7.6. Having an advantage of a high generality, this ansatz leads to nonstandard Lax equations. We get rid of this drawback by using other Lax representations in $\boldsymbol{g}\otimes\boldsymbol{g}\otimes\boldsymbol{g}$ (Sect. 7.7, 7.8), ideally suited for applying our general recipe for discretization. We perform the discretization of $\mathrm{RVL}_\pm(\alpha)$ in Sect. 7.10, 7.12, and find the corresponding localizing changes of variables in Sect. 7.11, 7.13. For example, the local discretization of $\mathrm{RVL}_+(\alpha)$ reads:

$$\left\{\begin{array}{l} \widetilde{\mathbf{u}}_k\,\dfrac{1+h\widetilde{\mathbf{v}}_{k-1}}{1-h\alpha\widetilde{\mathbf{u}}_k\widetilde{\mathbf{v}}_{k-1}} = \mathbf{u}_k\,\dfrac{1+h\mathbf{v}_k}{1-h\alpha\mathbf{u}_{k+1}\mathbf{v}_k}\;, \\[3ex] \widetilde{\mathbf{v}}_k\,\dfrac{1+h\widetilde{\mathbf{u}}_k}{1-h\alpha\widetilde{\mathbf{u}}_k\widetilde{\mathbf{v}}_{k-1}} = \mathbf{v}_k\,\dfrac{1+h\mathbf{u}_{k+1}}{1-h\alpha\mathbf{u}_{k+1}\mathbf{v}_k}\;. \end{array}\right. \tag{7.1.3}$$

This discretization belongs to the *modified relativistic Volterra* hierarchy $\mathrm{MRVL}(\alpha;h)$ with the modification parameter h. The simplest flow of the general $\mathrm{MRVL}(\alpha;\epsilon)$ hierarchy with the modification parameter ϵ reads (Sect. 7.14):

$$\left\{\begin{array}{l} \dot{u}_k = u_k(1+\epsilon u_k)\left(\dfrac{v_k+\alpha u_{k+1}v_k}{1-\epsilon\alpha u_{k+1}v_k} - \dfrac{v_{k-1}+\alpha u_k v_{k-1}}{1-\epsilon\alpha u_k v_{k-1}}\right)\;, \\[3ex] \dot{v}_k = v_k(1+\epsilon v_k)\left(\dfrac{u_{k+1}+\alpha u_{k+1}v_k}{1-\epsilon\alpha u_{k+1}v_k} - \dfrac{u_k+\alpha u_k v_{k-1}}{1-\epsilon\alpha u_k v_{k-1}}\right)\;. \end{array}\right. \tag{7.1.4}$$

This system possesses one local invariant Poisson structure (in agreement with the general expectations). Its discretization is performed in Sect. 7.15.

7.16 Bibliographical remarks

The relativistic Volterra lattice was introduced in Suris and Ragnisco (1999) along with the "first" and the "third" Lax representations (cf. Sect. 7.5 and 7.8). The discretization of this system was achieved (including the localizing change of variables) in Suris (1999c). The results on the modified relativistic Volterra lattice are new.

Chapter 8

Newtonian equations of the relativistic Toda type

Contents

8.1 Introduction

Recall that the relativistic Toda lattice invented by Ruijsenaars, was actually described by Newtonian equations of motion

$$\ddot{x}_k = (1+\alpha\dot{x}_{k+1})(1+\alpha\dot{x}_k)\frac{e^{x_{k+1}-x_k}}{1+\alpha^2 e^{x_{k+1}-x_k}} - (1+\alpha\dot{x}_k)(1+\alpha\dot{x}_{k-1})\frac{e^{x_k-x_{k-1}}}{1+\alpha^2 e^{x_k-x_{k-1}}} . \quad (8.1.1)$$

We will see that these Newtonian equations appear as a result of parametrizing the bracket $\{\cdot,\cdot\}_1 + \alpha\{\cdot,\cdot\}_2$ for the phase variables (a,b) of $\mathrm{RTL}_+(\alpha)$ via canonically conjugate variables. The same parametrization produces from $\mathrm{RTL}_-(\alpha)$ the Newtonian equations

$$\ddot{x}_k = (1-\alpha\dot{x}_{k+1})(1-\alpha\dot{x}_k)\frac{e^{x_{k+1}-x_k}}{1+\alpha^2 e^{x_{k+1}-x_k}} - (1-\alpha\dot{x}_k)(1-\alpha\dot{x}_{k-1})\frac{e^{x_k-x_{k-1}}}{1+\alpha^2 e^{x_k-x_{k-1}}} , \quad (8.1.2)$$

which differ only by the sign of α, or, what is equivalent, by the change of time direction.

If we look for the maps $\mathrm{dRTL}_\pm(\alpha)$ in the same parametrization, then we come to two systems of *discrete time Newtonian equations of motion*:

$$\frac{1+\alpha h^{-1}\Big(e^{\widetilde{x}_k-x_k}-1\Big)}{1+\alpha h^{-1}\Big(e^{x_k-\underset{\sim}{x}_k}-1\Big)}=\frac{\Big(1+\alpha^2 e^{x_{k+1}-x_k}\Big)\Big(1+\alpha(\alpha-h)e^{x_k-\widetilde{x}_{k-1}}\Big)}{\Big(1+\alpha^2 e^{x_k-x_{k-1}}\Big)\Big(1+\alpha(\alpha-h)e^{\underset{\sim}{x}_{k+1}-x_k}\Big)}\,, \tag{8.1.3}$$

and

$$\frac{1-\alpha h^{-1}\Big(e^{\widetilde{x}_k-x_k}-1\Big)}{1-\alpha h^{-1}\Big(e^{x_k-\underset{\sim}{x}_k}-1\Big)}=\frac{\Big(1+\alpha^2 e^{x_{k+1}-x_k}\Big)\Big(1+\alpha(\alpha+h)e^{x_k-\widetilde{x}_{k-1}}\Big)}{\Big(1+\alpha^2 e^{x_k-x_{k-1}}\Big)\Big(1+\alpha(\alpha+h)e^{\underset{\sim}{x}_{k+1}-x_k}\Big)}\,, \tag{8.1.4}$$

respectively. Remarkably, the relation between these two systems mimics the relation between their continuous time counterparts: one comes from (8.1.3) to (8.1.4) by changing the sign of either α or h.

But, as already mentioned, the tri–Hamiltonian nature of the $\mathrm{RTL}(\alpha)$ hierarchy is responsible for existence of a very vast set of different Newtonian systems hidden behind the flows $\mathrm{RTL}_\pm(\alpha)$. Namely, not only the above Poisson bracket may be parametrized via canonically conjugate variables, but also other linear combinations of the three invariant Poisson brackets. We list in this Section the most important results which can be obtained along this path. They always can be interpreted as α–perturbations of the corresponding results for the usual Toda lattice.

Parametrizing the linear bracket $\{\cdot,\cdot\}_1$ for the (a,b) variables, we find that this can be done in two dual ways, and in each one of these parametrizations the Newtonian equations for $\mathrm{RTL}_+(\alpha)$ and $\mathrm{RTL}_-(\alpha)$ do not coincide anymore. The first parametrization leads, as in the case of the usual Toda lattice, to equations with exponential interactions. We find for the $\mathrm{RTL}_+(\alpha)$ flow the Newtonian system

$$\ddot{x}_k=(1+\alpha\dot{x}_{k+1})\,e^{x_{k+1}-x_k}-(1+\alpha\dot{x}_{k-1})\,e^{x_k-x_{k-1}}-\alpha^2 e^{2(x_{k+1}-x_k)}+\alpha^2 e^{2(x_k-x_{k-1})}\,, \tag{8.1.5}$$

and for the $\mathrm{RTL}_-(\alpha)$ flow a substantially different system

$$\ddot{x}_k=(1-\alpha\dot{x}_k)^2\Big((1-\alpha\dot{x}_{k+1})\,e^{x_{k+1}-x_k}-(1-\alpha\dot{x}_{k-1})\,e^{x_k-x_{k-1}}\Big)\,. \tag{8.1.6}$$

The map $\mathrm{dRTL}_+(\alpha)$ in this parametrization delivers the following discretization for (8.1.5):

$$\begin{aligned} e^{\widetilde{x}_k-x_k}-e^{x_k-\underset{\sim}{x}_k} &= h\alpha e^{x_{k+1}-x_k}-h\alpha e^{x_k-x_{k-1}}\\ &\quad -\frac{h(\alpha-h)\,e^{\underset{\sim}{x}_{k+1}-x_k}}{1-h\alpha e^{\underset{\sim}{x}_{k+1}-x_k}}+\frac{h(\alpha-h)\,e^{x_k-\widetilde{x}_{k-1}}}{1-h\alpha e^{x_k-\widetilde{x}_{k-1}}}\,, \end{aligned} \tag{8.1.7}$$

while the map $\mathrm{dRTL}_-(\alpha)$ becomes a discretization for (8.1.6):

$$\begin{aligned} &\frac{1}{1-\alpha h^{-1}\Big(e^{\widetilde{x}_k-x_k}-1\Big)}-\frac{1}{1-\alpha h^{-1}\Big(e^{x_k-\underset{\sim}{x}_k}-1\Big)}=\\ &=\alpha(\alpha+h)e^{\underset{\sim}{x}_{k+1}-x_k}-\alpha(\alpha+h)e^{x_k-\widetilde{x}_{k-1}}-\alpha^2 e^{x_{k+1}-x_k}+\alpha^2 e^{x_k-x_{k-1}}\,. \end{aligned} \tag{8.1.8}$$

The dual parametrization of the linear bracket leads to Newtonian systems with rational interactions:

$$\ddot{x}_k=\dot{x}_k\,(x_{k+1}-2x_k+x_{k-1})+\frac{\alpha\dot{x}_{k+1}\dot{x}_k}{1+\alpha(x_{k+1}-x_k)}-\frac{\alpha\dot{x}_k\dot{x}_{k-1}}{1+\alpha(x_k-x_{k-1})}\,, \tag{8.1.9}$$

and

$$\ddot{x}_k=\dot{x}_k(1+\alpha^2\dot{x}_k)\left(\frac{x_{k+1}-x_k-\alpha\dot{x}_{k+1}}{1+\alpha(x_{k+1}-x_k)}-\frac{x_k-x_{k-1}-\alpha\dot{x}_{k-1}}{1+\alpha(x_k-x_{k-1})}\right)\,, \tag{8.1.10}$$

for $\mathrm{RTL}_+(\alpha)$ and $\mathrm{RTL}_-(\alpha)$, respectively. The corresponding discretizations appear from the maps $\mathrm{dRTL}_+(\alpha)$ and $\mathrm{dRTL}_-(\alpha)$ under the same parametrization:

$$\frac{\widetilde{x}_k - x_k}{x_k - \underset{\sim}{x}_k} = \frac{\Big(1 + h(\underset{\sim}{x}_{k+1} - x_k)\Big)}{\Big(1 + \alpha(\underset{\sim}{x}_{k+1} - x_k)\Big)} \cdot \frac{\Big(1 + \alpha(x_k - \widetilde{x}_{k-1})\Big)}{\Big(1 + h(x_k - \widetilde{x}_{k-1})\Big)} \cdot \frac{\Big(1 + \alpha(x_{k+1} - x_k)\Big)}{\Big(1 + \alpha(x_k - x_{k-1})\Big)}\,, \tag{8.1.11}$$

and

$$\frac{(\widetilde{x}_k - x_k)\Big(1 + \alpha(\alpha+h)h^{-1}(x_k - \underset{\sim}{x}_k)\Big)}{(x_k - \underset{\sim}{x}_k)\Big(1 + \alpha(\alpha+h)h^{-1}(\widetilde{x}_k - x_k)\Big)} =$$

$$= \frac{\Big(1 + \alpha(x_{k+1} - x_k)\Big)}{\Big(1 + \alpha(x_k - x_{k-1})\Big)} \cdot \frac{\Big(1 + (\alpha+h)(x_k - \widetilde{x}_{k-1})\Big)}{\Big(1 + (\alpha+h)(\underset{\sim}{x}_{k+1} - x_k)\Big)}\,, \tag{8.1.12}$$

respectively.

Looking for further invariant brackets of the $\mathrm{RTL}(\alpha)$ hierarchy allowing for a canonical parametrization, we come to the bracket $\{\cdot,\cdot\}_2$. As in the nonrelativistic case (where this bracket led to the modified exponential Toda lattice), it is more convenient to work with the Miura transformed (Volterra) lattice. Under the corresponding parametrization the flow $\mathrm{RVL}_+(\alpha)$ leads to the Newtonian system

$$\begin{aligned} \ddot{x}_k &= \dot{x}_k\Big(e^{x_{k+1}-x_k} - e^{x_k - x_{k-1}}\Big) \\ &\quad + \alpha\left(\dot{x}_{k+1}\dot{x}_k\,\frac{e^{x_{k+1}-x_k}}{1+\alpha e^{x_{k+1}-x_k}} - \dot{x}_k\dot{x}_{k-1}\,\frac{e^{x_k-x_{k-1}}}{1+\alpha e^{x_k-x_{k-1}}}\right), \end{aligned} \tag{8.1.13}$$

while the flow $\mathrm{RVL}_-(\alpha)$ gives rize to the following Newtonian system:

$$\ddot{x}_k = \dot{x}_k(1-\alpha\dot{x}_k)\left((1-\alpha\dot{x}_{k+1})\,\frac{e^{x_{k+1}-x_k}}{1+\alpha e^{x_{k+1}-x_k}} - (1-\alpha\dot{x}_{k-1})\,\frac{e^{x_k-x_{k-1}}}{1+\alpha e^{x_k-x_{k-1}}}\right). \tag{8.1.14}$$

These both systems may be considered as $O(\alpha)$–perturbations of the modified exponential Toda lattice.

Integrable discretizations of these systems are obtained via representing the maps $\mathrm{dRVL}_\pm(\alpha)$ in the same parametrization. The map $\mathrm{dRVL}_+(\alpha)$ leads to the discretization of (8.1.13):

$$\frac{\Big(e^{\widetilde{x}_k - x_k} - 1\Big)}{\Big(e^{x_k - \underset{\sim}{x}_k} - 1\Big)} = \frac{\Big(1 + he^{\underset{\sim}{x}_{k+1} - x_k}\Big)\Big(1 + \alpha e^{x_k - \widetilde{x}_{k-1}}\Big)\Big(1 + \alpha e^{x_{k+1}-x_k}\Big)}{\Big(1 + \alpha e^{\underset{\sim}{x}_{k+1} - x_k}\Big)\Big(1 + he^{x_k - \widetilde{x}_{k-1}}\Big)\Big(1 + \alpha e^{x_k - x_{k-1}}\Big)}\,, \tag{8.1.15}$$

while the map $\mathrm{dRVL}_-(\alpha)$ leads to the discretization of (8.1.14):

$$\frac{\Big(e^{\widetilde{x}_k - x_k} - 1\Big)}{\Big(e^{x_k - \underset{\sim}{x}_k} - 1\Big)} \cdot \frac{1 - \alpha h^{-1}\Big(e^{x_k - \underset{\sim}{x}_k} - 1\Big)}{1 - \alpha h^{-1}\Big(e^{\widetilde{x}_k - x_k} - 1\Big)} =$$

$$= \frac{\Big(1 + \alpha e^{x_{k+1}-x_k}\Big)}{\Big(1 + \alpha e^{x_k - x_{k-1}}\Big)} \cdot \frac{\Big(1 + (\alpha+h)\,e^{x_k - \widetilde{x}_{k-1}}\Big)}{\Big(1 + (\alpha+h)\,e^{\underset{\sim}{x}_{k+1} - x_k}\Big)}\,, \tag{8.1.16}$$

Further, we mention two Newtonian systems coming via a canonical parametrization of the bracket $\{\cdot,\cdot\}_{12}$ for the phase variables $(\boldsymbol{q},\boldsymbol{r})$ of the flows $\mathrm{MRTL}_\pm(\alpha;\epsilon)$. These systems are of the same type as the last two ones, but the presence of an additional parameter ϵ gives more freedom. The flow $\mathrm{MRTL}_+(\alpha;\epsilon)$ leads to the following Newtonian system:

$$
\begin{aligned}
\ddot{x}_k &= (1+\epsilon\dot{x}_k)\left((1+\alpha\dot{x}_{k+1})\,\frac{e^{x_{k+1}-x_k}}{1+\epsilon\alpha e^{x_{k+1}-x_k}}-(1+\alpha\dot{x}_{k-1})\,\frac{e^{x_k-x_{k-1}}}{1+\epsilon\alpha e^{x_k-x_{k-1}}}\right.\\
&\left.+\alpha(\epsilon-\alpha)\,\frac{e^{2(x_{k+1}-x_k)}}{1+\epsilon\alpha e^{x_{k+1}-x_k}}-\alpha(\epsilon-\alpha)\,\frac{e^{2(x_k-x_{k-1})}}{1+\epsilon\alpha e^{x_k-x_{k-1}}}\right). \qquad (8.1.17)
\end{aligned}
$$

Notice that this system interpolates between (8.1.5) (which corresponds to $\epsilon=0$) and (8.1.1) (corresponding to $\epsilon=\alpha$).

The same parametrization of the map $\mathrm{dMRTL}_+(\alpha;\epsilon)$ leads to the following discretization of (8.1.17):

$$
\begin{aligned}
&\frac{1+\epsilon h^{-1}\left(e^{\widetilde{x}_k-x_k}-1\right)}{1+\epsilon h^{-1}\left(e^{x_k-\underset{\sim}{x}_k}-1\right)}=\\
&=\frac{\left(1+\epsilon\alpha e^{x_{k+1}-x_k}\right)\left(1+h(\epsilon-\alpha)\,e^{\underset{\sim}{x}_{k+1}-x_k}\right)\left(1+\alpha(\epsilon-h)e^{x_k-\widetilde{x}_{k-1}}\right)}{\left(1+\epsilon\alpha e^{x_k-x_{k-1}}\right)\left(1+h(\epsilon-\alpha)\,e^{x_k-\widetilde{x}_{k-1}}\right)\left(1+\alpha(\epsilon-h)e^{\underset{\sim}{x}_{k+1}-x_k}\right)}. \qquad (8.1.18)
\end{aligned}
$$

Apart from the values $\epsilon=0$, $\epsilon=\alpha$, one more particular value is of a special interest, namely $\epsilon=h$. In this case we get a novel discretization for the flow (8.1.5):

$$
e^{\widetilde{x}_k-2x_k+\underset{\sim}{x}_k}=\frac{\left(1+h\alpha e^{x_{k+1}-x_k}\right)\left(1+h(h-\alpha)\,e^{\underset{\sim}{x}_{k+1}-x_k}\right)}{\left(1+h\alpha e^{x_k-x_{k-1}}\right)\left(1+h(h-\alpha)\,e^{x_k-\widetilde{x}_{k-1}}\right)}. \qquad (8.1.19)
$$

The same parametrization of the flow $\mathrm{MRTL}_-(\epsilon;\alpha)$ leads to the Newtonian system, interpolating between (8.1.6) and (8.1.2):

$$
\begin{aligned}
\ddot{x}_k &= (1-\alpha\dot{x}_k)\Big(1+(\epsilon-\alpha)\dot{x}_k\Big)\left((1-\alpha\dot{x}_{k+1})\,\frac{e^{x_{k+1}-x_k}}{1+\epsilon\alpha e^{x_{k+1}-x_k}}-\right.\\
&\left.-(1-\alpha\dot{x}_{k-1})\,\frac{e^{x_k-x_{k-1}}}{1+\epsilon\alpha e^{x_k-x_{k-1}}}\right). \qquad (8.1.20)
\end{aligned}
$$

From the map $\mathrm{dMRTL}_-(\alpha;\epsilon)$ we derive a discretization of the previous system:

$$
\begin{aligned}
&\frac{1-\alpha h^{-1}\left(e^{x_k-\underset{\sim}{x}_k}-1\right)}{1-\alpha h^{-1}\left(e^{\widetilde{x}_k-x_k}-1\right)}\cdot\frac{1+(\epsilon-\alpha)h^{-1}\left(e^{\widetilde{x}_k-x_k}-1\right)}{1+(\epsilon-\alpha)h^{-1}\left(e^{x_k-\underset{\sim}{x}_k}-1\right)}=\\
&=\frac{\left(1+\epsilon\alpha e^{x_k-x_{k-1}}\right)}{\left(1+\epsilon\alpha e^{x_{k+1}-x_k}\right)}\cdot\frac{\left(1+\epsilon(\alpha+h)\,e^{\underset{\sim}{x}_{k+1}-x_k}\right)}{\left(1+\epsilon(\alpha+h)\,e^{x_k-\widetilde{x}_{k-1}}\right)}. \qquad (8.1.21)
\end{aligned}
$$

This time a significant simplification occurs for $\epsilon=\alpha+h$.

As a last bracket in the relativistic Toda theory, clearly admitting the canonical parametrization, we take the invariant bracket $\{\cdot,\cdot\}_{23}$ for the phase space coordinates (y,z) of the $\mathrm{MRVL}(\alpha;\epsilon)$ hierarchy. Equivalently, one can speak about the linear combination $\{\cdot,\cdot\}_3+4\nu\{\cdot,\cdot\}_2$ of the cubic and the quadratic invariant brackets of the RTL(α) hierarchy, where $4\nu=\epsilon^{-1}$. In the corresponding canonical parametrization the flow MRVL$_+(\alpha;\epsilon)$ leads to the following Newtonian system:

$$\begin{aligned}\ddot{x}_k &= -\frac{1}{2}(\dot{x}_k^2-\nu^2)\left(\frac{\sinh 2(x_{k+1}-x_k)-\nu^{-1}\sinh(2\nu\alpha)\,\dot{x}_{k+1}}{\sinh^2(x_{k+1}-x_k)-\sinh^2(\nu\alpha)}-\right.\\ &\qquad\left.-\frac{\sinh 2(x_k-x_{k-1})-\nu^{-1}\sinh(2\nu\alpha)\,\dot{x}_{k-1}}{\sinh^2(x_k-x_{k-1})-\sinh^2(\nu\alpha)}\right),\end{aligned}\tag{8.1.22}$$

and the map dMRVL$_+(\alpha;\epsilon)$ leads to the following discretization of the above system:

$$\begin{aligned}&\frac{\sinh(\widetilde{x}_k-x_k+\nu h)}{\sinh(\widetilde{x}_k-x_k-\nu h)}\cdot\frac{\sinh(x_k-\underset{\sim}{x}_k-\nu h)}{\sinh(x_k-\underset{\sim}{x}_k+\nu h)}=\\ &\qquad=\frac{\sinh(x_{k+1}-x_k+\nu\alpha)}{\sinh(x_{k+1}-x_k-\nu\alpha)}\cdot\frac{\sinh(x_k-x_{k-1}-\nu\alpha)}{\sinh(x_k-x_{k-1}+\nu\alpha)}\times\\ &\qquad\times\frac{\sinh(\underset{\sim}{x}_{k+1}-x_k-\nu\alpha+\nu h)}{\sinh(\underset{\sim}{x}_{k+1}-x_k+\nu\alpha-\nu h)}\cdot\frac{\sinh(x_k-\widetilde{x}_{k-1}+\nu\alpha-\nu h)}{\sinh(x_k-\widetilde{x}_{k-1}-\nu\alpha+\nu h)}\,.\end{aligned}\tag{8.1.23}$$

The $\nu=0$ particular case of the previous equations corresponds to a canonical parametrization of the pure cubic bracket $\{\cdot,\cdot\}_3$, and results in the following Newtonian equations:

$$\begin{aligned}\ddot{x}_k &= -\dot{x}_k^2\Bigg(\coth(x_{k+1}-x_k)-\coth(x_k-x_{k-1})-\\ &\qquad-\frac{\alpha\dot{x}_{k+1}}{\sinh^2(x_{k+1}-x_k)}+\frac{\alpha\dot{x}_{k-1}}{\sinh^2(x_k-x_{k-1})}\Bigg),\end{aligned}\tag{8.1.24}$$

while the discretization reads:

$$\begin{aligned}&h\coth(\widetilde{x}_k-x_k)-h\coth(x_k-\underset{\sim}{x}_k)=\alpha\coth(x_{k+1}-x_k)-\alpha\coth(x_k-x_{k-1})-\\ &\qquad-(\alpha-h)\coth(\underset{\sim}{x}_{k+1}-x_k)+(\alpha-h)\coth(x_k-\widetilde{x}_{k-1})\,.\end{aligned}\tag{8.1.25}$$

There exists also another interesting limit case of (8.1.22), (8.1.23). One can first rescale in this equations $x_k\mapsto\nu x_k$, $\nu\mapsto\gamma\nu$, and *only then* send $\nu\to 0$. This results in the Newtonian equations:

$$\ddot{x}_k=-(\dot{x}_k^2-\gamma^2)\left(\frac{x_{k+1}-x_k-\alpha\dot{x}_{k+1}}{(x_{k+1}-x_k)^2-\gamma^2\alpha^2}-\frac{x_k-x_{k-1}-\alpha\dot{x}_{k-1}}{(x_k-x_{k-1})^2-\gamma^2\alpha^2}\right),\tag{8.1.26}$$

and their discretization

$$\begin{aligned}&\frac{(\widetilde{x}_k-x_k+\gamma h)}{(\widetilde{x}_k-x_k-\gamma h)}\cdot\frac{(x_k-\underset{\sim}{x}_k-\gamma h)}{(x_k-\underset{\sim}{x}_k+\gamma h)}=\frac{(x_{k+1}-x_k+\gamma\alpha)}{(x_{k+1}-x_k-\gamma\alpha)}\cdot\frac{(x_k-x_{k-1}-\gamma\alpha)}{(x_k-x_{k-1}+\gamma\alpha)}\times\\ &\qquad\times\frac{(\underset{\sim}{x}_{k+1}-x_k-\gamma\alpha+\gamma h)}{(\underset{\sim}{x}_{k+1}-x_k+\gamma\alpha-\gamma h)}\cdot\frac{(x_k-\widetilde{x}_{k-1}+\gamma\alpha-\gamma h)}{(x_k-\widetilde{x}_{k-1}-\gamma\alpha+\gamma h)}\,.\end{aligned}\tag{8.1.27}$$

The system (8.1.26) is dual to (8.1.24), in particular, it also corresponds to a canonical parametrization of the pure cubic bracket $\{\cdot,\cdot\}_3$.

Finally, one can combine both limits described above, i.e. either rescale $x_k\mapsto\nu x_k$ and then send $\nu\to 0$ in the equations (8.1.24), (8.1.25), or, equivalently, send $\gamma\to 0$ in the equations (8.1.26), (8.1.27). This leads to the simplest Newtonian equations, still corresponding to a canonical parametrization of the pure cubic bracket $\{\cdot,\cdot\}_3$:

$$\ddot{x}_k=-\dot{x}_k^2\left(\frac{1}{x_{k+1}-x_k}-\frac{1}{x_k-x_{k-1}}-\frac{\alpha\dot{x}_{k+1}}{(x_{k+1}-x_k)^2}+\frac{\alpha\dot{x}_{k-1}}{(x_k-x_{k-1})^2}\right)\,,\tag{8.1.28}$$

and to their discretization

$$\frac{h}{\widetilde{x}_k - x_k} - \frac{h}{x_k - \underset{\sim}{x}_k} = \frac{\alpha}{x_{k+1} - x_k} - \frac{\alpha}{x_k - x_{k-1}} - \frac{\alpha - h}{\underset{\sim}{x}_{k+1} - x_k} + \frac{\alpha - h}{x_k - \widetilde{x}_{k-1}} . \tag{8.1.29}$$

Let us stress once more that all this variety of Newtonian systems arises from different parametrizations of the (a, b) variables in no more than two continuous time flows and in their two integrable discretizations. It is the multi–Hamiltonian structure of the relativistic Toda hierarchy that finds its manifestation in these different parametrizations.

8.11 Bibliographical remarks

In the paper Ruijsenaars (1990) the term "relativistic Toda lattice" was used for the systems (8.1.1), (8.1.2) only.

The discretization of these systems was performed in Suris (1996a), where the systems equivalent to (8.1.3), (8.1.4) were derived, along with the $N \times N$ Lax representations and solutions in terms of matrix factorizations.

Further, in Suris (1997a) the Newtonian systems (8.1.5) and (8.1.6) were found, along with their discretizations (8.1.7) and (8.1.8), as well as the systems (8.1.13) and (8.1.14) with the discretizations (8.1.15) and (8.1.16), and the systems (8.1.17) and (8.1.20) with the discretizations (8.1.18) and (8.1.21). The "dual" systems (8.1.9) and (8.1.10) with the discretizations (8.1.11) and (8.1.12) appeared in Suris (1999a). Needless to say that the relation with the flows $\mathrm{RTL}_\pm(\alpha)$, resp. with the maps $\mathrm{dRTL}_\pm(\alpha)$ provided a complete understanding for all these systems, including the full set of integrals of motion, $N \times N$ and 2×2 Lax representations etc. Some of the 2×2 Lax representations were found independently by Deconinck (1996).

Along another line of research, Adler and Shabat (1997a,b) achieved a complete classification of "integrable" systems of the type

$$\ddot{x}_k = r(\dot{x}_k)\Big(\dot{x}_{k+1} f(x_{k+1} - x_k) - \dot{x}_{k-1} f(x_k - x_{k-1}) + g(x_{k+1} - x_k) - g(x_k - x_{k-1})\Big) .$$

Their notion of "integrability" is a clever and unexpected device, allowing to carry out a complete classification, but it has, *à priori*, nothing to do with the usual Liouville–Arnold integrability. Namely, they noticed that the above systems are always Lagrangian, and required that their form be retained under a sort of Legendre transformation. This allowed them to find all the continuous time Newtonian equations mentioned above, as well as the system (8.1.22) with its particular cases (8.1.24), (8.1.26), (8.1.28). Moreover, they also studied and classified coupled equations of the nonlinear Schrödinger type related to the above Newtonian lattices (cf. Bibliographical remarks to Chapt. 5). We provide the reader with the simplest example. Consider the lattice equivalent to (8.1.5) (dots stand for derivatives with respect to the time t):

$$\ddot{x}_k = \dot{x}_{k+1} e^{x_{k+1} - x_k} - \dot{x}_{k-1} e^{x_k - x_{k-1}} - e^{2(x_{k+1} - x_k)} + e^{2(x_k - x_{k-1})} ,$$

which is generated by the Hamilton function $\mathrm{H}_2 = \frac{1}{2}\sum p_k^2 + \sum p_k e^{x_{k+1} - x_k}$. Consider also a commutiong flow (with the time denoted by s) with the Hamilton function $\mathrm{H}_3 = \frac{1}{3}\sum p_k^3 + \sum (p_k + p_{k+1}) p_k e^{x_{k+1} - x_k} + \sum p_k e^{x_{k+2} - x_k}$. Denoting $u = e^{x_k}$ and $v = e^{-x_{k-1}}$, one arrives after some straightforward manipulations at the system

$$u_s = u_{tt} - 2u^2 v_t - 2u^3 v^2 , \qquad v_s = -v_{tt} - 2v^2 u_t + 2v^3 u^2 .$$

In the subsequent publication Adler (1999) used a similar approach to perform a classification of "integrable" discrete time Lagrangian systems of the relativistic Toda type. He found, in addition to the discretizations mentioned above, also the discrete time lattice (8.1.23) and its particular cases (8.1.25), (8.1.27), and (8.1.29). As we demonstrate in Sect. 8.7–8.10, these discrete time lattices are integrable also in the usual sense.

Various interrelations between the discrete time Toda and relativistic Toda lattices are explored in Suris (1995), Adler (2000a).

Chapter 9

Explicit discretizations for Toda–type systems

Contents

9.1 Introduction

Recall that the maps dRTL$_+(\alpha)$, dRVL$_+(\alpha)$ introduced in Chapt. 6, 7, depend actually on two parameters: the "relativistic" parameter α (entering into the notations explicitly) and the time step h (suppressed from the notations). If one takes an attentive look at the recurrent relations for the auxiliary quantities participating in the definitions of the maps dRTL$_+(\alpha)$, dRVL$_+(\alpha)$ etc., one sees that there exists a special value of the relativistic parameter α, for which these recurrent relations simplify dramatically. This is the value $\alpha = h$. More precisely, for this value the above mentioned relations cease to be recurrent. Instead, they immediately deliver closed expressions for the auxiliary variables. This allows eventually to put the equations of motion of the maps under consideration into a closed, and morover, an *explicit* form. The same conclusions hold for the maps dRTL$_-(\alpha)$, dRVL$_-(\alpha)$ etc., with $\alpha = -h$. Since the resulting simplified systems are basically the same (up to the change of time direction), we concentrate on the first case.

To be more concrete, the equations of motion of the map dRTL$_+(h)$ may be put in the form

$$\widetilde{\boldsymbol{b}}_k = \boldsymbol{b}_k + h(\boldsymbol{a}_k - \boldsymbol{a}_{k-1})\ , \qquad \widetilde{\boldsymbol{a}}_k = \boldsymbol{a}_k\,\frac{1+h\widetilde{\boldsymbol{b}}_{k+1}}{1+h\widetilde{\boldsymbol{b}}_k} \tag{9.1.1}$$

(the variables $(\boldsymbol{a},\boldsymbol{b})$ are related to (a,b) via a simple invertible change). These equations are explicit, since they allow us to calculate $(\widetilde{\boldsymbol{a}},\widetilde{\boldsymbol{b}})$, once $(\boldsymbol{a},\boldsymbol{b})$ are known. It is crucial that the relativistic parameter α for the map (9.1.1) coincides with the value of the time step h which implicitly enters the definition of the map dRTL$_+(\alpha)$. The map (9.1.1) may be considered as a disretization of the flow TL:

$$\dot b_k = a_k - a_{k-1}\ , \qquad \dot a_k = a_k(b_{k+1} - b_k)\ ,$$

but it belongs to the RTL(h) hierarchy rather than to the TL one. The equations (9.1.1) have to be compared with the implicit discretization

$$\widetilde{\mathbf{b}}_k = \mathbf{b}_k + h(\mathbf{a}_k - \widetilde{\mathbf{a}}_{k-1}) , \qquad \widetilde{\mathbf{a}}_k(1 + h\widetilde{\mathbf{b}}_k) = \mathbf{a}_k(1 + h\mathbf{b}_{k+1}) , \tag{9.1.2}$$

belonging to the (modified) nonrelativistic TL hierarchy. We see that the transition from (9.1.2) to (9.1.1) may be described in the following terms: the variables with the subscript $k-1$ are shifted backwards in time, while the variables with the subscript $k+1$ are shifted forwards in time.

Miura related to the map (9.1.1) is the map dMRTL$_+(h;\epsilon)$. Its most elegant form, achieved after an invertible change of variables $(q,r) \mapsto (\boldsymbol{q},\boldsymbol{r})$, reads:

$$1 + \epsilon\widetilde{\boldsymbol{r}}_k = (1 + \epsilon\boldsymbol{r}_k)\,\frac{1 + h\epsilon\boldsymbol{q}_k}{1 + h\epsilon\boldsymbol{q}_{k-1}} , \qquad \widetilde{\boldsymbol{q}}_k = \boldsymbol{q}_k\,\frac{1 + h\widetilde{\boldsymbol{r}}_{k+1}}{1 + h\widetilde{\boldsymbol{r}}_k} . \tag{9.1.3}$$

Clearly, this is an explicit discretization of the *nonrelativistic* MTL(ϵ) flow,

$$\dot{r}_k = (1 + \epsilon r_k)(q_k - q_{k-1}) , \qquad \dot{q}_k = q_k(r_{k+1} - r_k) .$$

This explicit discretization should be compared with the implicit one (3..17.3), which belongs to the (double) modified TL hierarchy.

Another map Miura related to (9.1.1) is the map dRVL$_+(h)$, whose equations of motion also have in an explicit form:

$$\widetilde{u}_k = u_k\,\frac{1 + hv_k}{1 + hv_{k-1}} , \qquad \widetilde{v}_k = v_k\,\frac{1 + h\widetilde{u}_{k+1}}{1 + h\widetilde{u}_k} . \tag{9.1.4}$$

(indeed, knowing (u,v), one can calculate first $\widetilde{u}$, then $\widetilde{v}$). Again, this map belongs to the RVL(h) hierarchy but serves as an explicit discretization of the flow

$$\dot{u}_k = u_k(v_k - v_{k-1}) , \qquad \dot{v}_k = v_k(u_{k+1} - u_k) ,$$

which is of course, nothing but the *nonrelativistic* VL.

Finally, a further Miura transformation leads to the map dMRVL$_+(h;\epsilon)$, whose equations of motion also are of an explicit nature:

$$\frac{\widetilde{y}_k}{1 + \epsilon\widetilde{y}_k} = \frac{y_k}{1 + \epsilon y_k}\cdot\frac{1 + hz_k}{1 + hz_{k-1}} , \qquad \frac{\widetilde{z}_k}{1 + \epsilon\widetilde{z}_k} = \frac{z_k}{1 + \epsilon z_k}\cdot\frac{1 + h\widetilde{y}_{k+1}}{1 + h\widetilde{y}_k} . \tag{9.1.5}$$

Once more are we confronted with the same phenomenon: this system belongs to to the relativistic MRVL($h;\epsilon$) hierarchy, but serves as an explicit discretization of the corresponding nonrelativistic flow MVL(ϵ):

$$\dot{y}_k = y_k(1 + \epsilon y_k)(z_k - z_{k-1}) , \qquad \dot{z}_k = z_k(1 + \epsilon z_k)(y_{k+1} - y_k) .$$

This discretization should be compared with the implicit one (4.25.2), belonging to the (double) modified Volterra hierarchy.

We begin this Chapter with formulating the most important results on the maps dRTL$_+(h)$, dMRTL$_+(h;\epsilon)$, dRVL$_+(h)$, and dMRVL$_+(h;\epsilon)$ in Sect. 9.2, 9.3, 9.4, and 9.5, respectively. Then, in Sect. 9.6, 9.7 we take a sidestep, and demonstrate how the explicit discretizations for TL and VL living in the relativistic hierarchies could be recovered from implicit discretizations living in the original (or, better, modified) nonrelativistic hierarchies. This requires a better understanding of the transition described above.

9.8 Bibliographical remarks

Actually, the explicit discretization of the Toda and Volterra lattices, (9.1.1) and (9.1.4), as well as of their modifications, (9.1.3) and (9.1.5), were found earlier than their implicit analogues, namely in the papers Hirota (1977a,b, 1978). Of course, they were found with the help of Hirota's bilinear formalism, without any reference to the relativistic systems and their discretizations. Also, the Hamiltonian properties were not discussed by Hirota.

The relation to the relativistic systems was observed, on the level of Hirota's bilinear representation and τ–functions, in Ohta *et al.* (1993). On the level of Lax representation an account very similar to our arguments in Sect. 9.6 is contained in Papageorgiou, Grammaticos and Ramani (1995). This approach, when one considers discrete–time lattice systems as systems on a two–dimensional lattice, and then tries to find new meaningful initial value problems for these two–dimensional lattice, is close in spirit also to constructions in Papageorgiou, Nijhoff and Capel (1990), Capel, Nijhoff and Papageorgiou (1991), Nijhoff *et al.* (1992), Papageorgiou and Nijhoff (1996). In Suris and Ragnisco (1999) this approach was formulated in an opposite direction, as a systematic procedure for finding integrable "relativistic" deformations for integrable lattice systems. Actually, the relativistic Volterra lattice was discovered in Suris and Ragnisco (1999) on this way. See a further discussion in Maruno and Oikawa (2000).

Chapter 10

Explicit discretizations for Newtonian Toda–type systems

Contents

10.1 Introduction

The Newtonian systems of Chapt. 8 simplify dramatically for certain special values of the relativistic parameter α. These values are: $\alpha = h$ for the equations related to $\mathrm{RTL}_+(\alpha)$ and $\alpha = -h$ for the equations related to $\mathrm{RTL}_-(\alpha)$. Since the resulting simplified systems are basically the same for the $\mathrm{RTL}_+(h)$ and $\mathrm{RTL}_-(-h)$ cases (up to a change of the time direction), we concentrate on the first case.

If one sets $\alpha = h$ in (8.1.3), one finds the following discrete time Newtonian system:

$$e^{\widetilde{x}_k - 2x_k + \underset{\sim}{x}_k} = \frac{1 + h^2 e^{x_{k+1} - x_k}}{1 + h^2 e^{x_k - x_{k-1}}} \; . \tag{10.1.1}$$

Obviously, this may be considered as a discretization of the usual (*nonrelativistic*) exponential Toda lattice

$$\ddot{x}_k = e^{x_{k+1} - x_k} - e^{x_k - x_{k-1}} \; . \tag{10.1.2}$$

The most remarkable feature of this discretization is its *explicit* character: the formulas (10.1.1) allow to calculate $\widetilde{x}$ explicitly, if $(x, \underset{\sim}{x})$ are known. This will be a common feature of all systems discussed in the present Chapter.

An alternative explicit discretization of (10.1.2) is generated by the system (8.1.7) at the parameter value $\alpha = h$:

$$e^{\widetilde{x}_k - x_k} - e^{x_k - \underset{\sim}{x}_k} = h^2\Big(e^{x_{k+1}-x_k} - e^{x_k - x_{k-1}}\Big)\;. \tag{10.1.3}$$

The dual system (8.1.11) by $\alpha = h$ turns into

$$\frac{\widetilde{x}_k - x_k}{x_k - \underset{\sim}{x}_k} = \frac{1 + h(x_{k+1} - x_k)}{1 + h(x_k - x_{k-1})}\;, \tag{10.1.4}$$

which is, clearly, a discretization of the dual (nonrelativistic) Toda lattice

$$\ddot{x}_k = \dot{x}_k(x_{k+1} - 2x_k + x_{k-1})\;. \tag{10.1.5}$$

The $\alpha = h$ particular case of the system (8.1.15) is the following discrete time Newtonian system:

$$\frac{e^{\widetilde{x}_k - x_k} - 1}{e^{x_k - \underset{\sim}{x}_k} - 1} = \frac{1 + he^{x_{k+1}-x_k}}{1 + he^{x_k - x_{k-1}}}\;, \tag{10.1.6}$$

which is, of course, an explicit discretization of the modified exponential Toda lattice

$$\ddot{x}_k = \dot{x}_k\Big(e^{x_{k+1}-x_k} - e^{x_k - x_{k-1}}\Big)\;. \tag{10.1.7}$$

A closely related discrete system appears as an $\alpha = h$ particular case of (8.1.18):

$$\frac{1 + \epsilon h^{-1}\Big(e^{\widetilde{x}_k - x_k} - 1\Big)}{1 + \epsilon h^{-1}\Big(e^{x_k - \underset{\sim}{x}_k} - 1\Big)} = \frac{1 + h\epsilon e^{x_{k+1}-x_k}}{1 + h\epsilon e^{x_k - x_{k-1}}}\;. \tag{10.1.8}$$

This serves as an explicit discretization of

$$\ddot{x}_k = (1 + \epsilon\dot{x}_k)\Big(e^{x_{k+1}-x_k} - e^{x_k - x_{k-1}}\Big)\;. \tag{10.1.9}$$

Finally, we come to four discrete systems which are the $\alpha = h$ specializations of the Newtonian equations for the dMRVL$_+(\alpha;\epsilon)$ flow. The most general of these systems is a specialization of (8.1.23):

$$\frac{\sinh(\widetilde{x}_k - x_k + \nu h)}{\sinh(\widetilde{x}_k - x_k - \nu h)}\cdot\frac{\sinh(x_k - \underset{\sim}{x}_k - \nu h)}{\sinh(x_k - \underset{\sim}{x}_k + \nu h)} = \frac{\sinh(x_{k+1} - x_k + \nu h)}{\sinh(x_{k+1} - x_k - \nu h)}\cdot\frac{\sinh(x_k - x_{k-1} - \nu h)}{\sinh(x_k - x_{k-1} + \nu h)}\;. \tag{10.1.10}$$

This plays the role of an explicit discretization of

$$\ddot{x}_k = -(\dot{x}_k^2 - \nu^2)\Big(\coth(x_{k+1} - x_k) - \coth(x_k - x_{k-1})\Big)\;. \tag{10.1.11}$$

The $\nu = 0$ particular case of the previous discrete time system (or, in other words, the $\alpha = h$ particular case of (8.1.25)) reads:

$$\coth(\widetilde{x}_k - x_k) - \coth(x_k - \underset{\sim}{x}_k) = \coth(x_{k+1} - x_k) - \coth(x_k - x_{k-1})\;, \tag{10.1.12}$$

which is an explicit discretization of

$$\ddot{x}_k = -\dot{x}_k^2\Big(\coth(x_{k+1} - x_k) - \coth(x_k - x_{k-1})\Big). \tag{10.1.13}$$

The $\nu \to 0$ limit of the system (10.1.10) with the variables scaled according to $x_k \mapsto \nu x_k$, $\nu \mapsto \gamma\nu$, which is the same as (8.1.27) for $\alpha = h$, reads:

$$\frac{(\widetilde{x}_k - x_k + \gamma h)}{(\widetilde{x}_k - x_k - \gamma h)}\cdot\frac{(x_k - \underset{\sim}{x}_k - \gamma h)}{(x_k - \underset{\sim}{x}_k + \gamma h)} = \frac{(x_{k+1} - x_k + \gamma h)}{(x_{k+1} - x_k - \gamma h)}\cdot\frac{(x_k - x_{k-1} - \gamma h)}{(x_k - x_{k-1} + \gamma h)}\;, \tag{10.1.14}$$

which serves as an explicit discretization of

$$\ddot{x}_k = -(\dot{x}_k^2 - \gamma^2)\left(\frac{1}{x_{k+1} - x_k} - \frac{1}{x_k - x_{k-1}}\right). \tag{10.1.15}$$

Finally, the common limit case of the latter two discrete time Newtonian systems can be characterized as (8.1.29) with $\alpha = h$, and reads:

$$\frac{1}{\widetilde{x}_k - x_k} - \frac{1}{x_k - \underset{\sim}{x}_k} = \frac{1}{x_{k+1} - x_k} - \frac{1}{x_k - x_{k-1}}. \tag{10.1.16}$$

This is an explicit discretization of

$$\ddot{x}_k = -\dot{x}_k^2\left(\frac{1}{x_{k+1} - x_k} - \frac{1}{x_k - x_{k-1}}\right). \tag{10.1.17}$$

If we compare the list of continuous time Newtonian systems from this Section with the list of Sect. 5.1, we shall see that these two lists are identical. On the other hand, the explicit discretizations listed here are easily obtainable from the implicit discretizations listed in Sect. 5.1, if we replace all occurences of $\underset{\sim}{x}_{k+1}$ by x_{k+1} and all occurences of $\widetilde{x}_{k-1}$ by x_{k-1}. Of course, this is not a mere coincidence. A deeper reason for this is obvious: all discrete time systems of the present Section are nothing but different parametrizations of the map $\mathrm{dRTL}_+(h)$ which is described by explicit equations of motion (see Chapter 9).

In the main text of this Chapter we complement the exhaustive list of explicit discrete time Newtonian equations hidden behind the map $\mathrm{dRTL}_+(h)$, with some additional attributes such as 2×2 Lax representations (when available). In other words, we extract different discrete time Newtonian equations of motion by means of considering various canonical parametrizations of the phase variables in the maps $\mathrm{dRTL}_+(h)$, $\mathrm{dRVL}_+(h)$, and $\mathrm{dMRTL}_+(h;\epsilon)$. However, we prefer not to do this anew, but rather to set $\alpha = h$ in the corresponding results of the previous Chapter. Therefore the present Chapter consists of a sequence of statements without proofs, – the specializations of the corresponding statements from Chapter 8.

10.11 Bibliographical remarks

The first explicit discretization of the exponential Toda lattice (10.1.2) was the system (10.1.3) found by Hirota (1978), along with the Lax pair slightly different from (10.3.8), (10.3.9).

The discretization (10.1.1) (and similar results for all generalized Toda lattices related to Lie algebras) was found in Suris (1990a,b). Also the $N \times N$ and 2×2 Lax representations were given in these papers. In Suris (1991) it was stressed that this map shares the integrals of motion and the Lax matrices with the relativistic Toda lattice.

This theme was developed further in Suris (1997b) where also the systems (10.1.4), (10.1.6), (10.1.8), and the "cubic" ones (10.1.12), (10.1.16) were derived. The systems (10.1.10), (10.1.14) were missed in the previous publications by the author.

A discretization of the two–dimensional Toda lattice closely related to (10.1.2), (10.1.1) is discussed in Hirota (1987), Kashaev and Reshetikhin (1997). A geometric interpretation of the two–dimensional Toda lattice as a sequence of Laplace transformations for a two–dimensional hyperbolic partial differential operator, or, what is the same, for a conjugate net in $\mathbb{R}^3$ (see Bibliographical remarks to Chapter 3), leads to a natural "geometric" discretization of this system. See Doliwa (1997c, 1999), Novikov and Dynnikov (1997), Adler and Startsev (1999).

A generalization in another direction, namely a q–difference analog of the above mentioned Hirota's discretizations, is derived in Kajiwara and Satsuma (1991), Kajiwara, Ohta and Satsuma (1993).

Chapter 11

Bruschi–Ragnisco lattice

Contents

11.1 Introduction

This short Chapter is devoted to a curious system which will be denoted BRL, after its inventors Bruschi and Ragnisco. They introduced it as a limiting case of the relativistic Toda lattice. Its Newtonian equations of motion read:

$$\ddot{x}_k = \frac{\alpha \dot{x}_{k+1}\dot{x}_k}{1+\alpha(x_{k+1}-x_k)} - \frac{\alpha \dot{x}_k \dot{x}_{k-1}}{1+\alpha(x_k-x_{k-1})} \; . \tag{11.1.1}$$

To get the flavour of its curiosity, let us point out the relation between the Toda lattice TL, the relativistic Toda lattice RTL, and the Bruschi–Ragnisco lattice BRL. Symbolically this relation may be represented as

$$\mathrm{RTL} = \mathrm{TL} + \mathrm{BRL} \; .$$

To explain what do we mean under this "formula", let us recall the Newtonian equations arising from the flows TL and $\mathrm{RTL}_+(\alpha)$ under the dual parametrization of the linear Poisson bracket. These equations read:

$$\ddot{x}_k = \dot{x}_k(x_{k+1} - 2x_k + x_{k-1}) \; , \tag{11.1.2}$$

and

$$\ddot{x}_k = \dot{x}_k(x_{k+1} - 2x_k + x_{k-1}) + \frac{\alpha \dot{x}_{k+1}\dot{x}_k}{1+\alpha(x_{k+1}-x_k)} - \frac{\alpha \dot{x}_k \dot{x}_{k-1}}{1+\alpha(x_k-x_{k-1})} \; , \tag{11.1.3}$$

respectively. We see that in the above–mentioned parametrization, the flow $\mathrm{RTL}_+(\alpha)$ of the relativistic Toda hierarchy may be represented in the form of a system of Newtonian equations, whose right–hand sides are the sums of the right–hand sides of the Newtonian equations for the flow TL (in the similar parametrization) and of the equations (11.1.1). It turns out that the discrete time versions of all these systems are related in the similar manner, but the right–hand sides must be multiplied rather than added. In symbolical form:

$$\mathrm{dRTL} = \mathrm{dTL} \times \mathrm{dBRL} \; .$$

As we shall demostrate in this Chapter, the discrete time version of (11.1.1) is:

$$\frac{\widetilde{x}_k - x_k}{x_k - \underset{\sim}{x}_k} = \frac{\Big(1+\alpha(x_{k+1}-x_k)\Big)}{\Big(1+\alpha(\underset{\sim}{x}_{k+1}-x_k)\Big)} \cdot \frac{\Big(1+\alpha(x_k-\widetilde{x}_{k-1})\Big)}{\Big(1+\alpha(x_k-x_{k-1})\Big)} . \tag{11.1.4}$$

This has to be compared with the discrete time version of (11.1.2):

$$\frac{\widetilde{x}_k - x_k}{x_k - \underset{\sim}{x}_k} = \frac{1+h(\underset{\sim}{x}_{k+1}-x_k)}{1+h(x_k-\widetilde{x}_{k-1})} , \tag{11.1.5}$$

and the discrete time version of (11.1.3):

$$\frac{\widetilde{x}_k - x_k}{x_k - \underset{\sim}{x}_k} = \frac{\Big(1+h(\underset{\sim}{x}_{k+1}-x_k)\Big)}{\Big(1+h(x_k-\widetilde{x}_{k-1})\Big)} \cdot \frac{\Big(1+\alpha(x_{k+1}-x_k)\Big)}{\Big(1+\alpha(\underset{\sim}{x}_{k+1}-x_k)\Big)} \cdot \frac{\Big(1+\alpha(x_k-\widetilde{x}_{k-1})\Big)}{\Big(1+\alpha(x_k-x_{k-1})\Big)} . \tag{11.1.6}$$

We apply in this Chapter our general recipe to derive the discretization (11.1.4). To this end we need first to include the system (11.1.1) in a general r–matrix approach. In doing this, we shall find out that this system is in fact trivial (linear) if considered properly. Nevertheless, the whole matter is very instructive and therefore worth working out.

11.7 Bibliographical remarks

11.1–11.2. The Bruschi–Ragnisco lattice was introduced in Bruschi and Ragnisco (1989c). In that paper different methods of the soliton theory, such as inverse scattering, were applied to this system. Also the bi–Hamiltonian structure was elaborated. An approach based on the notion of the master–symmetry, was used in Zhang *et al.* (1991).

11.3–11.4. The algebraic structure of the Bruschi–Ragnisco lattice was clarified in Suris (1993a). This resulted in a "trivialization" of the system.

11.5–11.6. The discretization of the Bruschi–Ragnisco lattice was performed in Suris (1999a), where also the curious "formulas" RTL = TL + BRL and dRTL = dTL $\times$ dBRL were pointed out.

Chapter 12

Multi–field Toda–like systems

Contents

12.1 Introduction

The privileged objects of our studies up to now were the Toda lattice TL and its various generalizations, such as the modified Toda lattice $\mathrm{MTL}(\epsilon)$ and the relativistic Toda lattice $\mathrm{RTL}(\alpha)$. All these lattices are *two–field systems*, i.e. their phase variables are naturally divided into two sorts (a_k and b_k, say) in such a way that equations of motions have different structure for each sort. It is easy to understand that the ultimate reason for this feature lies in the tri–diagonal structure of the Lax matrix for TL. If we consider Lax matrices with a larger number of nonvanishing diagonals, then inevitably we come to multi–field generalizations of the above–mentioned systems. More precisely, $(m+1)$–diagonal Lax matrices lead to m–field systems. The present Chapter is devoted to the study of these multi–field generalizations. (In principle, one could even send $m \to \infty$, but, following the general spirit of this book, devoted to finite–dimensional systems, we restrict ourselves here with the case of finite m).

The multi–field generalization of the Toda lattice is described by the following equations of motion:

$$\left\{ \begin{array}{l} \dot{b}_k = a_k^{(1)} - a_{k-1}^{(1)} \;, \\[2ex] \dot{a}_k^{(j)} = a_k^{(j)} \Big(b_{k+j} - b_k \Big) + \Big(a_k^{(j+1)} - a_{k-1}^{(j+1)} \Big) \;. \end{array} \right. \tag{12.1.1}$$

We obtain an m–field system, if we set $a_k^{(j)}=0$ for all $j\geq m$. The Lax representation of this system is introduced in Sect. 12.2. We give an r–matrix interpretation of this system in Sect. 12.3, 12.4, where the linear and the quadratic r–matrix structures are considered. These structures turn out to be exactly the same as for the usual TL. However, the cubic invariant Poisson bracket seems to be an exceptional feature of the two–field situation. Anyway, the Lax representation of (12.1.1) belongs to the general class considered in this book, so that our general recipe for integrable discretization is applicable. This is performed in Sect. 12.5, and the resulting discrete time equations of motion are, naturally, nonlocal. A localizing change of variables is found in Sect. 12.6, which leads to the following local discretization of (12.1.1):

$$\left\{\begin{array}{l} \widetilde{\mathbf{b}}_k+h\widetilde{\mathbf{a}}_{k-1}^{(1)}=\mathbf{b}_k+h\mathbf{a}_k^{(1)}\;, \\[2mm] \widetilde{\mathbf{a}}_k^{(j)}(1+h\widetilde{\mathbf{b}}_k)+h\widetilde{\mathbf{a}}_{k-1}^{(j+1)}=\mathbf{a}_k^{(j)}(1+h\mathbf{b}_{k+j})+h\mathbf{a}_k^{(j+1)}\;. \end{array}\right. \tag{12.1.2}$$

As usual, this discretization actually belongs to a modified hierarchy, with the modification parameter h.

We illustrate the general results with the three–field example in Sect. 12.7.

Equations of motion of the modified multi–field generalization of the Toda lattice, with a modification parameter ϵ, read:

$$\left\{\begin{array}{l} \dot{b}_k=(1+\epsilon b_k)\Big(a_k^{(1)}-a_{k-1}^{(1)}\Big)\;, \\[2mm] \dot{a}_k^{(j)}=a_k^{(j)}\Big(b_{k+j}-b_k+\epsilon a_{k+j-1}^{(1)}-\epsilon a_k^{(1)}\Big)+\Big(a_k^{(j+1)}-a_{k-1}^{(j+1)}\Big)\;. \end{array}\right. \tag{12.1.3}$$

Again, we obtain an m–field system, if we set $a_k^{(j)}=0$ for all $j\geq m$. The natural Lax representation of the modified system lives in $\boldsymbol{g}\otimes\boldsymbol{g}$ (see Sect. 12.8). The underlying quadratic r–matrix structure on this space is again exactly the same as in the two–field situation; it is elaborated in Sect. 12.9. Applying our general recipe, we discretize this system in Sect. 12.10, and find a localizing change of variables for the resulting nonlocal equations in Sect. 12.10. A local discretization of (12.1.3) reads:

$$\left\{\begin{array}{l} (1+\epsilon\widetilde{\mathbf{b}}_k)(1+h\epsilon\widetilde{\mathbf{a}}_{k-1}^{(1)})=(1+\epsilon\mathbf{b}_k)(1+h\epsilon\mathbf{a}_k^{(1)})\;, \\[2mm] \widetilde{\mathbf{a}}_k^{(1)}(1+h\widetilde{\mathbf{b}}_k)+h\widetilde{\mathbf{a}}_{k-1}^{(2)}=\mathbf{a}_k^{(1)}(1+h\mathbf{b}_{k+1})+h\mathbf{a}_k^{(2)}\;, \\[2mm] \widetilde{\mathbf{a}}_k^{(j)}(1+h\widetilde{\mathbf{b}}_k)(1+h\epsilon\widetilde{\mathbf{a}}_k^{(1)})+h\widetilde{\mathbf{a}}_{k-1}^{(j+1)}=\mathbf{a}_k^{(j)}(1+h\mathbf{b}_{k+j})(1+h\epsilon\mathbf{a}_{k+j-1}^{(j)})+h\mathbf{a}_k^{(j+1)} \end{array}\right. \tag{12.1.4}$$

(the last equation holds for $j\geq 2$). This discretization belongs to the double modified hierarchy, with the modification parameters ϵ,h. The differential equations of motion of the double modified system are also given in Sect. 12.11.

12.12 Bibliographical remarks

12.2. Actually, in the setting of infinite lattices ($N=\infty$) it is only natural to allow the number of fields m to become also infinite. The corresponding huge system is known under different names, such as the lattice, or discrete KP, or even simply as the Toda lattice, like in Ueno and Takasaki (1984). A systematic differential–difference calculus for such systems was developed in Kupershmidt (1985). An approach from the Grassmannian and vertex opeartors side is represented by Adler and van Moerbeke (1998, 1999a,b). An original bi–Hamiltonian approach is due to Magri and his school, see Casati *et al* (1997, 1998), Falqui, Magri and Pedroni (1998), Magri, Pedroni and Zubelli (1997). In the paper Dickey (1999) it is argued that the lattice KP hierarchy serves as a proper object for what can be called the modified KP hierarchy (we do not share this viewpoint). Anyway, an object constituted by the system (12.1.1) with $m=\infty$ and by the higher commuting flows, is extremely rich and is being actively studied in the soliton theory, so that an exhaustive bibliography seems impossible.

On the other hand, a finite–dimensional system corresponding to the case of open–end boundary conditions and $m=N-1$, is known under the name of the *full Kostant–Toda lattice*, and is also a popular object. See Deift *et al* (1986), Deift, Li and Tomei (1989), Deift and Li (1989) for the results on integrability and relation to matrix factorizations, Ercolani, Flaschka and Singer (1993) for the geometric study of the level sets, Damianou, Paschalis and Sophocleous (1995) for an approach to the tri–Hamiltonian structure based on master–symmetries, Kodama and McLaughlin (1996) and Kodama and Ye (1996) for an explicit integration of the flow and its asymptotic properties, Gekhtman and Shapiro (1999) for a generalization for an arbitrary Lie algebra. Almost all these results remain valid also for the case we are mostly interested in, namely that of the finite and fixed number of fields $m+1$.

12.3, 12.4. The Poisson brackets from Theorems 12.2, 12.3 were given in Kupershmidt (1985). The r–matrix approach to the linear bracket (12.3.3), (12.3.4) may be traced back at least to Reyman (1980), nevertheless, these formulas are not written explicitly in his paper. This was done, e.g., in Oevel and Ragnisco (1991), Blaszak and Marciniak (1994). These references contain also an r–matrix approach to the quadratic bracket (12.4.2)–(12.4.8), however Theorem 12.3 goes back to the work Suris (1993b) where the general quadratic bracket $\mathrm{PB}_2(\mathrm{A}_1,\mathrm{A}_2,\mathrm{S})$ was introduced. See also a review Oevel (1996). The systems of the type of TL_m may be considered as spatial discretizations of the Gelfand–Dickey systems. Various aspects of the corresponding continuous limit, including the limit of the Hamiltonian structures, were studied in Morosi and Pizzochero (1998a,b).

12.5. The formulas (12.5.2)–(12.5.4) and (12.5.5), defining the map dTL_{m+1}, were found in Gibbons and Kupershmidt (1992). However, they missed the interpretation (12.5.1) of this map in terms of matrix factorization, which did not allow them to find a natural explanation for the Poisson properties of this map. (Actually, only the invariance of the linear bracket was established in Gibbons and Kupershmidt (1992) by a direct computation.)

12.6. As a light irony of the history, Gibbons and Kupershmidt (1992) actually had in their hands all necessary for finding the localizaing change of variables (12.6.1), because it is nothing but the Miura map found in Kupershmidt (1983), (1985). However, the role of the Miura maps as localizing changes of variables was realized only in Suris (1999c).

12.8, 12.9. The modified system, its Lax representation and Hamiltonian structure are due to Kupershmidt (1985). The r–matrix approach to the quadratic Poisson bracket is a new result.

12.10, 12.11. The results of these Sections are new.

Chapter 13

Multi–field systems of the relativistic Toda type

Contents

13.1 Introduction

The multi–field analog of the Toda lattice TL_{m+1} studied in the previous Chapter, admits also a relativistic generalization, or, better, two different generalizations. Indeed, in Chapter 6 we introduced two different Lax representations of the $\mathrm{RTL}(\alpha)$ hierarchy. The multi–field analogs of these Lax representations do not lead to one and the same result. Rather, we get two different multi–field relativistic Toda hierarchies. The first construction is an application of the general ansatz from Sect. 6.2–6.4. The hierarchy appearing on this way contains a system which is a direct generalization of the flow $\mathrm{RTL}_{-}(\alpha)$:

$$\left\{\begin{aligned} \dot b_k &= \frac{a_k^{(1)}}{1+\alpha b_{k+1}} - \frac{a_{k-1}^{(1)}}{1+\alpha b_{k-1}}\;, \\ \dot a_k^{(j)} &= a_k^{(j)}\left(\frac{b_{k+j}}{1+\alpha b_{k+j}} - \frac{b_k}{1+\alpha b_k}\right) + \left(\frac{a_k^{(j+1)}}{1+\alpha b_{k+j+1}} - \frac{a_{k-1}^{(j+1)}}{1+\alpha b_{k-1}}\right). \end{aligned}\right. \tag{13.1.1}$$

However, the equations of motion of the "first" (polynomial) flow of this hierarchy are messy and less instructive. We introduce this hierarchy in Sect. 13.2, and elaborate an r–matrix interpretation of its invariant linear Poisson structure in Sect. 13.3. The quadratic invariant Poisson bracket is also granted by this construction, and turns out to coincide literally with the corresponding non–relativistic object, see Sect. 13.4. However, the cubic invariant bracket seems not to exist for this hierarchy (unless $m = 1$).

The first relativistic construction leads to Lax equations of the non–standard type. This drawback is absent in the second construction, which is described in Sect. 13.6. The hierarchy found on this way also contains only one "simple" flow, this time the "first" one, a direct generalization of the flow $\mathrm{RTL}_+(\alpha)$:

$$\left\{\begin{array}{l} \dot{b}_k = (1+\alpha b_k)\Big(a_k^{(1)} - a_{k-1}^{(1)}\Big)\,, \\[2mm] \dot{a}_k^{(j)} = a_k^{(j)}\Big(b_{k+j} - b_k + \alpha a_{k+j}^{(1)} - \alpha a_{k-1}^{(1)}\Big) + \Big(a_k^{(j+1)} - a_{k-1}^{(j+1)}\Big)\,. \end{array}\right. \tag{13.1.2}$$

The corresponding hierarchy lives naturally in $\boldsymbol{g}\otimes\boldsymbol{g}$, and its quadratic invariant Poisson bracket is obtained through the quadratic r–matrix construction on this space, see Sect. 13.7. The linear invariant Poisson bracket does not exist for the system (13.1.2) anymore.

The invertible change of variables $\mathbf{B}(\alpha)$ relating the $\mathrm{RTL}(-\alpha)$ and $\mathrm{RTL}(\alpha)$ hierarchies, also admits a multi–field generalization. However, now it relates two *different* hierarchies, described above. This link allows us to demonstrate the existence of the cubic invariant bracket for the second hierarchy. So, both multi–field relativistic Toda hierarchies turn out to be bi–Hamiltonian.

The Lax representation of (13.1.2) is ideally suited for applying our general recipe for integrable discretization. This is done in Sect. 13.9. The localizing change of variables for the resulting nonlocal discretization is found in Sect. 13.10. We find the following local discretization for (13.1.2):

$$\left\{\begin{array}{l} (1+\alpha\widetilde{\mathbf{b}}_k)(1+h\alpha\widetilde{\mathbf{a}}_{k-1}^{(1)}) = (1+\alpha\mathbf{b}_k)(1+h\alpha\mathbf{a}_k^{(1)})\,, \\[2mm] \widetilde{\mathbf{a}}_k^{(j)}(1+h\widetilde{\mathbf{b}}_k)(1+h\alpha\widetilde{\mathbf{a}}_{k-1}^{(1)}) + h\widetilde{\mathbf{a}}_{k-1}^{(j+1)} = \mathbf{a}_k^{(j)}(1+h\mathbf{b}_{k+j})(1+h\alpha\mathbf{a}_{k+j}^{(1)}) + h\mathbf{a}_k^{(j+1)}\,. \end{array}\right. \tag{13.1.3}$$

The Lax representation for (13.1.1) is of a nonstandard type, so that a direct application of our recipe is somewhat difficult. However, using the above mentioned invertible relation between both hierarchies, we find in Sect. 13.11 the following local discretization for (13.1.1):

$$\left\{\begin{array}{l} \widetilde{\mathbf{b}}_k + \dfrac{h\widetilde{\mathbf{a}}_{k-1}^{(1)}}{1+\alpha\widetilde{\mathbf{b}}_{k-1}} = \mathbf{b}_k + \dfrac{h\mathbf{a}_k^{(1)}}{1+\alpha\mathbf{b}_{k+1}}\,, \\[4mm] \widetilde{\mathbf{a}}_k^{(j)}\left(1+\dfrac{h\widetilde{\mathbf{b}}_k}{1+\alpha\widetilde{\mathbf{b}}_k}\right) + \dfrac{h\widetilde{\mathbf{a}}_{k-1}^{(j+1)}}{1+\alpha\widetilde{\mathbf{b}}_{k-1}} = \mathbf{a}_k^{(j)}\left(1+\dfrac{h\mathbf{b}_{k+j}}{1+\alpha\mathbf{b}_{k+j}}\right) + \dfrac{h\mathbf{a}_k^{(j+1)}}{1+\alpha\mathbf{b}_{k+j+1}}\,. \end{array}\right. \tag{13.1.4}$$

As usual, the local discretizations (13.1.3), (13.1.4) belong to the modified hierarchies with the modification parameter h; the corresponding continuoustime modified system is derived for the discretization (13.1.3) in Sect. 13.10.

Finally, it has to be noticed that Sect. 13.8 is devoted to illustrating the general results of this Chapter with the three–field example.

13.12 Bibliographical remarks

13.2–13.7. The "first" relativistic construction of Sect. 13.2 belongs to Gibbons and Kupershmidt (1990), Kupershmidt (1993). They found also the linear Hamiltonian formulation of the relativistic lattice KP hierarchy. All other results (the "second" ansatz of Sect. 13.6 and the r–matrix constructions of the linear and the quadratic bracket) are due to Suris (1999d).

13.9–13.11. All results on the discretization of the relativistic multifield systems are new.

Chapter 14

Belov–Chaltikian lattices

Contents

14.1 Introduction

We devote this Chapter to the study of the following multi–field system:

$$\dot{c}_k^{(j)} = c_k^{(j)}\left(c_{k+j}^{(1)} - c_{k-1}^{(1)}\right) + \left(c_k^{(j+1)} - c_{k-1}^{(j+1)}\right) , \tag{14.1.1}$$

which will be called the Belov–Chaltikian lattice. It may be considered as a reduction of the multi–field analog of the relativistic Toda lattice $\mathrm{RTL}_{m+1}^{(+)}(\alpha)$, so that many results may be readily formulated as transcriptions of the statements of the previous Chapter. In Sect. 14.2 we give a transcription of the previously obtained results concerning the bi–Hamiltonian nature of the system (14.1.1) and its Lax representation, and in Sect. 14.3 – of those results concerning the intebrable discretization of (14.1.1), including the localizing change of variables. In particular, the local discretization is governed by the following equations of motion:

$$\widetilde{\mathbf{c}}_k^{(j)}\left(1 + h\widetilde{\mathbf{c}}_{k-1}^{(1)}\right) + h\widetilde{\mathbf{c}}_{k-1}^{(j+1)} = \mathbf{c}_k^{(j)}\left(1 + h\mathbf{c}_{k+j}^{(1)}\right) + h\mathbf{c}_k^{(j+1)} . \tag{14.1.2}$$

As usual, this local discretization belongs to the modified hierarchy. The modified system for (14.1.1), with the modification parameter ϵ, reads:

$$\dot{c}_k^{(j)} = c_k^{(j)}\left(c_{k+j}^{(1)} - c_{k-1}^{(1)} + \epsilon c_{k+j}^{(1)}c_{k+j-1}^{(1)} - \epsilon c_k^{(1)}c_{k-1}^{(1)} + \epsilon c_{k+j-1}^{(2)} - \epsilon c_{k-1}^{(2)}\right) + \left(c_k^{(j+1)} - c_{k-1}^{(j+1)}\right). \tag{14.1.3}$$

We study this system on its own rights, starting with Sect. 14.4. As it usually happens with the modified systems, this system has one local invariant Poisson structure less than (14.1.1), i.e. it possesses only one such structure. We find a Lax representation of this modified system, which turns out to have the standard form, so that our recipe for integrable discretization may be applied. The corresponding discretization is performed in Sect. 14.5, and a localizing change of variables is found in Sect. 14.6. The local discretization of the system (14.1.3) reads:

$$\frac{\widetilde{\mathbf{c}}_k^{(j)}\Big(1+h\widetilde{\mathbf{c}}_{k-1}^{(1)}\Big)\Big(1+h\epsilon\widetilde{\mathbf{c}}_{k-1}^{(2)}\Big)}{1-h\epsilon\widetilde{\mathbf{c}}_k^{(1)}\widetilde{\mathbf{c}}_{k-1}^{(1)}}+h\widetilde{\mathbf{c}}_{k-1}^{(j+1)}=\frac{\mathbf{c}_k^{(j)}\Big(1+h\mathbf{c}_{k+j}^{(1)}\Big)\Big(1+h\epsilon\mathbf{c}_{k+j-1}^{(2)}\Big)}{1-h\epsilon\mathbf{c}_{k+j}^{(1)}\mathbf{c}_{k+j-1}^{(1)}}+h\mathbf{c}_k^{(j+1)}\,. \qquad (14.1.4)$$

Interestingly enough, one can apply also a general relativistic ansatz to obtain another deformation of the system (14.1.1). The simplest flow of the corresponding relativistic hierarchy reads:

$$\dot{c}_k^{(j)}=c_k^{(j)}\left(\frac{c_{k+j}^{(1)}}{1+\alpha c_{k+j}^{(1)}}-\frac{c_{k-1}^{(1)}}{1+\alpha c_{k-1}^{(1)}}\right)+\frac{c_k^{(j+1)}}{1+\alpha c_{k+j}^{(1)}}-\frac{c_{k-1}^{(j+1)}}{1+\alpha c_{k-1}^{(1)}}\,, \qquad (14.1.5)$$

see Sect. 14.7. A new and unexpected phenomenon comes out in this connection. Namely, the modified and the relativistic hierarchies turn out to be connected via a certain invertible change of variables (Sect. 14.8). This allows us also to derive a local integrable discretization of the flow (14.1.5):

$$\widetilde{\mathbf{c}}_k^{(j)}\left(1+\frac{h\widetilde{\mathbf{c}}_{k-1}^{(1)}}{1+\alpha\widetilde{\mathbf{c}}_{k-1}^{(1)}}\right)+\frac{h\widetilde{\mathbf{c}}_{k-1}^{(j+1)}}{1+\alpha\widetilde{\mathbf{c}}_{k-1}^{(1)}}=\mathbf{c}_k^{(j)}\left(1+\frac{h\mathbf{c}_{k+j}^{(1)}}{1+\alpha\mathbf{c}_{k+j}^{(1)}}\right)+\frac{h\mathbf{c}_k^{(j+1)}}{1+\alpha\mathbf{c}_{k+j}^{(1)}}\,. \qquad (14.1.6)$$

This is done in Sect. 14.9. We close this Chapter by specializing its results for two simplest particular cases of a one–field reduction (which is nothing but the usual Volterra lattice), and of a two–field reduction. These two examples are dealt with in Sect. 14.10, 14.11, respectively.

14.12 Bibliographical remarks

The lattices BCL_m were introduced in Belov and Chaltikian (1993a,b). Their motivation lied in the construction of the so called lattice W_N algebras, which they realized as the Poisson brackets $\{\cdot,\cdot\}_3$. A physical motivation, as well as further developments can be found in Antonov, Belov and Chaltikian (1997). Also the Miura maps from Theorem 14.17 were found in these papers. Some additional results, including the 3×3 r-matrix formulation of BCL_2, see in Hikami and Inoue (1997) and further papers by these authors. All results concerning the discretization of BCL_m, its modification and relativistic deformation, are new. The particular case of the discretization of BCL_2 was published previously in Suris (1999c).

Chapter 15

Multi–field Volterra–like systems

Contents

15.1 Introduction

We have seen in Chapter 12 that factorizing the Lax matrix $T(b,a^{(1)},\ldots,a^{(m)},\lambda)$ of the multi–field Toda lattice TL_{m+1} (or, more precisely, the matrix $I+\epsilon T$) into just two factors gives rise to the modified multi–field Toda lattice. One of these factors is two-diagonal, and the other is m–diagonal in the $(m+1)$–field situation. For the two–field systems this exhausts the possibilities of factorization, so that the Volterra lattice (coming from factorizing T itself) is more or less equivalent to the modified Toda lattice. This is no more true in the multi–field situation, and the present Chapter is devoted to the case of the ultimate factorization, when all the factors are bi–diagonal. The corresponding generalization of the Volterra lattice is a system with m fields $v^{(j)}$, $1\le j\le m$, satisfying

$$\dot{v}_k^{(j)}=v_k^{(j)}\left(\sum_{i=1}^{j-1}\left(v_{k+1}^{(i)}-v_k^{(i)}\right)+\sum_{i=j+1}^{m}\left(v_k^{(i)}-v_{k-1}^{(i)}\right)\right). \tag{15.1.1}$$

Actually, this system may be generalized even further, and the most general system studied in this Chapter reads:

$$\dot{v}_k^{(j)} = v_k^{(j)}\left(\sum_{i=1}^{j-1}\left(v_{k+1}^{(i)} - v_k^{(i)}\right) + \epsilon_j\left(v_{k+1}^{(j)} - v_{k-1}^{(j)}\right) + \sum_{i=j+1}^{m}\left(v_k^{(i)} - v_{k-1}^{(i)}\right)\right), \tag{15.1.2}$$

where

$$(\epsilon_1, \ldots, \epsilon_m), \qquad \epsilon_j \in \{0, 1\}, \tag{15.1.3}$$

is an arbitrary m–tuple of zeros and ones with $\epsilon_1 = 0$. Evidently, (15.1.1) corresponds to the case when all $\epsilon_j = 0$. Both systems (15.1.1) and (15.1.2) are integrable, they are just the "first" flows of the corresponding hierarchies, and the most direct way to describe these hierarchies lies through the Lax representations in the "big" algebras $\boldsymbol{g}^{\otimes m}$, see Sect. 15.2, 15.7. These Lax representations and the corresponding invariant Poisson bracket admit a beautiful r–matrix interpretation, Sect. 15.3, 15.8. This, in turn, paves the way to applying our general recipe of integrable discretization, which is done in Sect. 15.4, 15.9. The localizing changes of variables are found in Sect. 15.5, 15.10. In particular, the local integrable discretization of the system (15.1.2) is:

$$\begin{aligned}
&\widetilde{\mathbf{v}}_k^{(j)} \prod_{i=1}^{j-1}\left(1 + h\widetilde{\mathbf{v}}_k^{(i)}\right) \cdot \left(1 + \epsilon_j h\widetilde{\mathbf{v}}_{k-1}^{(j)}\right) \cdot \prod_{i=j+1}^{m}\left(1 + h\widetilde{\mathbf{v}}_{k-1}^{(i)}\right) \\
&\quad = \mathbf{v}_k^{(j)} \prod_{i=1}^{j-1}\left(1 + h\mathbf{v}_{k+1}^{(i)}\right) \cdot \left(1 + \epsilon_j h\mathbf{v}_{k+1}^{(j)}\right) \cdot \prod_{i=j+1}^{m}\left(1 + h\mathbf{v}_k^{(i)}\right).
\end{aligned} \tag{15.1.4}$$

As usual, these local discretizations belong to modified hierarchies. The "first" flow of the modified hierarchy for (15.1.2) has the following equations of motion:

$$\begin{aligned}
\dot{\mathbf{v}}_k^{(j)} \;=\; & \mathbf{v}_k^{(j)}\left(1 + h\mathbf{v}_k^{(j)}\right)\Bigg(\prod_{i=1}^{j-1}\left(1 + h\mathbf{v}_{k+1}^{(i)}\right) \cdot \left(1 + \epsilon_j h\mathbf{v}_{k+1}^{(j)}\right) \cdot \prod_{i=j+1}^{m}\left(1 + h\mathbf{v}_k^{(i)}\right) \\
& - \prod_{i=1}^{j-1}\left(1 + h\mathbf{v}_k^{(i)}\right) \cdot \left(1 + \epsilon_j h\mathbf{v}_{k-1}^{(j)}\right) \cdot \prod_{i=j+1}^{m}\left(1 + h\mathbf{v}_{k-1}^{(i)}\right)\Bigg)\Big/ h.
\end{aligned} \tag{15.1.5}$$

Because of a high generality of constructions of the present Chapter, we found it useful to elaborate in detail on several particular cases, besides the most symmetric one (15.1.1) with all $\epsilon_j = 0$. So, Sect. 15.11 is devoted to an opposite (in a sense) case, when all $\epsilon_j = 1$ for $j > 1$. Further, we specify our results for all three possible triples $(\epsilon_1, \epsilon_2, \epsilon_3)$ in the tree–field case. This is done in Sect. 15.6, 15.12, and 15.13, respectively.

15.14 Bibliographical remarks

15.2–15.6. The system VL_m was introduced in Suris (1997d) as systems delivering the first nontrivial example of the nonlocal quadratic r–matrix Poisson bracket on direct products. A similar construction, however avoiding the r–matrix theory, was present in Kupershmidt (1985, 1989). The discretization, including the localizaing changes of variables, was performed in Suris (1999c).

15.7–15.13. The systems $\mathrm{VL}_m(\sigma)$ with nontrivial signatures σ serve as natural lattice generalizations of the so called *constrained KP hierarchies*, which are characterized by having a pseudeodifferential Lax operator *rational* (rather than polynomial) in ∂. These rational reductions were considered, e.g., in Krichever (1995), Dickey (1995a,b), Aratyn (1998), Helminck and van de Leur (1998). In particular, it should be noted that the *Lax trids* play an essential role also in this pseudodifferential situation. Further information on the Hamiltonian structure of the constrained KP hierarchies, their Miura maps and modifications, may be found in: Oevel (1993, 1994, 1996), Oevel and Strampp (1993), Liu (1994, 1998), Cheng (1995), Aratyn *et al* (1995, 1997), Feher and Marshall (1997), Delduc, Feher and Gallot (1998), Shaw and Tu (1997a,b), Huang, Shaw and Tu (1998).

The systems $\mathrm{VL}_m(\sigma)$ were first introduced, under the name "constrained lattice KP" in Suris (1999c). The discretization was also performed in that paper.

In the finite–dimensional (open–end) setting, a closely related system appeared in Faybusovich and Gekhtman (2000) by the description of low–dimensional orbits of the full Toda system.

Chapter 16

Multi–field relativistic Volterra–like systems

Contents

16.1 Introduction

Recall that there exist two different relativistic Volterra hierarchies, depending on which one of the two Miura maps is used to pull back the relativistic Toda hierarchies. In other words, one of these relativistic Volterra hierarchies corresponds to the relativistic deformation $\mathcal{F}^{-1}UV$ of the product UV of the basic Lax matrices of the Volterra lattice, while the second one of the relativistic Volterra hierarchies corresponds to the relativistic deformation $\mathcal{F}^{-1}VU$ of the product VU.

In the present Chapter we elaborate on the similar relativistic deformation of the multi–field Volterra lattices $\mathrm{VL}_m(\sigma)$ introduced in the previous Chapter:

$$\dot{v}_k^{(j)}=v_k^{(j)}\left(\sum_{i=1}^{j-1}\left(v_{k+1}^{(i)}-v_k^{(i)}\right)+\epsilon_j\left(v_{k+1}^{(j)}-v_{k-1}^{(j)}\right)+\sum_{i=j+1}^{m}\left(v_k^{(i)}-v_{k-1}^{(i)}\right)\right). \qquad (16.1.1)$$

(Recall that here $\epsilon_j\in\{0,1\}$, and the signature $\sigma=(\sigma_1,\dots,\sigma_m)$ consists of the numbers $\sigma_j=1-2\epsilon_j\in\{+1,-1\}$).

There exist two different constructions of relativistic deformations of (16.1.1). The first construction (Sect. 16.2) is connected with the Lax matrices $\mathcal{F}^{-1}T_j$, where

$$T_j=V_j^{\sigma_j}\cdot\ldots\cdot V_2^{\sigma_2}\cdot U_1^{\sigma_1}\cdot V_m^{\sigma_m}\cdot\ldots\cdot V_{j+1}^{\sigma_{j+1}}$$

are the monodromy matrices of $\mathrm{VL}_m(\sigma)$. Clearly, there arise, in principle, m *different* deformations corresponding to $1 \le j \le m$. However, we shall restrict ourselves only to *one* relativistic deformation for each $\mathrm{VL}_m(\sigma)$, namely to the one corresponding to $j = m$. Although the systems corresponding to other js are, in principle, non–equivalent, their study runs along the same lines, and the case $j = m$ allows for the most simple notations.

It turns out that the hierarchy related to $\mathcal{F}^{-1}T_m$ contains one simple flow, with the vector field depending rationally on coordinates:

$$\begin{aligned}\dot v_k^{(j)} &= v_k^{(j)}\left[\sum_{i=1}^{j-1}\left(\frac{v_{k+1}^{(i)}}{1+\alpha s_{k+1}} - \frac{v_k^{(i)}}{1+\alpha s_k}\right) + \epsilon_j\left(\frac{v_{k+1}^{(j)}}{1+\alpha s_{k+1}} - \frac{v_{k-1}^{(j)}}{1+\alpha s_{k-1}}\right)\right. \\ &\left.\quad + \sum_{i=j+1}^{m}\left(\frac{v_k^{(i)}}{1+\alpha s_k} - \frac{v_{k-1}^{(i)}}{1+\alpha s_{k-1}}\right)\right], \end{aligned} \tag{16.1.2}$$

where

$$s_k = \sum_{i=1}^{m} v_k^{(i)} . \tag{16.1.3}$$

Clearly, this equation is a rational $O(\alpha)$–deformation of (16.1.1). We shall use the abbreviation $\mathrm{RVL}_m^{(-)}(\sigma;\alpha)$ for the system (16.1.2). In case of $m = 2$ amd $(\epsilon_1,\epsilon_2) = (0,0)$ we get, using the notations $(v_k^{(1)}, v_k^{(2)}) = (u_k, v_k)$, the equations of motion

$$\left\{\begin{aligned} \dot u_k &= u_k\left(\frac{v_k}{1+\alpha(u_k+v_k)} - \frac{v_{k-1}}{1+\alpha(u_{k-1}+v_{k-1})}\right), \\ \dot v_k &= v_k\left(\frac{u_{k+1}}{1+\alpha(u_{k+1}+v_{k+1})} - \frac{u_k}{1+\alpha(u_k+v_k)}\right), \end{aligned}\right. \tag{16.1.4}$$

which is nothing but $\mathrm{RVL}_-(\alpha)$ flow, one of the two simplest flows (the "minus first" one) of the relativistic Volterra hierarchy.

The second construction of the multi–field relativistic Volterra lattice leads to a different albeit gauge equivalent hierarchy, which will be called, along with its simplest flow, $\mathrm{RVL}_m^{(+)}(\sigma;\alpha)$, see Sect. 16.3, 16.4. It appears, if one performs an invertible change of variables

$$\mathbf{B}_m(\alpha): \quad \bar v_k^{(j)} = \frac{v_k^{(j)}}{1-\alpha s_k}$$

in the $\mathrm{RVL}_m^{(-)}(\sigma;-\alpha)$ hierarchy. The simplest flow, which comes as the $\mathbf{B}_m(\alpha)$–image of (16.1.2) with the opposite sigh of α, is polynomial:

$$\dot v_k^{(j)} = v_k^{(j)}\left(\sum_{i=1}^{j-1}\left(v_{k+1}^{(i)} - v_k^{(i)}\right) + \epsilon_j\left(v_{k+1}^{(j)} - v_{k-1}^{(j)}\right) + \sum_{i=j+1}^{m}\left(v_k^{(i)} - v_{k-1}^{(i)}\right) + \alpha\Big(S_{k+1} - S_k\Big)\right), \tag{16.1.5}$$

where

$$S_k = \sum_{1\le i<\ell\le m} v_k^{(i)} v_{k-1}^{(\ell)} + \sum_{i=1}^{m} \epsilon_i v_k^{(i)} v_{k-1}^{(i)} . \tag{16.1.6}$$

(This "relativistic correction" term S_k depends, clearly, on the signature σ). This is another $O(\alpha)$–deformation of (16.1.1). In case of $m = 2$ and $(\epsilon_1,\epsilon_2) = (0,0)$ we find: $S_k = u_k v_{k-1}$, and (16.1.5) turns into

$$\left\{\begin{aligned} \dot u_k &= u_k(v_k - v_{k-1} + \alpha u_{k+1}v_k - \alpha u_k v_{k-1}), \\ \dot v_k &= v_k(u_{k+1} - u_k + \alpha u_{k+1}v_k - \alpha u_k v_{k-1}), \end{aligned}\right. \tag{16.1.7}$$

which is nothing but $\mathrm{RVL}_+(\alpha)$ flow, the "first" flow of the relativistic Volterra hierarchy.

The most valuable advantage of the second construction is that the Lax triads of this hierarchy are of the standard type, so that our general prescription for integrable discretization is applicable.

It is worked out in Sect. 16.5, and the localizing change of variables is found in Sect. 16.6. The corresponding local discretization is given in terms of the function

$$\mathbf{S}_k(\mathbf{v}^{(1)},\ldots,\mathbf{v}^{(m)}) = S_k(\mathbf{v}^{(1)},\ldots,\mathbf{v}^{(m)}) + O(h)\ ,$$

which is a polynomial of its arguments, depending on the signature σ, but not on the relativistic parameter α, and is defined by the formula

$$h^2\mathbf{S}_k = 1 + h\sum_{j=1}^{m}\mathbf{v}_k^{(j)}\cdot\prod_{i=1}^{j-1}\left(1+h\mathbf{v}_k^{(i)}\right)\cdot\left(1+\epsilon_j h\mathbf{v}_{k-1}^{(j)}\right)\cdot\prod_{i=j+1}^{m}\left(1+h\mathbf{v}_{k-1}^{(i)}\right) - \prod_{j=1}^{m}\left(1+h\mathbf{v}_k^{(j)}\right)\ . \quad (16.1.8)$$

The equations of motion of the local form of the integrable discretization of (16.1.5) read:

$$\frac{\widetilde{\mathbf{v}}_k^{(j)}\cdot\prod_{i=1}^{j-1}\left(1+h\widetilde{\mathbf{v}}_k^{(i)}\right)\cdot\left(1+\epsilon_j h\widetilde{\mathbf{v}}_{k-1}^{(j)}\right)\cdot\prod_{i=j+1}^{m}\left(1+h\widetilde{\mathbf{v}}_{k-1}^{(i)}\right)}{1-h\alpha\widetilde{\mathbf{S}}_k}$$

$$= \frac{\mathbf{v}_k^{(j)}\cdot\prod_{i=1}^{j-1}\left(1+h\mathbf{v}_{k+1}^{(i)}\right)\cdot\left(1+\epsilon_j h\mathbf{v}_{k+1}^{(j)}\right)\cdot\prod_{i=j+1}^{m}\left(1+h\mathbf{v}_k^{(i)}\right)}{1-h\alpha\mathbf{S}_{k+1}}\ . \quad (16.1.9)$$

In particular, for $m=2$ and $(\epsilon_1,\epsilon_2)=(0,0)$ one has, obviously, $\mathbf{S}_k(\mathbf{u},\mathbf{v})=\mathbf{u}_k\mathbf{v}_{k-1}$, and we recover the local discretization of the $\mathrm{RVL}_+(\alpha)$ flow.

The cases of the signatures $\sigma=(+1,+1,\ldots,+1)$ and $\sigma=(+1,-1,-1,\ldots,-1)$ are considered in somewhat more details in Sect. 16.8, 16.9. The particular systems with $m=3$ are given as illustrative examples.

A very remarkable situation takes place, if h, the time step of our discretization, coincides with α, the relativistic parameter, see Sect. 16.7. In this case it is possible to put our discretization in a local form without any localizing change of variables. The corresponding map is governed by the following equations:

$$\widetilde{v}_k^{(j)} = v_k^{(j)}\ \frac{1+h\sum_{i=1}^{j-1}v_{k+1}^{(i)}+h\epsilon_j\left(v_{k+1}^{(j)}+\widetilde{v}_k^{(j)}\right)+h\sum_{i=j+1}^{m}\widetilde{v}_k^{(i)}}{1+h\sum_{i=1}^{j-1}v_k^{(i)}+h\epsilon_j\left(v_k^{(j)}+\widetilde{v}_{k-1}^{(j)}\right)+h\sum_{i=j+1}^{m}\widetilde{v}_{k-1}^{(i)}}\ . \quad (16.1.10)$$

This is, of course, a discretization of the *non-relativistic* system $\mathrm{VL}_m(\sigma)$. Most remarkably, in the case when all $\epsilon_j=0$ this is an *explicit* discretization! Indeed, in this case one can explicitly calculate the updates of the fields $\widetilde{v}^{(m)}$, $\widetilde{v}^{(m-1)}$, ... , $\widetilde{v}^{(1)}$ (in this order). Even if some of $\epsilon_j\neq 0$, then finding the updates of the corresponding field $\widetilde{v}^{(j)}$ requires only for solving a system of N algebraic equations with N unknowns $\widetilde{v}_k^{(j)}$, $1\le k\le N$, so that the implicitness of this system is much weaker than the implicitness of the general discretization (16.1.9).

The same properties has the following discretization of the *non-relativistic* system $\mathrm{VL}_m(\sigma)$:

$$\widetilde{v}_k^{(j)} = v_k^{(j)}\ \frac{\prod_{i=1}^{j-1}\left(1+h\widetilde{v}_{k+1}^{(i)}\right)\cdot\left(1+\epsilon_j h\widetilde{v}_{k+1}^{(j)}\right)\cdot\prod_{i=j+1}^{m}\left(1+hv_k^{(i)}\right)}{\prod_{i=1}^{j-1}\left(1+h\widetilde{v}_k^{(i)}\right)\cdot\left(1+\epsilon_j hv_{k-1}^{(j)}\right)\cdot\prod_{i=j+1}^{m}\left(1+hv_{k-1}^{(i)}\right)}\ . \quad (16.1.11)$$

In the case when all $\epsilon_j=0$ this is an *explicit* discretization, since one can explicitly calculate the updates of the fields $\widetilde{v}^{(1)}$, ... , $\widetilde{v}^{(m-1)}$, $\widetilde{v}^{(m)}$ (in this order). We show in Sect. 16.10 that this discretization belongs (up to a simple invertible change of variables) to the relativistic hierarchy $\mathrm{RVL}_m^{(-)}(\sigma;h)$ obtained via the first construction. This will serve us as a new example of the relativistic hierarchy arrived at via the route connected with the integrable discretizations.

16.11 Bibliographical remarks

The relativistic system (16.1.2) with all $\epsilon_j = 1$ appeared in Hikami (1998b), with a discretization which is different and less elegant than our one. The "explicit" discretization (16.1.11) of the nonrelativistic system $\mathrm{VL}_m(\sigma)$ with the trivial signature (all $\sigma_j = +1$) was found, along the path described in Sect. 16.10, in Suris and Ragnisco (1999). All other results of the present Chapter are new.

Chapter 17

Bogoyavlensky lattices

Contents

17.1 Introduction

There are three basic families of integrable lattice systems carrying (somewhat ahistorically) the name of ***Bogoyavlensky lattices***. These systems are enumerated by integer parameters $m, p \geq 1$ ($p > 1$ for the third one) and read:

$$\dot{a}_k = a_k \left(\sum_{j=1}^{m} a_{k+j} - \sum_{j=1}^{m} a_{k-j} \right) , \tag{17.1.1}$$

$$\dot{a}_k = a_k \left(\prod_{j=1}^{p} a_{k+j} - \prod_{j=1}^{p} a_{k-j} \right) , \tag{17.1.2}$$

$$\dot{a}_k = \prod_{j=1}^{p-1} a_{k+j}^{-1} - \prod_{j=1}^{p-1} a_{k-j}^{-1} . \tag{17.1.3}$$

We shall call these systems BL1(m), BL2(p), and BL3(p), respectively. The lattices BL1(m) and BL2(p) are two different generalizations of the Volterra lattice (4.1.1). Indeed, the Volterra lattice is the particular case of both of them, corresponding to $m = 1$ and $p = 1$, respectively. The lattice BL3(p) after the change of variables $a_k \mapsto c_k = a_k^{-1}$ and $t \mapsto -t$ turns into

$$\dot{c}_k = c_k^2 \left(\prod_{j=1}^{p-1} c_{k+j} - \prod_{j=1}^{p-1} c_{k-j} \right) . \qquad (17.1.4)$$

This is a generalization of the *modified Volterra lattice*, more precisely, its particular case (4.15.1), which appears from the above system by $p = 2$. We shall see that also in general (17.1.4) is a particular case of the *modified* BL1(m), if $p = m + 1$.

The Lax representations of the Bogoyavlensky lattices are given in Sect. 17.2. We shall see that they have a standard form, so that the general recipe for integrable discretization is applicable. But before doing this, we establish an r–matrix interpretation of these Lax representations, see Sect. 17.3, 17.4, 17.5.

Application of our general recipe, accompanied by localizing changes of variables, delivers the following local discretizations for the systems (17.1.1)–(17.1.3):

$$\widetilde{\mathbf{a}}_k \prod_{j=1}^{m} (1 + h\widetilde{\mathbf{a}}_{k-j}) = \mathbf{a}_k \prod_{j=1}^{m} (1 + h\mathbf{a}_{k+j}) , \qquad (17.1.5)$$

$$\widetilde{\mathbf{a}}_k \left(1 + h \prod_{j=1}^{p} \widetilde{\mathbf{a}}_{k-j} \right) = \mathbf{a}_k \left(1 + h \prod_{j=1}^{p} \mathbf{a}_{k+j} \right) , \qquad (17.1.6)$$

$$\widetilde{\mathbf{a}}_k \left(1 + h \prod_{j=0}^{p-1} \widetilde{\mathbf{a}}_{k-j}^{-1} \right) = \mathbf{a}_k \left(1 + h \prod_{j=0}^{p-1} \mathbf{a}_{k+j}^{-1} \right) . \qquad (17.1.7)$$

The latter map, after the change of variables $\mathbf{a}_k \mapsto \mathbf{c}_k = \mathbf{a}_k^{-1}$ and $h \mapsto -h$ turns into the local discretization of (17.1.4):

$$\widetilde{\mathbf{c}}_k \left(1 - h \prod_{j=0}^{p-1} \widetilde{\mathbf{c}}_{k-j} \right)^{-1} = \mathbf{c}_k \left(1 - h \prod_{j=0}^{p-1} \mathbf{c}_{k+j} \right)^{-1} . \qquad (17.1.8)$$

This results constitute the content of Sect. 17.6–17.8.

Finally, in Sect. 17.10–17.12 we give an interpretation of the Bogoyavlensky lattices as multi–field systems (similar to VL_m), and derive the corresponding alternative Lax representations, which could also serve as a starting point for the discretization procedure.

17.13 Bibliographical remarks

17.1, 17.2. The Bogoyavlensky lattices were introduced in Bogoyavlensky (1988a, b, 1989) along with the Lax representations living in the algebra $\boldsymbol{g}$. However, certain generalizations of the Volterra lattice, including BL1 and some particular cases of BL2 and BL3, were given earlier in Narita (1982), Kupershmidt (1985) and Itoh (1987).

17.3–17.5. The Hamiltonian structure of the lattice BL1 was found in Kupershmidt (1985), as well as in the original papers by Bogoyavlensky (1988a,b, 1989). The corresponding results for BL2, BL3 in the case of infinite lattices were found in Zhang *et al.* (1991). The r–matrix structure of the Bogoyavlensky lattices was elaborated in Suris (1994a). Also the subtleties of the Hamiltonian structure in the periodic case were explained for the first time in that paper (they were overseen in Zhang *et al.* (1991); for example, their Poisson bracket for BL2 with $p = 3$ does not allow a proper reduction in the periodic case with $N \equiv 1 \pmod 3$).

17.6–17.8. The lattice BL1 was first discretized in Tsujimoto, Hirota and Oishi (1993). They found the local equations of motion (17.6.22) as a certain reduction of the discrete KP equation in the bilinear form, and also gave the discrete time Lax representation (without any hint on how it was obtained). The discretization of all three lattices BL1–BL3 was achieved almost simultaneously in Papageorgiou and Nijhoff (1996) and Suris (1996b). The approach of these two papers is completely different. In the first one the local equations of motion (17.6.22), (17.7.22), (17.8.19) are derived from an *ad hoc* ansatz for the Lax pair. The approach of the second paper (reproduced here) allows not only to find the equations of motion, but also delivers automatically the invariant Poisson structures, interpolating Hamiltonian flows, and the solution of the initial value problem in terms of matrix factorizations. The modified equations of motion (17.6.24), (17.7.25), (17.8.22), arising here as granted by–products, seem to be new.

17.10–17.12. The alternative approach to the Bogoyavlensky lattice BL1 was invented in Suris (1997d) and served as a first application of the nonlocal quadratic r–matrix structure with $m \geq 2$. The results concerning BL2 and BL3 are new. This approach delivers a nontrivial example of dual Lax representations, cf. Adams, Harnad and Hurtubise (1990a).

Chapter 18

Ablowitz–Ladik hierarchy

Contents

18.1 Introduction

The hierarchy considered in the present Chapter appeared in the early days of the soliton theory as a spatial discretization of one of the most famous hierarchies of soliton equations in partial derivatives, namely of the AKNS (Ablowitz–Kaup–Newell–Segur) hierarchy, see Sect. 18.2. The latter is most conveniently described as a hierarchy of equations attached to the ***Zakharov–Shabat spectral problem***:

$$\Psi_x = P\Psi\ , \qquad P = \begin{pmatrix} \zeta & q \\ r & -\zeta \end{pmatrix}. \tag{18.1.1}$$

Here $q = q(x,t)$, $r = r(x,t)$ are unknown fields, in terms of which the hierarchy is formulated, and ζ is the spectral parameter. Each equation of the hierarchy may be presented as a compatibility condition of (18.1.1) with a linear differential equation describing the evolution of the function Ψ in time:

$$\Psi_t = Q\Psi\ , \tag{18.1.2}$$

where the matrix Q polynomially depends on the spectral parameter. Explicitly the above mentioned compatibility condition takes the form of the ***zero curvature equation***

$$P_t - Q_x + [P,Q] = 0 \ , \tag{18.1.3}$$

which is, for suitable Q, a nonlinear evolution equation for q, r. In particular, we can get on this way the famous ***nonlinear Schrödinger equation*** (NLS):

$$-iq_t = q_{xx} \mp 2|q|^2 q \ , \tag{18.1.4}$$

and the not less famous ***modified Korteweg–de Vries equation*** (MKdV):

$$q_t = q_{xxx} \mp 6q_x q^2 \ . \tag{18.1.5}$$

The NLS is obtained in the reduction

$$r = \pm q^* \ . \tag{18.1.6}$$

(In the present Chapter the asterisque denotes the complex conjugation, or the Hermitean conjugation, when applied to matrices.) For MKdV the following reduction is relevant:

$$r = \pm q \ . \tag{18.1.7}$$

Notice also that NLS (18.1.4) (with the plus sign) is gauge equivalent to the Heisenberg magnetic model

$$S_t = S_{xx} \times S \ , \tag{18.1.8}$$

where $S \in \mathbb{R}^3$, $\|S\| = 1$.

Ablowitz and Ladik performed the discretization of the equations (18.1.4), (18.1.5) in two steps. First, they discretized the spatial variable x. We discuss this step in Sect. 18.3. To this end, they replaced the spectral problem (18.1.1) by

$$\Psi_{k+1} = L_k \Psi_k \ , \tag{18.1.9}$$

with the following 2×2 matrix L_k depending on the unknown fields $q(t) = \Big(q_k(t)\Big)_{k\in\mathbb{Z}}$, $r(t) = \Big(r_k(t)\Big)_{k\in\mathbb{Z}}$ (living now on the lattice) and on the spectral parameter λ:

$$L_k = L_k(q,r,\lambda) = \begin{pmatrix} \lambda & q_k \\ r_k & \lambda^{-1} \end{pmatrix} . \tag{18.1.10}$$

If the evolution of Ψ_k is governed by a linear equation

$$\dot{\Psi}_k = M_k \Psi_k \ , \qquad M_k = M_k(q,r,\lambda) \ , \tag{18.1.11}$$

then the compatibility condition of (18.1.9), (18.1.11) is the following ***semi–discrete zero curvature equation***:

$$\dot{L}_k = M_{k+1} L_k - L_k M_k \ . \tag{18.1.12}$$

By suitably choosen M_k the equation (18.1.12) is an evolution equation for q, r, which, by construction, can be solved with the help of the methods of the soliton theory. As examples of this construction, we mention two lattice equations, the famous Ablowitz–Ladik discretization of NLS:

$$-i\dot{q}_k = q_{k+1} - 2q_k + q_{k-1} \mp |q_k|^2 (q_{k+1} + q_{k-1}) \ , \tag{18.1.13}$$

and their discretization of MKdV:

$$\begin{aligned} \dot{q}_k \ &= \ (1 \mp q_k^2)\Big(q_{k+2} - 2q_{k+1} + 2q_{k-1} - q_{k-2} \\ &\quad \mp q_{k+1}(q_{k+2}q_{k+1} + q_{k+1}q_k) \pm q_{k-1}(q_k q_{k-1} + q_{k-1}q_{k-2})\Big). \end{aligned} \tag{18.1.14}$$

The first one of them is obtained in the reduction $r = \pm q^*$, while the second one holds under the reduction $r = \pm q$, in a complete analogy with the continuous case. In order to perform the continuous limit from (18.1.13) to (18.1.4), one has first to assume in (18.1.13) $q_k = \epsilon q(k\epsilon)$, to rescale the time $t \mapsto t/\epsilon^2$, and then to send $\epsilon \to 0$. The continuous limit from (18.1.14) to (18.1.5) is similar, and differs from the previous one only by another rescaling of the time, namely $t \mapsto t/(2\epsilon^3)$. Notice also that this limit becomes meaningful also on the level of Lax representations, if the spectral parameters are connected as $\lambda = 1 + \epsilon\zeta + O(\epsilon^2)$, for example, $\lambda = \exp(\epsilon\zeta)$.

Note also that the discrete NLS (18.1.13) (with the plus sign) is gauge equivalent to the following spatial discretization of the Heisenberg magnetic equation (18.1.8):

$$\dot{S}_k = \frac{2S_{k+1} \times S_k}{1 + \langle S_{k+1}, S_k \rangle} - \frac{2S_k \times S_{k-1}}{1 + \langle S_k, S_{k-1} \rangle}\,. \tag{18.1.15}$$

Ablowitz and Ladik performed also the second step of discretization – the time discretization, see Sect. 18.4 for details. The approach to this second step of the discretization process was fundamentally different (although they did not stress this explicitly enough). Namely, the spectral problem (18.1.9) was not modified any more, and only a suitable choice of a (discrete)–time evolution of the wave function Ψ_k was imposed:

$$\widetilde{\Psi}_k = V_k \Psi_k\,, \qquad V_k = V_k(q, r, \lambda)\,. \tag{18.1.16}$$

The compatibility condition of (18.1.9), (18.1.16) is the following ***fully discrete zero curvature equation***:

$$\widetilde{L}_k V_k = V_{k+1} L_k\,. \tag{18.1.17}$$

In the more modern language of this book, the construction by Ablowitz and Ladik was the first example of the search for integrable discretizations in the same hierarchies where the underlying continuous time systems live.

The outcome of the work by Ablowitz and Ladik consisted of excellent numerical schemes for NLS and MKdV. However, these schemes were not free of some drawbacks. The main drawback was the non-locality (which means that $\widetilde{q}_k$, the discrete time update of q_k, depends explicitly on all q_j's and $\widetilde{q}_j$'s with $j \le k$). This property is unpleasant from the aesthetical point of view as well as from the practical one (since it implies a large amount of computations by numerical realization). In the present Chapter we revisit the Ablowitz–Ladik discretizations and demonstrate how the drawback of nonlocality may be overcome.

The main idea of our construction is following. The flows (18.1.13), (18.1.14) may be considered as superpositions of some more simple and fundamental commuting flows of the AL hierarchy. It turns out that each one of these fundamental flows admits a *local* discretization. Superpositions of the local discretizations deliver local difference schemes for the original flows, as well. This idea is performed in Sect. 18.6–18.8 for the discretization of NLS, and in Sect. 18.9–18.10 for the discretization of MKdV. Finally, in Sect. 18.11 we discuss the relation of the AL hierarchy with the relativistic Toda hierarchy.

18.12 Bibliographical remarks

18.2. Zakharov–Shabat (1971) used the spectral problem carrying their name, to find and to solve the second (after KdV) system admitting solitons, namely NLS. A slightly more general spectral problem was introduced by Ablowitz, Kaup, Newell and Segur (1973, 1974) to integrate the third such system – the sine-Gordon equation, and then to define and to solve the whole AKNS hierarchy. A comprehensive discussion of the AKNS hierarchy and a large list of relevant references may be found, e.g., in Newell (1985).

The inverse scattering approach to the Heisenberg magnetic model was developed in Takhtajan (1977). The gauge equivalence of NLS and the Heisenberg magnetic was proved in Zakharov and Takhtajan (1979), see also Faddeev and Takhtajan (1987). For the geometrical interpretation of this model see Hasimoto (1971, 1972), Langer and Perline (1996), Bobenko and Suris (1999a).

18.3. The spatial discretization of the AKNS hierarchy, preserving integrability, was performed in Ablowitz and Ladik (1975, 1976a). This papers turned out to be very influential in the general development of the theory of differential–difference integrable equations. At the same time, somewhat curiously, the AL hierarchy itself enjoyes a relatively small number of publications. The inverse scattering approach was applied to this hierarchy in the original papers Ablowitz and Ladik (1975, 1976a). Finite gap solutions were constructed for the discrete version of NLS in Ahmad and Roy Chowdhury (1987) and in Miller *et al.* (1995). The open end boundary conditions were investigated in Vekslerchik (1993).

Restricted flows of the hierarchy were studied in Zeng and Rauch–Wojciechowski (1995b). In a series of papers Vekslerchik (1994, 1995, 1996) a remarkable universality of the AL hierarchy was pointed out. Namely, in the multi–time formalism many well–known integrable systems may be embedded into the AL hierarchy. Further studies in Veklerchik (1998, 1999) led to an interesting τ–functional formalism for this hierarchy.

Geometrical applications of the Ablowitz–Ladik systems were studied in Doliwa and Santini (1995).

The lattice Heisenberg magnetic model was introduced in Sklyanin (1982) and in Ishimori (1982). The latter paper contains also the proof of the gauge equivalence of this model to the Ablowitz–Ladik's discrete NLS. A duscussion of the geometric meaning of the discrete Heisenberg model see in Doliwa and Santini (1999), Bobenko and Suris (1999a).

It should be also mentioned that there exist also other spatial discretizations of NLS, the most remarkable being the one of Izergin and Korepin (1981). As an advantageous feature, it should be mentioned that this model shares the r–matrix with the continuous one (the linear r–matrix bracket of the latter being replaced by the quadratic Sklyanin bracket for the former). The Izergin–Korepin discretization is, in a sense, a reparametrization the lattice Heisenberg magnetic model. Therefore, the Ablowitz–Ladik and the Izegin–Korepin discretizations are gauge equivalent, see Hoffmann (2000).

The vector generalization of the Ablowitz–Ladik flows is an interesting model which resisted for a long times against all attempts to prove its integrablity, cf. Ablowitz, Ohta and Trubach (1999). It was finally proved recently in Tsuchida, Ujino and Wadati (1998, 1999).

18.4. The time discretization of the nonlinear Schrödinger equation was performed by Ablowitz and Ladik (1976b, 1977). These schemes, as well as numerical schemes for KdV and MKdV equations were further developed and extensively tested in Taha and Ablowitz (1984, 1988).

Different double discrete versions of NLS and some related equations are found also in Date *et al.* (1982, 1983), and the paper Quispel *et al.* (1984) contains lattice versions of NLS, KdV, MKdV and some further soliton equations. The relations between all these findings are still not completely clarified.

A number of papers was devoted to a symplectic (not necessary integrable) time discretization of the Ablowitz–Ladik system, and to important and interesting effects of the spurious chaos in such discretizations, see Herbst and Ablowitz (1989), Ablowitz and Herbst (1990), McLaughlin and Schober (1992), Herbst, Varadi and Ablowitz (1994), Ablowitz and Schober (1994a,b), Ablowitz, Herbst and Schober (1996), Calini *et al.* (1996), Tang *et al.* (1996, 1997), Schober (1999), Hairer and Schober (2000).

18.5–18.10. These Sections contain the results of the papers Suris (1997e,f). See also the discussion in Black, Weideman and Herbst (1999), Suris (2000b).

For the notion of the partitioned Runge–Kutta methods and for their symplectic properties see Hairer *et al.* (1987), Suris (1990c), Sanz–Serna (1991), Suris (1996d). The Ruth–Yoshida–Suzuki techniques for constructing higher order symplectic integrators was developed in Forest and Ruth (1990), Yoshida (1990), Suzuki (1992).

18.11. The relation between the Ablowitz–Ladik and the relativistic Toda hierarchies was discovered in Kharchev *et al.* (1997), its Hamiltonian interpretation was given in Suris (1997e). The results on the discrete time members of these hierarchies and their interpolating Hamiltonian flows are contained in Suris (1997e,f).

Part III

Systems of classical mechanics

Chapter 19

Peakons system

Contents

19.1 Introduction

Starting from this Chapter, we entry the area of integrable systems of classical mechanics and their integrable discretizations. Of course, the boarder between lattice (or differential–difference) systems and the systems of classical mechanics is somewhat fuzzy. For instance, the exponential Toda lattice certainly belongs to classical mechanics. However, in this third part of the book we consider systems which definitely do not belong to the class of differential–difference equations (although they may be defined for an arbitrary number N of degrees of freedom). Another common feature of the systems considered from now on, is that application of our general recipe of integrable discretization usualy is accompanied by serious difficulties, which have not been surpassed in the majority of cases. These difficulties are mainly connected with finding a reasonably simple description of factors $\Pi_{\pm}(F(L))$ for some suitable functions $F : \boldsymbol{g} \mapsto \boldsymbol{G}$. Therefore, obtaining integrable discretizations is connected with a certain amount of guesswork and cannot be presented in such a systematic manner as for lattice systems of Part II. A further remarkable peculiarity is the existence of alternative discretizations, not living in original hierarchies, i.e. possessing *deformed* integrals of motion. Finding these ones is a *totally* non–formalizable process.

One of the several happy cases when a systematic derivation is still possible is the peakons system considered in the present Chapter.

Many interesting integrable systems of classical mechanics may be viewed as describing certain special solutions of integrable partial derivatives equations. So, the famous Calogero–Moser systems describe certain singular solutions of the Kadomtsev–Petviashvili (KP) equation, their relativistic analogs describe solitons of the sine–Gordon equation, and so on. Recently, there appeared another remarkable system of that sort, namely a system describing peaked solitons of the so called Camassa–Holm shallow water equation. This system, called hereafter the ***peakons system***, is Hamiltonian on the canonical phase space $\mathbb{R}^{2N}(x,p)$ with the Hamilton function

$$\mathrm{H}(x,p) = \frac{1}{2}\sum_{j,k=1}^{N} p_j p_k e^{-|x_k - x_j|} \,. \tag{19.1.1}$$

Here x_k is the position of the peak of the kth soliton, and p_k is related to its amplitude. It may be proved that, generally speaking, the peakons retain their ordering. More precisely, the sign of $(x_k - x_j)$ can change at some $t = t_0$, only if p_k, p_j become infinite at $t = t_0$. Hence, restricting ourselves to non–singular solutions, or, better, to time intervals where solutions are non–singular, we can assume $x_1 > x_2 > \ldots > x_N$, and the Hamilton function takes the form

$$\mathrm{H}(x,p) = \frac{1}{2}\sum_{k=1}^{N} p_k^2 + \sum_{1\le j<k\le N} p_j p_k e^{x_k - x_j} . \tag{19.1.2}$$

The canonical equations of motion read:

$$\dot{x}_k = \sum_{j<k} p_j e^{x_k - x_j} + p_k + \sum_{j>k} p_j e^{x_j - x_k} , \tag{19.1.3}$$

$$\frac{\dot{p}_k}{p_k} = -\sum_{j<k} p_j e^{x_k - x_j} + \sum_{j>k} p_j e^{x_j - x_k} . \tag{19.1.4}$$

This system possesses a Lax representation in $\mathrm{gl}(N)$ which fits nicely in the r–matrix framework, see Sect. 19.2. Therefeore, our general recipe is applicable, and in Sect. 19.3 we derive the following integrable discretization for (19.1.3), (19.1.4):

$$e^{\widetilde{x}_k - x_k} = \frac{1}{1 - h\sum_{j<k}\widetilde{p}_j e^{x_k - \widetilde{x}_j}} + h\sum_{j\ge k} p_j e^{x_j - x_k} , \tag{19.1.5}$$

$$\frac{\widetilde{p}_k}{p_k} = \frac{1 - h\sum_{j<k}\widetilde{p}_j e^{x_k - \widetilde{x}_j}}{1 - h\sum_{j>k} p_j e^{x_j - \widetilde{x}_k}} . \tag{19.1.6}$$

The Lagrangean interpretation of these equations is achieved in Sect. 19.4.

19.5 Bibliographical remarks

19.1. About the origin of many–body systems from soliton equations see, e.g., Airault, McKean and Moser (1977), Ruijsenaars and Schneider (1986), Babelon and Bernard (1993), Krichever *et al.* (1995), Krichever and Zabrodin (1995), van Diejen (2000), Krichever (2000), Akhmetshin and Volvovsky (2000).

The peakons system was discovered and studied in Camassa and Holm (1993), Camassa, Holm and Hyman (1994). These authors derived it from a certain integrable shallow water equation, and studied the dynamics of peakons in some detail. The relation to billiard systems was studied in Alber *et al.* (1994, 1999).

A generalization of the peakons system was found by Calogero (1995), Calogero and Francoise (1996), who constructed further integrable Hamiltonians of the type $\mathrm{H}(x,p) = \frac{1}{2}\sum_{j,k=1}^{N} p_j p_k f(x_k - x_j)$.

19.2. The Lax representation of the peakons system was found in Camassa and Holm (1993), Camassa, Holm and Hyman (1994). The beautiful connection with the Toda hierarchy was pointed out in Ragnisco and Bruschi (1996). Another aspect of this connection can be traced in the inverse scattering solution of the peakons system given by Beals, Sattinger and Szmigielski (1999, 2000).

The r–matrix interpretation of the Lax representation of the peakons system (Theorem 19.2) was found in Bruschi and Ragnisco (1996). This statement may be reformulated so: the matrices (19.2.2) form a low–dimensional symplectic leaf of the bracket $\mathrm{PB}_1(\frac{1}{2}\mathrm{R})$ on $\mathrm{gl}(N)$. Similar symplectic leaves in all simple Lie algebras are constructed in Kamalin and Perelomov (1985), Alber, Camassa and Gekhtman (2000). The paper Bruschi and Ragnisco (1996) contains also an analogous result for the Calogero's generalizations (when the r–matrix becomes dynamical).

19.3–19.4. The discretization of the peakons system was given in Suris (1996c).

Chapter 20

Standard–like discretizations of scalar second–order equations

Contents

20.1 Introduction

In the present short Chapter we deal with one of the symplest possible objects: scalar second order equations

$$\ddot{x} = f(x) , \tag{20.1.1}$$

where $x \in \mathbb{R}$, and find their explicit integrable discretizations of the type

$$\widetilde{x} - 2x + \underset{\sim}{x} = h^2 F(x,h) . \tag{20.1.2}$$

Of course, in order for (20.1.2) to be a discretization of (20.1.1), one must have $F(x,0) = f(x)$. We call the discretizations of the type (20.1.2) ***standard–like***, after the famous *standard map*, which corresponds to $F(x,h) = -\sin x$. It is well known that the corresponding difference equation,

$$\widetilde{x} - 2x + \underset{\sim}{x} = -h^2 \sin x ,$$

is non–integrable. In general, integrability of nonlinear equations (20.1.2), even in the one–dimensional case, is an exceptional phenomenon. For example, such equations with a function $F(x,h) = f(x)$ independent on h are generically non–integrable, if $f(x)$ is an entire function (in the sense of the complex analysis). Nevertheless, in Sect. 20.2 we construct three families of integrable real–analytic standard–like discretizations (20.1.2) for scalar equations (20.1.1) with the right–hand side $f(x)$ being either a polynomial of degree ≤ 3, or a trigonometric polynomial of degree ≤ 2, or an "exponential polynomial" of degree ≤ 2. Moreover, in Sect. 20.3 we prove that our list of integrable scalar standard–like discretizations is, in some sense, exhaustive. A typical representative of the first family is

$$\widetilde{x} - 2x + \underset{\sim}{x} = -h^2 \frac{Bx + 2Dx^3}{1 + h^2 Dx^2} , \tag{20.1.3}$$

serving as an integrable discretization for the so called Duffing equation

$$\ddot{x} = -Bx - 2Dx^3 . \tag{20.1.4}$$

Typical representatives of the second family are

$$\widetilde{x} - 2x + \underset{\sim}{x} = -2\,\arctan \frac{\dfrac{h^2}{2}\sin x}{1+\dfrac{h^2}{2}\cos x}\,, \tag{20.1.5}$$

and

$$\widetilde{x} - 2x + \underset{\sim}{x} = -4\,\arctan \frac{\dfrac{h^2}{4}\sin x}{1+\dfrac{h^2}{4}\cos x}\,, \tag{20.1.6}$$

both serving as integrable discretizations of the mathematical pendulum equation

$$\ddot{x} = -\sin x\,. \tag{20.1.7}$$

Finally, the third family is essentially given by the formula

$$\widetilde{x} - 2x + \underset{\sim}{x} = \log \frac{1+h^2\Big(B\,e^{-x}+D\,e^{-2x}\Big)}{1+h^2\Big(A\,e^{x}+C\,e^{2x}\Big)}\,, \tag{20.1.8}$$

serving as an integrable discretization for the 4–parameter family of differential equations

$$\ddot{x} = -A\,e^{x} + B\,e^{-x} - C\,e^{2x} + D\,e^{-2x}\,. \tag{20.1.9}$$

20.4 Bibliographical remarks

The results of the present Chapter were found in Suris (1989). A closely related family of intagrable plane maps was introduced in Quispel, Roberts and Thompson (1989). Actually, all our maps may be embedded (after appropriate changes of variables) into their 18–parameter (!) family. A closely related family of standard–like maps in higher dimensions was proposed by Ishimori (1994).

The non–integrability of a generic standard–like map was proved by Lazutkin (1988). The exponentially small separatrix splitting for the standard map was deeply studied by Lazutkin and his school, see Lazutkin, Schachmannski and Tabanov (1989), Lazutkin (1993), Gelfreich, Lazutkin and Svanidze (1994), Gelfreich (1999). Notice also that perturbations of our integrable maps surve as convenient models in this circle of ideas, cf. Delshams and Ramirez-Ros (1997, 1998), Lomeli and Meiss (2000).

Chapter 21

Lie–algebraic generalizations of the Toda systems

Contents

21.1 Introduction

This Chapter is devoted to one more family of integrable multi–dimensional second–order mechanical systems

$$\ddot{q}_k = f_k(q)\ , \qquad 1\le k\le N\ , \tag{21.1.1}$$

admiiting integrable standard–like discretizations

$$\widetilde{q}_k - 2q_k + \underset{\sim}{q}_k = h^2 F_k(q;h)\ , \qquad 1\le k\le N\ . \tag{21.1.2}$$

This family consists of the Lie–algebraic generalizations of the *exponential Toda lattice*. Recall that for the usual Toda lattice with the open–end or periodic boundary conditions all right–hand sides (for $1\le k\le N$) are equal to

$$f_k(q) = e^{q_{k+1}-q_k} - e^{q_k-q_{k-1}}\ , \tag{21.1.3}$$

and it admits an integrable standard–like discretization with the right–hand sides

$$F_k(q;h)=h^{-2}\log\frac{1+h^2e^{q_{k+1}-q_k}}{1+h^2e^{q_k-q_{k-1}}}\,. \tag{21.1.4}$$

We shall see that these systems can be associated with the A_{N-1} series of simple Lie algebras (in the open–end case) and to the $A_{N-1}^{(1)}$ series of affine Lie algebras (in the periodic case), and that there exist similar systems corresponding to all other classical series of simple and affine Lie algebras. For the most general systems considered in this Chapter, the formula (21.1.3) holds for $2\le k\le N-2$, and two types of the functions $f_k(q)$, resp. $F_k(q;h)$ are admissible for both the left end and the right end of the lattice. Namely, we say that the lattice has the left end of the type BC with the parameters (β_0,γ_0), if the formula (21.1.3) holds for $k=2$, and

$$f_1(q)=e^{q_2-q_1}-\beta_0e^{q_1}-\gamma_0e^{2q_1}\,. \tag{21.1.5}$$

We say that the lattice has the left end of the type D, if

$$f_1(q) = e^{q_2-q_1}-e^{q_2+q_1}\,, \tag{21.1.6}$$

$$f_2(q) = e^{q_3-q_2}-e^{q_2-q_1}-e^{q_2+q_1}\,. \tag{21.1.7}$$

Similarly, the right end of the type BC with the parameters (β_N,γ_N) is characterized by (21.1.3) for $k=N-1$, and

$$f_N(q)=\beta_Ne^{-q_N}+\gamma_Ne^{-2q_N}-e^{q_N-q_{N-1}}\,, \tag{21.1.8}$$

while in case of the right end of the type D we have:

$$f_{N-1}(q) = e^{-q_N-q_{N-1}}+e^{q_N-q_{N-1}}-e^{q_{N-1}-q_{N-2}}\,, \tag{21.1.9}$$

$$f_N(q) = e^{-q_N-q_{N-1}}-e^{q_N-q_{N-1}}\,. \tag{21.1.10}$$

In particular, if the parameter γ_0 or γ_N vanishes, we speak about the left (resp. right) end of the type B, and if the parameter β_0 or β_N vanishes, we speak about the left (resp. right) end of the type C. It turns out that the a system with an arbitrary combination of the left end and the right end is completely integrable. For different combinations of the pure B, C, D ends we present the "big" Lax representations in Sect. 21.6, which is based on the general Lie–algebraic construction developed in Sect. 21.2, 21.3, and the 2×2 Lax representations in Sect. 21.9. For systems where the ends of the type BC are admissible, we have only 2×2 Lax representations.

We prove that the following functions $F_k(q;h)$ provide us with integrable discretizations for the above mentioned systems. For all systems, (21.1.4) holds for $3\le k\le N-2$. For the lattice with the left end of the type BC with the parameters (β_0,γ_0), the formula (21.1.4) holds for $k=2$, and

$$F_1(q;h)=h^{-2}\log\frac{1+h^2e^{q_2-q_1}}{1+h^2\beta_0e^{q_1}+h^2\gamma_0e^{2q_1}}\,, \tag{21.1.11}$$

and for the lattice with the left end of the type D, there holds:

$$F_1(q;h) = h^{-2}\log\frac{1+h^2e^{q_2-q_1}}{1+h^2e^{q_2+q_1}}\,, \tag{21.1.12}$$

$$F_2(q;h) = h^{-2}\log\frac{1+h^2e^{q_3-q_2}}{\Big(1+h^2e^{q_2-q_1}\Big)\Big(1+h^2e^{q_2+q_1}\Big)}\,. \tag{21.1.13}$$

Similarly, for the lattice with the right end of the type BC with the parameters (β_N,γ_N), the formula (21.1.4) holds for $k=N-1$, and

$$F_N(q;h)=h^{-2}\log\frac{1+h^2\beta_Ne^{-q_N}+h^2\gamma_Ne^{-2q_N}}{1+h^2e^{q_N-q_{N-1}}}\,, \tag{21.1.14}$$

and for the lattice with the right end of the type D, we have:

$$F_{N-1}(q;h) = h^{-2}\log\frac{\Big(1+h^2e^{-q_N-q_{N-1}}\Big)\Big(1+h^2e^{q_N-q_{N-1}}\Big)}{1+h^2e^{q_{N-1}-q_{N-2}}}\,, \tag{21.1.15}$$

$$F_N(q;h) = h^{-2}\log\frac{1+h^2e^{-q_N-q_{N-1}}}{1+h^2e^{q_N-q_{N-1}}}\,. \tag{21.1.16}$$

Again, for the lattices with pure B, C, D type ends, we find "big" Lax representations in Sect. 21.7, 21.8, and 2×2 Lax representations in Sect. 21.10, while if BC ends are admissible, we have only 2×2 Lax representations. Unlike the continuos time situation, we do not have an abstract Lie–algebraic (representation independent) formulation of our discrete time systems, and only in the A cases do we have an r–matrix interpretation of the "big" Lax representation (see Sect. 21.4). Nevertheless, the complete integrability is proved in all cases with the help of 2×2 Lax representations.

It is important to notice that in all cases associated with the simple and affine Lie algebras, the corresponding Toda lattice may be equivalently written as

$$\ddot{Q}_\beta = -\sum_{\alpha\in\Phi} C_{\beta\alpha} e^{Q_\alpha}\ , \qquad \beta\in\Phi\ ; \tag{21.1.17}$$

here Φ is the set of simple roots (resp. weights) of the corresponding algebra, which enumerates also the set $\{Q_\alpha\}_{\alpha\in\Phi}$ of dependent variables of the corresponding lattice, and $C_{\beta\alpha}$ are entries of the Cartan matrix of the corresponding algebra. This form of equations of motion holds not only for classical series of simple and affine Lie algebras, but also for exceptional ones. Our standard–like discretizations may be always put into the form

$$\widetilde{Q}_\beta - 2Q_\beta + \underset{\sim}{Q}_\beta = -\sum_{\alpha\in\Phi} C_{\beta\alpha}\log\Big(1+h^2e^{Q_\alpha}\Big)\ , \qquad \beta\in\Phi\ . \tag{21.1.18}$$

A natural hypothesis says then that the latter system should be also integrable for exceptional simple and affine Lie algebras. We do not have a general proof of this hypothesis, valid for all exceptional algebras. However, we give several additional arguments in favour of this hypothesis. Namely, we find integrable discretizations of the Toda lattices corresponding to the algebras G_2, $G_2^{(1)}$, and $D_4^{(3)}$. In particular, the latter two systems may be put in the nice form, resembling the two–particle system with both ends of the BC type. More precisely, the equations of motion of are of the form (21.1.1) with

$$\begin{aligned}
&\left\{\begin{array}{l} 6f_1(q) = e^{3(q_2-q_1)} - 2e^{6q_1}\ ,\\ 2f_2(q) = 2e^{-2q_2} - e^{3(q_2-q_1)}\ , \end{array}\right. \qquad \text{lattice } G_2^{(1)}\ ,\\
&\left\{\begin{array}{l} 2f_1(q) = 3e^{q_2-q_1} - 2e^{2q_1}\ ,\\ 2f_2(q) = 2e^{-2q_2} - e^{q_2-q_1}\ , \end{array}\right. \qquad \text{lattice } D_4^{(3)}\ .
\end{aligned} \tag{21.1.19}$$

By now not very surprisingly, the right–hand sides of the integrable discretizations of these systems (21.1.2) are given by:

$$\begin{aligned}
&\left\{\begin{array}{l} 6F_1(q;h) = \log\Big(1+h^2e^{3(q_2-q_1)}\Big) - 2\log\Big(1+h^2e^{6q_1}\Big)\ ,\\ 2F_2(q;h) = 2\log\Big(1+h^2e^{-2q_2}\Big) - \log\Big(1+h^2e^{3(q_2-q_1)}\Big)\ , \end{array}\right. \qquad \text{lattice } G_2^{(1)}\ ,\\
&\left\{\begin{array}{l} 2F_1(q;h) = 3\log\Big(1+h^2e^{q_2-q_1}\Big) - 2\log\Big(1+h^2e^{2q_1}\Big)\ ,\\ 2F_2(q;h) = 2\log\Big(1+h^2e^{-2q_2}\Big) - \log\Big(1+h^2e^{q_2-q_1}\Big)\ , \end{array}\right. \qquad \text{lattice } D_4^{(3)}\ .
\end{aligned} \tag{21.1.20}$$

These results are contained in Sect. 21.11–21.13. In the suitably introduced ccordinates Q_α the above equations of motion for the discrete time lattices G_2, $G_2^{(1)}$, and $D_4^{(3)}$ have exactly the form (21.1.18).

21.14 Bibliographical remarks

21.2, 21.3. As a general reference for the theory of Lie algebras and Lie groups we recommend Helgason (1978). The Lie–algebraic generalizations of the exponential Toda lattices were discovered by Bogoyavlensky (1976). A solution in terms of the representation theory was given by Kostant (1979) and by Olshanetsky and Perelomov (1979). The involutivity of integrals was demonstrated by Reyman and Semenov-Tian-Shansky (1979, 1981) and Reyman (1980) with the help of r–matrix approach, and by Kostant (1982). See also an r–matrix account in Belavin and Drinfeld (1982).

The systems with the ends of BC type were added by Sklyanin (1987). He used 2×2 Lax representations. Even more general boundary conditions assuring integrability were found by Inozemtsev (1989). Various results were found toward a complete characterization of integrable systems with exponential interactions, see Adler and van Moerbeke (1982), who used the notion of algebraic complete integrability; Kozlov and Treshchev (1989a,b) who studied the existence of polynomial integrals and Kowalewski numbers of such systems.

A large amount of literature is devoted to the two–dimensional generalized Toda lattices connected with Lie algebras (affine Toda fields). Among important early papers on this topic see Leznov and Saveliev (1979), Mikhailov (1979), Fordy and Gibbons (1980a, 1983), Mikhailov, Olshanetsky and Perelomov (1981), Wilson (1981), Kupershmidt and Wilson (1981), Drinfeld and Sokolov (1984). A geometric interpretation of all Lie–algebraic two–dimensional Toda systems was achieved in Doliwa (1997a,b).

21.4, 21.5. See bibliographical remarks in Chapt. 3, 5 and 10.

21.6. In our choice of matrix representations for the classical simple Lie algebras and affine Lie algebras we follow Drinfeld and Sokolov (1984).

21.7, 21.8. The results of these Sections were found in Suris (1990a, 1991). Notice that in all cases apart of A_{N-1}, $A_{N-1}^{(1)}$, our Lax matrices do not belong to the corresponding Lie group. Even worth: the algebraic nature of these Lax representations remains absolute obscure. It should be noticed that somewhat different Lax representations, living in the corresponding groups, were found in Hoffmann *et al.* (2000) for all simple Lie algebras, including the exceptional ones, but still not for the affine Lie algebras.

We mention here also some results concerning the discretization of the two–dimensional Toda lattices (Toda field equations). The A_{N-1} case was discretized by Hirota (1987), see also a study of a closely related model in Kashaev and Reshetikhin (1997). An attempt of discretizing Toda field equations for other Lie algebras, following Suris (1990a), was undertaken in Ward (1995). A completely different approach is that of Kuniba, Nakanishi and Suzuki (1994), Kuniba and Suzuki (1995a,b), Kuniba, Nakamura and Hirota (1996), Tsuboi and Kuniba (1996), Tsuboi (1997). The equations which can be considered as discretizations of Toda field systems appeared in these papers as functional relations satisfied by transfer matrices of integrable models of statistical quantum physics. See in this relation also Krichever, Lipan, Wiegmann and Zabrodin (1997), Krichever, Wiegmann and Zabrodin (1997).

21.9. 2×2 Lax representations and the r–matrix structure for Toda lattices with ends of the BC type were given by Sklyanin (1987) and generalized for the case of the D type ends by Kuznetsov and Tsyganov (1989).

21.10. 2×2 Lax representations and the r–matrix structure for all discrete time Toda lattices were found in Suris (1990b), and used there for the proof of complete integrability and for the definition of the relativistic Toda lattiices corresponding to all simple Lie algebras and affine Lie algebras, as well as for the case of the BC type ends.

21.11–21.13. The continuous time results here are well known, the discrete time ones are new.

Chapter 22

Garnier system

Contents

22.1 Introduction

In the present Chapter we study several closely related integrable mechanical systems with cubic nonlinearities, along with their integrable discretizations. The first one is the so called *anharmonic oscillator*:

$$\ddot{x}_k = -\omega_k x_k - 2x_k\langle x,x\rangle\ , \qquad 1\le k\le N\ , \tag{22.1.1}$$

where

$$\langle x,x\rangle = \sum_{j=1}^{N} x_j^2\ .$$

This system is completely integrable in the Liouville–Arnold sense, all integrals being quadratic in momenta. Therefore, it also solvable via the separation of variables method (which we do not discuss here). We give two different Lax representations of this system, as well as their r–matrix intepretations, which is the most direct way towards proving the involutivity of integrals. All this is contained in Sect. 22.3.

The anharmonic oscillator possesses several different integrable discretizations, and we consider here two most remarkable among them.

The first one (Sect. 22.6) is an implicit one:

$$(1+h^2\omega_k)^{1/2}(\widetilde{x}_k+\underset{\sim}{x}_k) = x_k\left((1-h^2\sigma)^{1/2}+(1-h^2\underset{\sim}{\sigma})^{1/2}\right), \quad 1\le k\le N\ , \tag{22.1.2}$$

where

$$\sigma = \langle\widetilde{x},\widetilde{x}\rangle+\langle x,x\rangle\ , \qquad \underset{\sim}{\sigma} = \langle x,x\rangle+\langle\underset{\sim}{x},\underset{\sim}{x}\rangle\ . \tag{22.1.3}$$

We demonstrate that this discrete time system of Newtonian equations admits a Lagrangian interpretation. Introducing the corresponding momenta, the main feature of (22.1.2) uncovers: namely, it shares the integrals of motion and, naturally, also the Lax matrices with its continuous time counterpart. In other words, it may be considered as a *Bäcklund transformation* of the continuous time system.

The second discretization (Sect. 22.9) does not have this attractive property, being instead more elegant and, moreover, an explicit one:

$$\widetilde{x}_k+\underset{\sim}{x}_k = \frac{(2-h^2\omega_k)x_k}{1+h^2\langle x,x\rangle}\ , \qquad 1\le k\le N\ . \tag{22.1.4}$$

This is also a completely integrable symplectic map, and its integrals of motion and Lax matrices, being different from their continuous time counterparts, tend to the latter in the limit $h\to 0$.

There exists also a generalization of the anharmonic oscillator, which corresponds to imposing additional inverse square potential upon all particles. The corresponding system, due to *Wojciechowski*, reads:

$$\ddot{x}_k = -\omega_k x_k + \frac{m_k^2}{x_k^3} - 2x_k\langle x,x\rangle\ , \qquad 1\le k\le N\ . \tag{22.1.5}$$

The integrability properties of this system constitute the subject of Sect. 22.4.

It turns out to be possible to generalize both discretizations above for the case of the Wojciechowski system. This generalization is obtained along the similar lines, and results for both discretizations in a certain modification of the left–hand side, while retaining the right–hand side intact. An implicit discretization sharing the integrals of motion and the Lax matrix with (22.1.5) reads (Sect. 22.7):

$$\widetilde{x}_k\left(1-\frac{h^2m_k^2}{\widetilde{x}_k^2x_k^2}\right)^{1/2}+\underset{\sim}{x}_k\left(1-\frac{h^2m_k^2}{x_k^2\underset{\sim}{x}_k^2}\right)^{1/2}=(1+h^2\omega_k)^{-1/2}x_k\left((1-h^2\sigma)^{1/2}+(1-h^2\underset{\sim}{\sigma})^{1/2}\right). \quad (22.1.6)$$

An explicit discretization of (22.1.5) is:

$$\widetilde{x}_k\left(1-\frac{h^2m_k^2}{\widetilde{x}_k^2x_k^2}\right)^{1/2}+\underset{\sim}{x}_k\left(1-\frac{h^2m_k^2}{x_k^2\underset{\sim}{x}_k^2}\right)^{1/2}=\frac{(2-h^2\omega_k)x_k}{1+h^2\langle x,x\rangle}. \quad (22.1.7)$$

This system is studied in detail in Sect. 22.10.

Actually, both the anharmonic oscillator and the Wojciechowski system turn out to be two different reductions of a more general system, known as the *Garnier system*. The latter describes a coupled system of two N–dimensional oscillators $(x,\xi)\in\mathbb{R}^{2N}$, interacting via a biquadratic potential $\langle\Omega x,\xi\rangle+\langle x,\xi\rangle^2$. We give a detailed account of this general system in Sect. 22.2, and study its two integrable discretizations in Sect. 22.5, 22.8.

22.11 Bibliographical remarks

22.2, 22.3. The Garnier system and the anharmonic oscillator were discovered by Garnier (1919). It should be mentioned that this paper contains one of the earliest examples of the Lax representation of a mechanical system. Namely, the "big" Lax matrix (22.2.20) is written there explicitly! This remarkable discovery remained forgotten for a long time, so that, e.g., Reyman (1980) considered finding a Lax representation for the anharmonic oscillator as an open problem. However, at that time it was already rediscovered by Choodnovsky and Choodnovsky (1979), who derived it from the Lax representation of the vector nonlinear Schrödinger equation, for which the anharmonic oscillator is a stationary flow. The r–matrix interpretation of this Lax representation was found by Reyman (1986).

Actually, the anharmonic oscillator may be included into a family of natural mechanical systems with polynomial potentials, admitting a separation of variables in elliptic coordinates. This family was defined by Wojciechowski (1985a), see also Marshall and Wojciechowski (1988), Antonowicz and Rauch-Wojciechowski (1992a).

The "small" 2×2 Lax representation was derived, as a by–product of the interpretation of the anharmonic oscillator as a restricted flow of the KdV hierarchy, in Antonowicz (1992), Antonowicz and Rauch-Wojciechowski (1992b). A bi–Hamiltonian structure of this and related systems was established in Antonowicz and Rauch-Wojciechowski (1990). See also related papers on the restricted flows of soliton hierarchies: Antonowicz, Fordy and Wojciechowski (1987), Antonowicz and Rauch-Wojciechowski (1991, 1993), Rauch-Wojciechowski (1991a,b), Ragnisco and Rauch–Wojciechowski (1992, 1994), Orlov and Rauch-Wojciechowski (1993), and Marciniak (1997). The r–matrix structures of the 2×2 Lax representations of the restricted flows were studied in Eilbeck, Enolskij, Kuznetsov and Leykin (1993), Eilbeck, Enolskij, Kuznetsov and Tsiganov (1994), Kulish, Rauch-Wojciechowski and Tsiganov (1996).

22.4. The system studied in here was introduced by Wojciechowski (1985b). This paper contains also a solution by separation of variables, the proof of complete integrability, and the "big" Lax representation. The dynamical r–matrix was found in Suris (1995b).

22.5–22.7. The Bäcklund transformations for the anharmonic oscillator and the Wojciechowski system were found in Hone, Kuznetsov and Ragnisco (1999) on the base of 2×2 Lax representations. We add here to their results the case of the Garnier system and the $N\times N$ Lax representations. Notice that these discrete time systems are nothing but the stationary "flows" of the vector analog of the Ablowitz–Ladik discrete nonlinear Schrödinger equation. Interestingly enough, the integrability of this latter system was established only recently by Tsuchida, Ujino and Wadati (1999). The corresponding Lax representation is *not* the nonstationary version of our one, in particular, the Lax matrix is not $N\times N$, but rather lives in the Clifford algebra $\mathcal{C}\ell_N$.

22.8–22.10. Explicit discretizations of the Garnier system and of the anharmonic oscillator were found in Suris (1994b) (a particular case $N=2$ was found earlier by McLachlan (1993), without proof of integrability). The explicit discretization of the Wojciechowski system was given in Suris (1994d).

Chapter 23

Lie–algebraic generalizations of the Garnier system

Contents

23.1 Introduction

We discuss in this Chapter a large family of multi–dimensional second–order integrable mechanical systems admitting integrable standard–like discretizations, which generalize the Garnier system and its reduced version, the anharmonic oscillator:

$$\ddot{x}_k = -\omega_k x_k - 2x_k \sum_{j=1}^{N} x_j^2 \,, \qquad 1 \le k \le N \,. \tag{23.1.1}$$

Recall that its integrable standard–like discretization found in Chapter 22 is a natural multi–dimensional analog of (20.1.3):

$$\widetilde{x}_k - 2x_k + \underset{\sim}{x}_k = -h^2 \, \frac{\omega_k x_k + 2x_k \sum_{j=1}^{N} x_j^2}{1 + h^2 \sum_{j=1}^{N} x_j^2} \,, \qquad 1 \le k \le N \,. \tag{23.1.2}$$

Putting $c_k = 1 - h^2\omega_k/2$, and rescaling $x_k \mapsto c_k^{1/2} x_k$, we find the symplectic version of (23.1.2):

$$\widetilde{x}_k + \underset{\sim}{x}_k = \frac{2c_k x_k}{1 + h^2 \sum_{j=1}^{N} c_j x_j^2}\;, \qquad 1 \le k \le N\;. \tag{23.1.3}$$

The anharmonic oscillator turns out to possess a series of generalizations related to symmetric spaces. All these generalized systems admit standard–like discretizations, so that we find a vast family of integrable multi–dimensional standard–like maps related to symmetric spaces. The abstract Lie–algebraic formulation of these generalized systems is given in Sect. 23.2. Integrable discretizations are discussed in a case–by–case manner.

One family of these generalized systems (Sect. 23.3) is described by the matrix motion equation

$$\ddot{x} = -\Omega x - x\mathrm{A} - 2xx^{\mathrm{T}}x\;, \tag{23.1.4}$$

where x is an $N \times M$ matrix, and $\Omega = \mathrm{diag}(\omega_1, \dots, \omega_N)$, $\mathrm{A} = \mathrm{diag}(\alpha_1, \dots, \alpha_M)$. So, the anharmonic oscillator is a $M = 1$ particular case of (23.1.4). A standard–like integrable discretization of (23.1.4) reads:

$$\widetilde{x} + \underset{\sim}{x} = 2Cx(D + h^2 x^{\mathrm{T}}Cx)^{-1} = h^{-2}\partial \log\det(D + h^2 x^{\mathrm{T}}Cx)/\partial x^{\mathrm{T}}, \tag{23.1.5}$$

where $C = \mathrm{diag}(c_1, \dots, c_N)$, $D = \mathrm{diag}(d_1, \dots, d_M)$, and the relation to the continuous–time parameters is as follows: $c_k = 1 - h^2\omega_k/2 + o(h^2)$, $d_k = 1 + h^2\alpha_k + o(h^2)$. In Sect. 23.3 we find a Lax representation of this discrete time system, and discuss its integrals of motion and the r–matrix structure. As pointed out in Sect. 23.4, if $M = N$ and $\Omega = \mathrm{A}$, resp. $C = D^{-1}$, then the above continuous and discrete time systems admit important reductions singled out by the condition of the matrix x being symmetric or skew–symmetric.

Finally, in Sect. 23.5 the following oscillator with a quartic potential is considered:

$$\ddot{x} = -\Omega x - x\alpha - 2x\langle x, x\rangle + Sx\langle x, Sx\rangle\;, \tag{23.1.6}$$

where α is some real number, $S = (\delta_{j,N-k+1})_{j,k=1}^{N}$, and the diagonal matrix Ω satisfies $S\Omega = -\Omega S$. An integrable standard–like discretization of this system reads:

$$\widetilde{x} + \underset{\sim}{x} = \frac{1}{\Delta}\Big(2Cxd + h^2 Sx\langle x, Sx\rangle\Big) = h^{-2}\partial\log\Delta/\partial x^{\mathrm{T}},\;, \tag{23.1.7}$$

where

$$\Delta = d^2 + h^2 d\langle Cx, x\rangle + \frac{h^4}{4}\langle x, Sx\rangle^2\;. \tag{23.1.8}$$

Here $d = 1 + h^2\alpha/2 + o(h^2)$, and the diagonal matrix $C = I - h^2\Omega/2 + o(h^2)$ satisfies $SC = C^{-1}S$.

23.6 Bibliographical remarks

23.2. Integrable systems with the quartic potential related to Hermitean symmetric spaces were introduced in Fordy, Wojciechowski and Marshall (1986). They appeared as stationary flows of the corresponding generalization of the nonlinear Schrödinger equation found by Fordy and Kulish (1983). It should be mentioned that this latter work was followed by a series of papers where some other scalar integrable PDEs were generalized in a similar manner: Fordy (1983b), Athorne and Fordy (1987). Returning to the quartic oscillators, we notice that their r–matrix interpretation was given by Reyman (1986) and Marshall (1988).

23.3–23.5. The discretizations given here were found, along with the Lax representations and r–matrix interpretation (when available) by Suris (1994c).

Chapter 24

Neumann system

Contents

24.1 Introduction

The subject of the present Chapter constitute several well–known integrable mechanical systems, all being close relatives of the famous Neumann system. They belong to the area of constrained mechanics, describing the motion of a particle (or of two interacting particles) on the surface of the sphere $S = \{x \in \mathbb{R}^N : \langle x, x\rangle = 1\}$.

The *Neumann system* (Sect. 24.3) describes the motion of a particle $x \in S$ under the influence of the harmonic potential $\frac{1}{2}\langle \Omega x, x\rangle$. The equations of motion read:

$$\ddot{x}_k = -\omega_k x_k - \alpha x_k \ , \qquad 1 \le k \le N \ , \tag{24.1.1}$$

where $\alpha = \alpha(x, \dot{x})$ is the Lagrange multiplier assuring the validity of the relations $\langle x, x\rangle = 1$, $\langle \dot{x}, x\rangle = 0$ during the evolution. It is easy to see that

$$\alpha = \langle \dot{x}, \dot{x}\rangle - \langle \Omega x, x\rangle \ . \tag{24.1.2}$$

This system can be given a Hamiltonian interpretation in terms of the Dirac Poisson bracket, and it turns out to be completely integrable in the Liouville–Arnold sense, all integrals being quadratic in momenta (which also implies that it is solvable via the separation of variables method). Actually, this system is in some sense a close relative of the anharmonic oscillator, and our presentation here follows closely that of the previous Chapter. We give two different Lax representations of the Neumann system, and discuss their r–matrix interpretations.

Further, we discuss three different integrable discretization of the Neumann system. (This system is probably the champion in possessing essentially different integrable discretizations.) The difference equations of the first one, due to Veselov, read:

$$\widetilde{x}_k + \underset{\sim}{x}_k = (1 + h^2\omega_k)^{-1/2}\beta x_k \ , \tag{24.1.3}$$

where β is the Lagrange multiplier, assuring that x remains on the sphere S during the discrete time evolution. It is easy to see that

$$\beta = \frac{2\Big\langle (I + h^2\Omega)^{-1/2}x, \underset{\sim}{x}\Big\rangle}{\Big\langle (I + h^2\Omega)^{-1}x, x\Big\rangle} = \frac{2\Big\langle (I + h^2\Omega)^{-1/2}x, \widetilde{x}\Big\rangle}{\Big\langle (I + h^2\Omega)^{-1}x, x\Big\rangle} = \frac{\Big\langle (I + h^2\Omega)^{-1/2}x, \widetilde{x} + \underset{\sim}{x}\Big\rangle}{\Big\langle (I + h^2\Omega)^{-1}x, x\Big\rangle} \ . \tag{24.1.4}$$

The first two expressions for β make this difference equation *explicit* for the evolution in both the positive and the negative directions of (discrete) time. The third expression makes it obvious that $\beta = 2 - h^2\alpha + o(h^2)$ with α from (24.1.2), so that we indeed have a discretization of (24.1.1). We show in Sect. 24.6 that this equation (24.1.3) is a discrete time Lagrangian system, which provides us with a systematic way of introducing the canonically conjugate momenta. Performing this, the main feature of (24.1.3) comes into the light: namely, it shares the integrals of motion and the Lax matrices with its continuous time counterpart. In other words, it is a *Bäcklund transformation* of the Neumann system.

The second discretization of the Neumann system, due to Ragnisco, is governed by the following difference equations:

$$\frac{\widetilde{x}_k}{\langle \widetilde{x}, x\rangle} - 2x_k + \frac{\underset{\sim}{x}_k}{\langle x, \underset{\sim}{x}\rangle} = -h^2\omega_k x_k + h^2\langle \Omega x, x\rangle x_k \ . \tag{24.1.5}$$

Again, this is a discrete time Lagrangian system. Introducing the canonically conjugate momenta in a proper way, we shall see in Sect. 24.8 that the integrals of motion and the Lax matrices for (24.1.5) are $O(h)$–perturbations of the corresponding objects of the continuous time Neumann system.

The same holds also for the third discretization of the Neumann system, due to V. Adler:

$$\frac{\widetilde{x}_k + x_k}{1 + \langle \widetilde{x}, x\rangle} + \frac{x_k + \underset{\sim}{x}_k}{1 + \langle x, \underset{\sim}{x}\rangle} = \frac{(2 - h^2\omega_k/2)\, x_k}{1 - h^2\,\langle \Omega x, x\rangle/4} \ . \tag{24.1.6}$$

We study this system in Sect. 24.9.

The next integrable system in the Neumann family is the *Rosochatius system*, see Sect. 24.4. It can be said that its relation to the Neumann system is approximately the same as the relation of the

Wojciechowski system to the anharmonic oscillator: it consists of adding an inverse square potential $\frac{1}{2}\sum_{k=1}^{N} m_k^2/x_k^2$. So, the equations of motion of the Rosochatius system read:

$$\ddot{x}_k = -\omega_k x_k + \frac{m_k^2}{x_k^3} - \alpha x_k\ , \qquad 1 \le k \le N\ . \tag{24.1.7}$$

Here the formula for the Lagrange multiplier, assuring $\langle x, x\rangle = 1$ and $\langle \dot{x}, x\rangle = 0$, reads:

$$\alpha = \langle \dot{x}, \dot{x}\rangle - \langle \Omega x, x\rangle + \left\langle \frac{m}{x}, \frac{m}{x}\right\rangle ,$$

where, of course, m/x stands for the vector with components m_k/x_k. The discretizations of the Neumann system can be generalized for the case of the Rosochatius system. We consider in detail only the generalization of (24.1.3), see Sect. 24.7. This generalization also mimics the corresponding construction of the previous Chapter. The discrete time Newtonian equations of motion are:

$$\widetilde{x}_k\left(1 - \frac{h^2 m_k^2}{\widetilde{x}_k^2 x_k^2}\right)^{1/2} + \underset{\sim}{x}_k\left(1 - \frac{h^2 m_k^2}{x_k^2 \underset{\sim}{x}_k^2}\right)^{1/2} = (1 + h^2\omega_k)^{-1/2}\beta x_k\ , \tag{24.1.8}$$

with the corresponding Lagrange multiplier β. Again, the Lagrangian nature of these equations leads to their Hamiltonian interpretation, and the latter makes it apparent that (24.1.8) shares the integrals of motion ans the Lax matrix with its continuous time analog (24.1.7).

Actually, we consider the Neumann system and the Rosochatius system as two different reductions of the more general one, the so called *double Neumann system.* The latter describes the motion of the particle (x, ξ) on the surface of the hyperboloid $\Sigma = \{(x,\xi) \in \mathbb{R}^{2N} : \langle x, \xi\rangle = 1\}$. See Sect. 24.2 for a detailed account of this system, and Sect. 24.5 for its discretization, whose particular cases (or, better, reductions) the systems (24.1.3) and (24.1.8) are.

The double Neumann system has not to be confused with the *coupled Neumann system*, which actually describes the motion of two particles x and y on the sphere S, under the bilinear potential of interaction $\langle \Omega x, y\rangle$. The latter system is considered in Sect. 24.10. The equations of motion read:

$$\ddot{x}_k = -\omega_k y_k - \alpha x_k\ , \quad \ddot{y}_k = -\omega_k x_k - \gamma y_k\ , \qquad 1 \le k \le N\ . \tag{24.1.9}$$

Here α and γ are two *different* Lagrange multipliers, whose proper choice guarantees that the relations $\langle x, x\rangle = \langle y, y\rangle = 1$ are fulfilled for all times. This proper choice is:

$$\alpha = \langle \dot{x}, \dot{x}\rangle - \langle \Omega x, y\rangle\ , \qquad \gamma = \langle \dot{y}, \dot{y}\rangle - \langle \Omega x, y\rangle\ . \tag{24.1.10}$$

The system (24.1.9), (24.1.10) is completely integrable: it possesses a Lax representation, admitting an r–matrix interpretation. We consider two (probably) integrable discretizations of the coupled Neumann system. The first one is a direct generalization of the Ragnisco's discrete time Neumann system (24.1.5):

$$\frac{\widetilde{x}_k}{\langle \widetilde{x}, x\rangle} - 2x_k + \frac{\underset{\sim}{x}_k}{\langle x, \underset{\sim}{x}\rangle} = -h^2\omega_k y_k + h^2\langle \Omega x, y\rangle x_k\ , \tag{24.1.11}$$

$$\frac{\widetilde{y}_k}{\langle \widetilde{y}, y\rangle} - 2y_k + \frac{\underset{\sim}{y}_k}{\langle y, \underset{\sim}{y}\rangle} = -h^2\omega_k x_k + h^2\langle \Omega x, y\rangle y_k\ . \tag{24.1.12}$$

The second one is a direct generalization of the V. Adler's discrete time Neumann system (24.1.5):

$$\frac{\widetilde{x}_k + x_k}{1 + \langle \widetilde{x}, x\rangle} + \frac{x_k + \underset{\sim}{x}_k}{1 + \langle x, \underset{\sim}{x}\rangle} = \frac{2x_k - h^2\omega_k y_k/2}{1 - h^2\langle \Omega x, y\rangle/4}\ , \tag{24.1.13}$$

$$\frac{\widetilde{y}_k + y_k}{1 + \langle \widetilde{y}, y\rangle} + \frac{y_k + \underset{\sim}{y}_k}{1 + \langle y, \underset{\sim}{y}\rangle} = \frac{2y_k - h^2\omega_k x_k/2}{1 - h^2\langle \Omega x, y\rangle/4}\ . \tag{24.1.14}$$

More precisely, we demonstrate in Sect. 24.11 that the system of difference equations (24.1.11), (24.1.12) possesses at least N integrals of motion, quadratic in momenta, which are $O(h)$–perturbations of the corresponding integrals of the continuous time coupled Neumann system. For the system of difference equations (24.1.13), (24.1.14) we give only one integral of motion. A construction of Lax representations of these discretizations, as well as finding lacking integrals (N for the first one, and $2N-1$ for the second one), remains an interesting open problem.

24.12 Bibliographical remarks

24.2–24.4. The systems studied here were discovered in XIXth century, in Neumann (1859) and Rosochatius (1877) (of course, in the "physical" dimension $N = 3$ only). In these classical papers the systems were integrated in hyperelliptic functions with the help of separation of variables. The modern period in studying these systems starts with the work of Moser (1980a,b). He formulated the Hamiltonian structure in terms of the constrained Dirac bracket, and gave explicitly all integrals of motion. Further, Moser discussed an isomorphism between the solutions of the Neumann system and the finite–gap potentials of the Schrödinger operator. This isomorphism was established in McKean and Trubowitz (1976), and developed further also in Veselov (1980). An isomorphism between the Neumann system and the geodesic flow on an ellipsoid was also discovered by Moser (1980a,b), and further developed in Knörrer (1982). It was extended to the case of the Rosochatius system in Kubo *et al.* (1999).

The algebro–geometric study of the Neumann system was pushed forward by by Mumford (1984) and by Flashka (1984) (who seems to be the inventor of what we call the double Neumann system), see also Ercolani and Flaschka (1985). In the case of Rosochatius system such study was performed in Gagnon *et al.* (1985). The Flashka's approach was further developed by Schilling (1987, 1989, 1992a,b), who introduced Neumann–like systems related to different spectral problems.

An interpretation of the complete integrability of the Neumann and Rosochatius systems in the framework of Euler–Poisson systems on semidirect product Lie algebras was given by Ratiu (1981, 1982).

The progress in Moser (1980a,b) was mainly due to the discovery of the "big" ($N \times N$) Lax representation. The interpretation of this Lax representation for the Neumann system in terms of the Adler–Kostant–Symes scheme was given soon by Adler and van Moerbeke (1980) and Reyman (1980). See a modern (r–matrix) presentation in Avan, Talon (1990), Reyman and Semenov-Tian-Shansky (1994). The 2×2 Lax representation of the Neumann system is present already in Mumford (1984). The existense of such different Lax representations is a particular case of the general "duality" explored in Adams, Harnad and Hurtubise (1990a), see also Schilling (1991). The "small" Lax representation was interpreted in terms of momentum maps into loop algebras, and generalized to the Rosochatius system, in Adams, Harnad and Previato (1988). An extensive development of this approach is given in Adams, Harnad and Hurtubise (1990b, 1993, 1997). See also Avan and Talon (1991a), Kuznetsov (1992). This works contain also (more or less explicitly) an r–matrix intepretation of the 2×2 Lax representation of the unconstrained version of the Neumann system. The dynamical r–matrix structure for the constrained version appeared in Ragnisco and Suris (1996). The dynamical r–matrix for the "big" Lax representation of the Rosochatius system was derived in Suris (1995b).

A whole family of natural mechanical systems on a sphere, integrable via separation of variables, was constructed in Wojciechowski (1985a). This is related to the interpretation of the Neumann system as a restricted flow of the KdV hierarchy, given in Antonowicz and Rauch-Wojciechowski (1991, 1992b), which implies also a new interpretation of the 2×2 Lax representation, cf. Eilbeck *et al.* (1994). Of course, this is closely related to the above mentioned Moser–Trubowitz isomorphism with finite–gap Schrödinger operators.

24.5–24.7. The discretization of the Neumann system was achieved for the first time in Veselov (1988), its ralation to the matrix factorization was established in Veselov (1991a), Moser and Veselov (1991). An attempt (in our view, unsuccessfull) to relate this factorization problem to the one connected with the standard decomposition of the loop algebra into subalgebras spanned by negative, resp. nonnegative powers of ν was undertaken in Krešić-Jurić (1999). A generalization to the Rosochatius system was given in Suris (1994d). A different approach to these discretizations, based on the interpretation of the Neumann system as a restricted flow of the KdV hierarchy, and therefore on 2×2 Lax representations, was proposed in Hone, Kuznetsov and Ragnisco (1999). This approach led also to the interpretation of these discretizations as Bäcklund transformations.

24.8. This discretization was proposed by Ragnisco (1992), and was also based on the technique of restricted flows. Its r–matrix interpretation was given in Ragnisco (1995), Ragnisco and Suris (1996). A closely related discretization of the double Neumann system, which however does not admit the reduction $\xi = x$, was proposed in Ragnisco and Rauch–Wojciechowsi (1996).

24.9. This discretization was proposed by V. Adler (2000b), as a result of his discretization of the Landau–Lifshits equation. He found also two integrals of motion in the case $N = 3$. The integrability in the general case was proved in Suris (2000c).

24.10, 24.11. The coupled Neumann system was invented by Reyman (1980) and independently by Veselov (1984). Also the "big" Lax representation and its r–matrix interpretation was given in Reyman (1980). The "small" Lax representation seems not to appear in the literature previously. The discretizations of the coupled Neumann system are proposed here for the first time.

Chapter 25

Some integrable cases of rigid body dynamics

Contents

25.1 Introduction

The rigid body dynamics are rich with problems interesting from the mathematical point of view, in particular, with integrable problems. Certainly, the most famous ones are the three integrable cases of the rotation of a heavy rigid body about a fixed point in a homogeneous gravity field, named after Euler, Lagrange, and Kovalevsky. They were discovered in the 18th and the 19th century, and can be called classical. However, the list of integrable problems of the rigid body dynamics is by far not exhausted by these ones. Another important example is the rotation of a rigid body with a fixed point in an arbitrary quadratic potential. The integrability of this problem is a much more recent finding due to Bogoyavlensky and Reyman.

In the present Chapter we study those problems for which integrable discretizations are known. The list includes the Euler top, the Lagrange top, and the rigid body in a quadratic potential. Moreover, we discuss multidimensional generalizations of the first and the third of these systems. In all cases the equations of motion have a form of (generalized) Euler–Poisson equations. They can be interpreted as Lie–Poisson equations on duals to certain semidirect product Lie algebras, and the most natural way to derive them is through the Lagrangian reduction. We present such a derivation for all the above mentioned systems and for their discrete time analogs.

We start our presentation in Sect. 25.2 with the n–dimensional analog of the Euler top (free rigid body with a fixed point). The equations of motion are in terms of a skew–symmetric matrix of the kinetic momenta $M \in \mathrm{so}(n)$:

$$\dot{M} = [M, \Omega] , \tag{25.1.1}$$

where the angular velocity matrix $\Omega \in \mathrm{so}(n)$ is defined by the relation

$$M = J\Omega + \Omega J \quad \Leftrightarrow \quad \Omega_{jk} = M_{jk}/(J_j + J_k) , \tag{25.1.2}$$

where $J = \mathrm{diag}(J_1, \dots, J_n)$ is the inertia tensor. We present a Lagrangian derivation of these equations and their Hamiltonian structure. Moreover, we formulate a bi–Hamiltonian property of (25.1.1), which could serve as a route to establishing a complete integrability of this system. However, we follow another route, namely through the Lax representation and its r–matrix interpretation.

Then, in Sect. 25.3, we present a discrete time Lagrange function which leads to an integrable discretization of (25.1.1). The role of the angular velocity is played now by the *group element* $W_k \in \mathrm{SO}(n)$, such that $W_k \approx I + h\Omega$, $W_k^{\mathrm{T}} \approx I - h\Omega$. The equations of motion read:

$$M_{k+1} = W_k^{\mathrm{T}} M_k W_k , \tag{25.1.3}$$

while the relation between the kinetic momenta and the angular velocity also becomes less simple than (25.1.2):

$$M_k = \frac{1}{h}(W_k J - J W_k^{\mathrm{T}}) . \tag{25.1.4}$$

(Notice that in the present Chapter we found it more convenient to use the index notation for discrete time systems, so that the subscript k refers to the discrete time $t_k = kh$). The Lagrangian derivation leads to a simple proof of a Poisson property of the map $M_k \mapsto M_{k+1}$. However, this map is multivalued: as opposed to (25.1.2), the relation (25.1.4), considered as an equation for W_k with a given M_k, has more than one solution (although only one of them has correct asymptotic properties by $h \to 0$). Nevertheless, any branch of the multi–valued map $M_k \mapsto M_{k+1}$ thus obtained is completely integrable. This is demonstrated with the help of a discrete time analog of the Lax representation. By a sort of a wonderful coincidence, the Lax matrix for this map is exactly the same as for its continuous time counterpart. Hence the discrete time Euler top shares the integrals of motion with the continuous time one.

An important generalization of the Euler top is connected with putting it in an arbitrary quadratic potential. The resulting integrable system is studied in Sect. 25.4. It is described by the following generalized Euler–Poisson equations:

$$\left\{ \begin{array}{l} \dot{M} = [M, \Omega] + [P, J] , \\[2ex] \dot{P} = [P, \Omega] , \end{array} \right. \tag{25.1.5}$$

Here $M \in \mathrm{so}(n)$ is the matrix of kinetic momenta, while the symmetric matrix $P \in \mathrm{Symm}(n)$ comprises the information both about the position of the body and about the coefficients of the quadratic potential. The meaning of the symbols J and Ω is the same as before. In particular, there holds (25.1.2). Again, our Lagrangian derivation leads automatically to the Hamiltonian interpretation of this system. Its complete integrability is demonstrated with the help of a Lax representation and its r–matrix interpretation.

In Sect. 25.5 we present a discrete time Lagrangian system which is an integrable discretization of a rigid body in a quadratic potential. Needless to say that the Lagrangian derivation automatically assures Poisson property of the resulting (multi–valued) map $(M_k, P_k) \mapsto (M_{k+1}, P_{k+1})$. The equations of motion may be presented as

$$\left\{ \begin{array}{l} M_{k+1} = W_k^{\mathrm{T}} M_k W_k + \dfrac{h}{2}\,[P_{k+1}, J W_k + W_k^{\mathrm{T}} J] \\[2ex] P_{k+1} = W_k^{\mathrm{T}} P_k W_k . \end{array} \right. \tag{25.1.6}$$

Here the matrix of angular velocity $W_k \in \mathrm{SO}(n)$, $W_k \approx I + h\Omega$, has to be determined from the following equation which generalizes (25.1.4) and also has (25.1.2) as its continuous limit:

$$M_k = \frac{1}{h}(W_k J - JW_k^{\mathrm{T}}) - \frac{h}{2}(P_k W_k J - JW_k^{\mathrm{T}} P_k) . \tag{25.1.7}$$

Again, this equation has several solutions for W_k in terms of (M_k, P_k), but only one of them has a correct asymptotic behavior as $h \to 0$. A complete integrability of the map (25.1.6), (25.1.7) is established with the help of a Lax representation and its r–matrix structure. Remarkably, this time the Lax matrix does not coincide with the Lax matrix of the continuous time system, but is its $O(h^2)$–perturbation. The same holds, therefore, for integrals of motion of our discretization.

Finally, in Sect. 25.7 we study the Lagrange case of the heavy top motion. Also this system admits multi–dimensional generalizations, but we have an integrable discretization only for the classical, three–dimensional, one. In the frame firmly attached to the body the equations of motion of a heavy top are:

$$\left\{ \begin{array}{l} \dot{M} = M \times \Omega + P \times A \ , \\[2ex] \dot{P} = P \times \Omega \ . \end{array} \right. \tag{25.1.8}$$

Here all quantities are vectors from $\mathbb{R}^3$: M is the vector of kinetic momenta of the rigid body, P is the unit vector along the direction of the gravity field, as seen from the body, and A is the (constant, when seen from the body) vector pointing from the fixed point to the center of mass. The angular velocity vector Ω is related to M via

$$M = \mathcal{J}(\Omega) = (\mathrm{I}_1\Omega_1, \mathrm{I}_2\Omega_2, \mathrm{I}_3\Omega_3)^{\mathrm{T}} \ , \tag{25.1.9}$$

where I_k are the moments of inertia. The Lagrange top is characterized by the following data: $\mathrm{I}_1 = \mathrm{I}_2$, which means that the body is rotationally symmetric with respect to the third coordinate axis, and $A_1 = A_2 = 0$, which means that the fixed point lies on the symmetry axis. Choosing units properly, we may assume that

$$\mathrm{I}_1 = \mathrm{I}_2 = 1 \ , \qquad \mathrm{I}_3 = \alpha \ , \qquad A = (0,0,1)^T \ . \tag{25.1.10}$$

The integrability of this case is due to the symmetry with respect to rotations about the vector A, which is responsible for the additional integral $M_3 = \langle M, A \rangle$, lacking for a general heavy top. This same symmetry leads also to a less known circumstance, namely that the Lagrange top admits a nice Euler–Poisson–type description also in the rest frame. The corresponding equations of motion read:

$$\left\{ \begin{array}{l} \dot{m} = p \times a \ , \\[2ex] \dot{a} = m \times a \ , \end{array} \right. \tag{25.1.11}$$

and do not contain the anisotropy parameter α at all. Here the vectors m, p, a are the same as M, P, A, respectively, but seen from the rest frame (in particular, the gravity vector p is constant now, and the motion of the symmetry axis of the top a is of a prime interest). Sect. 25.7 contains a systematic Lagrangian derivation of all these facts, as well as Lax representations of the Lagrange top, both in the body frame and in the rest frame.

Next, in Sect. 25.8 we find a discrete time Lagrangian system, which serves as an integrable discretization of the Lagrange top. We derive the discrete time analogs of both the body frame and the rest frame Euler–Poisson equations. The rest frame ones are the most beautiful:

$$\left\{ \begin{array}{l} m_{k+1} = m_k + hp \times a_k \ , \\[2ex] a_{k+1} = a_k + \dfrac{h}{2} m_{k+1} \times (a_k + a_{k+1}) \ . \end{array} \right. \tag{25.1.12}$$

It is very important that, unlike the previous two cases, the equations of motion determine the updates (m_{k+1}, a_{k+1}) uniquely in terms of (m_k, a_k). Our approach leads automatically to Poisson properties of this map. Its integrability is demonstrated directly (by finding the full set of integrals in involution), and also the Lax representation is found, together with an r–matrix interpretation. The

Lax matrix and the integrals turn out to be $O(h)$–perturbations of their continuous time counterparts. Similar results hold for the system in the body frame.

Finally, in Sect. 25.9 we display the relations between the Lagrange top and the Heisenberg magnetic, which were instrumental in finding the integrable discretization of the former.

Appendix A contains some useful technical details about the group SU(2) we work with by studying the Lagrange top.

25.10 Bibliographical remarks

25.1. There exists a rich literature on the various aspects of the three classical integrable cases of the rigid body dynamics, which cannot be reviewed here. We restrict ourselves to mentioning two nice books. A classical but somewhat old–fashioned presentation is contained in Golubev (1953), a modern treatment is given by Audin (1996).

What concerns more special themes of a relevance for our presentation, but not covered by detailed references that follow, one should mention an r–matrix approach to top–like systems covering almost all known integrable examples, due to Reyman (1980), see also Reyman and Semenov-Tian-Shansky (1994). A more detailed treatment of this approach for the Kovalevsky case is given by Bobenko, Reyman and Semenov-Tian-Shansky (1989) and Marshall (1998). An interesting integrable case of the dynamics of two interacting rigid bodies was found and studied by Perelomov, Ragnisco and Wojciechowski (1986). A Lax pair for the Goryachev–Chaplygin case was found by Bobenko and Kuznetsov (1988), and for the Appelrot case by Dragović and Gajić (1999).

25.2. According to Bogoyavlensky (1992), the multidimensional generalization of the Euler top is due to Frahm (1875) and Schottky (1891), and appeared later by Weyl (1923), and by Bottema and Beth (1951). A modern period of the interest in this system started with the general theory of geodesic flows on Lie groups, due to Arnold (1966), as well as with the paper by Manakov (1976), who found the n–dimensional Euler top as a reduction of the so called n wave equations, along with a spectral parameter dependent Lax representation which we reproduce in Theorem 25.2. Far reaching generalizations of this Lax representation and the corresponding sets of commuting integrals were developed by Mishchenko, Fomenko and the school of the latter; a detailed account of these results see, e.g., in Trofimov and Fomenko (1994, 1995). A detailed discussion of the bi–Hamiltonian structure, as formulated in Theorem 25.1, and its consequences is contained in Morosi and Pizzocchero (1996b).

25.3. A discretization of the Euler top is due to Veselov (1988), an interpretation in terms of matrix factorization was given in Moser and Veselov (1991), Veselov (1991a). An attempt to include these matrix factorizations into the general r–matrix framework was undertaken in Deift, Li and Tomei (1992). However, an artificial R–operator they used is apparently different from the standard R–operator from Theorem 25.3, and therefore the underlying Poisson structure of the Euler top is presumably also different from the standard ones.

We mention also two different integrable discretization of the 3–dimensional Euler top, which deal only with systems on the Lie algebra so(3), rather than on all of the Lie group SO(3): Bobenko, Lorbeer and Suris (1998) and Hirota and Kimura (2000). It is an interesting open question, whether these system may be interpreted as reductions of Lagrangian systems on SO(3). Also the possibility of multidimensional generalizations of these results is not studied.

25.4. The integrability of a rigid body in an arbitrary quadratic potential was discovered by Reyman (1980), who gave also the r–matrix interpretation, and independently by Bogoyavlensky (1984), who integrated this system in terms of Prym theta functions.

25.5. The integrable discretization of the top in a quadratic potential was found by Suris (2000a).

25.6. The integrable case of the motion of a 3–dimensional rigid body in an ideal fluid was found by Clebsch (1870). Formally the same system appeared by de Brun (1893) in the context of rotation of a rigid body in the field of a gravitationg plane, which may be approximated by a quadraitic potential. The Clebsch's work was a development of results of Kirchhoff (1869) who gave a Lagrangian derivation of equations of motion in teh general case, and pointed out an integrable case connected with a rotational symmetry. The integration in terms of hyperelliptic theta functions was performed by Weber (1878) and Kötter (1892). The multi–dimensional generalization was found by Perelomov (1981). The integrable discretization is due to Suris (2000a).

25.7. The Lagrange integrable case of the heavy top motion was discovered by him in 1788. A modern approach to its integration based on the Lax representation and algebro–geometric technique see in Ratiu and van Moerbeke (1982), Gavrilov and Zhivkov (1998). The bi–Hamiltonian structure was elaborated in Médan (1996). A very thorough study of the geometry of invariant tori, including the monodromy phenomenon, is given in Cushman and Bates (1997).

As for the multidimensional case, there are two possible ways to generalize the notion of a heavy top, and, correspondingly, two different generalizations of the Lagrange top. The first one, living on $\mathrm{so}(n) \ltimes \mathbb{R}^n$, was proposed in Belyaev (1981), see also Reyman and Semenov-Tian-Shansky (1994). The second one, living on $\mathrm{so}(n) \ltimes \mathrm{so}(n)$, was introduced by Ratiu (1982b).

25.8, 25.9. The integrable discretization of the Lagrange top was performed in Bobenko (1999), Bobenko and Suris (1999a). See these papers also for the motivation, which lies in the theory of elastic curves and their discrete analogs.

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