A nonstandard analysis characterization of submanifolds in Euclidean space

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Summary. A nonstandard analysis characterization for standard submanifolds in a Euclidean space is given.

1. Introduction

The aim of this note is to prove a nonstandard analysis characterization for submanifolds in Euclidean n-space. This characterization features a purely geometric approach that might be useful in the context of "infinitesimal geometry" (cf.[4]). In Stroyan's fundamental paper [9], a weaker statement than the one we are going to formulate is proven — even though some remarks strongly suggested the characterization given here[3)]. In the author's opinion, it should be possible to give similar characterizations for higher order differentiable submanifolds. However, the elementary methods used here seem not to be appropriate to produce a rigorous proof.

We are going to formulate our statement (and its proof) using Nelson's approach [5] to nonstandard analysis (cf.[7]): thus, we work with the usual objects of "classical" (standard) mathematics, and all statements of classical mathematics are valid in our setting. We just deal with an additional attribute "standard" for mathematical objects that allows to define terms like "infinitely close", or "infinitesimal". However, this attribute has to be used with care as it behaves different than the attributes one is used to from classical mathematics. For more

[3)] This misled the author to erronously cite Stroyan's paper for the theorem proven in the present paper. And, after noticing this error, the author was very surprised that he could not find any alternative reference for our theorem below.

details, the reader is referred to [7], [4], or the original paper [5] by E. Nelson[4)] Before formulating our theorem, we give a useful

Notation. *We say that "$q \in I\!\!R^n$ is infinitely close to $p \in I\!\!R^n$ mod μ", $q \simeq_\mu p$, if $\frac{|q-p|}{\mu} \leq \varepsilon$ for all standard $\varepsilon > 0$ (or, $q = p$ in case $\mu = 0$).*
In case $\mu = 1$, we write $q \simeq p$, and say that "q is infinitely close to p".

With this, we formulate the main

Theorem. *A standard subset $M^m \subset I\!\!R^n$, $n \ll \infty$ is a C^1-submanifold if and only if there exists a standard tangent plane map $T : M \to G(m,n)$ into the set of affine m-planes such that for every near standard (in M) point $p \in M$,*

(a) *p lies on its tangent plane[5)], $p \in T_p$;*

(b) *the orthogonal projection $\pi_p : M \to T_p$ is an infinitesimal bijection[6)];*

(c) *the angle that the secant line through p and any infinitely close point $q \in M$, $q = p + dp \simeq p$, forms with T_p is infintesimal: $|q - \pi_p(q)| \simeq_{|dp|} 0$.*

2. Some linear algebra

A key issue in the proof of the theorem is to control the "angle" between nearby tangent planes. As we are dealing with arbitrary dimension and codimension, it seems necessary to first clarify this notion of angle: to that end, note that the Euclidean scalar product on $I\!\!R^n$ uniquely extends to the Grassmann algebra $\Lambda I\!\!R^n = \oplus_{k=0}^n \Lambda^k I\!\!R^n$ of $I\!\!R^n$ via $\langle v_1 \wedge \ldots \wedge v_k, w_1 \wedge \ldots \wedge w_k \rangle = \det(\langle v_i, w_j \rangle)_{ij}$, and $\Lambda^m I\!\!R^n \perp \Lambda^k I\!\!R^n$ for $m \neq k$.

Lemma & Definition. *Let $V, W \subset I\!\!R^n$ be two m-dimensional subspaces, and choose bases $(v_1, \ldots, v_m)$ and $(w_1, \ldots, w_m)$. Then, the number $\alpha \in [0, \frac{\pi}{2}]$ with*

$$|v_1 \wedge \ldots \wedge v_m|\,|w_1 \wedge \ldots \wedge w_m| \cos\alpha = |\langle v_1 \wedge \ldots \wedge v_m, w_1 \wedge \ldots \wedge w_m \rangle|$$

does not depend on the choice of bases for V and W. We will call α the "intersection angle of V and W".

Proof. First note that α is well defined, by the Cauchy-Schwarz inequality. To understand that α is independent of the choice of bases, consider an endomorphism $A \in End(V)$, $\tilde{v}_i = \sum_{j=1}^m a_{ij} v_j$. Then,

$$\begin{aligned} \langle \tilde{v}_1 \wedge \ldots \wedge \tilde{v}_m, w_1 \wedge \ldots \wedge w_m \rangle &= \det(\textstyle\sum_{j=1}^m a_{ij} \langle v_j, w_k \rangle)_{ik} \\ &= \det(A) \cdot \langle v_1 \wedge \ldots \wedge v_m, w_1 \wedge \ldots \wedge w_m \rangle \end{aligned}$$

which implies the claim. ◁

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4) Also, there exists an excellent part of an unwritten book by E. Nelson [6], that the reader may find on E. Nelson's web site.

5) As this is a classical property, it will hold for *all* $p \in M$ as soon as it holds for all standard p, by transfer.

6) This means: for $M \ni q, \tilde{q} \simeq p$, $\pi_p(q) = \pi_p(\tilde{q})$ implies $q = \tilde{q}$ (i.e. π_p is an infinitesimal injection), and for any $T_p \ni x \simeq p$ there is a $q \in M$, $q \simeq p$, with $\pi_p(q) = x$ (π_p is an infinitesimal surjection).

Using that description of the just defined intersection angle of subspaces as an ordinary angle of two vectors in a higher dimensional vector space, we obtain a triangle inequality:

1. Lemma. *Given three m-dimensional subspaces $V_i \subset \mathbb{R}^n$, $i = 1, 2, 3$, with intersection angles α_{ij}, we have $\alpha_{13} \leq \alpha_{12} + \alpha_{23}$.*

Proof. Thinking of the subspaces V_i as lines in $\Lambda^m \mathbb{R}^n \cong \mathbb{R}^{\binom{n}{m}}$, this just is the triangle inequality in $\mathbb{R}P^{\binom{n}{m}}$ equipped with the spherical metric. ◁

On the other hand, it will be important to relate the intersection angle α between two subspaces with angles of vectors spanning them.

2. Lemma. *If $V, W \subset \mathbb{R}^n$ are two m-dimensional subspaces, then there exist orthonormal bases $(v_1, \ldots, v_m)$ and $(w_1, \ldots, w_m)$ of V and W, respectively, such that $\langle v_i, w_j \rangle = \delta_{ij} \cos\alpha_i$ with angles $\alpha_i \in [0, \frac{\pi}{2}]$.*

Proof. First, choose an orthonormal basis $(v_1, \ldots, v_k)$ of $V \cap W$, and let $w_i := v_i$ for $i = 1, \ldots, k$. Then, denote $S := \{x \in \mathbb{R}^n \mid |x| = 1 \,\&\, x \perp V \cap W\}$, and consider the function[7] $f : (V \cap S) \times (W \cap S) =: T \to \mathbb{R}$, $(v, w) \mapsto f(v, w) = \langle v, w \rangle$. As T is compact, there are unit vectors $v_{k+1} \in V$ and $w_{k+1} \in W$ such that $f(v_{k+1}, w_{k+1}) < 1$ is maximal. Now, let $w \in W$, $w \perp w_{k+1}$; then, $w \perp v_{k+1}$: since $s \mapsto f(v_{k+1}, w_{k+1} \cos s + w \sin s)$ becomes maximal for $s = 0$, its derivative $\langle v_{k+1}, w \rangle = 0$. By symmetry, the same is true for $v \in V$, $v \perp v_{k+1}$. Thus, taking $S = \{x \in \mathbb{R}^n \mid |x| = 1 \,\&\, x \perp V \cap W, v_{k+1}, w_{k+1}\}$, we may reduce the dimension by 2×1. Repeating this construction, we get bases of V and W as desired. ◁

In particular[8], $\cos\alpha = \prod_{k=1}^{m} \cos\alpha_k$. As an immediate consequence, we obtain the following

3. Lemma. *The orthogonal projection $\pi : \mathbb{R}^n \to V$ satisfies $|\pi(w)| \geq |w| \cos\alpha$ for all $w \in W$, where α denotes the intersection angle of V and W.*

And[9], the restriction $\pi|_W : W \to V$ of π yields an isomorphism as soon as $\cos\alpha > 0$, or $\alpha \neq \frac{\pi}{2}$:

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[7] Note that $f(v, w) = \cos\angle(v, w)$; thus, maximizing f is equivalent to minimizing the (spherical) distance. Also, note that $f(v, -w) = -f(v, w)$: thus, since $w \in S \cap W \Leftrightarrow -w \in S \cap W$, the maximum of f is ≥ 0, and the sperical distance is $\leq \frac{\pi}{2}$.

[8] Thus, the intersection angle of two subspaces is *not* the Hausdorff distance of the two $(m-1)$-spheres $S^{n-1} \cap V$ and $S^{n-1} \cap W$, i.e. it is, generally, not the "worst angle" between V and W. However, in the cases of dimension $m = 1$, or codimension $n - m = 1$, the angle α is exactly what one expects to be the intersection angle of two lines resp. hyperplanes.

[9] Another proof is by noticing that, given *any* orthonormal bases $(v_1, \ldots, v_m)$ and $(w_1, \ldots, w_m)$ of V and W, respectively, the matrix $(\langle w_i, v_j \rangle)_{ij}$ is exactly the representation matrix of $\pi|_W : W \to V$. Hence, $\mathrm{rk}(\pi|_W) = m$ if and only if $\alpha < \frac{\pi}{2}$.

4. Lemma. *Let $V, W \subset \mathbb{R}^n$ be two m-dimensional linear subspaces with intersection angle $\alpha \in [0, \frac{\pi}{2}]$, and denote $\pi : \mathbb{R}^n \to V$ the orthogonal projection of $\mathbb{R}^n$ onto V; then, the restriction $\pi|_W : W \to V$ is an isomorphism iff $\alpha < \frac{\pi}{2}$.*

Finally, we will need two rather technical estimates that will take care of some substantial part of the proof of our theorem. We prove them here.

5. Lemma. *Let $V, W \subset \mathbb{R}^n$ be two m-dimensional linear subspaces with intersection angle $\alpha < \frac{\pi}{4}$, and denote $\pi : \mathbb{R}^n \to V$ the orthogonal projection of $\mathbb{R}^n$ onto V; let $x \in \mathbb{R}^n$ make the angle $\beta < \frac{\pi}{4}$ with its orthogonal projection onto W, and denote*[10] *$v := \pi(x)$ and $w := \pi|_W^{-1}(\pi(x))$. Then,*

(i) $|x - w| \leq \frac{1}{\cos^2\alpha(1-\tan\alpha\cdot\tan\beta)} \tan\beta \cdot |v|$, *and*

(ii) $|x - v| \leq \tan(\alpha + \beta) \cdot |v|$.

Proof. Denote $Y := (V \oplus W)^\perp$ and $Z := (V \oplus W) \cap (V \cap W)^\perp$; note that Z has even dimension and $\dim(V \cap Z) = \dim(W \cap Z) = \frac{1}{2}\dim Z =: k$. Thus, as above, we can choose bases $(v_1, \ldots, v_k)$ and $(w_1, \ldots, w_k)$ of $V \cap Z$ and $W \cap Z$ such that $\mathrm{span}\{v_1, \ldots, v_k, w_1, \ldots, w_k\} = Z$ and $\langle v_i, w_j \rangle = \cos\alpha_j \cdot \delta_{ij}$. Then, we have $\cos\alpha = \prod_{j=1}^k \cos\alpha_j$. Further, let $v_j^\perp$ and $w_j^\perp$, $j = 1, \ldots, k$, denote vectors such that $w_j = \cos\alpha_j \cdot v_j + \sin\alpha_j \cdot v_j^\perp$ and $w_j^\perp = -\sin\alpha_j \cdot v_j + \cos\alpha_j \cdot v_j^\perp$. With this setup, we write

$$\begin{array}{rcl} x &=& x_0 + \sum_{j=1}^k \langle x, v_j \rangle v_j + \sum_{j=1}^k \langle x, v_j^\perp \rangle v_j^\perp + y \\ v &=& x_0 + \sum_{j=1}^k \langle x, v_j \rangle v_j \\ w &=& x_0 + \sum_{j=1}^k \langle x, v_j \rangle v_j + \sum_{j=1}^k \tan\alpha_j \langle x, v_j \rangle v_j^\perp \end{array}$$

since $\pi|_W^{-1}(v_j) = v_j + \tan\alpha_j \cdot v_j^\perp$. With $\tan^2\beta = \frac{|y|^2 + \sum \langle x, w_j^\perp \rangle^2}{|x_0|^2 + \sum \langle x, w_j \rangle^2}$ we compute

$$\begin{array}{rcl} |x - w|^2 &=& |y|^2 + \sum_{j=1}^k \langle x, -\tan\alpha_j \cdot v_j + v_j^\perp \rangle^2 \\ &=& \tan^2\beta \cdot |\tilde\pi(x)|^2 + \sum_{j=1}^k \tan^2\alpha_j \langle x, w_j^\perp \rangle^2 \\ &\leq& (1 + \tan^2\alpha)\tan^2\beta \cdot |\tilde\pi(x)|^2 \end{array}$$

where $\tilde\pi(x) = x_0 + \sum_{j=1}^k \langle x, w_j \rangle w_j$ is the orthogonal projection of x onto W. Since $w_j = \frac{1}{\cos\alpha_j} v_j + \tan\alpha_j \cdot w_j^\perp$, we deduce $(|\tilde\pi(x)| - |w|)^2 \leq |\tilde\pi(x) - w|^2 = \sum_{j=1}^k \tan^2\alpha_j \langle x, w_j^\perp \rangle^2 \leq \tan^2\alpha \cdot \tan^2\beta \cdot |\tilde\pi(x)|^2$ which gives

$$|\tilde\pi(x)| \leq \frac{1}{1-\tan\alpha\cdot\tan\beta}|w| \leq \frac{1}{(1-\tan\alpha\cdot\tan\beta)\cos\alpha}|v|$$

to obtain the first estimate. The second is obtained from

$$\begin{array}{rcl} |w - v|^2 &=& \sum_{j=1}^k \tan^2\alpha_j \langle x, v_j \rangle^2 \\ &\leq& \tan^2\alpha \sum_{j=1}^k \langle x, v_j \rangle^2 \\ &\leq& \tan^2\alpha \cdot |v|^2 \end{array}$$

via the triangle inequality $|x - v| \leq |x - w| + |w - v|$. ◁

[10] Note that $\pi|_W : W \to V$ is an isomorphism as $\alpha < \frac{\pi}{2}$.

However, we consider the affine tangent plane map of a submanifold. Thus, we have to clarify what the angle between two *affine* m-planes in $I\!R^n$ should be — here, one major issue is that two affine m-planes need not intersect.

Definition. *Let $p+V, q+W \subset I\!R^n$ be two affine m-planes with direction vector spaces V and W, respectively. We define the "angle $\alpha \in [0, \frac{\pi}{2}]$ between $p+V$ and $q+W$" to be the intersection angle of V and W.*

Note that the angle defined this way does not depend on the choice of base points p and q, hence defines an "angle between the two m-planes", indeed. The properties of the intersection angle of two linear subspaces proven above carry over for the angle between two (affine) m-planes — in some cases, minor and obvious reformulations are required that we leave to the reader.

3. Proof of the Theorem

First, let $M \subset I\!R^n$, M and n standard, be a C^1-submanifold. By transfer, the (affine) tangent plane map $p \mapsto T_p$ (and hence, the map $p \mapsto N_p$ assigning to each point the affine normal space of M) is standard — taking standard values at standard points. Clearly, the tangent plane map T satisfies (a).

Now, let $q \in M$ be near standard, and $p \in M$ standard with $p \simeq q$. As a C^1-submanifold, M can locally be represented as a graph over its tangent plane[11)] at p, i.e. there exists a locally defined C^1-function $g : T_p \supset U \to U^\perp \subset N_p$ such that

$$M \cap (U \times U^\perp) = \{(x, g(x)) \in (T_p \times N_p) \cong I\!R^n \mid x \in U\}.$$

As the existence of g is a classical statement with standard parameters p, M, T_p and N_p, the function g (and hence, U and $U^\perp$) can be assumed to be standard, by transfer. In particular, for every $q \in M$, $q \simeq p$, there is a unique $x \in T_p$ such that $q = (x, g(x))$, i.e. $\pi_p(q) = x$.

Since M is C^1, we can, after possibly making U smaller, assume that the angle[12)] $\alpha_{p,q}$ between any tangent plane T_q, $q \in M \cap (U \times U^\perp)$, and T_p is less than $\frac{\pi}{4}$. This is a classical statement, such that U can still be assumed to be standard. Then, by the triangle inequality (Lemma 1), the angle between any two tangent planes at points in $M \cap (U \times U^\perp)$ is less than $\frac{\pi}{2}$. Hence, $M \cap (U \times U^\perp)$ intersects the fibres of any orthogonal projection $\pi_q : I\!R^n \to T_q$ transversally, and, consequently, $\pi_q|_{M \cap (U \times U^\perp)} : M \cap (U \times U^\perp) \to T_q$ is a bijection onto its image. Thus, for $q \simeq p$, π_q yields an infinitesimal bijection. This proves (b).

Finally, to show (c), let $q + dq \in M$ be infinitely close to q, $q + dq \simeq q$, and write $q = (x, g(x))$ and $q + dq = (x + dx, g(x + dx))$ in terms of the above local

11) This is a straightforward application of the implicit mapping theorem.

12) Note that the map $q \mapsto \alpha_{p,q}$ is continuous since g is continuously differentiable.

representation of M as a standard graph. Since $\pi_q(q+dq)\in T_q$ minimizes, in T_q, the distance to $q+dq$ we have

$$\begin{array}{rcl} |(q+dq)-\pi_q(q+dq)| & \leq & |(x+dx,g(x+dx))-((x,g(x))+(dx,d_xg(dx)))| \\ & = & |g(x+dx)-g(x)-d_xg(dx)| \\ & = & |\int_0^1 d_{x+t\cdot dx}g(dx)-d_xg(dx)dt| \\ & \leq & (\int_0^1 |d_{x+t\cdot dx}g-d_xg|dt)\cdot|dx| \quad \simeq_{|dx|} \quad 0 \end{array}$$

since g is continuously differentiable, i.e. $\max_{t\in[0,1]}\{|d_{x+t\cdot dx}g-d_xg|\}\simeq 0$.

On the other hand, $dq=(x+dx,g(x+dx))-(x,g(x))\simeq_{|dx|}(dx,d_xg(dx))$ such that $0\ll\frac{|dq|}{|dx|}$; hence, $(\ldots)\simeq_{|dx|}0\Rightarrow(\ldots)\simeq_{|dq|}0$, and (c) follows.

Conversely, assume that a standard subset $M\subset\mathbb{R}^n$, $n\ll\infty$, satisfies conditions (a) through (c) of the theorem. Let $p\in M$ be standard.

First, we want to show that M can be written as the graph of a standard function g over its tangent plane T_p. Denoting $B_\delta(p)\subset T_p$ and $B_\delta^\perp(p)\subset N_p$ the open δ-ball around p in the (affine) tangent and normal space of M at p, respectively, (b) tells us that the orthogonal projection $\pi_p:\mathbb{R}^n\to T_p$ is a bijection from $M\cap(B_\delta(p)\times B_\delta^\perp(p))$ onto its image for any infinitesimal $\delta>0$. By (c), for $q=p+dp\in M\cap(B_\delta(p)\times B_\delta^\perp(p))$, we have $q-\pi_p(q)\simeq_{|dp|}0$: hence, $|q-\pi_p(q)|<\delta$, as $|dp|<\sqrt{2}\,\delta$, and $\pi_p:M\cap(B_\delta(p)\times B_\delta^\perp(p))\to B_\delta(p)$ is onto. By the overspill principle, there is a standard $\delta>0$ such that, with $U:=B_\delta(p)$ and $U^\perp:=B_\delta^\perp(p)$, $\pi_p:M\cap(U\times U^\perp)\to U$ is a bijection. Thus, we can write $M\cap(U\times U^\perp)$ as a graph of the function $g=\pi_p|_{M\cap(U\times U^\perp)}^{-1}:U\to U^\perp$, i.e.

$$M\cap(U\times U^\perp)=\{(x,g(x))\,|\,x\in U\}.$$

Next, we want to show that g is C^1. To that end, we first show that the (standard, as p is standard) map $q\mapsto\alpha_{p,q}$, assigning to each point $q\in M$ the angle between the tangent planes T_q and T_p, is continuous. Thus, let $q\in M$ be standard and $q+dq\in M$, $q+dq\simeq q$. Since, by the spherical triangle inequality (Lemma 1), $\alpha_{p,q+dq}\leq\alpha_{p,q}+\alpha_{q,q+dq}$ as well as $\alpha_{p,q}\leq\alpha_{p,q+dq}+\alpha_{q+dq,q}$, it suffices to show that $\alpha_{q,q+dq}\simeq 0$ whenever q is standard and $dq\simeq 0$:

6. Lemma. *The angle between the tangent planes T_q and T_p at two infinitely close, near standard points $p,q\in M$ is infinitesimal.*

Proof. Denote $\delta:=|q-p|$, choose some direction in T_q, and let $\tilde{q}\in M$ be such that $|\tilde{q}-q|=\delta$ and $t_q:=\pi_q(\tilde{q})-q$ yields the chosen direction (this is possible by the bijection assumption (b)). Further, decompose $t_q=t_p+n_p$ into components t_p and n_p parallel and orthogonal to T_p, respectively. This way we have, by (c), $\tilde{q}\simeq_\delta\pi_q(\tilde{q})$ as well as $\tilde{q}\simeq_\delta\pi_p(\tilde{q})$ since $0\simeq\frac{|\tilde{q}-\pi_p(\tilde{q})|}{|\tilde{q}-p|}\geq\frac{|\tilde{q}-\pi_p(\tilde{q})|}{|\tilde{q}-q|+|q-p|}=\frac{|\tilde{q}-\pi_p(\tilde{q})|}{2\delta}$. Then, $q+t_q\simeq_\delta\pi_q(\tilde{q})\simeq_\delta\tilde{q}\simeq_\delta\pi_p(q+t_q)$ as $|\pi_p|\leq 1$ and $\tilde{q}\simeq_\delta q+t_q$. Using the decomposition $t_q=t_p+n_p$ and the fact that π_p is an affine map, we arrive at

$\pi_p(q+t_q) = \pi_p(q) + t_p \simeq_\delta q + t_p$. Thus, $t_q \simeq_\delta t_p$. For the angle formed by t_q and t_p, we conclude that $\frac{\langle t_p, t_q\rangle}{|t_p|\,|t_q|} = \frac{|t_p|}{|t_q|} \simeq 1$ as $\delta = |\tilde{q} - q| \simeq_\delta |t_q| \simeq_\delta |t_p|$.

But, t_q was an arbitrary direction in T_q; hence, the above basis representation (Lemma 2) for the angle yields the claim[13]: $\cos\alpha = \prod_{j=1}^m \frac{\langle (t_q)_i, (t_p)_i\rangle}{|(t_q)_i|\,|(t_p)_i|} \simeq 1$. $\triangleleft$

Hence, we can assume that $\alpha_{p,q} < \frac{\pi}{4}$ for all $q \in U$, while keeping the assumption of U being standard (by transfer, as before). As a consequence, the orthogonal projection $\pi_p|_{T_q} : T_q \to T_p$ is an isomorphism for any $q = (x, g(x)) \in M$, $x \in U$. Therefore, we can write its inverse $\pi_p|_{T_q}^{-1}(x+v) = (x+v, g(x)+l_x(v))$ with some linear map $l_x : T_xT_p \to T_{g(x)}N_p$ — we want to show that, whenever $x \in U$ is near standard $g(x+dx) - g(x) - l_x(dx) =: \varepsilon \simeq_{|dx|} 0$ for all $dx \simeq 0$, i.e. that g is continuously differentiable[14] with differential $l_x = d_x g$ at x.

The proof mainly relies on the estimates in Lemma 5. First, let $x \in U$ be standard, and $dx \simeq 0$; denote $q := (x, g(x))$ and $q + dq = (x+dx, g(x+dx))$. Then, Lemma 5(ii) implies[15] that

$$|g(x+dx) - g(x)| = |dq - (dx, 0)| \leq \tan(\alpha+\beta)\cdot|dx| \simeq 0$$

where $\beta \simeq 0$ denotes the angle that dq makes with the tangent plane T_q at q. Thus, g is continuous. Now, let $x \in U$ be near standard, dx, q, and dq as before. Since g is continuous, $q = (x, g(x))$ is near standard (in M). Consequently, Lemma 5(i) yields

$$\begin{array}{rcl} |g(x+dx) - g(x) - l_x(dx)| & = & |dq - (dx, l_x(dx))| \\ & \leq & \frac{1}{\cos^2\alpha_{p,q}(1-\tan\alpha_{p,q}\tan\beta)}\tan\beta\cdot|dx| \\ & \simeq_{|dx|} & 0 \end{array}$$

as $\beta \simeq 0$ and $\alpha_{p,q} \ll \frac{\pi}{2}$. Thus, g is shown to be continuously differentiable, concluding the proof of the theorem

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13) This is an obvious point where we need that $m = \dim M \ll \infty$.

14) This condition is sometimes called "uniform differentiability"; for standard maps, it implies continuous differentiability.

15) Here, as well as below, we consider q as "0", dq, $(dx, 0)$, and $(dx, l_x(dx))$ as vectors in the corresponding linear space.

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