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Volker Bach, Jochen Brüning,
Georg Lang, and Bianca Toltz

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**Abstract**

**Address:**
FB Mathematik 8-5, TU Berlin,
Straße des 17. Juni 136, D-10623 Berlin, Germany;
**email:**
sfb288@sfb288.math.tu-berlin.de,
vbach@mathematik.uni-mainz.de,
bruening@mathematik.hu-berlin.de;
**Homepage:**
http://www-sfb288.math.tu-berlin.de/

# Contents

# I Abstracts of the Plenary Lectures

The plenary lectures of the conference were given by J. Avron, W. Ballmann, J.-P. Bourguignon, J. Cheeger, J.-M. Combes, L. Faddeev, J. Fröhlich, A. Jaffe, E. Lieb, W. Müller, S. Novikov, B. Simon, and C.-L. Terng. The abstracts of these plenary lectures are given below, as far as they are available.

## I.1 Joseph Avron: *Geometry of Dissipating Quantum Pumps*

## I.2 Werner Ballmann: *Spaces of Nonpositive Curvature: Rank and Rank Rigidity*

Bounds on the sectional curvature of a Riemannian manifold can be described by the shape of small geodesic triangles. This leads to the notion of curvature bounds for general metric spaces and allows to use methods from Riemannian geometry in the investigation of such metric spaces with curvature bounds. Of particular interest are spaces of nonpositive curvature, since their universal covering spaces are contractible.
In the talk I address some geometric problems for such spaces which are related to problems in dynamical systems and in geometric group theory.

## I.3 Jean Pierre Bourguignon: *Spaces of Kähler Potentials: Multi-Faceted Objects Relevant for Geometry*

## I.4 Jeff Cheeger: *Analysis on Metric Measure Spaces*

A classical theorem of Rademacher asserts that a Lipschitz function, $f : \mathbb{R}^n \to \mathbb{R}$, is differentiable almost everywhere. The notion of "Lipschitz" requires a metric and "almost everywhere" a measure. We show that for metric measure spaces, $(Z, \mu)$, satisfying a doubling condition and a (suitably formulated) Poincaré inequality, Rademacher's theorem has a good generalization. This, in turn, provides strong constraints on the infinitesimal structure of such spaces (which, none-theless, can have fractional Hausdorff dimension). Natural examples, to which our discussion applies, include measured Gromov-Hausdorff limits of Riemannian manifolds with a fixed lower Ricci curvature bound, Carmot-Carathéodory spaces, and the boundaries at infinity of certain hyperbolic buildings.

## I.5 Jean-Michel Combes: *Bloch Decomposition and Band Structure for Absorptive Photonic Crystals*

We present some recent developments of photonic crystal theory with emphasis on dissipative and absorptive systems. We discuss their relevance to the basic question of gap opening. However, we present a general theorem about non-existence of gaps in the absorptive case. This can be explained in particular from a deconvolution formalism leading to a classical form of a quantum electrodynamics Hamiltonian. It also provides a pertubation framework in which it is convenient to analyze through complex scaling the effect of the classical "photon-matter" coupling. Low-order pertubation theory suggests that, although gaps disappear under the effect of this coupling, pseudo-gaps of the density of states should show up when absorption is small.

## I.6 Ludwig Faddeev: *String-like Solitons and their possible Applications in Physics*

The term "soliton" in the title means simply the localized finite energy solutions of the nonlinear equation of motion. No reference to integrability is meant. Such solutions do exist in some models in 3+1 dimensional space-time. Most notable are the Skyrme model and the t'Hooft-Polyakov magnetic monopole. Both these solutions are concentrated around a point in space, so they can be called particle line. The topological charges are induced by $\pi_3(\mathbb{S}^3)$ and $\pi_2(\mathbb{S}^2)$, correspondingly. In my talk I advertise for one more model, admitting solitons. In contradistinction to the models mentioned above, the soliton is concentrated around a closed line, which can be knotted. This is why I call them string-like. The topological invariant is produced by $\pi_3(\mathbb{S}^2)$ and is called Hopf invariant.

The first part of the talk is dedicated to the description of the model and numerical data, confirming the existence of the knot-like soliton. In the second part I discuss possible applications in QCD and plasma physics.

## I.7 Jürg Fröhlich: *The Newtonian Limit of Large Systems of Bosons*

## I.8 Arthur Jaffe: *Twist Fields and Constructive Quantum Field Theory*

## I.9 Elliott Lieb: *The Bose Gas: A subtle Many-Body Problem*

Three topics concerning the ground state of a gas of bosons will be discussed.

**A. (with Jakob Yngvason) The homogenous gas of $N$ particles in a large volume of density $\rho = N/V$.**

We assume that the two-body potential is non-negative and finite range (compact support). The Hamiltonian is

$$H \;=\; -\mu \sum_{i=1}^{N} \Delta_i + \sum_{1\le i<j\le N} v(x_i - x_j).$$

We prove that for low density and 3 dimensions, the ground state energy per particle, $e_0$, satisfies

$$e_0(\rho) \;\sim\; 4\pi\mu\,\rho\,a,$$

where $a$ is the scattering length of $v$. This asymptotic formula, $4\pi\mu\,\rho\,a$, was conjectured by Bogolubov in 1947 but proved by us only in 1998. Dyson observed an upper bound of this form in 1957 and our contribution was to get the lower bound. We have verified the formula conjectured in 1971 by Schick,

$$e_0(\rho) \;\sim\; \frac{4\pi\,\rho}{|\ln(\rho a^3)|}, \qquad \text{as } \rho \to 0.$$

**B. (with Robert Seiringer and Jakob Yngvason) Particles in a trap (as in recent experiments).**

The Hamiltonian is

$$H \;=\; \mu \sum_{i=1}^{N} \big[ -\Delta_i - V(x_i) \big] + \sum_{1\le i<j\le N} v(x_i - x_j),$$

where $V(x)$ is an external (trap) potential. We study the Gross-Pitaevskii functional

$$\mathcal{E}^{GP}(\Phi) \;=\; \int_{\mathbb{R}^3} \Big[ \mu|\nabla\Phi|^2 + V\Phi^2 + 4\pi\mu a\Phi^4 \Big] d^3x$$

which one minimizes over all $\Phi$ satisfying $\int \Phi^2 = N$. This minimum is denoted $E^{GP}$. We show that if $ra$ is fixed and $r \to \infty$ then

$$\lim_{N\to\infty} \frac{E^{GP}}{E^{QM}} \to 1,$$

where $E^{QM}$ is the true quantum mechanical ground state energy. We also show convergence of the quantum particle density to the GP density $|\Phi^{GP}(r)|^2$.

**C. (with Jan Philip Solovej) We consider the one-component charged Bose gas with a neutralizing background (jellium).**

In 1961, Foldy used Bogolubov's 1947 method to derive

$$e_0(\rho) \sim -0.803 \left(\frac{2\pi}{3\mu}\right)^{1/4} \rho^{1/4} ,$$

for *large* $\rho$. We prove this asymptotic formula. This is the first example (for more than one dimension) in which Bogolubov's pairing idea has been rigorously verified.

## I.10 Werner Müller: *Automorphic Forms and Scattering Theory*

Harmonic analysis on locally symmetric spaces of finite volume plays an important role in the modern theory of automorphic forms. A locally symmetric space of finite volume is defined to be the quotient $\Gamma\backslash X$ of a symmetric space $X = G/K$ by a discrete subgroup $\Gamma \subset G$ of cofinite volume. Of particular interest are quotients by arithmetic subgroups. An important problem in this context is the study of the spectral decomposition of the center $Z(\mathfrak{g}_{\mathbb{C}})$ of the universal enveloping algebra of $G$, acting in the Hilbert space of square integrable sections of a given locally homogeneous vector bundle $E \to \Gamma\backslash X$. The continuous part of the spectrum can be described in terms of Langlands' theory of Eisenstein series.

In the talk we shall explain the connection between Eisenstein series and the stationary approach to scattering theory. In the case of the upper half-plane this connection was first studied by L.D. Faddejev. A locally symmetric space $\Gamma\backslash X$ of the type that we are considering has a natural compactification $\overline{\Gamma\backslash X}$ as a manifold with corners. This is the Borel-Serre compactification of $\Gamma\backslash X$. The boundary of $\overline{\Gamma\backslash X}$ has a natural stratification and each stratum determines a certain subspace in the absolute continuous subspace. Each stratum $\Gamma_P\backslash X_P$ is a fibre bundle over a lower dimensional locally symmetric space $\Gamma_M\backslash X_M$. The Eisenstein series may be regarded as generalized eigenfunctions of the algebra of invariant differential operators. They are associated to eigenfunctions on the base $\Gamma_M\backslash X_M$ of some stratum $\Gamma_P\backslash X_P$. The asymptotic behaviour of the rank one cuspidal Eisenstein series near the boundary at infinity determines the scattering matrix. Eisenstein series and the scattering matrices are meromorphic functions of several complex variables. Using the stationary approach to scattering theory we prove a number of results concerning the order of these meromorphic functions and we estimate the number of singular hyperplanes intersecting a given ball. From this we deduce a number of results about automorphic L-functions that occur as factors of scattering matrices associated to rank one cuspidal Eisenstein series.

## I.11 Sergej Novikov: *Discrete Integrable Systems*

## I.12 Barry Simon: *An Elementary Proof of the Local Borg-Marchenko Theorem*

## I.13 Chun-Lian Terng: *Submanifold Geometry and Integrable Systems*

Let $G/K$ be a rank $n$ symmetric space, $\tau$ conjugate linear and $\sigma$ $\mathbb{C}$-linear involutions of $G_{\mathbb{C}}$ such that $G$ = fixed points of $\tau$ and $K$ fixed points of $\tau$ in $G$, and $\mathcal{G} = \mathcal{K} + \mathcal{P}$ the Cartan decomposition. Let $\mathcal{A} \subset \mathcal{P}$ be a maximal abelian subalgebra, and $a_1, \ldots, a_n$ a basis of $\mathcal{A}$ consisting of regular elements, i.e., $\mathrm{Ad}[K](a_i)$ is a principal orbit for all $i$. The $G/K$-system is the PDE for $v : \mathbb{R}^n \longrightarrow P \cap \mathcal{A}^{\perp}$:

$$(*) \qquad \left[a_i, \frac{\partial v}{\partial x_j}\right] - \left[a_j, \frac{\partial v}{\partial x_i}\right] = \Big[[a_i, v], [a_j, v]\Big], \quad i \neq j$$

where $\mathcal{A}^{\perp}$ is the orthogonal complement of $\mathcal{A}$ w.r.t. the Killing form. The system is the collection of the first $n$ flows in the $G/K$-hierarchy of solition equations, and also occurs as the Gauss-Codazzi equation for submanifolds with various geometric properties.
The $G/K$-system $(*)$ has a Lax pair, i.e., $v$ is a solution of $(*)$ iff $\theta_\lambda = \sum_{i=1}^{n} (a_i\lambda + [a_i, v])\, dx_i$ is flat for all $\lambda \in \mathbb{C}$.
Hence, there exists $E(x, \lambda)$ such that

$$E^{-1}dE = \Theta_\lambda , \quad E(0, \lambda) = I$$

for all $\lambda$. Moreover, $E(x, \lambda)$ is holomorphic for $\lambda \in \mathbb{C}$.
In joint work with M. Brück, X. Du, J. Park, we find the corresponding submanifold geometry for $O(m+n)/O(m) \times O(n)$ and $\frac{O(m+n,1)}{O(m)\times O(n,1)}$ systems: This includes isometric inversions of space forms in space forms, isothermic surfaces, constant mean curvature surfaces in $\mathbb{R}^3, \ldots$, etc. The dressing action of a meromorphic map on the space of solutions of the $G/K$-system of one and two poles gives rises to geometric Bäcklund and Ribaucour transformations. The inverse scattering gives the smooth version of the Cartan-Kähler theorem for solving the initial value problem. The relation between the $\frac{O(4,1)}{O(3)\times O(1,1)}$ – system, isothermic surfaces in $\mathbb{R}^3$ and conformal geometry was first studied by Burstall, Jeromin-Hertrich, Pedit, and Pinkall. Burstall generalized these to $\frac{O(n+1,1)}{O(n)\times O(1,1)}$ – systems and isothermic surfaces in $\mathbb{R}^n$. Ferus and Pedit studied the curve flat system, which is gauge equivalent to the $G/K$-system.

# II Abstracts of the Sectional Talks

## II.1 Section on "Differential Geometry and Geometric Analysis"

**Arsan, Güler Gürpinar**

*Chebyshev Nets Formed By The Laguerre Lines On Centro-Surfaces*[1]

Let $S$ be a surface of class $C^4$ in Euclidean 3-space with vector equation $\vec{x} = \vec{x}(u, v)$ and let $C$ be a line on $S$. $C$ is said to be a Laguerre line ($G$-line), if at every point $P$ on $C$, the normal plane of $S$ containing the tangent $PT$ to $C$, cuts the surface $S$ in a line superosculated by its circle of curvature at P. A Laguerre line is characterized by the equation

$$L = \rho'_n - 2\rho_g \tau_g = 0 \quad (\rho'_n = \frac{d\rho_n}{ds})$$

where $L$ is the Laguerre's function of direction and $\rho_n,\ \rho_g,\ \tau_g$ and $s$ are, respectively, the normal curvature, the geodesic curvature , the geodesic torsion and the arclength of $C$.
The surfaces $S_1$ and $S_2$ defined, respectively, by the vector equations

$$S_1 : \vec{y} + \alpha\vec{n}$$

$$S_2 : \vec{\bar{y}} + \beta\vec{n}$$

are called centro-surfaces of $S$ where $\alpha$ and $\beta$ are the radii of principal curvatures and $\vec{n}$ is the unit normal vector of $S$ at the point $P$.
A 2-net on $S$ is said to be semi-Chebyshev, if the tangent vector fields of one family of the net undergoes a parallel displacement along each curve of the other family. If the tangent vector field of one family undergoes a parallel displacement along the curves of the other family, then the 2-net is called a semi-Chebyshev net.
In this work, we proved the following theorems:
**Theorem 1.** If the two families of Laguerre lines of $S_1$ form a conjugate Chebyshev net, then $S$ and $S_1$ are general cylinders.
**Theorem 2.** If the two families of Laguerre lines of $S_1$ and $S_2$ correspond and if two families of Laguerre lines on $S_1$ form a conjugate semi-Chebyshev net, then $S$ is a general cylinder and $S_1$ is a cone.

**Boos, Herman**

*Some Exact Results for the Three-Layer Zamolodchikov Model*

We discuss our recent results for the three-layer three-dimensional Zamolodchikov model with modified boundary conditions which is equivalent to the $sl(3)$-chiral Potts model at $q^2 = -1$. We derive the pair of functional relations and obtain the Bethe ansatz equations from them. The spectrum of transfer matrices is analysed in connection with the solution to the Bethe ansatz equations. In thermodynamic limit we accept a two-string hypothesis and derive the integral equations for the distribution densities. The solution to these equations is found exactly. We calculate the partition function in thermodynamic limit. We also generalize these results on the one-string regime.

**Bär, Christian**

*Unique Continuation and the Localization Problem for Fermions*

[1] see also Subsection III.1

Many (but not all) elliptic partial differential equations have the so-called unique-continuation-property meaning that every solution vanishing on some open subset of a connected domain must vanish everywhere. We show that this unique-continuation-property is true not only for solutions of these PDEs or for suitable partial differential inequalities but much more generally for sections over Riemannian manifolds satisfying a spectral inequality of Atiyah-Patodi-Singer type. This is in particular true for the Dirac equation and we explain how this is related to interpretational difficulties in relativistic quantum mechanics, the localization problem for fermions.

**Çivi, Gülçin**

*A Function Of Direction In A Generalized Weyl Hypersurface*[2]

An $n$-dimensional differentiable manifold $W_n^*$ having an asymmetric connection $\nabla^*$ and an asymmetric conformal metric tensor $g^*$ preserved by $\nabla^*$ is called a generalized Weyl space. In local coordinates we have the compatibility condition

$$\overset{*}{\nabla}_k g^*_{ij} - 2\,T^*_k\, g^*_{ij} = 0$$

where $T^*_k \quad (k = 1, 2, ..., n)$ are the components of the covariant vector field $T^*$ called the complementary vector field of the generalized Weyl space which will be denoted by $W_n^*(\nabla^*, g^*, T^*)$.
Let $W_n^*(\nabla^*, g^*, T^*)$ be a hypersurface with coordinates $u^i$ $(i = 1, 2, ..., n)$ of a generalized Weyl space $W^*_{n+1}(\nabla^*, g^*, T^*)$ with coordinates $x^a$ $(a = 1, 2, ..., n, n+1)$. The prolonged covariant derivative of the satellite $A$, with weight $p$, relative to $\nabla^*$ which will be denoted by $\dot{\nabla}^*_k A$, is defined as

$$\dot{\nabla}^*_k A = \nabla^*_k A - p\,T^*_k\, A$$

where $\nabla^*_k A$ is the usual covariant derivative.
The metrics of $W_n^*(\nabla^*_k, g^*_{ij}, T^*_l)$ and $W^*_{n+1}(\nabla^*_a, g^*_{bc}, T^*_d)$ are given, respectively, by $g_{ij}\, du^i\, du^j$ and $g_{ab}\, dx^a\, dx^b$ which are connected by the relations

$$g_{ij} = g_{ab}\, x^a_i\, x^b_j \quad (i, j = 1, 2, ..., n\ ;\ a, b = 1, 2, ..., n+1)$$

where $x^a_i$ denotes the covariant derivative of $x^a$ with respect to $u^i$. In this work, we choose $n$ congruences of an orthogonal ennuple of the vector fields $\underset{r}{v}$ $(r = 1, 2, ..., n)$ which are normalized by $g_{ij}\, \underset{r}{v}^i\, \underset{r}{v}^j = 1$ in the hypersurface $W_n^*(\nabla^*_k, g^*_{ij}, T^*_l)$ of the generalized Weyl space $W^*_{n+1}(\nabla^*_a, g^*_{bc}, T^*_d)$ . Let $\underset{rr}{\kappa}^*$ be the normal curvature of the hypersurface in the direction of the vector of components $\underset{r}{v}^i$. In this work, it is shown that the expression

$$-\,\underset{r}{v}^h\,(\,\nabla^*_h\, \underset{rr}{\kappa}^*\,) + \overset{s}{\underset{r}{\eta}}\,(\underset{rs}{\kappa}^* + \underset{sr}{\kappa}^*) + \underset{r}{P}\; \underset{rr}{\kappa}^*$$

is a function of direction where $\overset{s}{\underset{r}{\eta}}$ are the geodesic curvature of the orthogonal ennuple and $\underset{rs}{\kappa}^*$ are the invariant of the geodesic torsion of the curvature of the congruence with tangent vector $\underset{r}{v}^i$.

**Driuma, Valerii**

*Riemann Geometry in Theory of Second Order ODE's*

Properties of the second order ODE's

$$y'' + a_1 y'^3 + 3a_2 y'^2 + 3a_3 y' + a_4 = 0$$

with arbitrary coefficients $a_i = a_i(x, y)$ with help of four-dimensional Riemann spaces with the metrics in form:

$$ds^2 = 2z(a_3 dx^2 + 2a_2 dxdy + a_1 dy^2) - 2t(a_4 dx^2 + 2a_3 dxdy + a_2 dy^2) + 2dxdz + 2dydt$$

[2] see also Subsection III.2

are considered.

**Huebschmann, Johannes**

*Kähler quantization commutes with reduction*

Exploiting an appropriate notion of Kähler space with singularities, we show that, in the world of "singular Poisson-Kähler geometry", reduction after quantization coincides with quantization after reduction: For a stratified symplectic space, exploiting the notion of Lie-Rinehart algebra, we introduce the concept of stratified polarization; the latter encapsulates polarizations on the strata and, moreover, the behaviour of the polarizations across the strata. We isolate the notions of stratified Kähler polarization and show that symplectic reduction carries a Kähler manifold to a normal complex analytic stratified Khler space in such a way that the sheaf of germs of polarized functions thereupon coincides with the ordinary sheaf of germs of holomorphic functions. Using reduction of prequantum modules we show that, for a Kähler manifold, reduction after quantization coincides with quantization after reduction in the sense that not only the reduced and unreduced quantum phase spaces correspond but the unreduced and reduced quantum observables as well. We illustrate our approach with a number of examples of reduced spaces with singularities arising from angular momentum zero.

**McIntosh, Ian**

*Algebraic harmonic tori*

I will describe a way of looking at harmonic tori in symmetric spaces which highlights the algebraic geometry of the situation and raise a few questions concerning the existence of completely algebraic examples.

**Ma, Xiaonan**

*Rigidity and vanishing theorem for elliptic genera in K-Theory*

Witten considered the indices of elliptic operators on the free loop space $\mathcal{L}M$ of a spin manifold $M$. In particular the index of the formal signature operator on the loop space is exactly the elliptic genus of Landweber-Stong, Ochanine. Recall that the elliptic genus was used by them to define a new cohomology theory, so called “Elliptic cohomology”. Motivated by physics, in 1986, Witten made the conjectures about the rigidity of these elliptic operators which says that their $S^1$-equivariant indices on $M$ are independent of $g \in S^1$. These conjectures were proved by Taubes, Bott-Taubes. Hirzebruch and Krichever proved Witten's conjectures for almost complex manifolds. In 1992, Kefeng Liu gave a simple and unified proof of the above conjecture by using the modular invariance argument, as well as various further generalizations.

In many situations in geometry, it is rather natural and necessary to generalize the above rigidity and vanishing theorems to the family case.

To be more precise, let $M$, $B$ be two compact smooth manifolds, and $\pi : M \to B$ be a smooth fibration with compact fiber $X$. Assume a compact Lie group $G$ acts fiberwise on $M$. Let $P$ be a family of $G$-equivariant elliptic operators along the fiber, then the family index of $P$ is defined to be

$$\mathrm{Ind}(P) = \mathrm{Ker}P - \mathrm{Coker}P \in K_G(B)$$

Note that $\mathrm{Ind}(P)$ is a virtual $G$-representation. We denote by $(\mathrm{Ind}(P))^G \in K(B)$ the $G$-invariant part of $\mathrm{Ind}(P)$.

A family of elliptic operator $P$ is said to be *rigid on the equivariant Chern character level* with respect to this $G$-action, if the equivariant Chern character $\chi_g(\mathrm{Ind}(P)) \in H^*(B)$ is independent of $g \in G$. If $\chi_g(\mathrm{Ind}(P))$ is identically zero for any $g$, then we say $P$ has *vanishing property on* *the equivariant Chern character level.* More generally, we say that $P$ is *rigid on the equivariant $K$-theory level,* if $\mathrm{Ind}(P) = (\mathrm{Ind}(P))^G$. If this index is identically zero in $K_G(B)$, then we say that $P$ has *vanishing property on the equivariant $K$-theory level.* To study rigidity and vanishing of $\mathrm{Ind}(P)$, it is clear that we only need to restrict to the case when $G = S^1$.

I and Kefeng LIU proved several family rigidity and vanishing theorems on the equivariant Chern character level, which can be considered as rigidity and vanishing theorems for families of elliptic operators on loop space. To prove them, we used the modular invariance argument, together with a scaling trick and some properties of Jacobi forms. Such theorems contains higher level vanishing terms which will be useful for our understanding of the relationship between group actions and fundamental groups.
Very recently, I and Kefeng LIU and WEiping Zhang start the program to understand the possible torsions of the index bundle. As a first step, we prove a refined $K$-theory version of the fixed point formula for circle action. This fixed point formula is more subtle than the usual index theorem for Chern character, since Chern character kills the torsion elements included in the index bundle.
By using this refined fixed point formula for $S^1$-action, we were able to prove several rigidity and vanishing theorems for the index bundle on the equivariant $K$-theory level.
Our proof is self-contained and the arguments even in the case where the base $B$ is a point, are different from the ones in the papers of Bott-Taubes, Liu and Taubes.

**Marvan, Michal**

*Some integrable classes of Gauss-Mainardi-Codazzi equations*

Regardless of a coordinate system chosen, Gauss-Mainardi-Codazzi (GMC) equations are known to posses a nonparametric $sl_2$-valued zero-curvature representation (ZCR). Under a constraint (considering the so called reduced GMC equations), the ZCR may admit a parameter, which then can play the role of the spectral parameter of IST and make the equations integrable in the sense of IST. Presented will be some results of systematic search for integrable reduced GMC equations, in part obtained with I.S. Krasil'shchik.

**Maschler, Gideon**

*Kähler Metrics with a Distinguished Killing Vector Field*

The extremal Kähler metrics of Calabi are equipped with a Killing vector field associated to the scalar curvature. In dimension four, examples are known that are also conformal to Einstein metrics. We describe solutions by ODE methods to two related metric types: central, where the Killing field is associated instead to the determinant of the Ricci endomorphism (joint with Hwang), and a local classification of Kähler metrics conformal to Einstein metrics in higher dimensions (joint with Derdzinski).

**Matveev, Vladimir S.**

*Quantum integrability of the Beltrami-Laplace operator for geodesically equivalent metrics*

Let all geodesics of two Riemannian metrics $g$ and $\bar{g}$ on $M^n$ coincide (as unparameterized curves). We construct differential operators $\mathcal{I}_0, \mathcal{I}_1,..., \mathcal{I}_{n-1} : C^2(M^n) \to C^0(M^n)$ such that they commute with the Beltrami-Laplace operator $\Delta_g$ of the first metric and pairwise commute. If the manifold is closed and if all eigenvalues of $g^{-1}\bar{g}$ are different at least at a point then the manifold can be covered by the product of spheres and the variables in the equation $\Delta_g(f) = \lambda f$ can be globally separated: the $n$-dimensional Schroedinger equation $\Delta_g(f) = \lambda f$ can be reduced to a system of $n$ 1-dimensional Schroedinger equations.

**Myrzakulov, Ratbay**

*On the soliton geometry in multidimensions*[3]

The connection between multidimensional soliton equations and differential geometry of surfaces is discussed. A classes integrable surfaces (and/or curves) corresponding to the well known (2+1)-dimensional soliton equations are constructed. Examples of such equations include: the Davey-Stewartson, Zakharov, Kadomtseva-Petviashvili, KdV, mKdV and Knizhnik-Novikov-Veselov equations. Using the presented geometrical formalism the equivalence between these soliton equations and their spin counterparts are established.

**Pflaum, Markus**

*Length space structures on stratified spaces*

The metric aspects of stratified spaces are of broad interest in geometric analysis. It is shown in this talk that for every stratified space with a smooth structure and which is regular in the sense of Tougeron there exists an appropriately defined Riemannian metric such that the geodesic distance with respect to this Riemannian metric is complete, hence the space becomes a length space. After introducing a certain notion of tameness for stratified spaces with a smooth structure it is shown that a deRham-Theorem holds for Whitney-functions on such spaces.

**Rademacher, Hans-Bert**

*Twistor Spinors*

Twistor spinors are solutions of a conformally invariant field equation. We construct solutions of this equation with zeros and obtain a classification result in this case using methods from conformal geometry.

**Vrancken, Luc**

*Special classes of 3-dimensional Lagrangian submanifolds in $CP^3$*

The purpose of this lecture is to show how examples of minimal 3-dimensional Lagrangian submanifolds in the complex projective space can be constructed starting from special surface equations. In particular we show how to associate with -a minimal surface in $S^5$ (with ellipse of curvature a circle), -a solution of the 2-dimensional sinh-Gordon equation, a nontrivial (not coming from a minimal Lagrangian immersion in $CP^2$) minimal Lagrangian immersion into $CP^3$. We also give a characterisation of these different classes and obtain some explicit examples of torii.

[3] see also Subsection III.4

## II.2 Section on "Partial Differential Equations and Mathematical Physics

**Blanchard, Philippe**

*What can we learn about Selforganized Criticality from Dynamical Systems theory?*

We develop a dynamical system approach for the Zhang's model of Self-Organized Criticality, for which the dynamics can be described either in terms of Iterated Function Systems, or as a piecewise hyperbolic dynamical system of skew-product type. In this setting we describe the SOC attractor, and discuss its fractal structure. We show how the Lyapunov exponents, the Hausdorff dimensions, and the system size are related to the probability distribution of the avalanche size, via thte Ledrappier-Young formula.

**Demuth, Michael**

*Integral conditions for the stability of the continuous spectra*

Let $A$ and $B$ two selfadjoint operators bounded below in $L^2(R^d)$. Assume that $A$ and $B$ generate ultracontractive semigroups $e^{-tA}$ and $e^{-tB}$ with $t \geq 0$. Take $t = 1$ and define

$$D(x) := \int\limits_{R^d} |e^{-A}(x,y) - e^{-B}(x,y)| dy.$$

We will study condition for that comparison function $D(.)$ such that the absolutely continuous spectra of $A$ and $B$ are the same. If $A$ is the Laplacian one can find integral conditions for $D(.)$ such that the singularly continuous spectrum of $B$ is empty. Several examples taken from stochastic analysis will illustrate the results.

**Dobrokhotov, Serguei**

*Global semiclassical description of the spectrum of certain 2 D Schrödinger operators with periodic magnetic and electric field*

We consider the spectral theory of the 2-D Schrödinger operator for a charged particle in a (strong) periodic magnetic field and a (small) periodic electric field. The related classical problem is analized first by means of the Krylov-Bogoljubov-Alfven and Neishtadt averaging methods. It allows us to show "almost integrability" of the original 2-D classical Hamilton system, and to reduce it to a problem with phase space the twodimensional torus. Using topological methods for integrable Hamiltonian systems, and elementary facts from Morse theory, we can classify the classical motion. According to this classification, the classical motion is separated into different regimes with the different topological characteristics (like rotation numbers and Maslov indices). Using these regimes, the semiclassical approximation, the Bohr-Sommerfeld rule, and the correspondence principle we give the general asymptotic description of the (band) spectrum of the original Schrödinger operator and, in particular, estimates for the number of the bands in near each Landau level. From our point of view, the regimes are the classical preimages of "spectral series" of the Schrödinger operator. We also discuss the relationship between this spectrum and the spectrum of the 1-D difference operator.

**Eckhoff, Michael**

*The Low-lying Spectrum of a Class of Markov Chains and Metastability*

The Glauber dynamics of a large class of (disordered) mean field models are closely related to Markov chains on the state space of the order parameter. Their metastable (or tunneling) behavior can be described in terms of the free energy, the rate function of the reversible measure for the chain. We show that the low-lying spectrum of the chain up to errors on an exponentially smaller scale is given by a collection of ground state energies of appropriately chosen reference problems of Dirichlet type, which correspond via a sharp uncertainty principle to exponentially large expected transition (or tunneling) times. This is a consequence of the representation of the eigenfunctions in terms of Laplace transforms of transition times. The expected transition times can be calculated to leading order by a variational principle. The spectral results lead to very precise information on the distribution function for the transition times.

**Feinberg, Joshua**

*'Single Ring' Theorem and the Disk-Annulus Phase transition*

Recently, an analytic method was developed to study in the large $N$ limit non-hermitean random matrices that are drawn from a large class of circularly symmetric non-Gaussian probability distributions, thus extending the existing Gaussian non-hermitean literature. One obtains an explicit algebraic equation for the integrated density of eigenvalues from which the Green's function and averaged density of eigenvalues could be calculated in a simple manner. Thus, that formalism may be thought of as the non-hermitean analog of the method due to Brèzin, Itzykson, Parisi and Zuber for analyzing hermitean non-Gaussian random matrices. A somewhat surprising result is the so called "Single Ring" theorem, namely, that the domain of the eigenvalue distribution in the complex plane is either a disk or an annulus. In this talk I will report on simple new explicit expressions for the radii of the eigenvalue distiobution and for the value of the eigenvalue density at the edges of the eigenvalue distribution of the non-hermitean matrix in terms of moments of the eigenvalue distribution of the associated hermitean matrix. We then present several numerical verifications of the previously obtained analytic results for the quartic ensemble and its phase transition from a disk shaped eigenvalue distribution to an annular distribution. Finally, we demonstrate numerically the "Single Ring" theorem for the sextic potential, namely, the potential of lowest degree for which the "Single Ring" theorem has non-trivial implications.

**Hempel, Rainer**

*Spectral Properties of Periodic or Random Media in the Limit of High Contrast*

We investigate the band-gap structure and the integrated density of states for a class of periodic divergence type operators which, in the simplest case, are given by

$$T_\lambda = -\nabla \cdot (1 + \lambda \chi_\Omega)\nabla, \qquad \lambda >> 1,$$

acting in $L_2(\mathbf{R}^m)$ with $m \geq 2$. We assume in the periodic case that $\Omega$ is an open, connected, periodic subset of $\mathbf{R}^m$ and that the complement $M$ of $\Omega$ does not intersect the boundary of the fundamental period cell $Q_0$. Operators of this type occur in simple models for heat conduction (or propagation of acoustic waves) in a metal with impurities like grains of sand or air bubbles, and in connection with photonic crystals.
Among other results, we find that $T_\lambda$ will always have at least one open gap, for $\lambda$ large (except for trivial cases) and we establish a connection between the band-gap structure of $T_\lambda$ and the Dirichlet eigenvalue problem on $M_0 = M \cap Q_0$.
We also discuss some extensions of the above results to suitable random models.
(Joint work with K. Lienau, Braunschweig)

**Hoppe, Jens**

*Relativistic Membrane Dynamics*

(Hyper-)Surfaces moving in such a way that the manifold swept out in space-time has vanishing mean curvature, are related to hydrodynamical equations of motion, Born Infeld equations, and - via a replacement of functions by matrices - to (supersymmetric) matrix models which, after quantizations, yield a candidate for a unification of string theories.

**Korotyaev, Evgeni**

*Scattering, resonances, conformal mappings and inverse problem for Schrödinger operator*

We consider the Schrödinger operator on the half line, with a real finite support potential. We prove the following results: i) using the S-matrix for this operator we construct the conformal mapping from the upper half plane onto some domain on the plane, (it is some analogue of the quasimomentum for periodic potentials which was introduced by Marchenko and Ostrovski ) ii) we study the properties of this mapping and we express this conformal mapping in terms of the resonances, iii) we obtain some properties of the resonances. iv) we express the trace formular in terms of the resonances, v) we show that the mapping: a potential with the finite support - the Jost function (some class of entire functions) is one-to-one and onto. Moreover, we prove that such potential is uniquely determined by its bound states and resonances. vi) we do the same for the Schrödinger operator on the real line and for Zakharov-Shabat system.

**Kostrykin, Vadim**

*Concavity of Eigenvalue Sums and The Spectral Shift Function*

It is well known that the sum of negative (positive) eigenvalues of some finite Hermitian matrix $V$ is concave (convex) with respect to $V$. Using the theory of the spectral shift function we generalize this property to self-adjoint operators on separable Hilbert space with an arbitrary spectrum. More precisely, we prove that the spectral shift function integrated with respect to the spectral parameter from $-\infty$ to $\lambda$ (from $\lambda$ to $+\infty$) is concave (convex) with respect to trace class perturbations. The case of relative trace class perturbations is also considered.

**Kunze, Markus**

*Adiabatic Limit of the Maxwell-Lorentz Equations*

We consider the classical Maxwell-Lorentz model of a charged particle subject to its self-generated electric and magnetic fields. An important issue for such models is to derive an “effective equation” for the particle position $q(t) \in R^3$, i.e., an ODE for $q$ alone, with the contributions from Maxwell's equations incorporated in this ODE. By formal calculations it can be seen that the resulting equation is of type

$$(m_0 + \epsilon^{-1} m_e)\ddot{q} = F(q, \dot{q}) + aq^{(3)} \tag{1}$$

with $a > 0$, $m_0$ and $m_e$ are the rest mass and the effective (electromagnetic) mass of the particle, respectively; $F$ denotes the external forces. This is the well-known Lorentz-Dirac equation which (due to the term with the third derivative) has undesirable ”run-away”-solutions, i.e., solutions that go off to infinity exponentially fast. There have been quite some attempts to single out those data that lead to physically relevant solutions of (1), and to interpret them as approximations of the full problem, but no satisfactory general answer has been known yet.

We outline a solution of this problem (and a rigorous derivation of (1)). It will be shown that the physically relevant solutions are exactly those which build up a normally hyperbolic manifold for (1) as a singular perturbation problem. In addition, the solutions are good approximations of the solution to the full Maxwell-Lorentz equations. The true "effective equation" then is the corresponding equation on the center-like manifold, and this equation has only physically reasonable solutions. (Joint work with H. Spohn)

**Lenz, Daniel**

*Hierarchies in Sturmian dynamical systems and uniform spectral properties of one-dimensional quasicrystals*

We discuss recent results on uniform spectral properties of one-dimensional Schrödinger operators with Sturmian potentials. These results include uniform absence of eigenvalues for all (irrational) rotation numbers as well as uniform alpha-continuity for rotation numbers of bounded density. This is joint work with D. Damanik resp. D. Damanik and R. Killip.

**Myrzakul, Kuralay**

*Soliton equations in multidimensions as exact reductions of the Self-Dual Yang-Mills equation*[4]

One of the most interesting and important multidimensional soliton equations is the Self-Dual Yang-Mills equation (SDYME). At present, it is well known that the SDYME consists many integrable soliton equations as particular reductions. In this report we show that some well known (2+1)-dimensional soliton equations such as the Davey-Stewartson, Zakharov, Kadomtseva-Peviashvili and other equations are exact reductions of the SDYME.

**Neumaier, Nikolai**

*A Fedosov approach to star products with separation of variables on Kaehler manifolds*

On Kaehler manifolds one can define special deformation quantizations that are the natural generalizations of the well-known Wick-product that arises from the usual creation and annihilation operator quantization on $C^n$ using the normal ordering. In this talk we give a concrete construction of such deformation quantizations by slightly modifying Fedosov's procedure. Moreover we show that by this construction one can find a representant of any possible isomorphy class of these star products. These isomorphy classes are shown to be characterized by formal power series of closed two-forms of type (1,1).

**Paul, Thierry**

*Toda lattice in the limit of large number of particles*

We present some results on the dispersionless limit of the Toda lattice equations viewed as the semi-classical limit of an equation involving certain Toeplitz operators. For the non-periodic case the phase space is the Riemann sphere, while in the periodic case it is the torus. This implies estimates on the dispersionless limit. The results here extend previous ones known in the literature. In the nonperiodic case we also present a method of solution for the dispersionless Toda lattice equations for a certain class of (linear) initial data.

**Potthoff, Jürgen**

*White noise approach to parabolic stochastic PDE's*

[4]see also Subsection III.3

It is shown how the tools of white noise analysis, viz. generalized random variables, S-transform and characterization theorems, can be used to solve stochastic partial differential equations of parabolic type.

**Protogenov , Alexander**

*Chern-Simons Correlations in the Model of Discrete (2+1)D Nonlinear Schrödinger Equation*

The structure of zero modes in the model of gauged nonlinear Schrödinger equation on (2+1)D lattice and their contribution to the critical value of the number of particles are considered. It is shown that due to Chern-Simons correlations there exists an additional universal attraction between field configurations.

**Schmidt, Karl Michael**

*Eigenvalue asymptotics of critically perturbed periodic Sturm-Liouville operators.*

As Kneser's classical oscillation criterion shows, potentials of asymptotic decay $c/x^2$ present a critical case in the spectral theory of Sturm-Liouville operators, since there is a finite or infinite number of negative eigenvalues depending on whether $c > -1/4$ or $c < -1/4$. In the latter case, the asymptotic distribution of the eigenvalues at their point of accumulation differs from the semiclassical formula.
In this talk it is shown that similar coupling constant thresholds exist for each edge of a spectral gap of a periodic Sturm-Liouville operator with a perturbation of quadratic decay (e.g., an angular momentum term). The method used also yields the asymptotic distribution of the eigenvalues, with corrections to the semiclassical formula reflecting both the critical character of the perturbation and the underlying periodic background.

**Solomyak , Michael**

*On the Floquet - Bloch theory for non-smooth periodic elliptic operators*

Let $\Omega \subset R^d$ be an unbounded domain, periodic along a chosen direction (a waveguide-type domain) and $\mathcal{P}$ be a self-adjoint elliptic second order operator in $L_2(\Omega)$, periodic along the same direction. We suppose that $\mathcal{P}$ is an operator in divergency form, with the bounded measurable coefficients. Let $\mathcal{P}_\xi$ be the operators with compact resolvent, appearing in the Floquet - Bloch theory for $\mathcal{P}$. We are interested in the analytic properties of the eigenvalues and the eigenfunctions of the operators $\mathcal{P}_\xi$, as functions of $\xi$. These properties are formulated in a standard way, however the proofs are rather non-trivial, due to the weak assumptions on the coefficients of $\mathcal{P}$.
In the talk we are going to outline these proofs. The results are joint with M. Sh. Birman.

**Steudel, Heinz**

*Darboux transformations, N-soliton formulas and Vandermonde-like determinants*

N–fold Darboux transformations for scattering problems of AKNS or more general types can be expressed in terms of Vandermonde–like determinants [1,2]. It has been shown that these determinants are provided with remarkable structural properties [3,4]. In particular, they are expressible as sums over products of genuine Vandermonde determinants. By this way N–soliton formulas both over vacuum and over finite background can be written in a rather compact and transparent form which as well is suited for numerical computations.
As an example we derive such a formula for the Derivative Nonlinear Schrö-dinger equation which

has applications in plasma physics and in nonlinear optics. Computer pictures of N–soliton collisions up to N=8 are presented.

[1] G. Neugebauer and R. Meinel, Phys. Lett. **100 A** (1984) 467.
[2] V. B. Matveev and M. A. Salle, Darboux transformations and solitons (Springer–Verlag, Berlin 1991).
[3] H. Steudel, R. Meinel and D. J. Kaup, J. Mod. Opt. **44** (1997) 287.
[4] H. Steudel, R. Meinel and G. Neugebauer, J. Math. Phys. **38** (1997) 4692.

**Veselić, Ivan**

*Integrated density of states for random Schrödinger operators on manifolds*[5]

We consider a Riemannian manifold $X$ admitting a compact quotient $X/\Gamma$, i.e. $\Gamma$ is a cocompact subgroup of isometries acting property discontinuously on $X$.
We show, under certain conditions on $\Gamma$, that it is possible to define a reasonable integrated density of states for $\Gamma$-ergodic random Schrödinger operators on $X$. These conditions are, e.g., satisfied if $\Gamma$ has polynomial growth.
The diferential geometric aspect of our work is inspired mainly by Adachi/Sunada: Density of states in spectral geometry, Comment. Math. Helvetici 68 (1993) while the mathematical physics background can be found e.g. in Kirsch/Martinelli: On the density of states of Schrödinger operators with a random potential, J. Phys. A15 (1982)
(Joint work with N. Peyerimhoff, Bochum)

[5] see also Subsection III.5

## II.3 Section on "Quantum Mechanics and Quantum Field Theory

**Baumgaertel, Hellmut**

*Abstract link between global and local gauge group in special Quantum Field Theories*

The construction of Hilbert systems for the global gauge group T using the implementation of loop goups in Fock representations of fermion algebras is presented where the local gauge group (w.r.t. the spectrum of the center of the fixed point algebra) is characterized as the stabilizer of the fixed point algebra.

**Brunetti, Romeo**

*On a local construction of Wick polynomials of free fields*

As part of a recent paper with Prof. Fredenhagen, I shall discuss about a new construction of Wick polynomials of free fields which uses the language of microlocal analysis and aims at a local and state-independent construction.

**Exner, Pavel**

*Edge-type currents in a plane with point perturbations*

We study a charged two-dimensional particle in a homogeneous magnetic field interacting with a periodic array of point obstacles. We show that while Landau levels remain to be infinitely degenerate eigenvalues, between them the system has bands of absolutely continuous spectrum and exhibits thus a transport along the array. In distinction to the usual edge states, this is a purely quantum effect. We compute the band functions and the corresponding probability current.

**Feinberg, Joshua**

*The Periodic Table of Static Fermion Bags in the 1+1 Dimensional Gross-Neveu Model*

Examination of the asymptotic spatial behavior of the fermion current expectation value in a static localized inhomogeneous scalar (or pseudoscalar) background reveals that back-reaction of the fermions will not render the background time dependent only if the associated Dirac operator is reflectionless. This observation is independent of the large $N$ limit, and explains in simple terms the classical results due to Dashen, Hasslacher and Neveu. Thus, static inhomogeneous saddle point configurations of the scalar condensate (i.e., the fermion bag) which govern the path integral, must be reflectionless. Based on that, we construct the full spectrum of static fermion bags in the Gross-Neveu model, including bag profiles and masses.

**Geyler, Vladimir**

*Berry phase in a magnetic field for moving point potentials (P. Exner, V. Geyler)*

We consider a two-dimensional charged particle interacting with a magnetic field, in general non-homogeneous, perpendicular to the plane, located in a zero-range well. If the latter moves adiabatically along a loop the state corresponding to an isolated eigenvalue acquires a Berry phase. We derive an expression for it and evaluate it in several examples such as a homogeneous field, the field of an Aharonov-Bohm solenoid, a particle confined at a ring or in quantum dots, a parabolic and a zero-range one. We also discuss the behavior of the lowest Landau level in this setting obtaining an explicit example of the Wilczek-Zee phase for an infinitely degenerated eigenvalue.

**Griesemer, Marcel**

*Asymptotic completeness for massive Pauli-Fierz Hamiltonians: A new proof*

Consider a quantum particle bound by a potential and coupled linearly to a quantized field of massive relativistic bosons. We derive asymptotic completeness for this system form the fact that the mean square diameter of states away from thresholds and eigenvalues diverges like $t^2$ as $t \to \infty$.

**Guido, Daniele**

*The spin and statistics connection for curved spacetimes*

I will describe some results concerning the relation between the statistics of a superselection sector and the univalence of the associated representation of the group of spacetime symmetries in the framework of algebraic quantum field theory.

**Haba, Zbigniew**

*Decoherence in QED and in quantum gravity*

By means of approximate calculations in finite temperature QED and quasiclassical approximation in quantum mechanics a decay of off-diagonal matrix elements is shown. Similar calculations are performed in linearized quantum gravity showing that gravitons have a decohering effect on particles described by non-relativistic quantum mechanics.

**Helffer, Bernard**

*Semiclassical problems in superconductivity*

In this talk, we intend to describe some spectral problems appearing in superconductivity, which may be treated through semi-classical analysis techniques.

**Kovarik, Hynek**

*Transport properties of the local Iwatsuka model*

We analyze the spectrum of the “local” Iwatsuka model, i.e. a two-dimensional charged particle interacting with a magnetic field which is homogeneous outside a finite strip. We give two new sufficient conditions for absolute continuity of the spectrum.

**Krasovsky, Igor**

*Bloch electron in a magnetic field and the Ising model*

The spectral determinant $\det(H - \varepsilon I)$ of the Azbel-Hofstadter Hamiltonian $H$ is related to Onsager's partition function of the 2D Ising model for any value of magnetic flux $\Phi = 2\pi P/Q$ (in the Hofstadter model) through an elementary cell, where $P$ and $Q$ are coprime integers. The band edges of $H$ correspond to the critical temperature of the Ising model; the spectral determinant at these (and other points defined in a certain similar way) is independent of $P$. A connection of the mean of Lyapunov exponents to the asymptotic (large $Q$) bandwidth is indicated.

**Krejcirik, David**

*Bound states in curved quantum layers*

We consider a quantum particle constrained to a curved layer of a constant width built over an infinite smooth surface. We suppose that the latter is asymptotically planar in a suitable sense and that the layer has the hard-wall boundary. Under these assumptions we prove that the particle Hamiltonian possesses geometrically induced bound states.

**Leandre, Remi**

*Quotient of a loop group and Witten genus*

We define over the quotient of a loop group an infinite dimensional Dirac operator. We give the conjecture that its equivariant Index is equal to the Witten genus of the quotient of the group by a torus. We justify the conjecture by using a small time asymptotic expansion of the operator which allow to arrive at an operator at ‘the manner of Taubes, where the computation can be done by hand.

**Lehmann, Detlef**

*Resummation of Feynman Diagrams and the Inversion of Matrices*

In many field theoretical models one has to resum two and four legged subdiagrams in order to determine their behaviour. In this talk we would like to present a novel formalism which does this in a nice way. It is based on the central limit theorem of probability and an inversion formula for matrices which is obtained by repeated application of the Feshbach projection method. We discuss applications to the Anderson model and to the many-electron system.

**Lüst, Dieter**

*Supersymmetric 3-cycles in string theory*

In this talk we will talk about supersymmetric brane configurations which are described by supersymmetric 3-cycles. We will discuss their relations to supersymmetric gauge theories and the influence of a magnetic field on the deformation of supersymmetric $n$-cycles. The talk is based on the following two papers: hep-th/9912065 and hep-th/9810264.

**Pavlyk, Oleksandr**

*Generalized integrable hierarchies of Drinfeld-Sokolov and solutions of WDVV equations*

We consider integrable hierarchies associated to primitive regular conjugacy class of Weyl group of underlying Lie algebra g. We define the notion of dispersionless limit for these hierarchies and prove that one can always extract an algebraic solution of WDVV equations. As a byproduct, a new representation of Weyl group W(g) is found.

**Teufel, Stefan**

*Semiclassical limit for the Schrödinger equation with a short scale periodic potential*

We consider the dynamics generated by the Schrödinger operator $H = -1/2\Delta + V(x) + W(\varepsilon x)$, where $V$ is a lattice periodic potential and $W$ an external potential which varies slowly on the scale set by the lattice spacing. We prove that in the limit $\varepsilon \to 0$ the time dependent position operator and, more generally, semiclassical observables converge strongly to a limit which is determined by the semiclassical dynamics.

**Verch, Rainer**

*Passivity and microlocal spectrum condition*

The microlocal spectrum condition has over the past several years been recognized as an appropriate generalization of the flat-space spectrum condition for quantum field theory in curved spacetime. I report on recent work which shows that the microlocal spectrum condition may be viewed as a consequence of 'passivity', i.e. the validity of the second law of thermodynamics for quantum states, and which establishes the connection between the microlocal spectrum condition and the Hadamard condition for fields that are sections in arbitrary vectorbundles over globally hyperbolic spacetimes, in quasifree states fulfilling the CCR or CAR.

**Zograf, Peter**

*Point fluxes in quantum Hall fluids*

The forces that act on a point flux in quantum Hall fluids are related to the adiabatic curvature associated with Landau Hamiltonians. The forces due to external fields (Lorenz and Magnus forces), and the forces due to the mutual interaction of point fluxes are calculated.

# III Contributed Papers

## III.1 Güler Gürpinar Arsan: *Chebyshev Nets formed by the Laguerre Lines on Centro-Surfaces*

**Abstract.** Let $S$ be a surface of class $C^4$ in Euclidean 3-space and let $S_1$ and $S_2$ be the centro-surfaces of $S$. In this paper, it is shown that if the two families of Laguerre lines of $S_1$ form a conjugate Chebyshev net, then $S$ and $S_1$ are general cylinders. Furthermore, it is proved that if the two families of Laguerre lines of $S_1$ and $S_2$ correspond and if two families of Laguerre lines of $S_1$ form a conjugate semi-Chebyshev net, then $S$ is a general cylinder and $S_1$ is a cone.

**1. Introduction.** Let $S$ be a surface of class $C^4$ in Euclidean 3-space with vector equation $\vec{x} = \vec{x}(u, v)$ and let $C$ be a line on $S$. $C$ is said to be a Laguerre line ($G$-line), if and only if at every point $P$ on $C$, the normal plane of $S$ containing the tangent $PT$ to $C$ cuts the surface $S$ in a line $\bar{C}$ superosculated by its circle of curvature at $P$ [1],i.e. if $\bar{C}$ has contact of third order with its circle of curvature at $P$ [2]. A Laguerre line is characterized by the equation

$$\mathcal{L} = \rho_n' - 2\rho_g\tau_g = 0 \qquad (\rho_n' = \frac{d\rho_n}{ds}) \tag{1.1}$$

where $\mathcal{L}$ is the Laguerre's function of direction ([3],[4]) and $\rho_n,\ \rho_g,\ \tau_g$ and $s$ are respectively, the normal curvature, the geodesic curvature, the geodesic torsion and the arclength of $C$ [1].
Let a Laguerre line on a surface $S$ be given in function of any parameter $t$, by

$$u = u(t)\ ,\ v = v(t)$$

$u$ and $v$ being the parameters on $S$. Then the differential equation (1.1) can be expressed in function of the coefficients of the two fundamental forms of $S$ and their partial derivatives, namely

$$\begin{gathered}\left\{L_u - 2\,(L\,\Gamma_{11}^1 + M\,\Gamma_{11}^2)\right\} du^3 + 3\left\{L_v - 2\,(L\,\Gamma_{12}^1 + M\,\Gamma_{12}^2)\right\} du^2\,dv \\ +3\left\{N_u - 2\,(M\,\Gamma_{12}^1 + N\,\Gamma_{12}^2)\right\} du\,dv^2 + \left\{N_v - 2\,(M\,\Gamma_{22}^1 + N\,\Gamma_{22}^2)\right\} dv^3 = 0\end{gathered} \tag{1.2}$$

where $L, M$ and $N$ are the coefficients of the second fundamental form of $S$ and $\Gamma_{ij}^k$ $(i, j, k = 1, 2)$ are its Christoffel symbols [5].
The surfaces $S_1$ and $S_2$ defined, respectively, by the vector equations

$$S_1 :\ \vec{y} = \vec{x} + \alpha\vec{n} \tag{1.3}$$

$$S_2 :\ \vec{\bar{y}} = \vec{x} + \beta\vec{n} \tag{1.4}$$

are called centro-surfaces of $S$ where $\alpha$ and $\beta$ are the radii of principal curvatures and $\vec{n}$ is the unit normal vector of $S$ at the point $P$.
A 2-net on $S$ is said to be a Chebyshev net, if the tangent vector fields of either family of the net undergoes a parallel displacement along each curve of the other family. If the tangent vector field of one family undergoes a parallel displacement along the curves of the other family, then the 2-net is called a semi-Chebyshev net [6].
The unit vector field $\vec{e}$ making an angle $\theta$ with the unit tangent vector field $\vec{t}$ of a curve $C$ on $S$ will undergo a parallel displacement, in the sense of Levi-Civita, along $C$ provided that

$$\frac{d\theta}{ds} = -\rho_g\ , \tag{1.5}$$

where $\theta$ is measured from $\vec{t}$ to $\vec{e}$ in the positive sense about the normal of $S$, $s$ and $\rho_g$ being, respectively, the arclength and geodesic curvature of $C$ at the point considered ([7],[8]).

Without loss of generality, we may take any two families of curves on $S$, as the parametric lines $v=$ const. , $u=$ const.,respectively. The necessary and sufficient conditions for the parametric lines to form a Chebyshev net are, according to (1.5),

$$g^{*}+\delta_1=0\ ,\quad g^{**}-\delta_2=0 \tag{1.6}$$

where $g^{*}$ , $g^{**}$ and $\delta$ are respectively the geodesic curvatures of $v=$ const. , $u=$ const and the angle between the parametric lines and the indices 1,2 are the indices of invariant differentiation in the direction of $v=$ const. and $u=$ const., respectively.
The conditions (1.6) are equivalent to

$$E=E(u)\quad,\quad G=G(v)\,. \tag{1.7}$$

On the other hand, in the most general coordinate system the Mainardi-Codazzi equations and Gauss characteristic equation [2] can be written as

$$\begin{aligned} L_v-M_u&=L\Gamma^1_{12}+M(\Gamma^2_{12}-\Gamma^1_{11})-N\Gamma^2_{11}\\ M_v-N_u&=L\Gamma^1_{22}+M(\Gamma^2_{22}-\Gamma^1_{12})-N\Gamma^2_{12}\quad (1.8)\end{aligned}$$

$$K=\frac{L\,N-M^2}{H^2}=\frac{1}{2H}\Big\{\big[\frac{F}{EH}\,E_v-\frac{1}{H}\,G_u\big]_u+\big[\frac{2}{H}\,F_u-\frac{1}{H}\,E_v-\frac{F}{EH}\,E_u\big]_v\Big\}\,. \tag{1.9}$$

**2. Chebyshev net formed by the Laguerre lines on $\mathbf{S_1}$ .**

Let the lines of curvature on $S$ be taken as the parametric lines. In virtue of (1.3), the fundamental magnitudes for $S_1$ [9] are given by

$$\begin{aligned} &\bar E=\alpha_u^2\ ,\quad \bar F=\alpha_u\,\alpha_v\ ,\quad \bar G=\alpha_v^2+G\,(1-\tfrac{\alpha}{\beta})^2\\ &\bar L=-\varepsilon\,\sqrt{E}\,\tfrac{\alpha_u}{\alpha}\ ,\quad \bar M=0\ ,\quad \bar N=\tfrac{\varepsilon\,\alpha\,G}{\sqrt{E}}\,\tfrac{\beta_u}{\beta^2}\quad (\varepsilon=\mp 1)\quad (2.1)\end{aligned}$$

Since $\bar M=0$ it follows that the parametric lines on $S_1$ form a conjugate system.
On the other hand, using (1.2) the differential equation of the Laguerre lines of $S_1$ is found to be

$$\begin{aligned} &\big(\bar L_u-2\,\bar L\,\bar\Gamma^1_{11}\big)\,du^3+3\,\big(\bar L_v-2\,\bar L\,\bar\Gamma^1_{12}\big)\,du^2\,dv\\ &+3\,\big(\bar N_u-2\,\bar N\,\bar\Gamma^2_{12}\big)\,du\,dv^2+\big(\bar N_v-2\,\bar N\,\bar\Gamma^2_{22}\big)\,dv^3=0\quad (2.2)\end{aligned}$$

where $\bar L$ , $\bar M$ and $\bar N$ are the coefficients of the second fundamental form of $S_1$ and $\bar\Gamma^k_{ij}$ $(i,j,k=1,2)$ are its Christoffel symbols.
Suppose that the curves $v=$ const. and $u=$ const. are the Laguerre lines on $S_1$ and that these curves form a conjugate Chebyshev net. Then, from the differential equation (2.2) we obtain

$$\bar L_u-2\,\bar L\,\bar\Gamma^1_{11}=0 \tag{2.3}$$

$$\bar N_v-2\,\bar N\,\bar\Gamma^2_{22}=0 \tag{2.4}$$

and from (1.7) we get

$$\bar E=\bar E(u)\quad,\quad \bar G=\bar G(v)\ .$$

In this case, the equation (2.3) reduces to

$$\bar L_u-2\bar L\,\frac{\alpha_{uu}}{\alpha_u}=0\ . \tag{2.5}$$

The general solution of the differential equation (2.5), is of the form

$$\ln\bar L-2\ln|\alpha_u|=\ln A(v)\ ,\quad (\bar L>0\ ,\ A(v)>0)\ .$$

According to (2.1) we have

$$\bar L=A(v)\,\alpha_u^2=-\varepsilon\,\sqrt{E}\,\frac{\alpha_u}{\alpha}\qquad(\alpha_u\neq 0)$$

or

$$\sqrt{E} = -\frac{1}{\varepsilon}\,\alpha\,\alpha_u\,A(v)\ . \tag{2.6}$$

On the other hand, using the Mainardi Codazzi equations for $S_1$ we get

$$\begin{aligned}\bar{L}_v &= \bar{L}\,\bar{\Gamma}^1_{12} - \bar{N}\,\bar{\Gamma}^2_{11}\\ \bar{N}_u &= -\bar{L}\,\bar{\Gamma}^1_{22} + \bar{N}\,\bar{\Gamma}^2_{12}\ .\end{aligned} \tag{2.7}$$

If $\bar{\Gamma}^2_{11} = \bar{\Gamma}^1_{12} = 0$ is substituted in (2.7) we have

$$\bar{L}_v = 0 \quad \Rightarrow \quad \bar{L} = \bar{L}(u)\ ,$$

and

$$A(v) = \frac{\bar{L}(u)}{\alpha_u^2} = \frac{\bar{L}(u)}{\bar{E}(u)} = \text{const.} \tag{2.8}$$

Using the Mainardi-Codazzi equations for $S$ [10] which is referred to its lines of curvature we get

$$(\frac{1}{\alpha})_v + (\frac{1}{\alpha} - \frac{1}{\beta})\,(\ln\sqrt{E})_v = 0 \tag{2.9}$$

$$(\frac{1}{\beta})_u + (\frac{1}{\beta} - \frac{1}{\alpha})\,(\ln\sqrt{G})_u = 0\ . \tag{2.10}$$

By using (2.6), (2.8) and (2.9) we obtain $\alpha_v = 0$. In this case, from (2.1) we get $\bar{F} = 0$ so that the lines of curvature of $S$ and $S_1$ correspond.
If the values of $\bar{E} = \bar{E}(u)$ , $\bar{G} = \bar{G}(v)$ , $\bar{F} = \bar{M} = 0$ are substituted in Gauss equation (1.9) we get

$$\bar{K} = -\frac{1}{\sqrt{\bar{E}\bar{G}}}\Big[\frac{\partial}{\partial u}\,(\frac{1}{\sqrt{\bar{E}}}\,\frac{\partial\sqrt{\bar{G}}}{\partial u}) + \frac{\partial}{\partial v}\,(\frac{1}{\sqrt{\bar{G}}}\,\frac{\partial\sqrt{\bar{E}}}{\partial v})\Big] = 0\ .$$

From (2.1), it is easy to see that $\beta_u = 0$. Since $\alpha_v = \beta_u = 0$, $S$ is a general cylinder and $S_1$ is a developable surface. We have thus proved the

**Theorem.** If the two families of Laguerre lines of $S_1$ form a conjugate Chebyshev net, then $S$ and $S_1$ are general cylinders.

## 3. Semi-Chebyshev net formed by the Laguerre lines on $\mathbf{S_1}$ .

The fundamental magnitudes for $S_2$ are obtained from (1.4) as

$$\begin{aligned}&\bar{E}' = \beta_u^2 + E\,(1 - \tfrac{\beta}{\alpha})^2\ , \quad \bar{F}' = \beta_u\,\beta_v\ , \quad \bar{G}' = \beta_v^2\\ &\bar{L}' = \tfrac{\varepsilon'\,\beta\,E}{\sqrt{G}}\,\tfrac{\alpha_v}{\alpha^2}\ , \qquad \bar{M}' = 0\ , \quad \bar{N}' = -\varepsilon'\,\sqrt{G}\,\tfrac{\beta_v}{\beta} \qquad (\varepsilon' = \mp 1)\end{aligned} \tag{3.1}$$

where $\varepsilon'$ is equal to $\mp 1$ according as $\beta_v\,(1 - \frac{\beta}{\alpha})$ is negative or positive [9].
We now proceed to determine the surface $S$ if the two families of Laguerre lines of $S_1$ and $S_2$ correspond and if the two families of Laguerre lines on $S_1$ form a conjugate semi-Chebyshev net. Let $S$ be referred to its lines of curvature. Using (3.1), the differential equation of Laguerre lines of $S_2$ is found to be

$$\begin{aligned}&(\,\bar{L}'_u - 2\,\bar{L}'\,\bar{\Gamma}'^1_{11}\,)\,du^3 + 3\,(\,\bar{L}'_v - 2\,\bar{L}'\,\bar{\Gamma}'^1_{12}\,)\,du^2\,dv\\ &+3\,(\,\bar{N}'_u - 2\,\bar{N}'\,\bar{\Gamma}'^2_{12}\,)\,du\,dv^2 + (\,\bar{N}'_v - 2\,\bar{N}'\,\bar{\Gamma}'^2_{22}\,)\,dv^3 = 0\ .\end{aligned} \tag{3.2}$$

Let the curves $v =$ const. and $u =$ const. be taken as the two families of Laguerre lines on $S_1$. Then, we have the equations (2.3) and (2.4). Furthermore, since the two families of Laguerre lines of $S_1$ and $S_2$ correspond, we must have

$$\bar{L}'_u - 2\,\bar{L}'\,\bar{\Gamma}'^1_{11} = 0 \tag{3.3}$$

$$\bar{N}'_v - 2\,\bar{N}'\,\bar{\Gamma}'^2_{22} = 0\ . \tag{3.4}$$

Suppose that the two families of Laguerre lines of $S_1$ form a conjugate semi-Chebyshev net. Then, using the fact that the curves $v=$ const. are geodesics on $S_1$, from (1.6) we obtain

$$\bar{\delta}_1 = -\bar{g}^* = 0\ , \tag{3.5}$$

where $\bar{g}^*$ is the geodesic curvature of $v=$ const. and $\bar{\delta}$ is the angle between the parametric lines on $S_1$.
On the other hand, inserting the fundamental magnitudes of $S_1$ and $S_2$, given by (2.1) and (3.1), in the equations (2.3), (2.4), (3.3) and (3.4) we find, respectively,

$$(-\varepsilon\sqrt{E}\,\frac{\alpha_u}{\alpha})_u - 2(-\varepsilon\sqrt{E}\,\frac{\alpha_u}{\alpha})\,\frac{\alpha_{uu}}{\alpha_u} = 0\ , \tag{3.6}$$

$$\begin{aligned}(\frac{\varepsilon\,\alpha\,G}{\sqrt{E}}\ \frac{\beta_u}{\beta^2})_v - 2(\frac{\varepsilon\,\alpha\,G}{\sqrt{E}}\ \frac{\beta_u}{\beta^2})\Big\{(\ln\sqrt{G})_v + [\ln(1-\frac{\alpha}{\beta})]_v&\\ + \frac{\alpha_v}{\alpha_u}\,(\ln\sqrt{G})_u + \frac{\alpha_v}{\alpha_u}\,[\ln(1-\frac{\alpha}{\beta})]_u\Big\} &= 0\ ,\end{aligned} \tag{3.7}$$

$$\begin{aligned}(\frac{\varepsilon'\beta E}{\sqrt{G}}\ \frac{\alpha_v}{\alpha^2})_u - 2(\frac{\varepsilon'\beta E}{\sqrt{G}}\ \frac{\alpha_v}{\alpha^2})\Big\{(\ln\sqrt{E})_u + [\ln(1-\frac{\beta}{\alpha})]_u&\\ + \frac{\beta_u}{\beta_v}\,(\ln\sqrt{E})_v + \frac{\beta_u}{\beta_v}\,[\ln(1-\frac{\beta}{\alpha})]_v\Big\} &= 0\ ,\end{aligned} \tag{3.8}$$

$$(-\varepsilon'\sqrt{G}\,\frac{\beta_v}{\beta})_u - 2(-\varepsilon'\sqrt{G}\,\frac{\beta_v}{\beta})\,\frac{\beta_{vv}}{\beta_v} = 0\ . \tag{3.9}$$

From (3.6) and (3.9), we obtain

$$\begin{aligned}\frac{\alpha\,\alpha_u}{\sqrt{E}} &= \mu(v)\quad ,\qquad (\mu(v)>0)\\ \frac{\beta\,\beta_v}{\sqrt{G}} &= \gamma(u)\quad ,\qquad (\gamma(u)>0)\ .\end{aligned} \tag{3.10}$$

Taking the equation $\cos\bar{\delta} = \frac{\bar{F}}{\sqrt{\bar{E}\bar{G}}}$ and (3.10) into consideration we have

$$\cos\bar{\delta}(v) = \mp\frac{\alpha_v}{\sqrt{\alpha_v^2 + [\,\frac{\beta\,\beta_v}{\gamma(u)}\,(1-\frac{\alpha}{\beta})\,]^2}} \tag{3.11}$$

On the other hand, we can write the Mainardi-Codazzi equations of $S$ in the form

$$\begin{aligned}\alpha_v &= \left(\frac{\alpha_{uv}}{\alpha_u} - \frac{\mu'(v)}{\mu(v)}\right)(\beta-\alpha)\\ \beta_u &= \left(\frac{\beta_{uv}}{\beta_v} - \frac{\gamma'(u)}{\gamma(u)}\right)(\alpha-\beta)\end{aligned} \tag{3.12}$$

where the primes indicate the differentiation with respect to arguments. In this case, from (3.7) and (3.8) we obtain

$$\begin{aligned}3\alpha_v\,\beta_u - (\alpha-\beta)\,\beta_{uv} - 2\beta_u\,\beta_v &= 0\\ 3\alpha_v\,\beta_u - (\beta-\alpha)\,\alpha_{uv} - 2\alpha_u\,\alpha_v &= 0\ .\end{aligned} \tag{3.13}$$

Differentiating (3.11), with respect to $u$, we have

$$2\,\alpha_v\,A\,\{\alpha_{vu}-\alpha_v\,A\,[\alpha_v\,\alpha_{vu}+\frac{\beta\,\beta_v}{\gamma(u)}\,(1-\frac{\alpha}{\beta})\,B]\}=0 \tag{3.14}$$

where $A$ and $B$ are given by

$$A=A(u,v)=\frac{1}{\alpha_v^2+[\frac{\beta\,\beta_v}{\gamma(u)}\,(1-\frac{\alpha}{\beta})]^2}\quad ,$$

$$B=B(u,v)=\left[\frac{\beta_u\,\beta_v+\beta\,\beta_{uv}}{\gamma(u)}-\frac{\beta\,\beta_v\,\gamma'(u)}{\gamma^2(u)}\right](1-\frac{\alpha}{\beta})-\frac{\beta_v\,\alpha_u}{\gamma(u)}+\frac{\alpha\,\beta_u\,\beta_v}{\gamma(u)\,\beta}\ .$$

From (3.14) the following two cases occur:
**Case I.** $\alpha_v=0$ , **Case II.** $\alpha_{vu}-\alpha_v\,A\,[\alpha_v\,\alpha_{vu}+\frac{\beta\,\beta_v}{\gamma(u)}\,(1-\frac{\alpha}{\beta})\,B]=0$ , $\alpha_v\neq 0$ .
In **Case I**, from the first equations of (3.12) and (3.13) we obtain respectively

$$\begin{gathered}\mu(v)=\mu=\text{const.}\,,\\ (\alpha-\beta)\,\beta_{uv}+2\,\beta_u\,\beta_v=0\ . \quad (3.15)\end{gathered}$$

With the aid of (3.15) the second equation of (3.12) takes the form

$$3\,\beta_u+(\alpha-\beta)\,\frac{\gamma'(u)}{\gamma(u)}=0\ . \tag{3.16}$$

Differentiating (3.16) with respect to $v$, we get

$$3\,\frac{\beta_{vu}}{\beta_v}=\frac{\gamma'(u)}{\gamma(u)}$$

or

$${\beta_v}^3=\gamma(u)\,C(v)\ ,\quad (C(v)>0)\ .$$

In this case, from (3.15) and (3.16) we obtain $\beta_u=0$. So, we conclude that $S$ is a cylinder and $S_1$ and $S_2$ are the developable surfaces. The fact that the two families of Laguerre lines on a developable surface form a semi-Chebyshev net, implies that the developable surface is a cone [11]. This shows that $S_1$ is a cone.
In **Case II**, taking $\beta_v\neq 0$ and values of $A$ , $B$ into consideration we obtain

$$(\beta-\alpha)\,(\alpha_{vu}\,\beta_v-\alpha_v\,\beta_{uv})-\beta_v\,\alpha_v\left[(\beta-\alpha)\,(\frac{\beta_u}{\beta}-\frac{\gamma'(u)}{\gamma})-(\alpha_u-\frac{\alpha\,\beta_u}{\beta})\right]=0\ . \tag{3.17}$$

On the other hand, using (3.13) and (3.17) we get

$$3\alpha_v\,\beta_u-\alpha_u\,\beta_v+\beta_v\,\frac{\gamma'(u)}{\gamma(u)}\,(\beta-\alpha)=0\ . \tag{3.18}$$

In virtue of (3.12) and (3.13), the equation (3.18) take the form

$$3\beta_u=\alpha_u$$

or

$$\beta=\frac{1}{3}\left(\alpha+d(v)\right)\ .$$

In this case, from (3.13) we get

$$\begin{aligned}7\,\alpha_u\,\alpha_v-\big(2\,\alpha-d(v)\big)\,\alpha_{uv}-2\,\alpha_u\,d'(v)&=0\quad ,\\ -3\,\alpha_u\,\alpha_v+\big(2\,\alpha-d(v)\big)\,\alpha_{uv}&=0 \qquad (3.19)\end{aligned}$$

or

$$\alpha_u \left(2\,\alpha_v - d'(v)\right) = 0\ .$$

For $\alpha_u \neq 0$, we obtain $\alpha = \frac{1}{2}\,d(v) + e(u)$. Substituting this value of $\alpha$ in (3.19) we find that $\alpha_u = 0$ which is a contradiction. So, only the **Case I** may occur.
Summing up what we found above we obtain the
**Theorem.** If the two families of Laguerre lines of $S_1$ and $S_2$ correspond and if two families of Laguerre lines on $S_1$ form a conjugate semi-Chebyshev net, then $S$ is a general cylinder and $S_1$ is a cone.

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## III.2 Gülçin Çivi and Güler Gürpınar Arsan : *A Function of Direction in a Generalized Weyl Hypersurface*

In [1], the function of direction $\frac{\partial}{\partial s_k} w_{kk} - 2\sum_{h=1}^{n} w_{kh}\,\gamma_{hkk}$ in a hypersurface $V_n$ of the Riemannian surface $V_{n+1}$ is obtained where the symbol $\frac{\partial}{\partial s_k}$ denotes the differentiation in the direction of the vector of components $\lambda^i_k$ and that $w_{kh}$ $(k \neq h)$ and $\gamma_{lhk}$ $(l, h, k = 1, 2, ..., n)$ are, respectively, the invariants of the geodesic torsion of the curve of the congruence with unit tangent vector of components $\lambda^i_k$ and Ricci's coefficients of rotation of the orthogonal ennuple. In this paper, in a hypersurface $\overset{*}{W}_n$ belonging to a generalized Weyl space $\overset{*}{W}_{n+1}$ we choose $n$ congruences of an orthogonal ennuple of the vector field $\underset{r}{v}$ $(r = 1, 2, ..., n)$ which are normalized by the condition $\overset{*}{g}_{ij}\, \underset{r}{v}^i\, \underset{r}{v}^j = 1$ and we prove that the expression

$$\underset{r}{v}^h\,(\nabla^*_h\, \underset{rr}{\kappa^*}) - \underset{r}{\overset{s}{\eta}}\,(\underset{rs}{\kappa^*} + \underset{sr}{\kappa^*}) + \underset{r}{P}\, \underset{rr}{\kappa^*}$$

is a function of direction where $\underset{r}{\overset{s}{\eta}}$ are the geodesic curvatures of the curves belonging to the orthogonal ennuple and that $\underset{rr}{\kappa^*}$ and $\underset{rs}{\kappa^*}$ $(r \neq s)$ are, respectively, the normal curvature of the hypersurface in the direction of the vector of components $\underset{r}{v}^i$ and the invariant of the geodesic torsion of the curve $\underset{r}{C}$of the congruence with tangent vector $\underset{r}{v}^i$.

### INTRODUCTION

An $n$-dimensional differentiable manifold having a symmetric connection $\nabla$ and a symmetric conformal metric tensor $g$ preserved by $\nabla$ is called a Weyl space. In local coordinates we have the compatibility condition

$$\nabla_k\, g_{ij} - 2\,T_k\, g_{ij} = 0\ , \tag{1}$$

where $T_k\,(k = 1, 2, ..., n)$ are the components of the covariant vector field $T$ called the complementary vector field of the Weyl space. Such a Weyl space will be denoted by $W_n(\nabla, g, T)$.
Let $\Gamma^i_{jk}$ denote the coefficients of the connection $\nabla$ . Then, from the compatibility condition given by (1) we get

$$\Gamma^i_{jk} = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} - (\,\delta^i_j\, T_k + \delta^i_k\, T_j - g^{li}\, g_{jk}\, T_l\,)\ . \tag{2}$$

An $n$-dimensional differentiable manifold having an asymmetric connection $\overset{*}{\nabla}$ and an asymmetric conformal metric tensor $g^*$ preserved by $\overset{*}{\nabla}$ is called a generalized Weyl space. In local coordinates, we then have

$$\overset{*}{\nabla}_k\, g^*_{ij} - 2\,T^*_k\, g^*_{ij} = 0\ , \tag{3}$$

where $T^*_k\ (k = 1, 2, ..., n)$ are the components of the covariant vector field $\overset{*}{T}$ called the complementary vector field of the generalized Weyl space which will be denoted by $W^*_n(\overset{*}{\nabla}, g^*, T^*)$. The Weyl space $W_n(\nabla, g, T)$ is called the associate space to the generalized Weyl space $W^*_n(\overset{*}{\nabla}, \overset{*}{g}, T)$, where $g$ is the symmetric part of $\overset{*}{g}$.
The coefficients $L^i_{jk}$ of the connection $\overset{*}{\nabla}$ are obtained from the compatability condition as [2]

$$L^i_{jk} = \Gamma^i_{jk} + \frac{1}{2}\,[\,\Omega^h_{kl}\, g^*_{(jh)} + \Omega^h_{jl}\, g^*_{(hk)} + \Omega^h_{jk}\, g^*_{(hl)}\,]\, g^{*(li)} \tag{4}$$

or, putting

$$Q^i_{jk} = \frac{1}{2}\,[\,\Omega^h_{kl}\, g^*_{(jh)} + \Omega^h_{jl}\, g^*_{(hk)} + \Omega^h_{jk}\, g^*_{(hl)}\,]\, g^{*(li)} \tag{5}$$

we have

$$L^i_{jk} = \Gamma^i_{jk} + Q^i_{jk} \tag{6}$$

where $\Omega^i_{jk} = L^i_{jk} - L^i_{kj}$ are the components of the torsion tensor of the connection $\nabla^*$.
An object $A$ is called a satellite with weight $\{p\}$ of the fundamental tensor $g$, if under a renormalization of $g$ of the form $\widetilde{g} = \lambda^2\, g$ it changes according to the rule $\widetilde{A} = \lambda^p\, A$ [3].

The prolonged covariant derivatives of the satellite $A$ relative to $\nabla$ and $\overset{*}{\nabla}$ denoted respectively by $\dot{\nabla} A$ and $\dot{\overset{*}{\nabla}} A$ are defined by

$$\dot{\nabla}_k A = \nabla_k A - p T_k A \tag{7}$$

and

$$\dot{\overset{*}{\nabla}}_k A = \overset{*}{\nabla}_k A - p T^*_k A \tag{8}$$

where $\nabla_k$ and $\overset{*}{\nabla}_k$ are the operators of usual covariant derivatives [2],[4].

Let $W^*_n(\nabla^*, g^*, T^*)$ be a hypersurface with coordinates $u^i$ $(i = 1, 2, ..., n)$ of a generalized Weyl space $W^*_{n+1}(\nabla^*, g^*, T^*)$ with coordinates $x^a$ $(a = 1, 2, ..., n, n+1)$.

Suppose that the metrics of $\overset{*}{W}_n$ and $\overset{*}{W}_{n+1}$ are elliptic and that they are given, respectively, by $g_{ij}\, du^i\, du^j$ and $g_{ab}\, dx^a\, dx^b$ which are connected by the relation

$$g^*_{ij} = g^*_{ab}\, x^a_i\, x^b_j \quad (i, j = 1, 2, ..., n\ ;\ a, b = 1, 2, ..., n+1)\ , \tag{9}$$

from which it follows that

$$g^*_{(ij)} = g^*_{(ab)}\, x^a_i\, x^b_j \quad , \quad g^*_{[ij]} = g^*_{[ab]}\, x^a_i\, x^b_j \tag{10}$$

where $x^a_i$ denotes the covariant derivative of $x^a$ with respect to $u^i$ and $g^*_{(ij)}$ and $g^*_{[ij]}$ are the symmetric and anti-symmetric parts of $g^*_{ij}$, respectively.

It is easy to see that the prolonged covariant derivative of $A$, relative to $\overset{*}{W}_n$ and $\overset{*}{W}_{n+1}$ are related by

$$\dot{\overset{*}{\nabla}}_k A = x^c_k\, \dot{\overset{*}{\nabla}}_c A\ . \tag{11}$$

Let $n^a$ be the contravariant components of the vector field in $\overset{*}{W}_{n+1}$ normal to $\overset{*}{W}_n$, and let it be normalized by the condition $\overset{*}{g}_{ab}\, n^a\, n^b = 1$. Then, we have,

$$\overset{*}{g}_{(ab)}\, n^a\, n^b = 1\ . \tag{12}$$

The moving frame $\{x^i_a, n_a\}$ on $\overset{*}{W}_n$ reciprocal to the moving frame $\{x^a_i, n^a\}$ is defined by [5]

$$n_a\, n^a = 1 \quad , \quad n_a\, x^a_i = 0 \quad , \quad n^a\, x^i_a = 0 \quad , \quad x^a_i\, x^j_a = \delta^j_i\ . \tag{13}$$

For the covariant derivative of $x^a_i$, using (6) and (8) we get

$$\begin{aligned} \dot{\overset{*}{\nabla}}_k\, x^a_i = \overset{*}{\nabla}_k\, x^a_i &= \frac{\partial x^a_i}{\partial u^k} + L^a_{bc}\, x^b_i\, x^c_k - L^j_{ik}\, x^a_j \\ &= \nabla_k\, x^a_i + Q^a_{bc}\, x^b_i\, x^c_k - Q^j_{ik}\, x^a_j\ . \end{aligned} \tag{14}$$

Separating the tangential and normal parts, we can write

$$\dot{\overset{*}{\nabla}}_k\, x^a_i = A_{ik}\, n^a + B^j_{ik}\, x^a_j \tag{15}$$

which yields, with help of (12) and (13)

$$A_{ik} = \overset{*}{g}_{(ab)}\, (\dot{\overset{*}{\nabla}}_k\, x^a_i)\, n^b \quad , \quad B^j_{ik} = x^j_a\, (\dot{\overset{*}{\nabla}}_k\, x^a_i)\ . \tag{16}$$

If we take the prolonged covariant derivative of (12) relative to $\overset{*}{\nabla}$ we get

$$\overset{*}{g}_{(ab)}\, n^a\, \dot{\overset{*}{\nabla}}_k\, n^b = 0 \tag{17}$$

so that $\dot{\overset{*}{\nabla}}_k\, n^b$ regarded as a vector in $\overset{*}{W}_n$ is orthogonal to $n^a$. Let us express it as

$$\dot{\overset{*}{\nabla}}_k\, n^a = x^a_i\, \psi^i_k\ . \tag{18}$$

## A FUNCTION OF DIRECTION IN $\overset{*}{W}_n$

We choose $n$ congruences of an orthogonal ennuple of the vector fields $\underset{r}{\vec{v}}$ $(r = 1, 2, ..., n)$ in $\overset{*}{W}_n$, normalized by the condition $\overset{*}{g}_{ij}\,\underset{r}{v}^{\,i}\,\underset{r}{v}^{\,j} = 1$ . Then, we have,

$$\overset{*}{g}_{(ij)}\,\underset{r}{v}^{\,i}\,\underset{r}{v}^{\,j} = 1 \ . \tag{19}$$

Let $\underset{r}{v}^{\,a}$ and $\underset{r}{v}^{\,i}$ be respectively, the contravariant components of $\underset{r}{v}$ relative to $\overset{*}{W}_{n+1}$ and $\overset{*}{W}_n$. We then have

$$\underset{r}{v}^{\,a} = x_i^a\,\underset{r}{v}^{\,i} \tag{20}$$

and

$$\overset{*}{g}_{(ab)}\,\underset{r}{v}^{\,a}\,\underset{s}{v}^{\,b} = \delta_s^r \quad , \quad \overset{*}{g}_{(ab)}\,n^a\,\underset{r}{v}^{\,b} = 0 \ . \tag{21}$$

The generalized absolute derivative of the tangent vector field $\underset{r}{v}$ of the curve $\underset{r}{C}$ in $\overset{*}{W}_n$ is defined to be the vector field

$$\underset{r}{v}^{\,k}\,\dot{\overset{*}{\nabla}}_k\,\underset{r}{v}^{\,i} \tag{22}$$

which we called it the geodesic vector field of $\underset{r}{C}$relative to $\overset{*}{W}_n$ . The vector field $\underset{r}{v}^{\,k}\,\dot{\overset{*}{\nabla}}_k\,\underset{r}{v}^{\,i}$ can be written as

$$\underset{r}{v}^{\,k}\,\dot{\overset{*}{\nabla}}_k\,\underset{r}{v}^{\,i} = \underset{r}{\overset{s}{\eta}}\,\underset{s}{v}^{\,i} \ . \tag{23}$$

If we take the absolute derivative of the vector field $\underset{r}{v}$ relative to $\overset{*}{W}_{n+1}$ in the direction of $\underset{p}{v}$ and use (15) and (20) we find that

$$\begin{aligned} \underset{p}{v}^{\,k}\,\dot{\overset{*}{\nabla}}_k\,\underset{r}{v}^{\,a} &= \underset{p}{v}^{\,k}\,\dot{\overset{*}{\nabla}}_k\,(x_i^a\,\underset{r}{v}^{\,i}) \\ &= (\,A_{ik}\,\underset{r}{v}^{\,i}\,\underset{p}{v}^{\,k}\,)\,n^a + (\,B_{ik}^j\,\underset{r}{v}^{\,i}\,\underset{p}{v}^{\,k} + \underset{p}{v}^{\,k}\,\dot{\overset{*}{\nabla}}_k\,\underset{r}{v}^{\,j}\,)\,x_j^a \\ &= \underset{rp}{\kappa^*}\,n^a + (\,B_{ik}^j\,\underset{r}{v}^{\,i}\,\underset{p}{v}^{\,k} + \underset{rp}{a^j}\,)\,x_j^a \end{aligned} \tag{24}$$

and, by (11)

$$\underset{p}{v}^{\,c}\,\dot{\overset{*}{\nabla}}_c\,\underset{r}{v}^{\,a} = \underset{rp}{\kappa^*}\,n^a + (\,B_{ik}^j\,\underset{r}{v}^{\,i}\,\underset{p}{v}^{\,k} + \underset{rp}{a^j}\,)\,x_j^a \qquad (r \neq p) \tag{25}$$

where

$$\underset{p}{v}^{\,k}\,\dot{\overset{*}{\nabla}}_k\,\underset{r}{v}^{\,i} = \underset{rp}{a^i}$$

is the first curvature vector field of $\underset{r}{C}$ relative to $\overset{*}{W}_n$ and the functions

$$\underset{rp}{\kappa^*} = A_{ik}\,\underset{r}{v}^{\,i}\,\underset{p}{v}^{\,k} \tag{26}$$

may be regarded as the invariants of the geodesic torsion of the curve $\underset{r}{C}$belonging to the congruence with tangent vector $\underset{r}{v}$. In particular, taking $r = p$ in (26) we get

$$\underset{rr}{\kappa^*} = A_{ik}\,\underset{r}{v}^{\,i}\,\underset{r}{v}^{\,k} \tag{27}$$

which we call it the normal curvature of $\overset{*}{W}_n$ in the direction of the vector of components $\underset{r}{v}^{\,k}$. On the other hand, taking the absolute derivative of each side of the second equation in (21), in the direction of $\underset{p}{v}$, and using (25) we obtain

$$\overset{*}{g}_{(ab)}\,\left(\underset{p}{v}^{\,k}\,\dot{\overset{*}{\nabla}}_k\,n^a\right)\underset{r}{v}^{\,b} = -\,\underset{rp}{\kappa^*} \tag{28}$$

and, for $r = p$ we have

$$\overset{*}{g}_{(ab)} \left(\underset{r}{v}^{k}\, \dot{\overset{*}{\nabla}}_k\, n^a\right)\underset{r}{v}^{b} = -\underset{rr}{\kappa}^{*} . \tag{29}$$

We are now able to obtain the function of direction which we intended to find. If we take the absolute derivative of $\underset{rr}{\kappa}^{*}$, with weight $\{-1\}$, in direction of $\underset{r}{v}^{h}$, with help of (18), (21), (23), (25) and (26) we obtain

$$\begin{aligned}
-\underset{r}{v}^{h}\, \dot{\overset{*}{\nabla}}_h\, \underset{rr}{\kappa}^{*} &= \underset{r}{v}^{h}\, \dot{\overset{*}{\nabla}}_h \left(\overset{*}{g}_{(ab)} \left(\underset{r}{v}^{k}\, \dot{\overset{*}{\nabla}}_k\, n^a\right) \underset{r}{v}^{b}\right)\\
&= \overset{*}{g}_{(ab)}\, \overset{s}{\underset{r}{\eta}}\, \underset{s}{v}^{k} \left(\dot{\overset{*}{\nabla}}_k\, n^a\right) \underset{r}{v}^{b} + \overset{*}{g}_{(ab)}\, \underset{r}{v}^{k} \left(\underset{r}{v}^{h}\, \dot{\overset{*}{\nabla}}_h \dot{\overset{*}{\nabla}}_k\, n^a\right) \underset{r}{v}^{b}\\
&\quad + \overset{*}{g}_{(ab)}\, \underset{r}{v}^{k} \left(\dot{\overset{*}{\nabla}}_k\, n^a\right) \left[\underset{rr}{\kappa}^{*}\, n^b + \left(B^j_{il}\, \underset{r}{v}^{i}\, \underset{r}{v}^{l} + \overset{s}{\underset{r}{\eta}}\, \underset{s}{v}^{j}\right) x^b_j\right] \qquad (30)\\
&= -\overset{s}{\underset{r}{\eta}} \left(\underset{rs}{\kappa}^{*} + \underset{sr}{\kappa}^{*}\right) + \overset{*}{g}_{(mj)}\, \psi^m_k\, \underset{r}{v}^{k}\, \underset{r}{v}^{i}\, \underset{r}{v}^{l}\, B^j_{il} + \overset{*}{g}_{(ab)}\, \underset{r}{v}^{k}\, \underset{r}{v}^{h}\, \underset{r}{v}^{b}\, \dot{\overset{*}{\nabla}}_h \dot{\overset{*}{\nabla}}_k\, n^a .
\end{aligned}$$

On the other hand, after some calculations, we get

$$\overset{*}{g}_{(ab)}\, \underset{r}{v}^{k}\, \underset{r}{v}^{h}\, \underset{r}{v}^{b}\, \dot{\overset{*}{\nabla}}_h \dot{\overset{*}{\nabla}}_k\, n^a = \overset{*}{g}_{(ab)}\, \underset{r}{v}^{b}\, \underset{r}{v}^{c}\, \underset{r}{v}^{d}\, \dot{\overset{*}{\nabla}}_d \dot{\overset{*}{\nabla}}_c\, n^a + \overset{*}{g}_{(ab)}\, \underset{r}{v}^{b}\, \underset{r}{v}^{h}\, \underset{r}{v}^{j}\, B^i_{jh}\, \psi^m_i\, x^a_m . \tag{31}$$

and also, using (8) we have

$$\underset{r}{v}^{h}\, \dot{\overset{*}{\nabla}}_h\, \underset{rr}{\kappa}^{*} = \underset{r}{v}^{h}\, \overset{*}{\nabla}_h\, \underset{rr}{\kappa}^{*} + \underset{r}{v}^{h}\, \overset{*}{T}_h\, \underset{rr}{\kappa}^{*} . \tag{32}$$

By means of (31) and (32), (30) reduces to

$$\underset{r}{v}^{h}\, \overset{*}{\nabla}_h\, \underset{rr}{\kappa}^{*} - 2\, \overset{s}{\underset{r}{\eta}}\, \underset{(rs)}{\kappa}^{*} + \underset{r}{P}\, \underset{rr}{\kappa}^{*} = \underset{r}{v}^{i}\, \underset{r}{v}^{k}\, \underset{r}{v}^{l} \left[2\, B^j_{il}\, A_{(jk)} - \overset{*}{g}_{(ab)} \left(\dot{\overset{*}{\nabla}}_d \dot{\overset{*}{\nabla}}_c\, n^a\right) x^b_i\, x^c_k\, x^d_l\right] \tag{33}$$

where $\underset{r}{P} = \underset{r}{v}^{h}\, \overset{*}{T}_h$ and $A_{jk} = g^*_{(jl)}\, \psi^l_k$ . From (33) we obtain the

THEOREM: The expression

$$\underset{r}{v}^{h} \left(\overset{*}{\nabla}_h\, \underset{rr}{\kappa}^{*}\right) - 2\, \overset{s}{\underset{r}{\eta}}\, \underset{(rs)}{\kappa}^{*} + \underset{r}{P}\, \underset{rr}{\kappa}^{*}$$

is a function of direction in $\overset{*}{W}_n$ .

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## III.3 Kurala Myrzakul and Ratbay Myrzakulov: *Soliton Equations in Multidimensions as Exact Reductions of the Self-Dual Yang-Mills Equation*

It is shown that some well-known soliton equations in 2+1 dimensions are exact reductions of the Self-Dual Yang-Mills equation or its higher hierarchy. Also the corresponding geometrical interpretation in terms of surfaces are presented.

**1. Introduction** One of the most interesting and important multidimensional integrable equations is the self-dual Yang-Mills (SDYM) equation [11-12]. This four - dimensional equation arises in particular in relatiivity and in field theory. The SDYM equations describe a connection for a bundle over the Grassmannian of two-dimensional subspaces of the twistor space. Integrability for a SDYM connection means that its curvature vanishes on certain two-planes in the tangent space of the Grassmannian. This allows one to characterize SDYM connections in terms of the spliting problem for a transition function in a holomorphic bundle over the Riemann sphere, i.e. the trivialiization of the bundle.

Resently it has been shown that practically all known soliton equations in 1+1 and 2+1 dimensions may be obtained by reductions of the SDYM equation [2-3, 13-14] (see also the book [1] and references therein). On the other hand, it is well known that almost all known integrable equations may be obtained from the some equations of soliton geometry by reductions. These equations are the integrability conditions of the system describing the moving orthogonal trihedral of a curve or surface [4-10,15-17]. For example, in 2+1 dimensions, the role of the such geometrical equations play the (2+1)-dimensional Gauss-Weingarten (GW) and Gauss-Mainardi-Codozzi (GMC) equations. In this note we will show that some well-known (2+1)-dimensional integrable systems are exact reductions of the SDYM equation.

**2. Soliton geometry in $d = 4$ dimensions**. It is well known that there exist several equations describing the 4 - dimensional curves/"surfaces" in n - dimensional space. One of these equations has the form

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}_{\xi_1} = \lambda \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}_{\xi_3} + (A_1 - \lambda A_3) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix} \tag{1a}$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}_{\xi_2} = \lambda \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}_{\xi_4} + (A_2 - \lambda A_4) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}. \tag{1b}$$

Hence as the compatibility condition we obtain the SDYM equation

$$A_{2\xi_1} - A_{1\xi_2} + [A_2, A_1] = 0, \quad A_{4\xi_3} - A_{3\xi_4} + [A_4, A_3] = 0 \tag{2a}$$

$$A_{1\xi_4} - A_{4\xi_1} + [A_1, A_4] = A_{2\xi_3} - A_{3\xi_2} + [A_2, A_3] \tag{2b}$$

or

$$F_{\xi_1\xi_2} = 0, \quad F_{\xi_3\xi_4} = 0, \quad F_{\xi_4\xi_1} - F_{\xi_3\xi_2} = 0, \quad F_{\xi_i\xi_k} = A_{k\xi_i} - A_{i\xi_k} + [A_k, A_i]. \tag{3}$$

The SDYM equation (2) on a connection $A$ are the self-duality conditions on the curvature under the Hodge star operation

$$F = *F \quad (F_{\mu\nu} = \epsilon_{\mu\nu\rho\delta} F^{\rho\delta}) \tag{4}$$

where $*$ is the Hodge operator, $\epsilon_{\mu\nu\rho\delta}$ stands for the completely antisymmetric tensor in four dimensions with the convention: $\epsilon_{1234} = 1$. The SDYM equation is integrable by the Inverse

Scattering Transform (IST) method. The Lax representation (LR) of the SDYM equation has the form [11, 12]

$$\Phi_{\xi_1} - \lambda\Phi_{\xi_3} = (A_1 - \lambda A_3)\Phi, \quad \Phi_{\xi_2} - \lambda\Phi_{\xi_4} = (A_2 - \lambda A_4)\Phi. \tag{5}$$

Usually for the SDYM equation (2) the spectral parameter $\lambda$=constant. But in general it satisfies the following set of nonlinear equations

$$\lambda_{\xi_1} = \lambda\lambda_{\xi_3}, \quad \lambda_{\xi_2} = \lambda\lambda_{\xi_4}. \tag{6}$$

These equations have the following particular solutions

$$\lambda = \frac{n_1\xi_3 + n_3}{n_4 - n_1\xi_1}, \quad \lambda = \frac{m_1\xi_4 + m_3}{m_4 - m_1\xi_2}. \tag{7}$$

While the set (6) its has the following solution

$$\lambda = \frac{n_1\xi_3 + n_3 + m_1\xi_4}{n_4 - n_1\xi_1 - m_1\xi_2} \tag{8}$$

where $m_i, n_i = constants$. As known the corresponding solution of the SDYM equation (2) is called the breaking (overlapping) solutions.

2.1. *The 3-dimensional SDYM equation.* If $A_k$ are independent of $\xi_3$, from (2) we get the 3-dimensional SDYM equation

$$A_{2\xi_1} - A_{1\xi_2} + [A_2, A_1] = 0, \quad -A_{3\xi_4} + [A_4, A_3] = 0 \tag{9a}$$

$$A_{1\xi_4} - A_{4\xi_1} + [A_1, A_4] = -A_{3\xi_2} + [A_2, A_3]. \tag{9b}$$

We note that for the 3-dimensional SDYM equation (9) the corresponding geometrical equation has the form

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}_{\xi_1} = (A_1 - \lambda A_3) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}, \tag{10a}$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}_{\xi_2} = \lambda \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}_{\xi_4} + (A_2 - \lambda A_4) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}. \tag{10b}$$

So that the curve or "surface" corresponding to the equation (10) is the integrable. The LR of the (2+1)-dimensional SDYM equation (9) reads as

$$\Phi_{\xi_1} = (A_1 - \lambda A_3)\Phi, \quad \Phi_{\xi_2} - \lambda\Phi_{\xi_4} = (A_2 - \lambda A_4)\Phi. \tag{11}$$

with the spectral parameter satisfing the equations

$$\lambda_{\xi_1} = 0, \quad \lambda_{\xi_2} = \lambda\lambda_{\xi_4}. \tag{12}$$

**3. The Zakharov equation as the exact reduction of the SDYM equation**. Now let us we consider the gauge condition $A_4 = 0$. Then the 3-dimensional SDYM equation takes the form

$$A_{2\xi_1} - A_{1\xi_2} + [A_2, A_1] = 0, \quad A_{3\xi_4} = 0, \quad A_{1\xi_4} = -A_{3\xi_2} + [A_2, A_3]. \tag{13}$$

Moreover if we take $A_3 = const$ then the 3-dimensional SDYM equation takes the form

$$A_{2\xi_1} - A_{1\xi_2} + [A_2, A_1] = 0, \quad A_{1\xi_4} = [A_2, A_3]. \tag{14}$$

Now we consider the case $n = 3$. And we take the following representations for the connections $A_1, A_2, A_3$

$$A_1 = \begin{pmatrix} 0 & i(\phi - r^2\bar{\phi}) & (\phi + r^2\bar{\phi}) \\ -i(\phi - r^2\bar{\phi}) & 0 & 0 \\ -(\phi + r^2\bar{\phi}) & 0 & 0 \end{pmatrix} \tag{15a}$$

$$A_2 = \begin{pmatrix} 0 & (\phi + r^2\bar{\phi})_y & -i(\phi - r^2\bar{\phi})_y \\ -(\phi + r^2\bar{\phi})_y & 0 & -v \\ i(\phi + r^2\bar{\phi})_y & v & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \tag{15b}$$

and let $\xi_1 = x, \quad \xi_2 = t, \quad \xi_4 = y$. So that in our case the equation (14) takes the form

$$A_{2x} - A_{1t} + [A_2, A_1] = 0, \quad A_{1y} = [A_2, A_3] \tag{16}$$

or in elements

$$i\phi_t = \phi_{xy} + v\phi, \quad v_x = 2r^2\partial_y|\phi|^2. \tag{17}$$

It is the Zakharov equation (ZE) [20]. We note that in this case the corresponding geometrical equation (10) looks like

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = (A_1 - \lambda A_3)\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_t = \lambda\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_y + A_2\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}. \tag{18}$$

Hence we get the following LR of the ZE

$$\Phi_x = (A_1 - \lambda A_3)\Phi, \quad \Phi_t - \lambda\Phi_y = A_2\Phi \tag{19}$$

where $A_i$ are $\mathbf{so(3)}$ or $\mathbf{so(2,1)}$ matrices. It is convenient to use the well known isomorphism $\mathbf{so(3)} \simeq \mathbf{su(2)}$ or $\mathbf{so(2,1)} \simeq \mathbf{su(1,1)}$ and to rewrite the equations (19) in terms of $2\times2$ matrices. As result we have the following standard LR of the ZE (17)

$$\Psi_x = U\Psi, \quad \Psi_t = \lambda\Psi_y + V\Psi, \tag{20}$$

where

$$U = \frac{i\lambda}{2}\sigma_3 + G, \quad G = \begin{pmatrix} 0 & \phi \\ -r^2\bar{\phi} & 0 \end{pmatrix}, \quad V = i\sigma_3(\frac{1}{2}vI + G_y). \tag{21}$$

In addition, in this case for the spectral parameter we have the following information

$$\lambda_x = 0, \quad \lambda_t = \lambda\lambda_y, \quad \lambda = \frac{n_1 y + n_3}{n_4 - n_1 t}. \tag{22}$$

Also we note that in the (1+1)-dimensional case, i.e. when $y = x$ instead of the equations (18) and (17) we obtain respectively the (1+1)-dimensional equation

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = (A_1 - \lambda A_3)\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_t = [\lambda(A_1 - \lambda A_3) + A_2]\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \tag{23}$$

and the nonlinear Schrödinger equation

$$i\phi_t = \phi_{xx} + 2r^2|\phi|^2\phi = 0. \tag{24}$$

At last we note that if $n > 3$ then we get the n-component generalisation of the Zakharov equation

$$i\phi_{jt} = \phi_{jxy} + v\phi_j = 0, \quad v_x = 2r^2(\sum_{k=1}^{n}|\phi_k|^2)_y. \tag{25}$$

**4. Integrable spin systems and the SDYM equation**. Integrable spin systems in 2+1 dimensions also can be considered as exact reductions of the SDYM equation. As example we show that the M-I equation is the reduction of the SDYM equation. To this end, we consider the gauge condition $A_1 = A_2 = 0$. In this case the 3-dimensional SDYM equation takes the form

$$-A_y + [A_4, A_3] = 0, \quad -A_{4x} = -A_{3t}. \tag{26}$$

Let the connections $A_4, A_3$ have the forms

$$A_3 = \begin{pmatrix} 0 & rS_1 & -irS_2 \\ -rS_1 & 0 & S_3 \\ irS_2 & -S_3 & 0 \end{pmatrix}, \qquad A_4 = \left(A_4^{(1)}\ ,\ A_4^{(2)}\ ,\ A_4^{(3)}\right), \tag{27a}$$

where

$$A_4^{(1)} := \begin{pmatrix} 0 \\ ir[2iS_3S_{2y} - 2iS_2S_{3y} + iuS_1] \\ r[2S_3S_{1y} - 2S_1S_{3y} - uS_2 \end{pmatrix}, \tag{27b}$$

$$A_4^{(2)} := \begin{pmatrix} -ir[2iS_3S_{2y} - 2iS_2S_{3y} + iuS_1] \\ 0 \\ ir^2(S^+S_y^- - S^-S_y^+) - uS_3 \end{pmatrix}, \tag{27c}$$

$$A_4^{(2)} := \begin{pmatrix} -r[2S_3S_{1y} - 2S_1S_{3y} - uS_2] \\ -[ir^2(S^+S_y^- - S^-S_y^+) - uS_3] \\ 0 \end{pmatrix}. \tag{27d}$$

Substituting (27a–d) into (26) we get the M-I equation [18]

$$iS_t = ([S, S_y] + 2iuS)_x, \quad u_x = -\frac{1}{2i}tr(S[S_x, S_y]), \tag{28a}$$

$$S = \begin{pmatrix} S_3 & S^- \\ S^+ & -S_3 \end{pmatrix}, \quad S^\pm = S_1 \pm iS_2. \tag{28b}$$

For the M-I equation the corresponding geometrical equation (10) looks like

$$\begin{pmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{pmatrix}_x = -\lambda A_3 \begin{pmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{pmatrix} \quad \begin{pmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{pmatrix}_t = \lambda \begin{pmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{pmatrix}_y - \lambda A_4 \begin{pmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{pmatrix}. \tag{29}$$

The LR of the M-I equation follows from the LR of the SDYM equation and has the form

$$\Phi_x' = -\lambda A_3 \Phi', \quad \Phi_t' - \lambda \Phi_y' = -\lambda A_4 \Phi'. \tag{30}$$

As for the ZE, we can rewrite the LR of the M-I equation in the standart form, in terms of 2×2 - matrices

$$\Psi_x' = U'\Psi', \quad \Psi_t' = \lambda\Psi_y' + V'\Psi', \quad U' = \frac{i\lambda}{2}S, \quad V' = \frac{\lambda}{4}([S, S_y] + 2iuS). \tag{31}$$

In the 1+1 dimensions ($y = x$) instead of the equations (29) and (28), we get respectively the (1+1)-dimensional equation

$$\begin{pmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{pmatrix}_x = -\lambda A_3 \begin{pmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{pmatrix}_t = -(\lambda^2 A_3 + \lambda A_4) \begin{pmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{pmatrix} \tag{32}$$

and the Landau-Lifshitz equation $iS_t = \frac{1}{2}[S, S_{xx}]$.

**5. Gauge equivalence**. It is well known that the ZE (17) and the M-I equation (28) are G-equivalent and L-equivalent to each other [22]. In our case this fact realize by the transformation

$$\begin{pmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{pmatrix} = G \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \tag{33}$$

or $\Phi' = h^{-1}\Phi, \quad (\Psi' = h^{-1}\Psi)$ where $h$ is the solution of the equations (19) or (20) as $\lambda = 0$.

**6. Hierarchy of the M-LXVIII and Self-Dual Yang-Mills equations**. The higher hierarchy of the geometrical generating equation (1) we can write in the form

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}_{\xi_1} = \sum_{i=0}^{k_1} A_i\lambda^i \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}_{\xi_3} + \sum_{i=0}^{k_2} B_i\lambda^i \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix} \tag{34$a$}$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}_{\xi_2} = \sum_{i=0}^{k_3} C_i\lambda^i \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}_{\xi_4} + \sum_{i=0}^{k_4} D_i\lambda^i \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}. \tag{34$b$}$$

The compatibility condition of these equations yields the higher hierarchy SDYM equation. As example, we consider the 3-dimensional case, work the notation (16) and $n = 3$. Then instead of (10) we obtain the M-LXVII equation in the form

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = \sum_{i=0}^{k_2} B_i\lambda^i \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \tag{35$a$}$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_t = \sum_{i=0}^{k_3} C_i\lambda^i \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_y + \sum_{i=0}^{k_4} D_i\lambda^i \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}. \tag{35$b$}$$

It is remarkable that using the equations (35a–b) we can show that some known (2+1)-dimensional soliton equations are exact reductions of the higher hierarchy of the SDYM equation. Here we present some of them.

*i) The (2+1)-dimensional mKdV equation.* Let us now in the equation (35) we assume that $k_2 = 1, k_3 = 2, k_4 = 2, C_2 = 1, \quad C_1 = C_0 = 0$. Thus in this case the geometrical equation takes the form

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = (A_1 - \lambda A_3) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & -(q+p) & i(q-p) \\ q+p & 0 & 0 \\ -i(q-p) & 0 & 0 \end{pmatrix} \tag{36$a$}$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_t = \lambda^2 \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_y + (D_2\lambda^2 + D_1\lambda + D_0) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \tag{36$b$}$$

where $A_3, D_k$ are some matrices. Then the complex functions $q, p$ satisfy the (2+1)-dimensional complex mKdV equation

$$q_t + q_{xxy} - (qv_1)_x - qv_2 = 0, \quad v_{1x} = 2E(\bar{q}q)_y, \quad v_{2x} = 2E(\bar{q}q_{xy} - \bar{q}_{xy}q). \tag{37}$$

If $p = q$ is real, we get the following (2+1) - dimensional mKdV equation

$$q_t + q_{xxy} - (qv_1)_x = 0, \quad v_{1x} = 2E(q^2)_y = 4Eqq_y. \tag{38}$$

*ii) The (2+1)-dimensional derivative NLSE.* In (35) now we put $k_2 = k_3 = k_4 = 2; C_2 = 2c, C_1 = C_0 = 0, \quad B_2 = -cA_3, \quad B_2 = 2cA_1$. So that the equation (35) takes the form

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = (A_3\lambda^2 + A_1\lambda) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \tag{39a}$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_t = \lambda^2 \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_y + (D_2\lambda^2 + D_1\lambda + D_0) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \tag{39b}$$

where $A_1$ is given by (36). Then the complex functions $q, p$ satisfy the (2+1)-dimensional derivative NLSE

$$iq_t = q_{xy} - 2ic(vq)_x, \quad -ip_t = p_{xy} + 2ic(vq)_x, \quad v_x = 2(pq)_y \tag{40}$$

which is the Strachan equation [19].
*iii) The M-III$_q$ equation.* Now we consider the case

$$k_2 = k_3 = k_4 = 2, \quad C_2 = 2cI, \quad C_1 = 2dI, \quad C_0 = 0, \tag{41a}$$

$$B_2 = -cA_3, \quad B_1 = -dA_3 + 2cA_1, \quad B_0 = dA_1, \tag{41b}$$

for which the geometrical equation (34) has the form

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = [A_3(c\lambda^2 + d\lambda) + A_1(2c\lambda + d)] \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \tag{42a}$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_t = 2(c\lambda^2 + d\lambda) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_y + (D_2\lambda^2 + D_1\lambda + D_0) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \tag{42b}$$

where $A_1$ is given by (36). Then the complex functions $q, p$ satisfy the (2+1)-dimensional M-III$_q$ equation

$$iq_t = q_{xy} - 2ic(vq)_x + d^2vq, \quad -ip_t = p_{xy} + 2ic(vq)_x + d^2vp, \quad v_x = 2(pq)_y. \tag{43}$$

The M-III$_q$ equation (43) admits two integrable reductions: the Strachan equation (40) as $d = 0$ and the ZE (17) as $c = 0$. If we rewrite the equations (42) in terms of $2 \times 2$ - matrices, we get the following LR of the M-III$_q$ equation

$$\Psi_x = U\Psi, \quad \Psi_t = 2(c\lambda^2 + d\lambda)\Psi_y + V\Psi \tag{44}$$

where

$$U = i[(c\lambda^2 + d\lambda)\sigma_3 + (2c\lambda + d)G], \quad G = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}, \quad V = B_2\lambda^2 + B_1\lambda + B_0 \tag{45}$$

*iv) The M-XXII$_q$ equation.* Now let $k_2 = k_3 = k_4 = 2. B_2 = A_3, B_1 = A_1, B_0 = \frac{pq}{4}A_3, C_2 = 2I, C_1 = C_0 = 0$. Then the functions $q, p$ satisfy the following $M - XXII_q$ equations ($p = r^2\bar{q}$)

$$iq_t + q_{yx} + \frac{i}{2}[(v_1q)_x - v_2q - qpq_y] = 0, \quad v_{1x} = (pq)_y, \quad v_{2x} = p_{yx}q - pq_{yx}. \tag{46}$$

This set of equations is the G- and L-equivalent counterpart of the M-XXII$_s$ equation (spin system) [22]. The LR of this equation has the form

$$\Psi_{2x} = \{-i(\lambda^2 - \frac{pq}{4})\sigma_3 + \lambda Q\}\Psi_2, \quad \Psi_{2t} = 2\lambda^2\Psi_{2y} + (\lambda^2 B_2 + \lambda B_1 + B_0)\Psi_2 \tag{47}$$

with

$$Q = \begin{pmatrix} 0 & q \\ -p & 0 \end{pmatrix}, B_2 = \frac{i}{2}v_1\sigma_3, \quad B_1 = i\sigma_3 Q_y - \frac{1}{2}v_1 Q, \quad B_0 = \frac{1}{4}v_2 - \frac{i}{8}pqv_1. \tag{48}$$

Now let us consider the following transformation

$$q' = q\exp[-\frac{i}{2}\partial_x^{-1}(pq)], \quad p' = p\exp[\frac{i}{2}\partial_x^{-1}(pq)]. \tag{49}$$

Then the new variables $p', q'$ satisfy the Strachan equation [19]

$$iq'_t + q'_{xy} + i(v'q')_x = 0, \quad ip' - p'_{xy} + i(v'p')_x = 0, \quad v'_x = E(p'q')_y. \tag{50}$$

We see that the M-XXII$_q$ equation (46) and the Strachan equation (40) is gauge eqivalent to each other. The tranformation (49) induces the following tranformation of the Jost function$\Psi_1$ $\Psi_1 = f^{-1}\Psi_2$, where $\Psi_1$ is the solution of the equations (44) as $d = 0$ and $f = \exp(-\frac{i}{4}\partial_x^{-1} \mid q \mid^2 \sigma_3) = \Psi_1^{-1} \mid_{\lambda=0}$ . Then the new Jost function $\Psi_2$ satisfies the following set of equations

$$\Psi_{2x} = \{-i\lambda^2\sigma_3 + \lambda Q'\}\Psi_2, \quad \Psi_{2t} = 2\lambda^2\Psi_{2y} + \{\lambda^2 B'_2 + \lambda B'_1 + B'_0)\}\Psi_2, \tag{51a}$$

$$Q = \begin{pmatrix} 0 & q' \\ -p' & 0 \end{pmatrix} \tag{51b}$$

and $B'_j$ are some matrices. Note that the Strachan (40) and (46) equations are the simplest (2+1)-dimensional extensions of the following known NLSE

$$iq_t + q_{xx} + i(pq^2)_x = 0, \quad ip_t - p_{xx} + i(qp^2)_x = 0 \tag{52}$$

$$iq_t + q_{xx} + ipqq_x = 0, \quad ip_t - p_{xx} + ipqp_x = 0 \tag{53}$$

respectively. It is well known that these equations are gauge equivalent to each other [10].

**8. On the soliton geometry in $d = 3$ dimensions**. Soliton geometry in $d = 3$ dimensions is one of actively developing and still so for from completed stady branches of mathemaics and may be theoretical physics. No having (for today) possibility to explore this problem in whole, in next we consider one of the particular example, namely, the so-called M-LXI equation. This equation in $d = 3$ dimensions has the form

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}_{\xi_1} = A \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}_{\xi_2} = B \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}, \tag{54a}$$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}_{\xi_3} = C \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}. \tag{54b}$$

The compatibility condition of the equations (54) gives the 3-dimensional equation

$$A_{\xi_2} - B_{\xi_1} + [A, B] = 0, \quad A_{\xi_3} - C_{\xi_1} + [A, C] = 0, \quad C_{\xi_2} - B_{\xi_3} + [C, B] = 0. \tag{55}$$

This equation is the particular case of the Bogomolny equation (BE) [1]

$$\Psi_{\xi_3}+[\Psi,C]+A_{\xi_2}-B_{\xi_1}+[A,B]=0,\quad \Psi_{\xi_2}+[\Psi,B]+A_{\xi_3}-C_{\xi_1}+[A,C]=0$$
$$\Psi_{\xi_1}+[\Psi,A]+C_{\xi_2}-B_{\xi_3}+[C,B]=0. \tag{56}$$

In fact, frome hence as $\Psi=0$ we obtain the M-LXII equation (55). As well known that the BE (56) is integrable (see, e.g. the book [1]). As the particular case of the integrable BE (56), the equation (55) is also integrable. The corresponding LR has the form

$$\Phi_{\xi_1}-\lambda\Phi_{\xi_3}=[-iC-\lambda(A+iB)]\Phi,\quad \Phi_{\xi_2}-\lambda\Phi_{\xi_4}=[A-iB-\lambda iC]\Phi. \tag{57}$$

Let us consider the case $n=3,\xi_1=x,\xi_2=y,\xi_3=t$. Then the (54) and (55) equations, respectively, take the form

$$\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}_x=A\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix},\quad \begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}_y=B\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix},\quad \begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}_t=C\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix} \tag{58}$$

and

$$A_y-B_x+[A,B]=0,\quad A_t-C_x+[A,C]=0,\quad C_y-B_t+[C,B]=0 \tag{59a}$$

$$A=\begin{pmatrix}0 & k & -\sigma\\ -\beta k & 0 & \tau\\ \beta\sigma & -\tau & 0\end{pmatrix},\quad B=\begin{pmatrix}0 & m_3 & -m_2\\ -\beta m_3 & 0 & m_1\\ \beta m_2 & -m_1 & 0\end{pmatrix}, \tag{59b}$$

$$D=\begin{pmatrix}0 & \omega_3 & -\omega_2\\ -\beta\omega_3 & 0 & \omega_1\\ \beta\omega_2 & -\omega_1 & 0\end{pmatrix}. \tag{59a}$$

The equation (55) contents many (and perhaps all?) known soliton equations (see, e.g. the refs.[15-17, 21-22]). Some examples as follows.
*1) The Ishimori and DS equations.* Let us we show that the Ishimori (IE) and DS equations can be obtain from the (58) and (59) ($\sigma=0$) equations as some exact reductions. The IE reads as

$$\mathbf{S}_t=\mathbf{S}\wedge(\mathbf{S}_{xx}+\alpha^2\mathbf{S}_{yy})+u_x\mathbf{S}_y+_y\mathbf{S}_x,\quad u_{xx}-\alpha^2u_{yy}=-2\alpha^2\mathbf{S}\cdot(\mathbf{S}_x\wedge\mathbf{S}_y). \tag{60}$$

We take the following identification $\mathbf{S}=\mathbf{e}_1$. In this case we have

$$m_1=\partial_x^{-1}[\tau_y-\frac{\beta}{2\alpha^2}M_2^{Ish}u],\quad m_2=-\frac{1}{2\alpha^2k}M_2^{Ish}u, \tag{61a}$$

$$m_3=\partial_x^{-1}[k_y+\frac{\tau}{2\alpha^2k}M_2^{Ish}u] \tag{61b}$$

$$M_2^{IE}u=u_{xx}-\alpha^2u_{yy},\quad \omega_1=\frac{1}{k}[-\omega_{2x}+\tau\omega_3], \tag{62a}$$

$$\omega_2=-k_x-\alpha^2(m_{3y}+m_2m_1)+im_2u_x,$$
$$\omega_3=-k\tau+\alpha^2(m_{2y}-m_3m_1)+iku_y+im_3u_x. \tag{62b}$$

Now let us introduce two complex functions $q,p$, according to the formulae

$$q=a_1e^{ib_1},\quad p=a_2e^{ib_2}. \tag{63}$$

Let $a_j,b_j$ have the forms

$$\begin{aligned} a_1^2 &= \frac{1}{4}k^2+\frac{|\alpha|^2}{4}(m_3^2+m_2^2)-\frac{1}{2}\alpha_Rkm_3-\frac{1}{2}\alpha_Ikm_2,b_1\\ &= \partial_x^{-1}\{-\frac{\gamma_1}{2ia_1^2}-(\bar{A}-A+D-\bar{D})\} \end{aligned} \tag{64a}$$

$$a_2^2 = \frac{1}{4}k^2 + \frac{|\alpha|^2}{4}(m_3^2 + m_2^2) + \frac{1}{2}\alpha_R k m_3 - \frac{1}{2}\alpha_I k m_2,$$
$$b_2 = \partial_x^{-1}\{-\frac{\gamma_2}{2ia_2^2} - (A - \bar{A} + \bar{D} - D)\} \tag{64b}$$

where

$$\gamma_1 = i\{\frac{1}{2}k^2\tau + \frac{|\alpha|^2}{2}(m_3 k m_1 + m_2 k_y) -$$
$$\frac{1}{2}\alpha_R(k^2 m_1 + m_3 k\tau + m_2 k_x) + \frac{1}{2}\alpha_I[k(2k_y - m_{3x}) - k_x m_3]\}. \tag{65a}$$
$$\gamma_2 = -i\{\frac{1}{2}k^2\tau + \frac{|\alpha|^2}{2}(m_3 k m_1 + m_2 k_y) +$$
$$\frac{1}{2}\alpha_R(k^2 m_1 + m_3 k\tau + m_2 k_x) + \frac{1}{2}\alpha_I[k(2k_y - m_{3x}) - k_x m_3]\}. \tag{65b}$$

Here $\alpha = \alpha_R + i\alpha_I$. In this case, $q, p$ satisfy the following DS equation

$$iq_t + q_{xx} + \alpha^2 q_{yy} + vq = 0, \quad -ip_t + p_{xx} + \alpha^2 p_{yy} + vp = 0,$$
$$v_{xx} - \alpha^2 v_{yy} + 2[(pq)_{xx} + \alpha^2 (pq)_{yy}] = 0. \tag{66}$$

It is means that the IE (60) and the DS equation (66) are L-equivalent [10] to each other. As well known that these equations are G-equivalent to each other. A few comments are in order. i) From these results, we get the Ishimori I and DS-I equations as $\alpha_R = 1, \alpha_I = 0$; ii) the Ishimori II and DS-II equations as $\alpha_R = 0, \alpha_I = 1$; iii) For DS-II equation we have $pq =| q |^2 =| p |^2$; iv) at the same time, for the DS-I equation we obtain

$$pq \neq | q |^2 \neq | p |^2, \quad | q |^2 = | p |^2 - km_3, \quad pq = (pq)_R + i(pq)_I \tag{67}$$

so that $pq$ is the complex quantity.

*2) The KP and M-X equations.* The (2+1)-dimensional equation (54) in plane has the form

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}_x = A_p \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}_y = B_p \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}_t = D_p \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} \tag{68}$$

where already

$$A_p = \begin{pmatrix} 0 & k \\ -\beta k & 0 \end{pmatrix}, \quad B_p = \begin{pmatrix} 0 & m_3 \\ -\beta m_3 & 0 \end{pmatrix}, \quad D_p = \begin{pmatrix} 0 & \omega_3 \\ -\beta\omega_3 & 0 \end{pmatrix}. \tag{69}$$

In the plane case the equation (55) takes the following simple form

$$k_y = m_{3x}, \quad k_t = \omega_{3x}, \quad m_{3t} = \omega_{3y}. \tag{70}$$

Hence we get $m_3 = \partial_x^{-1} k_y$. Many (2+1)-dimensional integrable equations such as the Kadomtsev-Petviashvili, Novikov-Veselov (NV), mNV, KNV, (2+1)-KdV, mKdV equations are the integrable reductions of the equation (55). For example, let us show that the KP and mKP equations are exact reductions of the equation (55). Consider the M-X equation

$$\mathbf{S}_t = \frac{\omega_3}{k}\mathbf{S}_x, \quad \omega_3 = -k_{xx} - 3k^2 - -3\alpha^2\partial_x^{-1} m_{3y}. \tag{71}$$

As above if we put $\mathbf{S} = \mathbf{e}_1$ then from (70) we obtain the L-equivalent counterpart of the M-X equation which is the KP equation

$$k_t + 6kk_x + k_{xxx} + 3\alpha^2 m_{3y} = 0, \quad m_{3x} = k_y. \tag{72}$$

As known the LR of this equation is given by

$$\alpha\psi_y + \psi_{xx} + k\psi = 0, \quad \psi_t + 4\psi_{xxx} + 6k\psi_x + 3(k_x - \alpha m_3)\psi = 0. \tag{73}$$

*3) The Zakharov and M-IX equations.* Now we find the connection between the M-IX equation and the curves. The M-IX equation reads as

$$\mathbf{S}_t = \mathbf{S} \wedge M_1 \mathbf{S} + A_2 \mathbf{S}_x + A_1 \mathbf{S}_y, \quad M_2 u = 2\alpha^2 \mathbf{S}(\mathbf{S}_x \wedge \mathbf{S}_y) \tag{74}$$

where $\alpha, b, a=$ consts and

$$M_1 = \alpha^2 \frac{\partial^2}{\partial y^2} + 4\alpha(b-a)\frac{\partial^2}{\partial x \partial y} + 4(a^2 - 2ab - b)\frac{\partial^2}{\partial x^2}, \quad M_2 = \alpha^2 \frac{\partial^2}{\partial y^2} -$$

$$2\alpha(2a+1)\frac{\partial^2}{\partial x \partial y} + 4a(a+1)\frac{\partial^2}{\partial x^2}, \quad A_1 = i\{\alpha(2b+1)u_y - 2(2ab+a+b)u_x\},$$

$$A_2 = i\{4\alpha^{-1}(2a^2 b + a^2 + 2ab + b)u_x - 2(2ab+a+b)u_y\}. \tag{75}$$

The M-IX equation is integrable. It admits several integrable reductions: 1) the Ishimori equation as $a = b = -\frac{1}{2}$; 2) the M-VIII equation as $a = b = -1$; 3) the M-XXXIV equation $a = b = -1, x = t$; and so on. In this case we have

$$m_1 = \partial_x^{-1}[\tau_y - \frac{\beta}{2\alpha^2} M_2 u], \quad m_2 = -\frac{1}{2\alpha^2 k} M_2 u, \quad m_3 = \partial_x^{-1}[k_y + \frac{\tau}{2\alpha^2 k} M_2 u] \tag{76}$$

and

$$\omega_1 = \frac{1}{k}[-\omega_{2x} + \tau\omega_3], \tag{77a}$$

$$\omega_2 = -4(a^2 - 2ab - b)k_x - 4\alpha(b-a)k_y - \alpha^2(m_{3y} + m_2 m_1) + m_2 A_1, \tag{77b}$$

$$\omega_3 = -4(a^2 - 2ab - b)k\tau - 4\alpha(b-a)km_1 + \alpha^2(m_{2y} - m_3 m_1) + kA_2 + m_3 A_1. \tag{77c}$$

Functions $q, p$ are given by (63) with

$$a_1^2 = \frac{|a|^2}{|b|^2} a_1'^2 = \frac{|a|^2}{|b|^2}\{(l+1)^2 k^2 + \frac{|\alpha|^2}{4}(m_3^2 + m_2^2) - (l+1)\alpha_R k m_3 - (l+1)\alpha_I k m_2\} \tag{78a}$$

$$b_k = \partial_x^{-1}\{-\frac{\gamma_k}{2ia_k'^2} - (\bar{A} - A + D - \bar{D})\} \tag{78b}$$

$$a_2^2 = \frac{|b|^2}{|a|^2} a_2'^2 = \frac{|b|^2}{|a|^2}\{l^2 k^2 + \frac{|\alpha|^2}{4}(m_3^2 + m_2^2) - l\alpha_R k m_3 + l\alpha_I k m_2\} \tag{78c}$$

$$\gamma_1 = i\{2(l+1)^2 k^2 \tau + \frac{|\alpha|^2}{2}(m_3 k m_1 + m_2 k_y) - (l+1)\alpha_R[k^2 m_1 + m_3 k\tau + m_2 k_x] +$$

$$(l+1)\alpha_I[k(2k_y - m_{3x}) - k_x m_3]\}, \quad \gamma_2 = -i\{2l^2 k^2 \tau + \frac{|\alpha|^2}{2}(m_3 k m_1 + m_2 k_y) -$$

$$l\alpha_R(k^2 m_1 + m_3 k\tau + m_2 k_x) - l\alpha_I[k(2k_y - m_{3x}) - k_x m_3]\}. \tag{79}$$

Here $\alpha = \alpha_R + i\alpha_I$. In this case, $q, p$ satisfy the following Zakharov equation

$$iq_t + M_1 q + vq = 0, \quad ip_t - M_1 p - vp = 0, \quad M_2 v = -2M_1(pq) \tag{80}$$

which is integrable and admits several particular cases. As well known the M-IX equation admits several reductions: 1) the M-IXA equation as $\alpha_R = 1, \alpha_I = 0$; 2) the M-IXB equation as $\alpha_R = 0, \alpha_I = 1$; 3) the M-VIII equation as $a = b = 1$; 4) the IE $a = b = -\frac{1}{2}$ and so on.

**8. The Chern-Simons theory**. At last, we wish discuss the connection of some above mentioned equations with the Chern-Simons (CS) theory in 2+1 dimensions. To this purpose, we consider the action of the CS theory [21]

$$S[J] = \frac{k}{4\pi}\int_M tr(J \wedge dJ + \frac{2}{3} J \wedge J \wedge J) \tag{81}$$

where $J$ is a 1-form gauge connection with values in the Lie algebra $\hat{g}$ of a (compact or noncompact) non-Abelian simple Lie group $\hat{G}$ on an oriented closed 3-dimensional manifold $M$, $k$ is the coupling constant. The classical equation of motion is the zero-curvature condition

$$dJ + J \wedge J = 0, \quad J = Adx + Bdy + Cdt. \tag{82}$$

As shown in [21], hence we obtain the equation (59). Note that from this fact follows that the CS equation of motion (82) is exact reduction of the SDYM equation (2).

**9. Conclusion**. In this paper we analyzed the integrable systems connected with the geometrical equation (1). We have found the some integrable reductions of this equation. Also we have shown that the Zakharov and its spin counterpart the M-I equation are exact reductions of the SDYM equation. The higher hierarchy of the SDYM equation was introduced. Using this hierarchy it was shown that several simplest soliton equations in 2+1 dimensions such as mKdV, derivative NLS and M-$\mathrm{III}_q$ equations and so on, are its exact reductions.

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## III.4 Zh.Kh.Zhunussova, Kur.Myrzakul, R.N.Syzdykova, and R.Myrzakulov: *On the Soliton Geometry in Multidimensions*

Integrable reductions of curvilinear coordinate systems are considered. Some aspects of soliton immersions in multidimensions are discussed.

**1. Introduction**. The purpose of this note is to consider the relationship of some multidimensional spaces and some nonlinear partial differential equations (NPDE). The study of the deep relationship between differential geometry and NPDEs has a long and distinguished history, going back to the works of Darboux, Lie, Backlund, Goursat and E.Cartan. This relationship stems from the fact that most of the local properties of manifolds are expressed naturally in terms of NPDE [1-17].

There are several reasons for our interest in the study of the relationship between Riemann space and integrable NPDEs in (2+1) dimensions (see, e.g., [22-26,39]). One reason has been motivated by revealed connections between corresponding problems of classical differential geometry and modern problems of mathematical and theoretical physics. A more specific reason for interest is that the (1+1)-dimensional counterparts of the known integrable NPDE in (2+1) dimensions, such as the Kadomtsev-Petviashvili (KP) equation, its some modifications, the Davey-Stewartson (DS) equation, the three-dimensional three-wave resonant interaction system, and others, have been shown to have natural geometrical interpretations [1-13]. Hence it seems to us that the the study of the geometrical nature of multidimensional soliton equations is the very actual problem at present.

The classical differential geometry serves as an injector of many equations integrable in this or that sense [3-4]. Among them, integrable spin systems and their relativictic counterparts - sigma models play especially remarkable role [18,22-26,39]. It turns out that one of the central problems of the differential geometry: the problem of construction of $n$ - curvilinear coordinate systems (CCS) [19-20] is deeply connected with the theory of integrable NPDE, in particealer, with spin systems and sigma models [8]. In this note we shall consider some aspects of these problems. The details of the calculations will appear elsewhere.

**2. Flat Riemann space**. Let $V^n$ be the space endowed with the affine connection. In this space we introduce two systems of coordinates: $(x) = (x^1, x^2, ..., x^n)$ and $(y) = (y^1, y^2, ..., y^n)$. It is well known from the classical differential geometry that these coordinate systems are connected by the following set of equations of the second order [16]

$$\frac{\partial^2 y^k}{\partial x^i \partial x^j} = \Gamma^l_{ij}(x)\frac{\partial y^k}{\partial x^l} - \Gamma^k_{lm}(y)\frac{\partial y^l \partial y^m}{\partial x^i \partial x^j}. \tag{1}$$

In this paper we assume Einstein summation convention, i.e., the repeated indices are summed up. The curvature tensor has the form

$$R^i_{klm} = [\frac{\partial \Gamma_m}{\partial x^l} - \frac{\partial \Gamma_l}{\partial x^n} + \Gamma_l\Gamma_m - \Gamma_m\Gamma_l]^i_k. \tag{2}$$

Let the space $V^n$ to be flat and let the metric tensor in $V^n$ in the coordinate system $(y)$ to be diagonal one:

$$ds^2 = \mu_{ij}dy^i dy^j \tag{3}$$

with $\mu_{ij} = \pm 1$. Hence it follows that the corresponding connection and Riemann's curvature tensor are equal to zero:

$$\Gamma^i_{jk}(y) = 0, \quad R^i_{klm}(y) = 0. \tag{4}$$

Let for the coordinate system $(x)$ the metric has the form

$$ds^2 = g_{ij}dx_i dx_j. \tag{5}$$

As the curvature tensor has the law of transformation as the four rank tensor

$$R^i_{klm}(x) = \frac{\partial x^i}{\partial y^s}\frac{\partial y^r}{\partial x^k}\frac{\partial y^q}{\partial x^l}\frac{\partial y^p}{\partial x^n}R^i_{klm}(y) = 0, \quad R = \sum_{i,k,l,m=1}^{3} g^{il}g^{km}R_{iklm} = 0. \tag{6}$$

Hence,for the coordinate system $(x)$ we get the following system of three equations

$$\frac{\partial\Gamma_m}{\partial x^l} - \frac{\partial\Gamma_l}{\partial x^n} + \Gamma_l\Gamma_m - \Gamma_m\Gamma_l = 0, \quad \Gamma^k_{ij} = \frac{1}{2}g^{kl}(g_{il,j} + g_{jl,i} - g_{ij,l}). \tag{7}$$

Equations (7) are the compatibility condition of the following overdetermined system of the linear equations [21,33,38,39]

$$\Phi_{,k} = \Gamma_k\Phi. \tag{8}$$

Now the system (1) takes the form

$$\frac{\partial^2 y^k}{\partial x^i\partial x^j} = \Gamma^l_{ij}(x)\frac{\partial y^k}{\partial x^l}. \tag{9}$$

Let $\mathbf{r} = (y^1, y^2, ..., y^n) = \mathbf{r}(x^1, x^2, ..., x^n)$ is the position vector. Then, as it follows from (9), the position vector $,\mathbf{r}$, satisfies the following set of equations

$$\mathbf{r}_{,ij} = \Gamma^k_{ij}\mathbf{r}_{,k}. \tag{10}$$

Now we introduce the unit vectors (basis) by the way

$$\mathbf{e}_1 = \frac{\mathbf{r}_{,1}}{H_1}, \quad \mathbf{e}_l = \frac{\mathbf{r}_{,l}}{H_l} + \sum_{k=1,k\neq l}^{n} c_{kl}\mathbf{r}_{,k}. \tag{11}$$

Here $H_k =\mid \mathbf{r}_{,k} \mid, \quad \mathbf{e}_1^2 = \beta = \pm 1, \quad \mathbf{e}^2_{k\neq 1} = 1$. The equations for $\mathbf{e}_k$ take the form

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \cdots \\ \mathbf{e}_n \end{pmatrix}_{,k} = B_k \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \cdots \\ \mathbf{e}_n \end{pmatrix}, \quad B_k = (b^k_{ij}), \quad B_k \in so(p,q), \quad p+q = n. \tag{12}$$

The integrability condition of the system is

$$\frac{\partial B_i}{\partial x^j} - \frac{\partial B_j}{\partial x^i} + [B_i, B_j] = 0. \tag{13}$$

Now we shall consider the particular cases and present the integrable reductions of some one.

**3. The n=3 case**. Some integrable examples of 3-CCS connected with the (2+1)-dimensional soliton equations were considered in [38].

**4. The n=4 case**. Now we wish to consider the physically interesting case when $n = 4$. In this case the equation (12) takes the form

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \end{pmatrix}_{,k} = B_k \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \end{pmatrix} \tag{14}$$

where the matrices $B_k$ belong already to the Lie algebra $so(p,q)$, i.e., $B_k \in so(p,q), \quad p+q = 4$. Hence $B_j$ satisfies the equation (13).
It is usually convenient to reformulate equations (14) in terms of the matrices. To do so, we introduce the matrices $\hat{e}_k$ in such a way that the determinant of which is equal to 1:

$$\mathbf{e}_k = (e_{k1}, e_{k2}, e_{k3}, e_{k4}) \rightarrow \hat{e}_k = \begin{pmatrix} e_{k4} + i\epsilon_3 e_{k3} & -(\epsilon_1 e_{k1} + i\epsilon_2 e_{k2}) \\ \epsilon_1 e_{k1} - i\epsilon_2 e_{k2} & e_{k4} - i\epsilon_3 e_{k3} \end{pmatrix}. \tag{15}$$

Hence, and from the definition of the vectors $\mathbf{e}_k$, it follows that

$$det(\hat{e}_k) = \epsilon_1^2 e_{k1}^2 + \epsilon_2^2 e_{k2}^2 + \epsilon_3^2 e_{k3}^2 + e_{k4}^2 = 1 \tag{16}$$

where $\epsilon_k^2 = \pm 1$. The matrices $\hat{e}_k$ satisfy the set of 16 equations

$$\hat{e}_{i,j} = b^j_{ik}\hat{e}_k \tag{17}$$

where $b^j_{ik}$ are the elements of $B_j$.

4.1. *Integrable reductions of 4-CCSs.* In contrast with the $n = 3$ case, for the $n = 4$, we have only a few number of integrable systems in (3+1) dimensions such as the generalized and intrinsic generalized wave and sine-Gordon equations [31-32] (for the case of $n = 4$) and so on. Among them the SDYM equation takes the special place. We need the following form, e.g., the anti-SDYM equation [27,28,34]

$$(g^{-1}g_{z_1})_{\bar{z}_1} + (g^{-1}g_{z_2})_{\bar{z}_2} = 0 \tag{18}$$

where $g \in SL(2,C), \quad \det g = 1, \quad g > 0$ and $z_1 = \frac{1}{2}(x_4 + ix_3), \quad z_2 = \frac{1}{2}(x_2 + ix_1)$. Equation (18) is integrable by IST. The corresponding LR has the form

$$(\lambda\partial_{z_1} - \partial_{\bar{z}_2} + \lambda g^{-1}g_{z_1})\Phi = 0, \quad (\partial_{\bar{z}_1} + \lambda\partial_{z_2} + \lambda g^{-1}g_{z_2})\Phi = 0. \tag{19}$$

**Conjecture 1**. *If the matrix $\hat{e}_1$ (15) satisfies the anti-SDYM equation (18), i.e.,*

$$(\hat{e}_1^{-1}\hat{e}_{1z_1})_{\bar{z}_1} + (e_1^{-1}e_{1z_2})_{\bar{z}_2} = 0 \tag{20}$$

*then the equation with the matrices $B_k \in so(p,q)(p+q=4)$, is integrable.*

**5. On soliton immersions in multidimensions**. In this section we would like to discuss some aspects of simplest soliton immersions in multidimensions. The results of the study of this problem in the Fokas-Gelfand sense [1-2] will be reported elsewhere.

5.1. *$V^3$ in $R^{\mu=p+q=4}$*. Let $R^\mu, \quad \mu = (p,q), \quad p+q = 4$ to be the 4-dimensional space with the coordinate system $(y) = (y^k)$ and the metric

$$ds^2 = \mu_{ij}dy^i dy^j, \quad \mu_{ij} = \pm 1. \tag{21}$$

Let $V^3$ are 3-dimensional submanifolds of the $R^\mu$ with the coordinates $(x) = (x^k)$ and the metric

$$ds^2 = g_{ij}dx^i dx^j. \tag{22}$$

Let $F : \Omega \to R^\mu$ is an immersion of a domain $\Omega \in V^3$ into $R^\mu$. Let $(x) = (x^1 = x, x^2 = y, x^3 = t) \in \Omega$. Let $\mathbf{n}(x,y,t)$ is the normal vector field defined at each point of the three-dimensional submanifolds $F(x,y,t)$. The space $F(x,y,t)$ is defined by the first (22) and second fundamental forms

$$II = -dF \cdot \mathbf{n} = L_{ij}dx^i dx^j. \tag{23}$$

Then the system of vectors $(F_x, F_y, F_t, n)$ defines a basis of $V^3$ on $S$ parametrized by $F(x,y,t)$. Let $\mathbf{r} = (y^1, y^2, y^3, y^4) = \mathbf{r}(x^1, x^2, x^3)$ is the position vector. Then the position vector $\mathbf{r}$ satisfies the following Gauss-Weingarten (GW) equation

$$\mathbf{r}_{,ij} = \Gamma^k_{ij}\mathbf{r}_{,k} + L_{ij}\mathbf{n}, \quad \mathbf{n}_{,i} = p_{ij}\mathbf{r}_{,j} \tag{24}$$

Here

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}(g_{il,j} + g_{jl,i} - g_{ij,l}), \quad L_{ij} = (\mathbf{r}_{,ij} \cdot \mathbf{n}). \tag{25}$$

We can rewrite the GW equation (24) in the following form

$$Z_x = A_1 Z, \quad Z_y = A_2 Z, \quad Z_t = A_3 Z, \quad Z = (\mathbf{r}_x, \mathbf{r}_y, \mathbf{r}_t, \mathbf{n})^T \tag{26}$$

and

$$A_1 = \begin{pmatrix} \Gamma^1_{11} & \Gamma^2_{11} & \Gamma^3_{11} & L_{11} \\ \Gamma^1_{12} & \Gamma^2_{12} & \Gamma^3_{12} & L_{12} \\ \Gamma^1_{13} & \Gamma^2_{13} & \Gamma^3_{13} & L_{13} \\ p_{11} & p_{12} & p_{13} & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \Gamma^1_{21} & \Gamma^2_{21} & \Gamma^3_{21} & L_{21} \\ \Gamma^1_{22} & \Gamma^2_{22} & \Gamma^3_{22} & L_{22} \\ \Gamma^1_{23} & \Gamma^2_{23} & \Gamma^3_{23} & L_{23} \\ p_{21} & p_{22} & p_{23} & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} \Gamma^1_{31} & \Gamma^2_{31} & \Gamma^3_{31} & L_{31} \\ \Gamma^1_{32} & \Gamma^2_{32} & \Gamma^3_{32} & L_{32} \\ \Gamma^1_{33} & \Gamma^2_{33} & \Gamma^3_{33} & L_{33} \\ p_{31} & p_{32} & p_{33} & 0 \end{pmatrix}. \tag{27}$$

Integrability conditions for GW equations,

$$\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i} + [A_i, A_j] = 0 \tag{28}$$

are known as Gauss-Mainardi-Codazzi (GMC) equations.

5.1.1. *Integrable reductions.* First let us rewrite the equation (26) in terms of the vectors $\mathbf{e}_k$. We have

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \end{pmatrix}_x = B_1 \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \end{pmatrix}_y = B_2 \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \end{pmatrix}_t = B_3 \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \end{pmatrix}. \tag{29}$$

Here $B_k \in so(p,q), \quad p+q=4$. The compatibility condition of the linear equations (29) is

$$B_{1y} - B_{2x} + [B_1, B_2] = 0, B_{1t} - B_{3x} + [B_1, B_3] = 0, B_{2t} - B_{3y} + [B_2, B_3] = 0. \tag{30}$$

In terms of the matrices the equation (29) takes the form

$$\hat{e}_{jx} = b^1_{ji}\hat{e}_i, \quad \hat{e}_{jy} = b^2_{ji}\hat{e}_i, \quad \hat{e}_{jt} = b^3_{ji}\hat{e}_i \tag{31}$$

where $j = 1, 2, 3, 4$ $\hat{e}_k$ given by (15), $b^j_{ki}$ are elements of $B_j$. Now we give two examples of integrable cases.

i) **The Manakov-Zakharov-Mikhailov equation**. Let us consider the Manakov-Zakharov-Mikhailov (MZM) equation [35-36]

$$(g^{-1}g_t)_t - (g^{-1}g_\xi)_\eta = 0 \tag{32}$$

where $g$ is some $2 \times 2$ matrix and

$$\det g = 1, \quad \xi = \frac{1}{2}(x + \sigma y), \quad \eta = \frac{1}{2}(x - \sigma y), \quad \sigma^2 = \pm 1. \tag{33}$$

The MZM equation is equivalent to the compatibility condition for the two linear equations

$$(\lambda\partial_\eta - \lambda^{-1}\partial_\xi - \lambda^{-1}g^{-1}g_\xi + g^{-1}g_t)\Phi = 0, \quad (\partial_t + \lambda\partial_\eta + g^{-1}g_t)\Phi = 0 \tag{34a}$$

or [34]

$$(\partial_\xi + \lambda\partial_t + g^{-1}g_\xi)\Phi = 0, \quad (\lambda\partial_\eta + \partial_t + g^{-1}g_t)\Phi = 0. \tag{34b}$$

**Conjecture 2**. *If the matrix* $\hat{e}_1$ (15) *satisfies the MZM equation (32), i.e.,*

$$(\hat{e}_1^{-1}\hat{e}_{1t})_t - (\hat{e}_1^{-1}\hat{e}_{1\xi})_\eta = 0 \tag{35}$$

*then the equation (30) with the matrices* $B_k \in so(p,q), \quad p+q=4$ *is integrable.*

ii) **The generalized sigma model**. As a next example, we shall consider the following generalized sigma model [37]

$$(\Sigma_y)_x + (\Sigma_t)_t + \alpha[(\Sigma_t)_x - (\Sigma_x)_t] = 0. \tag{36}$$

This equation is integrated by the LR [37]

$$[\partial_y + (\alpha + k)\partial_t + (\Sigma_y + \alpha\Sigma_t)]\Phi = 0, \quad [-(\alpha + k)\partial_x + \partial_t + (-\alpha\Sigma_x + \Sigma_t)]\Phi = 0 \tag{37}$$

where $\Sigma_i = g^{-1}\partial_i g$. Hence as $\alpha = 0$, we get the Ward model.

**Conjecture 3**. *If the matrix $\hat{e}_1$ (15) satisfies the generalized sigma model (36), i.e.,*

$$\Sigma_i = \hat{e}_1^{-1}\partial_i \hat{e}_1 \tag{38}$$

*then the equation (30) with the matrices $B_k \in so(p,q)$, $\quad p+q=4$ is integrable.*
5.2. $V^4$ *in* $R^\mu$. Similarly, we can consider the manifold $V^4$ embedded in $R^\mu$, where $\mu=(p,q)$ with $p+q=n>4$. In this case the equations for $\mathbf{e}_k$ take the form

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \cdots \\ \mathbf{e}_n \end{pmatrix}_{,k} = B_k \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \cdots \\ \mathbf{e}_n \end{pmatrix} \tag{39}$$

where the matrices $B_k$ belong already to the Lie algebra, $so(p,q)$ with $p+q=n$. The matrices $B_k$ satisfy the system

$$B_{1y} - B_{2x} + [B_1, B_2] = 0, \quad B_{1z} - B_{3x} + [B_1, B_3] = 0 \tag{40a}$$

$$B_{1t} - B_{4x} + [B_1, B_4] = 0, \quad B_{2z} - B_{3y} + [B_2, B_3] = 0 \tag{40b}$$

$$B_{2t} - B_{4y} + [B_2, B_4] = 0, \quad B_{3t} - B_{4z} + [B_3, B_4] = 0. \tag{40c}$$

Let $\hat{e}_k$ are the matrix representation of the unit orthogonal vectors $\mathbf{e}_k$.
**Conjecture 4**. *If the matrix $\hat{e}_1$ satisfies the anti-SDYM equation (18), i.e.,*

$$(\hat{e}_1^{-1}\hat{e}_{1z_1})_{\bar{z}_1} + (e_1^{-1}e_{1z_2})_{\bar{z}_2} = 0 \tag{41}$$

*then the equation (40) with the matrices $B_k \in so(p,q)$, $\quad p+q=n$ is integrable.*

**6. Conclusion**. We see that the geometrical equations, describing $n$-CCSs in flat Euclidian and pseudo-Euclidian spaces, admit several integrable reductions. Moreover, we have conjectured that the immersions of 3-, and 4-dimensional manifolds arbitrarily embedded in $R^\mu$ admit integrable cases.
Concluding this work we would like to note that many problems remain still open. So this problem needs further study any development.

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## III.5 Norbert Peyerimhoff and Ivan Veselić: *Integrated Density of States for Random Schrödinger Operators on Manifolds*

We consider a Riemannian manifold $X$ admitting a compact quotient $X/\Gamma$, i.e., $\Gamma$ is a cocompact subgroup of the isometries acting properly discontinuously on $X$. We show, under certain conditions on $\Gamma$, that it is possible to define an integrated density of states for $\Gamma$-ergodic random Schrödinger operators on $X$ (see Theorem 7). These conditions are, e.g., satisfied if $\Gamma$ has polynomial growth.

### Physical and geometrical setting

The integrated density of states (IDS) is a quantity introduced in the quantum theory of solids corresponding to the number of electron states per unit volume in a given energy interval $]-\infty, E[$. The electron levels correspond to eigenvalues of a Schrödinger operator which models the motion of an electron in condensed matter. For the medium one assumes a certain form of homogeneity, which is mathematically encoded in the ergodicity of the electrostatic potential which is produced by the medium. For the physical and mathematical background see, e.g., [4, 8, 5, 13].
For Euclidean configuration spaces the IDS is introduced mathematically as follows:

1. Let $\Lambda_l \subset \mathbb{R}^d$ be a cube of lenght $l$ and centered at 0.
2. Consider a family of Schrödinger operators $H^\omega = -\Delta + V^\omega, \omega \in \Omega$, with ergodic potential $V^\omega$. (Precise definition of ergodicity will follow.) Note that we use the sign convention that $-\Delta$ is a positive operator.
3. Restrict $H^\omega$ to $\Lambda_l$ with Dirichlet boundary conditions to get $H_l^\omega$.
4. Call $N_l^\omega(E) := \#\{i \mid \lambda_i(H_\omega^l) < E\}$ the eigenvalue counting function of $H_l^\omega$. Here $\lambda_i(H_\omega^l)$ are the eigenvalues of $H_l^\omega$, enumerated in non-decreasing order and including multiplicities.
5. Under certain conditions on the family of ergodic potentials $V^\omega, \omega \in \Omega$ one can show that, for almost all $\omega$, say, in the set $\Omega'$ of full measure, the limit
$$\lim_{l\to\infty} \frac{N_l^\omega(E)}{l^d} = N^\omega(E) = N(E) \tag{34}$$
exists for all continuity points of $N$ and is independent of $\omega \in \Omega'$.

The limit (34) is called the IDS of the family $H^\omega$. Note that, by definition, the functions $N_l^\omega(\cdot)$ are non-negative, monotone increasing and left-continuous. Since, in the thermodynamic limit ($l \to \infty$), the randomness disappears, the IDS $N(E)$ is called self-averaging. Once the non-randomness of the IDS has been established, one relates it to the spectrum of the operators $H^\omega, \omega \in \Omega$. In the Euclidean case it is known [7] that the set of points of increase
$$\{\lambda \in \mathbb{R} \mid\ N(\lambda+\epsilon) - N(\lambda-\epsilon) > 0\ \forall \epsilon > 0\} \tag{35}$$
contains the spectrum of $H^\omega$, which is independent of $\omega$ almost surely.
The Euclidean Laplace operator and the associated fundamental operators from physics can be naturally generalized to Riemannian manifolds. The spectrum of these physical operators then also reflects local and global geometric properties of the underlying manifold. In 1992, Adachi and Sunada [2] carried over techniques developed by mathematical physicists (cf. e.g. [15, 9]) to define an IDS of an periodic Schrödinger operator on a Riemannian manifold. In this new situation one encounters the nontrivial problem how to replace the cubes $\Lambda_l$, $l \to \infty$, which exhaust the Euclidean space. Actually, in $\mathbb{R}^d$ one is allowed to choose much more general exhaustions to define the (same) IDS.
In [2] the following geometrical setting is considered:

- Let $X$ be a $d$-dimensional, non-compact, complete, connected Riemannian manifold.

- Let $\Gamma \subset$ be a discrete, amenable subgroup of the isometries acting freely and properly discontinously on $X$ such that the quotient $M := X/\Gamma$ is compact. Then $\Gamma$ is finitely generated.

Our geometric setting is the same as in [2]. However, instead of a single periodic operator we consider a random family of Schrödinger operators $H^\omega := -\Delta + V^\omega, \omega \in \Omega$, with the following properties.

- $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space with a group of ergodic measure perserving transformations $T_\gamma, \gamma \in \Gamma$. Ergodicity of the group of measure preserving transformations means that the only sets $A \in \mathcal{A}$ which are invariant under all $T_\gamma, \gamma \in \Gamma$ are those with $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.
- The potential is a jointly measurable stochastic process $V : \Omega \times X \to \mathbb{R}$ and, for technical reasons, we require $V^\omega \in C^\infty(X), \forall \omega$ and a uniform bound on the derivatives

$$\|\nabla^k V^\omega\|_\infty \leq C \text{ for all } \omega \in \Omega, k \leq \frac{1}{2}\dim(X) + 2 \ . \tag{36}$$

- The $\Gamma$-action on the potential and on $X$ are consistent in the following sense: For all $\gamma \in \Gamma$ we have

$$V^\omega(\gamma x) = V^{T_\gamma \omega}(x) \ . \tag{37}$$

  This property is called ergodicity of the stochastic process $V^\omega$.

If we define $U_\gamma f(x) := f(\gamma x)$ as the unitary operator on $L^2(X)$ corresponding to the isometry $\gamma : X \to X$, (37) can be written in the form $U_\gamma V^\omega U_\gamma^* = V^{T_\gamma \omega}$. Since $-\Delta$ is invariant under the operators $U_\gamma$ one gets

$$U_\gamma H^\omega U_\gamma^* = H^{T_\gamma \omega}. \tag{38}$$

There are two approaches at hand to prove the convergence in (34). Either one uses the Laplace transform of the normalised eigenvalue counting function, which can be written in terms of the trace of the corresponding heat equation semigroup. In this case one applies a pointwise ergodic theorem for additive (w.r.t. domains) processes. Or one uses Dirichlet-Neumann bracketing to show that the eigenvalue counting functions form themselves a superadditive process. For such processes there are also ergodic theorems avaliable [3, 10]. However only in the abelian case they assure the pointwise existence of a limit. In our non-abelian setting we would get just a limit in mean.
Thus we rely on the first alternative and therefore need some tools from the theory of heat equation semigroups.

### Heat kernel bounds

Under the above assumptions $H^\omega$ is a densely defined selfadjoint operator on $L^2(X)$. By the spectral theorem we can define the operator $\exp(-tH^\omega)$ which has an integral kernel $k^\omega(t, \cdot, \cdot)$. Let $D \subset X$ be a regular domain, i.e., a connected subset with compact closure and smooth boundary. We denote the restriction of $H^\omega$ to $D$ with Dirichlet boundary conditions by $H_D^\omega$ and the corresponding heat kernel of $\exp(-tH_D^\omega)$ by $k_D^\omega(t, \cdot, \cdot)$.
Since the heat kernel depends monotonously on the domain (see [6, 16]), we have

$$0 \leq k_D^\omega(t, x, y) \leq k^\omega(t, x, y) \ \forall \ t > 0; x, y \in D \ . \tag{39}$$

We need, however, more information about the integral kernels. Firstly, we would like to bound the rhs of (39) by some constant independent on $x, y$ and $\omega$. Furthermore we would like to control the difference between the heat kernel on the whole manifold and the one restricted to the domain $D$. We collect this information in the following lemma.

**Lemma 1** *Let $k^\omega, k_D^\omega$ be the heat kernels as above. Then we have*
*(a) $k^\omega(t,x,y) \le C(t),\ \forall x,y;\omega,$*
*(b) $\forall t,\epsilon > 0\ \exists h = h(t,\epsilon) < \infty$ such that, for all regular domains $D$ and all $\omega \in \Omega$, we have*

$$0 \le k^\omega(t,x,y) - k_D^\omega(t,x,y) \le \epsilon \quad \forall x,y \in D \setminus \partial_h D, \tag{40}$$

*where $\partial_h D := \{x \in D \mid d(x,\partial D) \le h\}$.*

The proof of (a) for a fixed potential $V^\omega$ can be infered from section 3 in [11]. Actually the dependence of the heat kernel on the potential can be bounded by a function of the sup-norm of the potential only. Since our family of potentials has an uniform norm bound we can choose the constant $C(t)$ to be independent of $\omega$. The proof of (b) uses a result about unit propagation speed of the wave equation with potential, as it can be found, for example, in [16]. The bound on the difference of the two heat kernels is then obtained by an argument of U. Bunke as outlined in Theorem 2.26 of [12]. See also [14].

### Assumptions on the group $\Gamma$

In the Euclidean case, any monotone exhaustion of $\mathbb{R}^d$ by cubes $\Lambda_{l_1}, \Lambda_{l_2}, \ldots$ guaranteed the existence of a nonrandom IDS by an application of an abelian ergodic theorem. Actually there is much more freedom. One is even allowed to shift the cubes a little bit on each scale, so that the cubes do not contain the preceeding ones any more; one can choose balls instead of cubes or other convex domains as long as they satisfy some very reasonable growth conditions, cf. [9, 10].

On a Riemannian manifold one has to deal with the problem to find an adequate substitute for the cubes $\Lambda_l$. This problem already appeared in [2] for a single operator with periodic potential. If one considers a $\Gamma$-ergodic family of potentials and wants to prove existence of the IDS there is need for a non-abelian ergodic theorem. We will use an ergodic theorem due to Tempelman [17, 18]. This theorem is only applicable for exhaustions or summing sequences (the analoga of $\Lambda_{l_1}, \Lambda_{l_2}, \ldots$) which satisfy much more restrictive conditions than those imposed in abelian ergodic theorems. The following definition lays the grounds to apply Tempelman's ergodic theorem (in the version given in [10]).

**Definition 2** *A sequence of subsets $\{I_n\}_{n\in\mathbb{N}}$ of $\Gamma$ is called an admissible summing sequence if*

$$\begin{array}{ll} (P0) & I_n \subset I_{n+1},\ ,0 < |I_n| < \infty\ \forall n \in \mathbb{N}, \\ (P1) & \forall \gamma:\ |I_n \Delta I_n\gamma|/|I_n| \longrightarrow 0 \textit{ for } n \to \infty, \\ (P2) & \exists K < \infty:\ |I_n^{-1} I_n| \le K|I_n|\ . \end{array}$$

*Here $|A|$ denotes the cardinality of a finite set $A$.*

The condition (P1) is an formulation of Følner's condition for discrete groups. (P2) is a growth condition which can be understood best, e.g., in the particular case where the sets $I_n$ are combinatorial balls of radius $n$ (w.r.t. a word metric): Doubling the radius increases the cardinality of the ball by only a constant multiple. Admissible summing sequences are the main input in Tempelman's ergodic theorem applied to the group $T_\gamma, \gamma \in \Gamma$. In the next section, we construct, to a given admissible summing sequence, a corresponding sequence of subsets of the Riemannian manifold to which we will restrict our Schrödinger operators.

### Admissible exhaustions and the main result

As explained in section 3 of [2], there exists a smooth triangulation of the compact manifold $M$ and, by lifting the simplices to $X$ in a suitable manner, we obtain a connected polyhedral $\Gamma$-fundamental domain $\mathcal{F}$. Particularly $\mathcal{F}$ consists of finitely many smooth images of $d$-dimensional simplices which can overlap only at their boundaries.

**Definition 3** *For an admissible summing sequence $\{I_n\}_{n\in\mathbb{N}}$ we define*

$$D_n = \bigcup_{\gamma \in I_n} \gamma\mathcal{F} \tag{41}$$

*where $\mathcal{F}$ is a fundamental domain as explained above. We call $\{D_n\}_{n\in\mathbb{N}}$ an admissible exhaustion if, additionally,*

$$(P) \qquad \text{For all } h>0: \frac{vol(\partial_h D_n)}{vol(D_n)} \longrightarrow 0 \quad \text{for} n\to\infty \ . \tag{42}$$

It was shown in [1, 2] that the geometric condition (P) is automatically satisfied if $|I_nA^n - I_n| \le |I_n|/(n|A^n|)$, for all $n\in\mathbb{N}$, where $A\subset\Gamma$ is the following particular set of generators: $A=\{\gamma : \gamma\mathcal{F}\cap\mathcal{F}\neq\emptyset\}$.

**Lemma 4** *For an admissible exhaustion $\{D_n\}_{n\in\mathbb{N}}$ and any $\delta>0$ there exists a sequence $\{M_n\}_{n\in\mathbb{N}}$ of regular domains such that:*

$$D_n\setminus\partial_\delta D_n \subset M_n \subset D_n\ ,\ \partial M_n\subset\partial_\delta D_n\ ,\ \forall n, \tag{43}$$

*and, for $n\to\infty$, we have*

$$\left| |D_n|^{-1}\int_{D_n} k^\omega(t,x,x)dx - |M_n|^{-1}\int_{M_n} k^\omega_{M_n}(t,x,x)dx \right| \longrightarrow 0\ . \tag{44}$$

*We call $\{M_n\}_{n\in\mathbb{N}}$ a regularisation of $\{D_n\}_{n\in\mathbb{N}}$.*

For the proof of (44) one uses Lemma 1(b) to bound the difference of the kernels by some arbitrary small number $\epsilon>0$ away from the boundary of $D_n$. The region near the boundary, where this estimate is not valid, is small by property (P). There we use Lemma 1(a) and the fact that the symmetric difference of $M_n$ and $D_n$ is small, as well.
The ergodicity of $V^\omega$ implies (38). An analogous relation is valid for the spectral projections of $H^\omega$ as can be inferred from [19, Theorem 7.15] and the uniqueness of the spectral projection.

$$U_\gamma E(\lambda,\omega)U_\gamma^* = E(\lambda, T_\gamma\omega)\ . \tag{45}$$

This relation also holds for $\exp(-tH^\omega)$ and thus its kernel $k^\omega(t,x,y)$ is an ergodic stochastic process in the following sense

$$k^\omega(t,\gamma x,\gamma x) = k^{T_\gamma\omega}(t,x,x)\ . \tag{46}$$

This enables us to apply later the next lemma to the heat kernels.

**Lemma 5** *Let $\{D_n\}_{n\in\mathbb{N}}$ be an admissible exhaustion and $f:\Omega\times X\to\mathbb{R}$ be a $\Gamma$-ergodic stochastic process. Then*

$$\lim_{n\to\infty}|D_n|^{-1}\int_{D_n} f(\omega,x)dx = |\mathcal{F}|^{-1}\,\mathbb{E}\left[\int_{\mathcal{F}} f(\bullet,x)dx\right]\ . \tag{47}$$

Here $\mathbb{E}$ denotes the expectation with respect to the measure $\mathbb{P}$.

**Proof:** Set $F(\omega)=|\mathcal{F}|^{-1}\int_{\mathcal{F}} f(\omega,x)dx$. Now transform the term on the lhs of (47).

$$\begin{aligned}
|D_n|^{-1}\int_{D_n} f(\omega,x)dx &= |I_n|^{-1}\sum_{\gamma\in I_n}|\mathcal{F}|^{-1}\int_{\gamma\mathcal{F}} f(\omega,x)dx\\
&= |I_n|^{-1}\sum_{\gamma\in I_n}|\mathcal{F}|^{-1}\int_{\mathcal{F}} f(\omega,\gamma x)dx\\
&= |I_n|^{-1}\sum_{\gamma\in I_n}|\mathcal{F}|^{-1}\int_{\mathcal{F}} f(T_\gamma\omega,x)dx\\
&= |I_n|^{-1}\sum_{\gamma\in I_n}F(T_\gamma\omega) \longrightarrow \tilde{f}(\omega)
\end{aligned}$$

for $n\to\infty$ and a measurable $\tilde{f}:\Omega\to\mathbb{R}$. Here we used that

1. $D_n$ can be written as an union of translates of the fundamental domain $\mathcal{F}$,
2. $\gamma$ is an isometry and thus preserves the volume element $dx$,
3. we can transfer the action of $\gamma$ from the geometric to the random parameter by the ergodicity of the process $f$,
4. admissible summing sequences satisfy the conditions of Tempelman's ergodic theorem.

We additionally know from this ergodic theorem that the function $\tilde{f}$ is invariant under all $T_\gamma, \gamma \in \Gamma$. By assumption, these transformations are ergodic and, thus, $\tilde{f}$ is constant almost surely. The convergence calculated above implies that this constant has to be equal to $\mathbb{E}(F)$. □

In order to prove our main result (Theorem 7) we need, finally, the following lemma (see [15, Lemma 5.2]). It will be used to derive convergence of the normalized eigenvalue counting function from convergence of the corresponding heat kernels.

**Lemma 6 (Šubin)** *Let $N_n : \mathbb{R} \to \mathbb{R}^+$ be a sequence of left-continuous, monotone increasing functions such that*

1. *there exists a $c \in \mathbb{R}: \ N_n(\lambda) = 0 \quad \forall \lambda \le c, n \in \mathbb{N}$,*
2. *there exists a $C : \mathbb{R}^+ \to \mathbb{R} : \psi_n(t) := \int e^{-\lambda t} dN_n(\lambda) \le C(t) \quad \forall n \in \mathbb{N},\ t > 0$,*
3. *$\lim_{n\to\infty} \psi_n(t) =: \psi(t)$ exists for all $t > 0$.*

*Then the limit*

$$N(\lambda) := \lim_{n\to\infty} N_n(\lambda)$$

*exists at all continuity points. $N$ is a non-negative, monotone increasing, left-continuous function of $\lambda$, and its Laplace transform is $\psi$.*

**Theorem 7** *Let $\{D_n\}_{n\in\mathbb{N}}$ be an admissible exhaustion and $\{M_n\}_{n\in\mathbb{N}}$ a regularisation of $\{D_n\}$. Define*

$$\phi_n^\omega(\lambda) := \#\{ \text{ eigenvalues of } H^\omega_{M_n} \le \lambda\},\ N_n^\omega = |M_n|^{-1}\phi_n^\omega\ . \tag{48}$$

*Then, for almost all $\omega \in \Omega$, the limit*

$$N(\lambda) = \lim_{n\to\infty} N_n^\omega(\lambda) \tag{49}$$

*exists at all continuity points of $N(\cdot)$. Moreover, $N$ is a non-negative, monotone increasing, left-continuous function (independent of $\omega$).*

**Proof:** We check Šubin's hypotheses:
1. $\phi_n^\omega(\lambda) = 0$ for all $\omega \in \Omega, n \in \mathbb{N}$, if $\lambda < -\|V^\omega\|_\infty$.
2.

$$\begin{aligned}
|M_n|^{-1} \int e^{-t\lambda} d\phi_n^\omega(\lambda) &= |M_n|^{-1} \sum_{\lambda_n \in \sigma(H^\omega_{M_j})} e^{-t\lambda_n} \\
&= |M_n|^{-1} \ \mathrm{Tr}\ e^{-tH^\omega_{M_j}} \\
&= |M_n|^{-1} \int_{M_n} k^\omega_{M_n}(t,x,x)dx \\
&\le |M_n|^{-1} \int_{M_n} k^\omega(t,x,x)dx \\
&\le C(t).
\end{aligned}$$

3.

$$\begin{aligned}
\lim_{n\to\infty} |M_n|^{-1} \int e^{-t\lambda} d\phi_n^\omega(\lambda) &= \lim_{n\to\infty} |M_n|^{-1} \int_{M_n} k^\omega_{M_n}(t,x,x)dx \\
&= \lim_{n\to\infty} |D_n|^{-1} \int_{D_n} k^\omega(t,x,x)dx \\
&= |\mathcal{F}|^{-1}\, \mathbb{E}\left[\int_{\mathcal{F}} k^\bullet(t,x,x)dx\right] .
\end{aligned}$$

Here we used the Lemmata 4 and 5. □

**An Example**

We supply a non-abelian example of a pair $(X,\Gamma)$ which satisfies all necessary conditions to define an IDS in the above manner.
For any Riemannian manifold with a polynomially bounded cocompact discrete group of isometries an admissible exhaustion $\{D_n\}_{n\in\mathbb{N}}$ can be constructed. In particular, this is the case for nilpotent groups. Thus, an example is given by the Heisenberg group

$$X = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid x,y,z\in\mathbb{R} \right\} \tag{50}$$

with a left-invariant metric, and $\Gamma = X\cap M(3,\mathbb{Z})$. $X$ is diffeomorphic to $\mathbb{R}^3$, but non-abelian. Note that the discrete group $\Gamma$ acts, by isometries, from the left on $X$ and that the quotient $\Gamma\backslash X$ is compact.

An example for a $\Gamma$-ergodic potential on $(X,\Gamma)$ is the analogue of the one which is called alloy-type potential in the Euclidean setting.
Let $u : X\to\mathbb{R}$ be a smooth function supported in $\mathcal{F}$. We choose $\Omega = \bigotimes_{\gamma\in\Gamma}\mathbb{R}$, equipped with the product measure $\mathbb{P} := \bigotimes_{\gamma\in\Gamma}\mu$, where $\mu$ is a probability measure on $\mathbb{R}$. Consider the independent, identically distributed random variables $\pi_\gamma : \Omega\to\mathbb{R}, \pi_\gamma(\omega) := \omega_\gamma, \gamma\in\Gamma$ and the measure preserving transformations $(T_\gamma(\omega))_\beta := \omega_{\gamma\beta}$. Then the potential, given by

$$V^\omega(x) := \sum_{\gamma\in\Gamma} \omega_\gamma u(\gamma x), \tag{51}$$

is $\Gamma$-ergodic and satisfies the required regularity assumptions (36) to define the IDS. Actually, it seems that by using probabilistic estimates for the heat kernels, instead, these regularity assumptions can be relaxed to the "natural"ones. These are formulated in terms of the expectation of some local $L^p$-norm of $V^\omega$, cf. [9].

**Discussion**

The IDS is a quantity related to the restrictions of Schrödinger operators in a familiy $H^\omega, \omega\in\Omega$, to suitable exhaustions of a non-compact Riemannian manifold $X$. It should be possible to use the same functional analytic arguments as in the Euclidean case to establish the equality of its points of increase (35) with the spectrum of $H^\omega$.
The definition of an admissible exhaustion contains assumptions about both the sequence of subsets $I_n$ of the group $\Gamma$ as well as the geometry of the domains $D_n$. It would be desirable to formulate the conditions in terms of either group-theoretic or geometric properties only. Recently, we observed, in the case of a polynomially bounded group $\Gamma$, that Theorem 7 is also valid for a sequence $M_n\subset X$ of regularized geodesic balls with increasing radii satisfying condition (P). Thus for this choice one

obtains via (49) a non-random IDS. It is natural to ask for more general classes of exhaustions of $X$, for which the normalised eigenvalue counting functions converge and whether they give rise to the same IDS.

The remark in Krengel [10] after the statement of Tempelman's theorem implies the uniqueness of the IDS in the following sense. Let $I_n, n \in \mathbb{N}$ be a sequence of subsets of $\Gamma$, not necessarily admissible in our sense and $D_n, n \in \mathbb{N}$ as in (41). Assume that it satisfies condition (P) and that a regularisation $M_n, n \in \mathbb{N}$ exists. If the $N_n^\omega$ defined in (48) converge to a non-random limit $N$, then this has to be equal to the IDS defined in Theorem 7 by an admissible summing sequence. In particular any two admissible exhaustions give rise to the same IDS.

Note also that our choice of the admissible exhaustion depends on the pair $(X, \Gamma)$ only. This means particularly, that it is independent of the properties of the family of Schrödinger operators$H_\omega, \omega \in \Omega$. If one would allow for admissible exhaustions depending on the stochastic process $V^\omega$, too, the convergence (49) of the normalised eigenvalue counting functions to an IDS could be achieved in more general geometric situations.

Finally, it should be possible to study an ergodic family of operators where higher order parts of the operators depend on the random parameter $\omega$ and to carry through the same procedure as in this paper. Such a situation would occur if $\omega$ were related to the change of a magnetic field or to the change of the metric of the underlying Riemannian manifold.

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# IV List of Participants

**Agricola, Ilka** agricola@mathematik.hu-berlin.de
Inst. fuer Reine Mathematik, HU Berlin, 10099 Berlin

**Arsan, Güler Gürpinar** ggarsan@itu.edu.tr
Istanbul Technical University, Faculty of Science and Letters, Department of Mathematics, 80626 Maslak, Istanbul- Turkey

**Asch, Joachim** asch@cptsu5.univ-mrs.fr
UFR de Science et de Technique, Universite de Toulon et du Var, BP 132, F-82957 La Garde Cedex, France

**Avron, Joseph** avron@physics.technion.ac.il
Department of Physics, Technion, Haifa, 32000 Israel

**Babujian, Hrachya** babujian@physik.fu-berlin.de
FU-Berlin,Arnimallee 14,14195 Berlin

**Bach, Volker** vbach@mathematik.uni-mainz.de
FB Mathematik, Uni Mainz, 55099 Mainz

**Bahns, Dorothea** bahns@physik.uni-freiburg.de
Fakultät für Physik, Hermann-Herder-Str. 3, 79104 Freiburg

**Ballmann, Werner** ballmann@math.uni-bonn.de
Mathematisches Institut der Universität Bonn, Beringstraße 1, D-53115 Bonn

**Baum, Helga** baum@mathematik.hu-berlin.de
Humboldt-Universität, Institut für Mathematik, Sitz: Ziegelstrasse 13a, 10099 Berlin

**Baumgaertel, Hellmut** baumg@rz.uni-potsdam.de
Universitaet Potsdam, Institut fuer Mathematik

**Bayak, Igor** bayak@mail.ru
St, Kosmonavtov 10-3, Grodno, 230003, Belarus

**Birke, Lothar** lothar@itp.phys.ethz.ch
Rigistrasse 19, 8006 Zuerich, CH

**Blanchard, Philippe** blanchard@physik.uni-bielefeld.de
Universität Bielefeld, Fakultät für Physik, 33615 Bielefeld

**Bless, Michael Harald** bless@uni-erlangen.de
Mathematisches Institut, Bismarcksrt. 1 1/2, D-91054 Erlangen

**Bobenko, Alexander** bobenko@math.tu-berlin.de
FB Mathematik, TU Berlin

**Bohle, Christoph** bohle@sfb288.math.tu-berlin.de
Torstr. 75

**Boos, Herman** boos@th.physik.uni-bonn.de
Universitaet Bonn, Nussallee 12, 53115 Bonn, Germany

**Bourguignon, Jean Pierre** jpb@math.polytechnique.fr
Centre de Mathematiques, Ecole Polytechnique, F-91128 Palaiseau, France

**Brunetti, Romeo** brunetti@na.infn.it
Dip. di Scienze Fisiche, Universita' di Napoli "Federico II", Pad.19 Mostra d'Oltremare, I-80125 Napoli, Italy

**Buchholz, Detlev** buchholz@theorie.physik.uni-goettingen.de
Institut für Theoretische Physik, Universität Göttingen, 37073 Göttingen

**Bär, Christian** baer@math.uni-hamburg.de

Universität Hamburg, FB Mathematik, Bundesstr. 55, 20146 Hamburg

**Böcker, Stefan** stefan.boecker@mathphys.ruhr-uni-bochum.de
Universitätsstraße 150, 44780 Bochum

**Brüning, Jochen** bruening@mathematik.hu-berlin.de
Inst. fuer Reine Mathematik, HU Berlin, 10099 Berlin

**Cegla, Wojciech** cegla@ift.uni.wroc.pl
Institute of Theoretical Physics University of Wroclaw, 50-204 pl.M.Borna Wroclaw,Pl.

**Centazzo, Massimo** calvin@aei-potsdam.mpg.de
MPI fur Gravitationsphysik, Am Muhlenberg 1, 14476 Golm

**Cheeger, Jeff** cheeger@cims.nyu.edu
Courant Institute of Mathematical Sciences, New York University, 251 Mercer St., New York, NY 10012, U.S.A.

**Chen, Thomas** chen@itp.phys.ethz.ch
Theoretische Physik, ETH Hoenggerberg, 8093 Zuerich, Schweiz

**Civi, Gülcin** civi@itu.edu.tr
Istanbul Technical University Faculty of Science and Letters Department of Mathematics 80626 Maslak-Istanbul-TURKIYE

**Combes, Jean-Michel** combes@cptsu5.univ-mrs.fr
UFR de Science et de Technique, Universite de Toulon et du Var, BP 132, F-82957 La Garde Cedex, France

**del Rio, Rafael** delrio@irsamc2.ups-tlse.fr
7 rue Clemence Isaure 31000 Toulouse

**Demuth, Michael** demuth@math.tu-clausthal.de
Erzstr. 1, 38678 Clausthal-Zellerfeld

**Dobrokhotov, Serguei** dobr@ipmnet.ru
Institute for Problems in Mechanics, prosp.Vernadskogo 101 , Moscow, 117526, Russia.

**Driuma, Valerii** valery@gala.moldova.su
2028, Moldova, Kishinev, Academy str.5, Institute of Mathematics AS RM

**Duzaar, Frank** duzaar@mathematik.hu-berlin.de
Inst. fuer Reine Mathematik, HU Berlin, 10099 Berlin

**Eckhoff, Michael** meckhoff@math.uni-potsdam.de
Behaimstrasse 20

**Enß, Volker** enss@rwth-aachen.de
Inst. f. Reine u. Angew. Mathematik, RWTH Aachen, D - 52056 Aachen

**Exner, Pavel** exner@ujf.cas.cz
Department of Theoretical Physics, NPI AS, CZ-25068 Rez-Prague

**Faddeev, Ludwig** faddeev@pdmi.ras.ru
Steklov Math. Inst., Fontanka 27, 191011 St.Petersburg, RUSSIA

**Feinberg, Joshua** joshua@physics.technion.ac.il
Physics Department, Technion, Haifa 32000, Israel

**Ferus, Dirk** ferus@sfb288.math.tu-berlin.de
TU Berlin, Str. des 17. Juni 135, D-10623

**Friedrich, Thomas** friedric@mathematik.hu-berlin.de
Inst. fuer Reine Mathematik, HU Berlin, 10099 Berlin

**Fring, Andreas** fring@physik.fu-berlin.de
Arnimallee 14, D-14195 Berlin

**Fröhlich, Jürg** juerg@itp.phys.ethz.ch
Institut fuer Theoretische Physik, ETH Hoenggerberg, CH-8093 Zuerich, Schweiz

**Ganter, Nora** ganter@math.uni-bonn.de
Meckenheimer Allee 148, 53115 Bonn

**Geyler, Vladimir** geyler@mrsu.ru
Department of Mathematics, Mordovian State University, Saransk, 430000, Russia

**Gilles, Carron** gcarron@umpa.ens-lyon.fr
umpa ens lyon

**Griesemer, Marcel** marcel@math.uab.edu
UAB; 452 Campbell Hall; Birmingham, AL 35294; USA

**Grieser, Daniel** grieser@mathematik.hu-berlin.de
Humboldt-Universitaet Berlin, Unter den Linden 6, 10099 Berlin

**Grinshpun, Vadim** grinshpun@ilt.kharkov.ua
Ukraine, 310202 Kharkov, poste restante

**Gruber, Michael J.** mjg@math.mit.edu
Dept. of Mathematics, MIT 2-167, 77 Mass. Av., Cambridge MA 02139-4307, USA

**Guido, Daniele** guido@unibas.it
Dip. Matematica, Univ. Basilicata, via Nazario Sauro 85, I-85100 Potenza, Italy

**Haba, Zbigniew** zhab@ift.uni.wroc.pl
Theoretical Physics,50-204 Wroclaw,Plac M.Borna 9

**Helffer, Bernard** Bernard.Helffer@math.u-psud.fr
Bat 425, Departement de Mathematiques, Universite Paris-Sud, 91 405 Orsay Cedex France

**Hempel, Rainer** r.hempel@tu-bs.de
Institut für Analysis, TU Braunschweig, Pockelsstr. 14, 38106 Braunschweig, Germany

**Herrmann, Daniel** danielh@sci.kun.nl
TU Berlin,MA 7-2, Strasse des 17 Juni 135, 10623 Berlin

**Hoffmann, Tim** timh@sfb288.math.tu-berlin.de
Technische Universität Berlin Fachbereich Mathematik, MA 8-3, Straße des 17. Juni 136, D-10623 Berlin

**Hoppe, Jens** hoppe@aei-potsdam.mpg.de
Albert Einstein Institute, Am Muehlenberg 1, D 14476 Potsdam, Germany

**Huebschmann, Johannes** Johannes.Huebschmann@agat.univ-lille1.fr
USTL, UFR de Mathematiques, F-59655 Villeneuve d'ASCQ Cedex France

**Igor, Bayak** bayak@mail.ru
St, Kosmonavtov 10-3, Grodno, 230003, Belarus

**Jaffe, Arthur** jaffe@math.harvard.edu
27 Lancaster Street (private address), Cambridge, Massachusetts 02140, USA

**Janauschek, Barbara** janausch@rz.mathematik.uni-muenchen.de
Mathematisches Institut der Universität, Theresienstraße 39, 80333 München

**Jansen, Stefan** jansen@statistik.uni-dortmund.de
Wilhelmstr.13, 44137 Dortmund

**Jecko, Thierry** jecko@maths.univ-rennes1.fr
IRMAR, Campus Beaulieu, F-35042 Rennes Cedex, France

**Karcher, Hermann** UNM416@uni-bonn.de
Math. Inst. Beringstr.1, D-53115 Bonn

**Karowski, Michael** karowski@physik.fu-berlin.de
FU-Berlin, Arnimallee 14, 14195 Berlin

**Kath, Ines** kath@mathematik.hu-berlin.de
Institut fuer Mathematik der Humboldt-Universitaet zu Berlin, Sitz: Ziegelstr. 13a, 10099 Berlin

**Khadem-Al-Charieh, Samy** khadem@physik.tu-berlin.de
Köthener Straße 32, 10963 Berlin

**Kirchberg, Klaus-Dieter** kirchberg@mathematik.hu-berlin.de
Ziegelstr. 13A 10099 Berlin

**Klein, Markus** mklein@math.uni-potsdam.de
Am Neuen Palais 10, 14469 Potsdam

**Knauf, Andreas** knauf@mi.uni-erlangen.de
Mathematisches Institut,Bismarckstr. 1 1/2, 91054 Erlangen

**Korff, Christian** korff@physik.fu-berlin.de
Suarezstr. 55, 14057 Berlin

**Korotyaev, Evgeni** evgeni@math.uni-potsdam.de
Universitaet Potsdam, Institut fuer Mathematik

**Kostrykin, Vadim** kostrykin@ilt.fhg.de
ILT, Steinbachstrasse 15, D-52074 Aachen, Germany

**Kovarik, Hynek** kovarik@ujf.cas.cz
Dep. of Theoretical Physics, NPI, Academy of Sciences, 25068 Rez near Prague, Czechia

**Krasovsky, Igor** ivk@mpipks-dresden.mpg.de
Max-Planck-Institut fuer Mathematik, Inselstr. 22–26, D-04103 Leipzig

**Kreft, Christian** christian.kreft@ac.com
Ahornweg 2, 65193 Wiesbaden

**Krejcirik, David** krejcirik@ujf.cas.cz
Theory Group, Nuclear Physics Institute, Academy of Sciences, 25068 Rez near Prague, Czech Republic

**Kronz, Manfred** kronz@mathematik.hu-berlin.de
Mathematisches Institut, Humboldt-Universität zu Berlin, Ziegelstr.13a, 10099 Berlin

**Kunze, Markus** mkunze@mi.uni-koeln.de
Mathematisches Institut, Universitaet Koeln, Weyertal 86, 50931 Koeln

**Kutz, Nadja** nadja@math.tu-berlin.de
TU Berlin, Str. d. 17. Juni 136, MA 8-5

**Lang, Georg** lang@math.tu-berlin.de
Rubensstr. 94, 12157 Berlin

**Lehmann, Detlef** lehmann@math.tu-berlin.de
TU Berlin, FB MA 7-2, Strasse des 17. Juni 136, 10623 Berlin

**Leitner, Felipe** leitner@mathematik.hu-berlin.de
Ziegelstr. 13A 10099 Berlin

**Lenz, Daniel** dlenz@math.uni-frankfurt.de
Fb. Mathematik;J.W. Goethe Universitaet; 60054 Frankfurt/Main

**Leschke, Katrin** katrin@sfb288.math.tu-berlin.de
Dr. Katrin Leschke

**Lieb, Elliott** lieb@princeton.edu
Department of Physics, Jadwin Hall, Princeton University, P.O.B 708, Princeton, NJ 08544-0708, USA

**Luest, Dieter** luest@physik.hu-berlin.de
Institut fuer Physik, Humboldt-Universitaet, Invalidenstr. 110, 10115 Berlin

**Lusala, Tsasa** tsasa@sfb288.math.tu-berlin.de
TU-Berlin, Fachbereich 3 / Mathematik, Sekr. MA 8-3, D-10623 Berlin

**Ma, Xiaonan** xiaonan@mathematik.hu-berlin.de
Humboldt universitat zu Berlin, Institut fur mathematik, unter den Linden 6 , D-10099 Berlin.

**Malmendier, Andreas** malmendi@th.physik.uni-bonn.de
Kapellenstrasse 48, 53115 Bonn

**Marvan, Michal** Michal.Marvan@math.slu.cz
Silesian University, Bezrucovo nam. 13

**Maschler, Gideon** maschler@mpim-bonn.mpg.de
Max Planck Institut fur Mathematik, P.O. Box 7280, D-53072, Bonn Germany

**Matveev, Vladimir S.** vmatveev@ihes.fr
IHES, 35. Route de Chartres, 91440 France

**McCune, Catherine** cat@sfb288.math.tu-berlin.de
SFB 288, TU-Berlin, MA 8-5; Strasse des 17 Juni 136; D-10623 Berlin

**McIntosh, Ian** im7@york.ac.uk
Department of Mathematics, University of York, York YO10 5DD, UNITED KINGDOM

**Müller, Olaf** olaf@math.uni-bonn.de
Hainstr. 52, Zi. 221 , 53121 Bonn

**Müller, Werner** mueller@math.uni-bonn.de
Mathematisches Institut der Universität Bonn, Beringstraße 1, D-53115 Bonn

**Myrzakul, Kuralay** cnlpmyra@satsun.sci.kz
Institute of Physics and Technology, 480082, Alma-Ata-82, Kazakhstan

**Myrzakulov, Ratbay** cnlpmyra@satsun.sci.kz
Institute of Physics and Technology

**Neumaier, Nikolai** nine@physik.uni-freiburg.de
Fakultät für Physik, Universität Freiburg, Hermann-Herder-Str. 3, 79104 Freiburg i. Br.

**Novikov, Sergej** novikov@ipstmail.umd.edu
Mathematics Department, University of Maryland, College Park, MD 20742 - 4015, USA

**Paul, Thierry** paul@dmi.ens.fr
DMI, ENS, 45 rue d'Ulm, 75230 Paris, France

**Pavlyk, Oleksandr** pavlyk@yahoo.com
viale XX Settembre, 76, Trieste 34126, Italy

**Pedit, Franz** franz@gang.umass.edu
Paul Lincke Ufer 31, 10999 Berlin

**Peters, G. Paul** leiterer@math.tu-berlin.de
Stubbenkammer Str. 12

**Pfaeffle, Frank** pfaeffle@math.uni-hamburg.de
Mathematisches Seminar der Universität Hamburg, Bundesstr. 55, 20146 Hamburg

**Pflaum, Markus** pflaum@mathematik.hu-berlin.de
Humboldt-Universitaet, Unter den Linden 6, 10099 Berlin

**Pinkall, Ulrich** pinkall@math.tu-berlin.de
Sekr. MA 8-5, TU Berlin, Strasse des 17. Juni 136, 10623 Berlin

**Polthier, Konrad** polthier@math.tu-berlin.de

TU-Berlin

**Post, Olaf** o.post@tu-bs.de
Institut für Analysis, Pockelsstr. 14, 38106 Braunschweig

**Potthoff, Jürgen** potthoff@math.uni-mannheim.de
LS Math 5, Univ. Mannheim, 68131 Mannheim

**Protogenov , Alexander** alprot@appl.sci-nnov.ru
Institute of Applied Physics of the RAS

**Quintino, Aurea** aurea@ptmat.lmc.fc.ul.pt
Rua da Fonte 31, Negrais 2715-328 Almargem do Bispo

**Rademacher, Hans-Bert** rademacher@mathematik.uni-leipzig.de
Universität Leipzig, Mathematisches Institut, Augustusplatz 10/11, 04109 Leipzig

**Ramacher, Pablo** ramacher@mathematik.hu-berlin.de
Inst. fuer Reine Mathematik, Ziegelstr. 13 A, 10099 Berlin

**Recknagel, Andreas** anderl@aei-potsdam.mpg.de
Max-Planck-Institut fuer Gravitationsphysik, Am Muehlenberg 1, D-14467 Golm

**Reitebuch, Ulrich** fuli@sfb288.math.tu-berlin.de
Hildegardstrasse 3, 10715 Berlin Wilmersdorf

**Remi, Leandre** leandre@iecn.u-nancy.fr
Department of Mathematics. Universite de Nancy I. 54000. Vandoeuvre-les-Nancy. France

**Roggenkamp, Daniel** roggenka@th.physik.uni-bonn.de
Physikalisches Institut, Nussallee 12 (AVZ 106), 53115 Bonn

**Rosenberger, Elke** erosen@rz.uni-potsdam.de
Kurländer Allee 14, 14055 Berlin

**Sacchetti, Andrea** Sacchetti@unimo.it
Department of Mathematics, Via Campi 213/B 41100 Modena, Italy

**Samuel Kofi Quarcoo, Samuel Kofi** makocs@hotmail.com
U.S.T-University hall,Room 86A-Kumasi, Ghana

**Scheck, Florian** scheck@dipmza.physik.uni-mainz.de
Institut für Physik, Johannes Gutenberg-Universität, 55099 Mainz

**Schlein, Benjamin** schlein@itp.phys.ethz.ch
Theoretische Physik, ETH Hoenggerberg, 8093 Zuerich, Schweiz

**Schmidt, Karl Michael** kschmidt@rz.mathematik.uni-muenchen.de
Mathematisches Institut der Universitaet, Theresienstr. 39, 80333 Muenchen

**Schmidt, Martin Ulrich** Martin.Schmidt@physik.fu-berlin.de
Institut fuer Theoretische Physik, Arnimallee 15, 14195 Berlin

**schmoll, martin** mschmoll@physik.uni-bielefeld.de
Sinselbachstr.12 34212 Melsungen

**Schnee, Kai** schnee@itp.phys.ethz.ch
Swiss Federal Institute of Technology (ETH), Institute for Theoretical Physics, ETH-Hoenggerberg HPZ G5, CH-8093 Switzerland

**Schrader, Robert** schrader@physik.fu-berlin.de
Inst.f.Theor.Physik, FUB, Arnimallee 14, 14195 Berlin

**Schwertfeger, Karl-Heinz** khschwertfeger@t-online.de
10245-Berlin

**Seiler, Ruedi** seiler@math.tu-berlin.de

FB Mathematik, TU Berlin, MA 7-2

**Simon, Barry** bsimon@bigfoot.com
Department of Mathematics, California Institute for Technology, Pasadena, CA 91125

**Simon, Udo** simon@math.tu-berlin.de
TU-BERLIN, MA 8-3

**Solomyak , Michael** solom@wisdom.weizmann.ac.il
Dep. of Math., Weizmann Institute of Science, Rehovot 76100, Israel

**Springborn, Boris** springb@sfb288.math.tu-berlin.de
Sekr. MA 8-5, TU Berlin, Strasse des 17. Juni 136, 10623 Berlin

**Steudel, Heinz** steudel@physik.hu-berlin.de
Institut für Physik der Humboldt-Universität Berlin, Invalidenstr. 110, 10115 Berlin

**Strohmaier, Alexander** alexander.strohmaier@itp.uni-leipzig.de
Institut für theoretische Physik, Augustusplatz 10/11, 04109 Leipzig

**Suris, Juri** suris@sfb288.math.tu-berlin.de
TU Berlin, FB Mathematik, Sekr. MA 8-5, Str. des 17. Juni 136, 10623 Berlin

**Terng, Chuu-Lian** terng@neu.edu
Department of Mathematics, Northeastern University, Boston, MA 02115, USA

**Teufel, Stefan** teufel@ma.tum.de
TU Muenchen, Zentrum Mathematik, Gabelsbergerstr. 49, 80333 Muenchen

**Travaglia, Marcos Vinicio** marcos@mathematik.uni-mainz.de
Generaloberst-Beck-Strasse 5 App. 3.10 55129 Mainz

**Verch, Rainer** verch@theorie.physik.uni-goettingen.de
Inst. f. Theor. Phys., Bunsenstr. 9, 37073 Goettingen

**Veselic, Ivan** Ivan.Veselic@ruhr-uni-bochum.de
RUB,44780 Bochum, Mathematik, NA 3/27

**Vrancken, Luc** luc@sfb288.math.tu-berlin.de
Sekr. MA 8-3 Fachbereich Mathematik TU-Berlin

**Wiehe, Martin** wiehe@math.tu-berlin.de
TU-Berlin, MA 8-3

**Wienhard, Anna** wienhard@math.uni-bonn.de
Math. Institut, Beringstrasse 1, 53115 Bonn

**Yamada, Kotaro** kotaro@math.sci.kumamoto-u.ac.jp
Dept. Math, Kumamoto University, Kumamoto 860-8555 Japan/MA 8-3 Fachbereich Mathematik, TU-Berlin, Strasse des 17. Juni 135, 10623, Berlin

**Zenk, Heribert** zenk@mathematik.uni-mainz.de
Fachbereich Mathematik (17), Johannes Gutenberg Universitaet, 55099 Mainz

**Zograf, Peter** zograf@pdmi.ras.ru
Steklov Math. Inst., Fontanka 27, 191011 St.Petersburg RUSSIA

## IV.1 Organization of the Conference

Sonderforschungsbereich (SFB) 288
*Differentialgeometrie und Quantenphysik*

Volker Bach
FB Mathematik
Universität Mainz
D-55099 Mainz
email:
vbach@mathematik.uni-mainz.de

Jochen Brüning
Institut fuer Mathematik
Humboldt-Universitaet zu Berlin
Unter den Linden 6
D-10117 Berlin
email:
bruening@mathematik.hu-berlin.de

Georg Lang
FB Mathematik MA 7-2
TU Berlin
Straße des 17. Juni 136
D-10623 Berlin
email:
lang@math.tu-berlin.de

Bianca Toltz
FB Mathematik MA 8-5
TU Berlin
Straße des 17. Juni 136
D-10623 Berlin
email:
sfb288@sfb288.math.tu-berlin.de

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