POSITIVE TWISTOR BUNDLE OF A KÄHLER SURFACE

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ABSTRACT. The aim of this paper is to characterize Kähler surfaces in terms of their positive twistor bundle. We prove that a 4-dimensional Riemannian manifold (M,g) admits a compatible Kähler structure if and only if $Z^+(M)$ admits a vertical Killing vector field.

1. Introduction. In our paper [J-1] we have proved that for a self-dual Einstein 4-manifold (M,g) an $SO(3)$-principal fiber bundle $p: P^- \to M$ associated to the vector bundle $\wedge^- M$ of anti-selfdual forms is an $\mathcal{A}$-manifold. In [J-1] we have explicitly constructed a family of $\mathcal{A}$-manifolds on a $SO(3)$ bundle P^-. This family is parameterized by a real number $c > 0$ and every of these manifolds has a Ricci tensor with two constant eigenvalues. If the scalar curvature τ of (M,g) satisfies $|\tau| = 48$ and $c^2 = \frac{1}{2}$ then P^- admits a 3-K-contact structure and it is a S^1-principal bundle over the twistor bundle $Z(M)$ such that the natural projection is a Riemannian submersion. The manifold $Z(M)$ admits a natural metric and an orthogonal almost Kähler structure such that $(Z(M), g, J)$ is an $\mathcal{A}$-manifold with Hermitian Ricci tensor. The (negative) twistor bundle of Kähler surfaces was recently studied in [A-G-I]. In the present paper we study the positive twistor bundle of a 4-dimensional oriented Riemannian manifold and in particular the positive twistor bundle of a Kähler surface. We prove that the bundle (P^+, g_c) admits a vertical Killing field ξ which is invariant with respect to the action of the structural group $SO(3)$ of P^+ if and only if (M,g) admits a compatible Kähler structure. Equivalently (M,g) admits a compatible Kähler structure if and only if the twistor bundle $Z^+(M)$ admits a vertical Killing vector field. The field ξ determines a K-contact structure on P^+ if and only if (M,g) is a Kähler-Einstein space with nonzero scalar curvature or (M,g) is a Kähler surface with zero scalar curvature whose Ricci tensor has two constant eigenvalues $\lambda, -\lambda$ where $\lambda \in \mathbb{R}_+$. We prove that (P^+, ξ) is a Sasakian structure if and only if (M,g) is a Kähler-Einstein space with non-zero scalar curvature or (M,g) is a product of two Riemannian surfaces of opposite curvatures. We show that if (M,g) admits a compatible almost Hermitian structure then P^+ is a circle bundle over the product $M \times \mathbb{CP}^1$. We prove also that an $SO(3)$-principal fiber bundle $p: P^+ \to M$ associated to the vector bundle

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$\wedge^+ M$ of selfdual forms of the Kähler-Einstein surface (M, g, J) with non-zero scalar curvature τ is an $\mathcal{A}$-manifold whose Ricci tensor has three constant eigenvalues.

2. Preliminaries. Let (M, g) be an oriented Riemannian manifold, $\dim M = 4$ and $P = P(M, SO(4))$ be the principal fiber bundle of oriented orthonormal frames of TM. The bundle of 2-forms over M splits as the Whitney sum $\wedge^2 M = \wedge^+ M \oplus \wedge^- M$, $\wedge^+ M, \wedge^- M$ being the eigenspaces bundles of the Hodge star operator $* \in \mathrm{End}(\wedge^2 M)$. The curvature tensor R of a 4-dimensional manifold (M, g) determine an endomorphism $\mathbb{R}$ of the bundle $\wedge M$ defined by $g(\mathcal{R}(X \wedge Y), Z \wedge W) = \Re(X \wedge Y, Z \wedge W) = -R(X, Y, Z, W) = -g(R(X, Y)Z, W)$ where $R(X, Y)Z = ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})Z$. Let us define $B = \frac{1}{2}(\mathcal{R} - *\mathcal{R}*); W = \frac{1}{2}(\mathcal{R} + *\mathcal{R}*)_0 = \frac{1}{2}(\mathcal{R} + *\mathcal{R}*) - \frac{\tau}{12}Id; W^+ = \frac{1}{2}(W + *W); W^- = \frac{1}{2}(W - *W)$. Then (see ex. [B])

$$\mathcal{R} = \frac{\tau}{12}Id + B + W^+ + W^-.$$

The tensor W is called the Weyl tensor and its components W^+, W^- are called a self dual and anti-self dual Weyl tensors. The Weyl conformal tensor $W \in \mathrm{End}(\wedge^2 M)$ leaves $\wedge^+ M, \wedge^- M$ invariant and the restrictions W^+, W^- of W to $\wedge^+ M, \wedge^- M$ may be considered as $(0, 4)$ tensors operating trivially on $\wedge^- M, \wedge^+ M$ respectively. By P^+ we mean a principal $SO(3)$-bundle of oriented orthogonal bases $\{\omega, \eta, \gamma\} \subset \wedge^+ M_x$, $x \in M$ of $\wedge^+ M$ such that $g(\omega, \omega) = g(\eta, \eta) = g(\gamma, \gamma) = 2$. We have a homomorphism of principal fibre bundles $F : P \longrightarrow P^+$ defined by (see [D]):

$$F(e_1, e_2, e_3, e_4) = (e_1 \wedge e_2 + e_3 \wedge e_4, e_1 \wedge e_3 + e_4 \wedge e_2, e_1 \wedge e_4 + e_2 \wedge e_3)$$

which is equivariant under homomorphism

$$\Phi_+ : SO(4) \to SO(3)$$

where $\Phi_\pm$ are defined by $\Phi + \pm(A) = \rho(A) + \pm$, ($C_\pm = C|_{\wedge^\pm \mathbb{R}^4}$) and $\rho : SO(4) \to SO(\wedge^2 \mathbb{R}^4)$ is defined on simple 2-vectors by $\rho(A)(e_i \wedge e_j) = Ae_i \wedge Ae_j$. Since $\rho(A)* = *\rho(A)$, it is clear that $\rho(A)$ leaves invariant $\wedge^\pm \mathbb{R}^4$ and $\rho(A)_\pm$ are the corresponding endomorphisms of $\wedge^\pm$ respectively. The homomorphisms $\Phi_\pm$ induce the Lie algebras homomorphisms $\phi_\pm : \mathfrak{so}(4) \to \mathfrak{so}(3)$ and $\mathfrak{so}(4) = \ker \phi_+ \oplus \ker \phi_-$ is the simple sum of ideals both isomorphic to $\mathfrak{so}(3)$. If we identify the Lie algebra $\mathfrak{so}(4)$ with $\wedge^2 \mathbb{R}^4$ then $\ker \phi_- = \wedge^+ \mathbb{R}^4$ and $\ker \phi_+ = \wedge^- \mathbb{R}^4$. Note that

$$\wedge^2 \mathbb{R}^4 = \wedge^+ \mathbb{R}^4 \oplus \wedge^- \mathbb{R}^4. \tag{2.1}$$

Let us consider an isomorphism of vector spaces defined by

$$\alpha : \mathfrak{so}(4) \to \wedge^2 \mathbb{R}^4 \qquad \alpha(A) = \sum_{i<j} A^i_j e_i \wedge e_j$$

where e_i is the standard orthonormal basis of $\mathbb{R}^4$. On $\mathfrak{so}(4)$ we consider a bi-invariant metric $\langle X, Y \rangle = -\frac{1}{2}\mathrm{tr} X \circ Y = \sum_{i<j} X^i_j Y^i_j$. Then α is an isometry of $(\mathfrak{so}(4), \langle , \rangle)$ on $\wedge^2 \mathbb{R}^4$ furnished with the standard metric also denoted by $\langle , \rangle$ defined

on the basis $e_i \wedge e_j$ $i < j$ by $\langle e_i \wedge e_j, e_p \wedge e_q \rangle = \delta^i_p \delta^j_q$. Note that $\ker\phi_- = \mathfrak{so}(4)_+ = \{X \in \mathfrak{so}(4) : \alpha(X) \in \wedge^+\mathbb{R}^4\}$ and $\ker\phi_+ = \mathfrak{so}_-(4) = \{X \in \mathfrak{so}(4) : \alpha(X) \in \wedge^-\mathbb{R}^4\}$. If $u = (f_1, f_2, f_3, f_4) \in P = SO(M)$ is an orthonormal frame then we define an isomorphism $\alpha_u : \mathfrak{so}(4) \to \wedge^2 T_{\pi(u)}M$ by $\alpha_u = \sum_{i<j} A^i_j f_i \wedge f_j$ where $\pi : SO(M) \to M$ is a projection on the base of the principal bundle $SO(M)$. Then for $A, B \in \mathfrak{so}((4)$ we have $\langle A, B \rangle = g(\alpha_u(A), \alpha_u(B))$. We shall denote by v_+, v_- or v^+, v^- the components of v with respect to decomposition (1.1). By $\mathfrak{X}(M)$ we denote the Lie algebra of all local vector fields on M. If D is a vector bundle over M then by $\Gamma(D)$ we denote the set of all local sections of D. We also write $\mathcal{A}^k(M) = \Gamma(\wedge^k TM^*)$.

Let $(P, \bar{g})$, (M, g) be Riemannian spaces. A Riemannian submersion $p : P \to M$ (see [O'N]) is a submersion which preserves lengths of horizontal vectors. We shall denote by V the distribution of vertical vectors ($X \in V$ if X is tangent to a fiber $p^{-1}(x)$ for a certain $x \in M$) and by H the horizontal distribution which is an orthogonal complement of V. Let $\mathcal{H}$ and $\mathcal{V}$ denote the projections of the tangent space TM of M onto the subspaces H, V of horizontal and vertical vectors, respectively. By T and A we denote the O'Neill's tensors defined as follows (see [O'N]):

$$T_X Y = \mathcal{H}(\nabla_{\mathcal{V}X}\mathcal{V}Y) + \mathcal{V}(\nabla_{\mathcal{V}X}\mathcal{H}Y)$$
$$A_X Y = \mathcal{V}(\nabla_{\mathcal{H}X}\mathcal{H}Y) + \mathcal{H}(\nabla_{\mathcal{H}X}\mathcal{V}Y).$$

The twistor bundle $\pi_Z : Z(M) \to M$ over an oriented four manifold (M, g) is the sphere bundle of the vector bundle $\wedge^+ TM$:

$$Z(M) = \{\omega \in \wedge^+ TM :\parallel \omega \parallel = \sqrt{2}\}.$$

This bundle may be identified with the bundle of almost complex structures on M which preserve the orientation. With every $\omega \in Z(M)$ we associate the almost complex structure J_ω on $T_{p(\omega)}M$ by the formula

$$g(J_\omega X, Y) = g(X \wedge Y, \omega).$$

Let us recall that a Riemannian manifold (M, g) is called by definition an $\mathcal{A}$-manifold (see [G]) if its Ricci tensor ρ satisfies the condition

$$\nabla_X \rho(X, X) = 0 \tag{A}$$

for all $X \in TM$. Recall that every naturally reductive homogeneous space belongs to the class of $\mathcal{A}$-manifolds, but there also many examples of non-locally homogeneous $\mathcal{A}$-manifolds. In some way the $\mathcal{A}$-manifolds together with manifolds with harmonic curvature tensor are very close to the class of Einstein manifold (see [G]).

An almost Hermitian manifold (M, g, J) is called almost Kähler if the Kähler form $\Omega(X, Y) = g(JX, Y)$ is closed $d\Omega = 0$ and strictly almost Kähler if $\nabla\Omega \neq 0$ and $d\Omega = 0$. A Ricci tensor ρ of an almost hermitian manifold (M, g, J) is called Hermitian if $\rho(JX, JY) = \rho(X, Y)$ for every $X, Y \in \mathfrak{X}(M)$. In the sequel we shall consider 4-dimensional Kähler manifolds (M,g,J), i.e. such that $\nabla\Omega=0$.. Such manifolds are always oriented and we choose an orientation in such a way that Ω is self-dual form (i.e. $\Omega\in\wedge^+M$). The vector bundle of self dual forms admits a decomposition

$$\wedge^+M=\mathbb{R}\Omega\oplus LM \tag{2.2}$$

where by LM we denote the bundle of real J-skew invariant 2-forms (i.e $LM=\{\Phi\in\wedge M:\Phi(JX,JY)=-\Phi(X,Y)\}$). The bundle LM is a complex line bundle over M with the complex structure $\mathcal{J}$ defined by $(\mathcal{J}\Phi)(X,Y)=-\Phi(JX,Y)$. A Kähler manifold (M,g,J) is called a Kähler Einstein manifold if (M,g) is an Einstein manifold. The positive Weyl tensor of a Kähler surface have a particularly simple form

$$W^+\Omega=\frac{\tau}{6}\Omega, W^+_{|LM}=-\frac{\tau}{12}id_{|LM}. \tag{2.3}$$

For an Einstein-Kähler surface we have $*\mathcal{R}=\mathcal{R}*$ and in particular

$$\mathcal{R}(\Omega)=\frac{\tau}{4}\Omega, \mathcal{R}_{|LM}=0. \tag{2.4}$$

Let (M,g) be a Riemannian manifold and ξ be a unit Killing vector field on M. Let us define a tensor field ϕ by $\phi(X)=\nabla_X\xi$ and a 1-form $\eta(X):=g(\xi,X)$. Then we call (M,g,ξ,ϕ,η) a K-contact structure if the following relation is satisfied:

$$\phi^2=-id+\eta\otimes\xi. \tag{K}$$

Let us assume that we have a Riemannian manifold (P,g) which is a principal S^1-bundle P over a manifold (M,g_*) and the natural projection $P:P\rightarrow M$ is a Riemannian submersion. Let us assume also that the group S^1 acts on P by isometries and the fundamental field ξ_0 of the action is of constant length. We shall find the conditions under which the Killing vector field $\xi=c\xi_0$ where $c=\frac{1}{\|\xi\|}$ defines the K-contact metric structure. Let us denote by $H=\ker\eta=\{X:g(\xi,X)=0\}$ the distribution of horizontal vectors on P. Then

Lemma A. *Under the above assumptions the Killing vector field ξ gives the K-contact structure on P if and only if the tensor $J=\phi_{|H}$ is the almost complex structure on the bundle H i.e. $J^2=-id_{|H}$. If this is the case then J induces an almost complex structure J_* on M and (M,g_*,J_*) is the almost Kähler manifold.*

3. Vertical Killing vector fields on the twistor bundle $Z(M)$. In the sequel we shall need several results from [J-1] which we shall recall briefly. Let (M,g) be a Riemannian manifold and $P(M,SO(4))=P$ be the $SO(4)$-principal bundle of orthonormal oriented frames on M. By ω we denote the connection form $\omega\in\mathcal{A}^1(P)\otimes\mathfrak{so}(4)$ on P induced by the Levi-Civita connection of (M,g_0). By τ we shall denote the scalar curvature of (M,g). Let $F:P\longrightarrow P^+$ be a homomorphism of principal fibre bundles defined by (see [J-1]):

$$F(e_1,e_2,e_3,e_4)=(e_1\wedge e_2+e_3\wedge e_4, e_1\wedge e_3+e_4\wedge e_2, e_1\wedge e_4+e_2\wedge e_3).$$

where $p : P^+ \to M$ is a $SO(3)$-principal bundle of orthogonal normed frames of the vector bundle $\wedge^+ M$. F is equivariant under a homomorphism $\Phi_+ : SO(4) \to SO(3)$ of Lie groups. The connection Γ on P given by ω induces a connection Γ^+ on P^+ with a connection form ω^+ defined by $F^*\omega^+ = \phi_+\omega$, where $\phi_+ = d\Phi_+$ is a homomorphism of Lie algebras. We shall identify the Lie algebra of the structural group of P^+ with $\mathfrak{so}(4)_+$. Then ϕ_+ is just a projection $\phi_+ : \mathfrak{so}(4) \to \mathfrak{so}(4)_+$ with respect to the decomposition $\mathfrak{so}(4) = \mathfrak{so}(4)_- \oplus \mathfrak{so}(4)_+$. The curvature form of Γ^+ is Ω^+ such that $F^*\Omega^+ = \phi_+\Omega$. We shall denote by V the vertical distribution and by H the horizontal distribution on TP^+. Hence $H = \{X : \omega(X) = 0\}$. We define the metric g_c on P^+ by the formula

$$g_c = -\frac{1}{2}c^2 \ \mathrm{tr}\ (\omega^+(X) \circ \omega^+(Y)) + p^*g(X, Y). \tag{3.1}$$

A natural projection $p : P^+ \to M$ is a Riemannian submersion of $P_c^+ = (P^+, g_c)$ onto (M, g). The O'Neill's tensor $T = 0$ (the fibers of p are totally geodesic and homothetic to $SO(3)$). Let us recall that a 4-manifold (M, g) is Einstein if and only if the Riemannian curvature tensor $R \in End(\wedge^2 TM)$ satisfies a condition $*R = R*$ i.e. if $\mathcal{R}$ preserves the bundles $\wedge^+ M, \wedge^- M$. If (M, g_0) is Einstein then we shall write $\mathcal{R}_+ = \mathcal{R}_{|\wedge^+ M}, \mathcal{R}_- = \mathcal{R}_{|\wedge^- M}$. Let us recall here that for $U, V \in \mathfrak{so}(4)$ we denote $\langle U, V\rangle = -\frac{1}{2}$ tr $U \circ V$. Let us recall (see [J-1]) also that for a frame $u = (f_1, f_2, f_3, f_4) \in SO(M)$ we defined a map :

$$\alpha_u : \mathfrak{so}(4) \to \wedge^2 T_{\pi(u)} M$$

by $\alpha_u(A) = u(\alpha(A)) = \sum_{i<j} A^i_j f_i \wedge f_j$. Then for $A, B \in \mathfrak{so}(4)$ we have $\langle A, B\rangle = g(\alpha_u(A), \alpha_u(B))$. Note that

$$\alpha_u : \mathfrak{so}(4)_+ = \wedge^+\mathbb{R}^4 \to \wedge^+ M$$

is an isomorphism and if $\Phi_+(g) = id$ then $\alpha_{ug|\mathfrak{so}(4)_+} = \alpha_{u|\mathfrak{so}(4)_+}$. If $g \in SO(4)$ and $X \in \mathfrak{so}(4)$ then $Ad_g X = g \circ X \circ g^{-1}$. Note that under the isomorphism $\mathfrak{so}(4) = \wedge^2\mathbb{R}^4$ if $X = \sum A^i_j e_i \wedge e_j$ then $Ad_g X = \sum A^i_j g e_i \wedge g e_j$. In particular

$$\alpha_{ug}(X) = \alpha_u(Ad_g X), \tag{3.2}$$

Let ω^+ be the induced connection form on P^+. Note that if $F(u) = F(v)$ then $u = vg$ where $\Phi_+(g) = id$. Consequently the isomorphism $T : V \to \wedge^+ M$

$$V_{F(u)} \ni X \to \alpha_u(\omega^+_{F(u)}(X)) \in \wedge^+ M_{\pi(u)} \tag{3.3}$$

is well defined. It follows that there exists a unique vertical field ξ such that

$$\alpha_u(\omega^+_{F(u)}(\xi)) = \Omega_{\pi(u)}. \tag{3.4}$$

The field ξ is invariant with respect to the action of the group $SO(3)$ on P^+. In fact we have

$$\alpha_{ug}(\omega^+_{F(u)g}(R_g\xi)) = \alpha_{ug}(Ad(g^{-1})\omega^+_{F(u)}(\xi)) = \alpha_u(\omega^+_{F(u)}(\xi)) = \Omega_{\pi(u)}. \tag{3.5}$$

Let $v_1 = e_1\wedge e_2+e_3\wedge e_4, v_2 = e_1\wedge e_3+e_4\wedge e_2, v_3 = e_1\wedge e_4+e_2\wedge e_3$ where (e_1, e_2, e_3, e_4) is the standard basis of $\mathbb{R}^4$. Then (v_1, v_2, v_3) is an orthogonal basis of $\mathfrak{so}(4)_+$ and let $\{V_1, V_2, V_3\}$ be Killing vector fields corresponding to v_i. Since ξ is invariant with respect to the action of $SO(3)$ on P^+ it follows that $[V_i, \xi] = 0$. We can describe the field ξ also in another way. Let $U(M) \subset SO(M)$ be the $U(2)$-principal bundle of unitary frames of (M, g, J). The Levi-Civita connection from $SO(M)$ reduces to $(U(M)$ since (M, g, J) is Kähler. Consequently $L = F(U(M)$ is a $U(1)$-principle subbundle of P^+ and the connection Γ^+ on P^+ reduces to $L = F(U(M))$. Note that $(*)$ $\xi_{|L} = V_{1|L}$. Thus ξ is the only $SO(3)$-invariant vector field satisfying the condition (*). Let the forms Φ, Ψ be a local orthogonal basis of LM such that $||\Phi|| = ||\Psi|| = \sqrt{2}$. Then we can analogously define the vertical local vector fields ξ_Φ, ξ_Ψ such that

$$\alpha_u(\omega^+_{F(u)}(\xi_\Phi)) = \Phi_{\pi(u)},\ \alpha_u(\omega^+_{F(u)}(\xi_\Psi)) = \Psi_{\pi(u)}.$$

Note that the fields ξ_Φ, ξ_Ψ are $SO(3)$ invariant and consequently $[\xi_\Phi, V_i] = [\xi_\Psi, V_i] = 0$.

Next we shall prove

Proposition 1. *Let a Riemannian four-manifold (M, g) be a Kähler surface. Then the vector field ξ is an $SO(3)$-invariant Killing vector field. On the other hand if (P^+, g_c) admits a non-zero SO(3)-invariant vertical Killing vector field then (M, g) admits a compatible Kähler structure J.*

Proof. We start by showing that if (M, g) is a Kähler manifold then ξ is the Killing vector field of (P^+, g_c). We have to show that $g_c(\nabla_X\xi, Y) = -g_c(\nabla_Y\xi, X)$ for all $X, Y \in TP^+$. Let $X^* \in H_l$ and $p(X^*) = X$. Let us take a geodesic γ on M such that $\dot\gamma(0) = X$. Let l_t be a horizontal lift of the geodesic γ_t. Then $X^*_t = \dot{l}_t$ is a horizontal vector field along l_t and $\nabla_{X^*_t}X^*_t = 0$. We have $g_c(\xi, X^*_t) = 0$. Consequently $g_c(\nabla_{X^*}\xi, X^*) = 0$. It follows that $g_c(\nabla_{X^*}\xi, Y^*) + g_c(\nabla_{Y^*}\xi, X^*) = 0$ for all $X^*, Y^* \in H$. Note also that

$$\begin{aligned} g_c(\nabla_{V_i}\xi, V_j) &= g_c(\nabla_\xi V_i, V_j) = -g_c(\nabla_{V_j}V_i, \xi) = g_c(\nabla_{V_i}V_j, \xi) \\ &= -g_c(\nabla_\xi V_j, V_i) = -g_c(\nabla_{V_j}\xi, V_i), \end{aligned} \tag{3.6}$$

where we used $[\xi, V_i] = 0, \nabla_{V_i}V_j = -\nabla_{V_j}V_i$. The last case is to prove that for every horizontal field X^* we have

$$g_c(\nabla_{X^*}\xi, V_i) = g_c(\nabla_{V_i}\xi, X^*). \tag{3.7}$$

Note that $g_c(\nabla_{V_i}\xi, X^*) = 0$ since V is totally geodesic. Consequently we have to prove that $g_c(\nabla_{X^*}\xi, V_i) = 0$. Since $X^*g_c(\xi, V_i) = g_c(\nabla_{X^*}\xi, V_i) + g_c(\xi, \nabla_{X^*}V_i) = g_c(\nabla_{X^*}\xi, V_i)$ the last equation is equivalent to $X^*g_c(\xi, V_i) = 0$. It is clear that for $l = F(u) \in P^+$ we have $g_c(\xi, V_i)_l = c^2g(\alpha_u(\omega^+_l(\xi), \alpha_u(v_i)) = c^2g(\Omega, \alpha_u(v_i))$. Let x_t be a smooth path on M such that $\dot x(0) = p(X^*)$. Let l_t be a horizontal lift of x_t

such that $l_0 = l = F(u)$. It is obvious that we can take $l_t = F(u_t)$ where u_t is the horizontal lift of x_t in $SO(M)$. Thus

$$X^* g_c(\xi, V_i) = \frac{d}{dt} c^2 g(\Omega_{x_t}, \alpha_{u_t}(v_i)) = 0 \tag{3.8}$$

since $\nabla\Omega = 0$ and $\alpha_{u_t}(v_i)$ is parallel along x_t. Now let us assume that ξ is an invariant vertical Killing vector field on (P^+, g_c). Since the fibers F_x of $p : P^+ \to M$ are totally geodesic and isometric to $\mathbb{RP}^3$ with constant scalar curvature it follows that $\xi_{|F_x}$ is a right invariant Killing vector field on each F_x. In particular since $|\xi|$ is constant on F_x it follows that $\nabla_\xi \xi = 0$ on F_x and hence $\nabla_\xi \xi = 0$ on P^+. It means that $|\xi|$ is constant on the whole of P^+. We can assume that $g_c(\xi, \xi) = 2c^2$. Let us define the Kähler form on (M, g) by the formula (3.4). From (3.7) it follows that $g_c(\nabla_{X^*}\xi, V_i) = 0$ (the fibers are totally geodesic), thus (3.8) holds and consequently $\nabla\Omega = 0$ (note that $\alpha_u(v_i) : i = 1, 2, 3$ span $\wedge^+ M$). $\diamondsuit$

If $\pi_Z : Z(M) \to M$ is a twistor bundle of (M, g) then a field $X \in \mathfrak{X}(M)$ is called vertical if $\pi_Z(X) = 0$. We have

Proposition 2. *Let $(Z(M), g_c)$ be a positive twistor bundle of a Riemannian 4-manifold (M, g) with the standard metric $g_c = -\frac{1}{2}c^2 tr(\omega^+ \circ \omega^+) + p^* g$. Then (M, g) admits a compatible positive Kähler structure if and only if $(Z(M), g_c)$ admits a non-zero vertical Killing vector field ξ.*

Proof. Let ξ be a vertical Killing vector field on $Z(M)$. Then locally ξ lifts to Killing vector field ξ^* on P^+. Let V_1, V_2, V_3 be a basis of a Lie algebra $\mathfrak{so}(3)_+$ of the structural group of P^+ (see [J-1]). Twistor bundle $Z(M)$ is the image of P^+ by a projection $\Phi_1 : P^+ \to Z(M)$ given by $\Phi_1(\omega_1, \omega_2, \omega_3) = \omega_1$. Now it is easy to show that $[\xi^*, V_1] = 0$ since both ξ^*, V_1 are Killing fields and $\nabla_{V_1} V_1 = 0$. Note that on each fiber ξ is π_Z - related with a Killing right-invariant vector field on $SO(3)$ with bi-invariant metric (V_i corresponds to the left-invariant Killing fields). Thus our local lift is $SO(3)$-invariant and hence uniquely determined and the local lifts glue together to the global lift ξ^* which is a $SO(3)$-invariant non-zero global vector field and thus gives a Kähler structure on (M, g).$\diamondsuit$

Proposition 3. *Let us assume that (M, g, J) is a Kähler surface with Kähler form Ω and scalar curvature τ. The field ξ determines a K-contact structure on P^+ if and only if (M, g) is (up to homothety) a Kähler-Einstein space with nonzero scalar curvature or (M, g) is a Kähler surface with zero scalar curvature whose Ricci tensor has two constant eigenvalues $\lambda, -\lambda$ where $\lambda \in \mathbb{R}_+$.*

Proof. We shall describe the Ricci tensor of (P^+, g_c) using the O'Neill formulas for a Riemannian submersion. Note that the O'Neill tensor T vanishes since the fibers of $p : P^+ \to M$ are totally geodesic. To compute the tensor A let us recall that if $X, Y \in T_{F(u)}P^+$ are horizontal vector fields then

$$\omega^+([X, Y])_{Fu} = -\Omega^+(X, Y)_{F(u)} = -\phi_+\Omega(\bar{X}, \bar{Y})_u = -(\Omega(\bar{X}, \bar{Y})_u)^+ \tag{3.9}$$

where $\bar{X}, \bar{Y}$ are horizontal vector fields on $P = SO(M)$ such that $dF(\bar{X}) = X \circ F$. Hence $A_X Y$ is a vertical vector field such that $\omega^+(A_X Y) = -\frac{1}{2}\Omega^+(X, Y)$. If V is a vertical vector field and $dF(\bar V)=V\circ F$ then:

(3.10)
$$\begin{aligned} g_c(A_XV,Y)_{F(u)} &= -g_c(V,A_XY)_{F(u)} = -\frac{1}{2}c^2 \text{ tr } (\omega^+(V)_{F(u)}\circ\omega^+(A_XY)_{F(u)}) \\ &= \frac{1}{4}c^2 \text{ tr } (\omega^+(V)_{F(u)}\circ\Omega^+(X,Y)_{F(u)}) = \frac{1}{4}c^2 \text{ tr}(\phi_+\omega(\bar V)_u\circ\phi_+\Omega(\bar X,\bar Y)_u). \end{aligned}$$

We can assume that X,Y are horizontal lifts of vector fields $X_*,Y_*\in\mathfrak{X}(M)$. Consequently we obtain

$$g_c(A_XV,Y)_{F(u)} = \frac{1}{2}c^2\mathcal{R}(X_*\wedge Y_*,\alpha_u(\omega^+(V))) = \frac{1}{2}c^2g(X_*\wedge Y_*,\mathcal{R}(\alpha_u\omega^+(V))). \tag{3.11}$$

Note that $\mathcal{R}(\Omega)=\alpha\Omega+\rho^0$ where $\alpha=\frac{\tau}{4}$. The O'Neill's tensor A of the Riemannian submersion $p:P^+\to M$ satisfies the condition

$$g_c(A_X\xi,Y)_{F(u)} = \frac{1}{2}c^2\alpha\Omega(X_*,Y_*)_{\pi(u)} + \frac{1}{2}c^2\rho^0(X_*,JY_*) \tag{3.12}$$

where ρ^0 is the trace-less part of the Ricci tensor ρ of (M,g), X,Y are horizontal vectors and $\alpha=\frac{1}{4}\tau$. The field ξ is a unit field if $c^2=\frac{1}{2}$ and determines a K-contact structure if the tensor $\phi=\nabla\xi$ restricted to the distribution $H_1=\{X\in TP^+ : g_c(X,\xi)=0\}$ gives almost complex structure i.e. $(\phi_{|H_1})^2=-id_{|H_1}$. We shall assume in the sequel that $c^2=\frac{1}{2}$ and that P^+ is furnished with the metric $g_1=g_{\sqrt{\frac{1}{2}}}$. Note that if $c^2=\frac{1}{2}$ then $\phi_{|H}$ gives an almost complex structure in $V_1=\{X\in V : g_1(X,\xi)=0\}$. Since for the horizontal lifts X,Y of the fields $X_*,Y_*\in\mathfrak{X}(M)$ we have

$$g_1(\phi X,Y) = \frac{1}{4}\alpha\Omega(X_*,Y_*)_{\pi(u)} + \frac{1}{4}\rho^0(X_*,JY_*)$$

it follows that ϕ induces an almost Kähler structure on H if and only if $|\alpha|=4$ and $\rho^0=0$ which means that (M,g) is an Einstein manifold with the scalar curvature $\tau=\pm16$ or $\tau=0$ and ρ^0 is a Kähler form of some almost Hermitian structure on (M,g) opposite to the Kähler structure J. It is possible if and only if ρ has two constant eigenvalues $\lambda=\pm4$. From (3.11) it follows that for a Kähler-Einstein surface (M,g,J) we have $\phi_{|H}=J^*$ where J^* denotes the horizontal lift of the Kähler structure J i.e. $J^*X^*=(JX)^*$. Thus $\phi^2(X)=-X$ for all $X\in H$. To show that Lemma A is satisfied for $(P^(M),g_1,\xi)$ we have to show that $\phi_{|V_1}$ is an almost complex structure on V_1 where $V_1=\{X\in V : g_1(X,\xi)=0\}$ i.e. $H_*=H\oplus V_1$. It is obvious since the leaves of V are totally geodesic and isometric to $\mathbb{RP}^3$ with the metric of constant sectional curvature 1. Analogously we prove the second case. $\diamondsuit$

Now let us assume that (M,g,J) is a Kähler-Einstein surface. Since $\mathcal{R}(\Phi)=\mathcal{R}(\Psi)=0$ analogously we obtain for any unit horizontal vector fields X,Y

$$g_c(A_X\xi_\Phi,Y)_{F(u)} = g_c(A_X\xi_\Psi,Y)_{F(u)} = 0. \tag{3.13}$$

Thus

$$||A_X\xi||^2 = \frac{1}{4}c^4\alpha^2,\ ||A_X\xi_\Phi||^2 = 0 = ||A_X\xi_\Psi||^2, \tag{3.14}$$

and

$$||A_XY||^2 = \frac{1}{2c^2}g_c(A_XY,\xi)^2 = \frac{1}{8}c^2\alpha^2(\Omega(X_*,Y_*)^2. \tag{3.15}$$

Using O'Neill formulas for a Riemannian submersion we get the following results for the sectional curvatures $K_c(P_{XY})$ of the planes determined by vectors $X, Y \in TP^+$ with respect to the metric g_c. By K we denote a sectional curvature with respect to the metric g on the Kähler-Einstein surface (M,g,J). By X^*, Y^* we shall mean the horizontal lifts of vectors $X, Y \in TM$. Let $\{E_1, E_2, E_3, E_4\}$ be any local orthonormal basis of TM such that $E_2 = JE_1; E_4 = JE_3$. Then we have

$$K_c(P_{UW}) = \frac{1}{2c^2} \text{ for any } U, W \in V \text{ such that } g_c(U,W) = 0, \tag{3.16a}$$

$$K_c(P_{X^*\xi}) = \frac{1}{8}c^2\alpha^2, K_c(P_{X^*\xi_\Phi}) = K_c(P_{X^*\xi_\Psi}) = 0, \tag{3.16b}$$

$$K_c(P_{E_1^*E_2^*}) = K(P_{E_3^*E_4^*}) = K(P_{E_1E_2}) - \frac{3}{8}c^2\alpha^2, \tag{3.16c}$$

$$K_c(P_{E_1^*E_3^*}) = K(P_{E_2^*E_4^*}) = K(P_{E_1E_3}), \tag{3.16d}$$

$$K_c(P_{E_1^*E_4^*}) = K(P_{E_2^*E_3^*}) = K(P_{E_1E_4}). \tag{3.16e}$$

Thus we obtain the following formulas for the Ricci tensor ρ_c of (P^+, g_c)

$$\rho_c(\xi,\xi) = c^{-2} + \frac{1}{2}c^2\alpha^2, \tag{3.17a}$$

$$\rho_c(\xi_\Phi,\xi_\Phi) = \rho_c(\xi_\Psi,\xi_\Psi) = c^{-2}, \tag{3.17b}$$

$$\rho_c(E_i,E_i) = \frac{\tau}{4} - \frac{3}{8}c^2\alpha^2 + \frac{1}{8}c^2\alpha^2 = \alpha - \frac{1}{4}c^2\alpha^2. \tag{3.17c}$$

Using O'Neill formulas it is easy to show that ξ is an eigenfield of the Ricci tensor ρ_c. Consequently ρ_c has three eigenvalues $\lambda_0 = c^{-2} + \frac{1}{2}\alpha^2, \lambda_1 = c^{-2}, \lambda_3 = \alpha - \frac{1}{4}c^2\alpha^2$.

Let us consider the mapping of fibre bundles

$$\Phi_\xi : P^+(M) \longrightarrow \tilde{Z}(M) = M \times \mathbb{CP}^1 \tag{3.18}$$

given by $\Phi(F(u_0g)) = (\pi(u_0), Ad(g^{-1})v_1)$ where $u_0 \in U(M)$ and $g \in SO(4)$. Note that $P^+(M)$ is a principal S^1-bundle over $\tilde{Z}(M)$. The mapping Φ induces on $\tilde{Z}(M)$ the Riemannian metric g_{c*} such that Φ is a Riemannian submersion. This is possible since ξ is a Killing vector field and we can define $g_{c*}(X,Y)_\omega = g_c(X^*,Y^*)_z$ where $X, Y \in TZ(M)$ and X^*, Y^* denote horizontal lifts of X, Y to T_zP^+ (i.e. $X^*, Y^* \in H_* = [\xi]^\perp = ker\eta$ where $\eta(X) = g_c(\xi, X)$ and $\Phi(X^*) = X, \Phi(Y^*) = Y$.)

Since $L_\xi \eta = 0$ it follows that $[\xi, X^*] = 0$. Consequently $\xi g_c(X^*, Y^*) = 0$ and g_{c*} is well defined. Let us define by $\tilde{V}, \tilde{H}$ the vertical and horizontal distributions of the Riemannian submersion $\tilde{p} : \tilde{Z}(M) \to M$. Note that $\tilde{H} = \Phi(H)$ and $\tilde{V} = \Phi(V_1)$ where $V_1 = \{X \in V : g_1(X, \xi) = 0\}$ i.e. $H_* = H \oplus V_1$. We shall show that if (M, g) is a Kähler surface then $\tilde{Z}(M)$ is in fact a Riemannian product $(M, g) \times (\mathbb{CP}^1, h_c)$ where h_c is a metric on $\mathbb{CP}^1$ such that a submersion $\Phi_\xi : (\mathbb{RP}^3, g_c) \to (\mathbb{CP}^1, h_c)$ is a Riemannian submersion on each fibre (i.e. it is a metric of constant sectional curvature $2c^{-2}$).

Proposition 4. *Let us assume that (M, g) is a Kähler surface. Then for every $c \in \mathbb{R}_+$ the manifold $(\tilde{Z}(M), g_{*c})$ is a Riemannian product $\tilde{Z}(M) = M \times \mathbb{CP}^1$. If (M, g) is a Kähler-Einstein surface then the manifold (P^+, g_c) is an $\mathcal{A}$-manifold for every $c \in \mathbb{R}_+$.*

Proof. Note that $\Phi : (P^+, g_c) \to (\tilde{Z}(M), g_{c*})$ is a Riemannian submersion with geodesic fibers. Since the restriction $\Phi_\xi|_{p^{-1}(x)} : SO(3) \to \mathbb{CP}^1$ is a Riemannian submersion we have to show that $\Phi_\xi : H \to TM$ is an isometry. If X^* is a horizontal lift of the field $X \in \mathfrak{X}(M)$ with respect to p then $g_c(X^*, V_i) = 0$ and consequently $g_c([\xi, X^*], V_i) = 0$ which means that $[\xi, X^*] \in H$. On the other hand the fields ξ, X^* are respectively p-related with $0, X$ hence $[\xi, X^*]$ is vertical. We get $[\xi, X^*] = 0$. It means that X^* is also a horizontal lift of X with respect to Φ_ξ. Thus $\Phi_\xi : H^* \to TM$ is an isometry and $\tilde{Z}(M) = M \times \mathbb{CP}^1$ as a Riemannian product. Since $\tilde{Z}(M)$ is an $\mathcal{A}$-manifold as a product of two Einstein manifolds it follows that (P^+, g_c) is an $\mathcal{A}$-manifold(see [J-2]). It can also be verified directly. $\diamondsuit$

Proposition 5. *Let us assume that (M, g, J) is a Kähler manifold. Then (P^+, g_1, ξ) is a Sasakian structure if and only if (M, g) is a Kähler-Einstein space with scalar curvature $\tau = \pm 16$ or (M, g) is (locally) a product of two Riemannian surfaces of opposite curvatures $K = \pm 4$. In these cases tensor $\phi = \nabla \xi$ induces on $(\tilde{Z}(M), g_*)$ a Kähler structure which is a product Kähler structure of $\tilde{Z}(M) = M \times \mathbb{CP}^1$.*

Proof. From Prop.2 it follows that (P^+, g_1, ξ) is a K-contact structure if and only if (M, g) is (up to homothety) a Kähler-Einstein space with nonzero scalar curvature or (M, g) is a Kähler surface with zero scalar curvature whose Ricci tensor has two constant eigenvalues $\lambda, -\lambda$ where $\lambda \in \mathbb{R}_+$. Consequently in those cases ϕ induces an almost Kähler structure J_* on $(\tilde{Z}(M), g_*)$ and $(\tilde{Z}(M), g_*, J_*)$ is an almost Kähler manifold. It is easy to verify that in the first case the induced structure is a Kähler product structure of $M \times \mathbb{CP}^1$ and thus the structure is a Sasakian structure. In the second case the structure is Sasakian if and only if the opposite almost Kähler structure given by $\rho(X, JY)$ is a Kähler structure which holds only if (M, g, J) is (locally) a product of two Riemannian surfaces of constant opposite curvatures.$\diamondsuit$

Now we compute the Ricci tensor of the positive twistor bundle $(Z(M), g_{*c})$ of a Kähler-Einstein surface (M, g, J). We have

$$g_c(A_X V_i, Y)_{F(u)} = \frac{1}{2|\xi|^2} c^2 \alpha \Omega(X_*, Y_*) g_c(V_i, \xi). \tag{3.19}$$

Thus

$$||A_X V_i||^2 = \frac{1}{16}\alpha^2 g_c(V_i, \xi)^2 \tag{3.20}$$

and

$$||A_X Y||^2 = \frac{1}{2c^2} g_c(A_X Y, \xi)^2 = \frac{1}{8}c^2\alpha^2(\Omega(X_*, Y_*)^2. \tag{3.21}$$

Twistor bundle $(Z(M), g_{*c})$ is an image of P^+ by a Riemannian submersion $\Phi_1 : P^+ \to Z(M)$ given by the Killing field V_1 and defined by $\Phi_1(\omega_1, \omega_2, \omega_3) = \omega_1$. Let us denote by ρ_{*c} the Ricci tensor of $(Z(M), g_{*c})$. Let us denote $\phi = \nabla V_1$. Let us assume that $X, Y \in \{X \in TP^+ : g_c(X, V_1) = 0\}$ and let $X_* = \pi_Z(X), Y_* = \pi_Z(Y)$. Then we have the following formula:

$$\rho_c(X, Y) = \rho_{*c}(X_*, Y_*) - \frac{1}{c^2} g_c(\phi(X), \phi(Y)). \tag{3.22}$$

Note that

$$\rho_c(V_i, V_i) = \frac{1}{c^2} + \frac{1}{8}\alpha^2 c^{-2} g_c(V_i, \xi)^2. \tag{3.23}$$

Let us assume that $A \in V, g_c(A, V_1) = 0$ and $X \in H$. Then $\phi(X) = A_X V_1$ and we have

$$\phi(A) = J_{\mathbb{CP}^1}(A_*)^*, \phi(X) = \frac{1}{4}\alpha g_c(V_1, \xi) J^* X \tag{3.24}$$

where $J_{\mathbb{CP}^1}$ is the standard Kähler structure on $\mathbb{CP}^1 = \pi_Z^{-1}(x)$. Consequently from (3.22) we derive :

$$\begin{aligned} \rho_{*c}(A, A) &= \frac{2}{c^2} g_c(A, A) + \frac{1}{8}\alpha^2 c^{-2} g_c(A, \xi)^2 \\ \rho_{*c}(X, X) &= (\alpha - \frac{1}{4}c^2\alpha^2 + \frac{1}{16}\alpha^2 c^{-2} g_c(V_1, \xi)^2)|X|^2. \end{aligned} \tag{3.25}$$

It follows that the Ricci tensor of $(Z(M), g_c)$ has three eigenvalues $\lambda_1 = \frac{2}{c^2}, \lambda_2 = \frac{2}{c^2} + \frac{1}{8}\alpha^2 c^{-2}(2c^2 - \frac{1}{2c^2} g_c(\xi, V_1)^2), \lambda_3 = \alpha - \frac{1}{4}c^2\alpha^2 + \frac{1}{16}\alpha^2 c^{-2} g_c(V_1, \xi)^2$.

Corollary. *Let $(Z(M), g_c)$ be the positive twistor bundle of the Kähler-Einstein surface (M, g, J) with the standard Riemannian metric g_c. Then the eigenvalues of the Ricci tensor ρ_{c*} of $Z(M)$ are constant if and only if (M, g, J) is Ricci flat. In particular the positive twistor space $Z(M)$ of a Kähler-Einstein surface with non-zero scalar curvature can never be locally homogeneous.*

Remark. Note that the twistor space $(Z(M), g_c)$ of an oriented Riemannian 4-manifold (M, g) admits more than one vertical Killing vector field if and only if (M, g) is a Kähler surface with vanishing Ricci tensor and hence locally hyper-Kähler. In fact a Kähler surface can admit two positive Kähler structures only if $\rho = 0$ (see for ex. [D]). Note that in this case P^+ is locally a product $M \times SO(3)$ (since then $\omega^+ = 0$). Consequently the space of local vertical Killing fields on $Z(M)$ which is locally a product $M \times \mathbb{CP}^1$ is three dimensional which means that (M, g) is locally hyperKähler. Note also that the fact that $(Z(M), g_c)$ is homogeneous does not imply that there exist (even locally) a vertical Killing vector fields on $Z(M)$. For example the negative twistor space of $\mathbb{CP}^2$ is a homogeneous flag space $F_{1,2}$ (see ex. [S],[A-G-I]) but it is clear that $(\mathbb{CP}^2, g_{F-S})$ does not admit any local opposite Kähler structure.

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References.

[A-G-I] V. Apostolov, G. Grantcharov and S. Ivanov *Hermitian Structures on Twistor Spaces* Ann. Global Analysis and Geom. **16**, 291-308 ,1998.

[B] A. Besse 'Einstein Manifolds' *Springer-Verlag* Berlin Heidelberg (1987).

[D] A. Derdziński ,*Self-dual Kähler manifolds and Einstein manifolds of dimension four*, Compositio Math. 49, (1983), 405-433.

[G] A. Gray '*Einstein like manifolds which are not Einstein*' Geom. Dedicata. **7** , 259-280 (1978).

[J-1] W.Jelonek *Almost Kähler $\mathcal{A}$-Structures on Twistor Bundles* Ann. Global Analysis and Geom. **17**, 329-339 ,1999.

[J-2] W.Jelonek *K-contact $\mathcal{A}$-manifolds* Colloq. Math.**75** No 1,(1998), 97-103.

[O'N] B. O'Neill.: '*The fundamental equations of a submersion*' Mich. Math. J. **13**,(1966), 459-469.

[S] S. Salamon Riemannian geometry and holonomy groups *Pitman Research Notes in Math.* Ser.201, (1989)

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