CONFORMAL REPARAMETRIZATION OF A FREE BOUNDARY FLOW IN 2D HYDRODYNAMICS*

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Abstract. We study a 2D potential flow problem with both free and prescribed boundary conditions. A conformal reparametrization of the domain of the flow leads to a first order elliptic system with first order boundary conditions for the flow and the parametrization. We prove that this system in elliptic and give its index. The index computation uses properties of the Atiyah-Bott formula and index formulae for the classical Riemann-Hilbert boundary problem. In the case of the centrifuge with an annulus-type profile, we prove that the system admits a one dimensional kernel and cokernel.

Key words. potential flow, Riemann-Hilbert problem, free boundary problem

AMS subject classifications. 76M40, 35F30, 35J55, 35Q15

1. Introduction. Our purpose is to study the following

PROBLEM 1. *Let $G, \partial G \subseteq \mathbf{R}^2$ be a bounded domain with smooth boundary and $m < \operatorname{area} G$. Find*

(i) *a region $\Omega, \partial\Omega \subseteq \bar{G}$ with $\operatorname{area}\Omega = m$ and boundary components*

$$\partial\Omega = \bigcup_1^\sigma \partial_j\Omega$$

(ii) *speeds $\gamma_j \in \mathbf{R}^+$*

(iii) *a velocity field $\vec{w}$ on $\bar{\Omega}$*

with:

(i) $\operatorname{div} \vec{w} = 0$ *and* $\operatorname{curl} \vec{w} = 0$ *in* Ω

(ii) $\vec{w}$ *is tangential on* $\partial\Omega$

(iii) *either $\partial_j\Omega \subset \partial G$ or $\partial_j\Omega \subset G$ and $\left|\vec{w}\right| = \gamma_j$ on $\partial_j\Omega$.*

In §2 we prove that solutions vary smoothly if G is varied smoothly. An example of the above problem is the case of the unit disc $G = \{(x,y) \in \mathbf{R}^2 : x^2 + y^2 \leq 1\}$. This represents a centrifuge. The rotationally invariant solutions are

$$\Omega = \{r^2 < x^2 + y^2 < 1\}, \qquad \vec{w} = (\gamma r/(x^2+y^2))(\mp y, \pm x)$$

where $r^2 = 1 - \frac{m}{\pi}$. Here, we prove the following. There exists a one-parameter family of solutions containing these particular solutions. For a one-codimensional family of perturbations of G we show that the centrifuge admits a solution close to the rotationally invariant one.

Since the domain Ω of $\vec{w}$ and its free boundaries in the interior of G are not determined in Problem 1, we will study a new problem in which Ω and the velocity $\vec{w}$ are parametrized by conformal maps (x,y) and (u,v) on a fixed domain $A, \partial A \subseteq \mathbf{R}^2$. The fact that the velocity $\vec{w}$ is tangential to the boundary $\partial\Omega$ leads to a first-order

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boundary condition for the parameterization. This problem and its linearization are formulated in §2. We prove that it is an elliptic boundary value system in the sense of Lopatinski [6]. In §3 we compute the index of the associated Fredholm operator. The argument has several parts. First a calculation of the index in the specific case of the annulus-type profile of a centrifuge in Proposition 3, see also [1] for related arguments. Secondly we use properties of the Atiyah-Bott topological index formula for manifolds with boundary, see [2], under change of the boundary conditions. Finally we deform the boundary conditions so as to decouple the system into two Riemann-Hilbert boundary problems for holomorphic functions on the unit disc and compute their index, which was first given in [8].

Helmholtz [4] and Kirchoff [5] solved Problem 1 in the case of a piecewise linear boundary using complex methods including the Schwartz-Christoffel mapping theorem. The latter does not apply in our case. For applications of the Riemann-Hilbert boundary problem to other problems of fluid mechanics we refer to [7],[9], and [10].

2. A related elliptic Problem. Problem 1 is awkward because the domain of definition Ω is not known. We therefore introduce an auxiliary Problem 2 where $A, \partial A \subseteq \mathbf{R}^2$ with coordinates ξ, η is a given parameter space and seek a conformal mapping taking A to Ω as well as $\vec{w}$, a function on A, representing the velocity. This differs from the original Problem 1 in three respects.

(i) The conformal class of A and hence of the graph of $\vec{w}$ is prescribed.

(ii) We drop the mass constraint on Ω but fix the velocities γ_j on its free boundary components.

(iii) Two solutions of the auxiliary Problem 2 which differ by composition with a conformal automorphism of A yield the same solution of Problem 1.

Problem 2. *Given bounded domains $A, \partial A \subseteq \mathbf{R}^2$ where $\partial A = \bigcup_1^\sigma \partial_j A$ and $G, \partial G \subseteq \mathbf{R}^2$, and given constants $\gamma_j \neq 0$, find functions x, y, u, v on $\bar{A}$ such that the map $(\xi, \eta) \mapsto (x, y)$ is invertible,*

$$\frac{\partial x}{\partial \xi} - \frac{\partial y}{\partial \eta} = 0, \qquad \frac{\partial y}{\partial \xi} + \frac{\partial x}{\partial \eta} = 0,$$

$$\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta} = 0, \qquad \frac{\partial v}{\partial \xi} - \frac{\partial u}{\partial \eta} = 0$$

$$\partial(x,y)/\partial(\xi,\eta) \neq 0, \qquad (x,y) \in \bar{G}$$

in A, and the following conditions on ∂A:

$$(\partial x/\partial\theta, \partial y/\partial\theta) \times (u, v) = 0,$$

where θ denotes a local coordinate on ∂A compatible with its orientation, and for each j we have either $(x,y) \in \partial G$ on $\partial_j A$ or $(x,y) \in G$ and $u^2 + v^2 = \gamma_j^2$ on $\partial_j A$.

Theorem 1. *Problem 2 is an elliptic boundary system in the sense of Lopatinski.*

Proof. We linearize Problem 2 at a solution x, y, u, v. The first order variations $\delta x, \delta y, \delta u, \delta v$ are again functions on A and we obtain the following linear problem for these.

Problem 3. *Given x, y, u, v solving Problem 2, find δx, δy, δu, δv such that*

$$\frac{\partial \delta x}{\partial \xi} - \frac{\partial \delta y}{\partial \eta} = 0, \qquad \frac{\partial \delta y}{\partial \xi} + \frac{\partial \delta x}{\partial \eta} = 0,$$

$$\frac{\partial \delta u}{\partial \xi}+\frac{\partial \delta v}{\partial \eta}=0, \qquad \frac{\partial \delta v}{\partial \xi}-\frac{\partial \delta u}{\partial \eta}=0$$

on $\bar{A}$ and

$$u\frac{\partial \delta y}{\partial \theta}+\frac{\partial y}{\partial \theta}\delta u-v\frac{\partial \delta x}{\partial \theta}-\frac{\partial x}{\partial \theta}\delta v=0$$

on ∂A. In addition, for each j, either

$$\frac{\partial \rho}{\partial x}\delta x+\frac{\partial \rho}{\partial y}\delta y=0$$

or

$$u\delta u+v\delta v=0$$

on $\partial_j A$, where ρ is a local defining function for ∂G. To verify ellipticity for Problem 3 we fix a boundary point $p \in \partial A$ and a local coordinate n normal to ∂A such that $|dn|=|d\theta|$ at p, and $n=0$ on ∂A, $n>0$ in A. The equations in $\bar{A}$ then become

$$\frac{\partial \delta x}{\partial \theta}-\frac{\partial \delta y}{\partial n}=O(n), \qquad \frac{\partial \delta y}{\partial \theta}+\frac{\partial \delta x}{\partial n}=O(n),$$

$$\frac{\partial \delta u}{\partial \theta}+\frac{\partial \delta v}{\partial n}=O(n), \qquad \frac{\partial \delta v}{\partial \theta}-\frac{\partial \delta u}{\partial n}=O(n).$$

We now freeze coefficients at p and replace $\partial/\partial\theta$ by $i\alpha$, where $\alpha \neq 0$. This gives a system of linear coefficient ODE's with boundary conditions, and the Lopatinski condition is that no solution should grow exponentially in n. The system is

$$\frac{\partial \delta y}{\partial n}-i\alpha\delta x=0, \qquad \frac{\partial \delta x}{\partial n}+i\alpha\delta y=0,$$

$$\frac{\partial \delta v}{\partial n}+i\alpha\delta u=0, \qquad \frac{\partial \delta u}{\partial n}-i\alpha\delta v=0$$

and

$$i\alpha u\delta y+\frac{\partial y}{\partial \theta}\delta u-i\alpha v\delta x-\frac{\partial x}{\partial \theta}\delta v=0$$

when $n=0$, and either

$$\frac{\partial \rho}{\partial x}\delta x+\frac{\partial \rho}{\partial y}\delta y=0$$

or $u\delta u+v\delta v=0$ when $n=0$.

The unbounded solutions of this system are

$$\delta x=Ae^{\alpha n}, \qquad \delta y=Aie^{\alpha n}$$

$$\delta u=Be^{\alpha n}, \qquad \delta v=-Bie^{\alpha n}.$$

The boundary conditions give

$$A[-\alpha u - i\alpha v] + B[\partial y/\partial\theta + i\partial x/\partial\theta] = 0$$

and either

$$A[\partial\rho/\partial x + i\partial\rho/\partial y] = 0$$

or

$$B\alpha[u - iv] = 0.$$

Since $d\rho \neq 0$, we have $A = 0$ in the first case. But $(\partial x/\partial\theta, \partial y/\partial\theta) \neq 0$ as well because x, y, u, v was assumed to solve Problem 2 and we therefore have $B = 0$. In the second case, we see from Problem 2 that $(u, v) \neq (0, 0)$, and we conclude $B = 0$, and then $A = 0$. □

THEOREM 2. *The index of Problem 2 is given by* $4 - 2\sigma$*.* The proof of this result in general depends on the special case of a centrifuge, which will be considered in the next section.

3. The centrifuge example and the proof of Theorem 2. In this section we consider the example of a centrifuge described in the introduction. The centrifuge is the simplest example of a free boundary problem with curved barriers. It also plays a role in the general index calculation in the next section. We will prove the following.

PROPOSITION 3. *The centrifuge admits a one parameter family of solutions. There is a one codimensional family of perturbations of G that admits a solution depending smoothly on the perturbation.* In particular the index of the problem is zero, though both the kernel and cokernel are nontrivial. The fact that the index is zero follows from Theorem 2, whose proof, however, makes use of this special case.

Proof. The background solution is

$$x = \xi, \quad y = \eta, \qquad u = -\gamma r\frac{\eta}{\xi^2 + \eta^2}, \quad v = \gamma r\frac{\xi}{\xi^2 + \eta^2}$$

There is a slight change of notation here. γ is now the velocity in the the counterclockwise direction rather than the absolute speed. We must describe both the kernel and cokernel of the linear operator implicit in Problem 3. To do this we must introduce an inhomogeneous term in each equation. This leads to the system

$$\frac{\partial\delta x}{\partial\xi} - \frac{\partial\delta y}{\partial\eta} = \phi \qquad \frac{\partial\delta y}{\partial\xi} + \frac{\partial\delta x}{\partial\eta} = \psi$$

$$\frac{\partial\delta u}{\partial\xi} + \frac{\partial\delta v}{\partial\eta} = \alpha \qquad \frac{\partial\delta v}{\partial\xi} - \frac{\partial\delta u}{\partial\eta} = \beta$$

on $\bar{A}$,

$$\xi\delta x + \eta\delta y = \kappa, \qquad -\gamma r\left(\xi\frac{\partial\delta x}{\partial\theta} + \eta\frac{\partial\delta y}{\partial\theta}\right) + \xi\delta u + \eta\delta v = \lambda$$

on the circle C_1 of radius 1, and

$$\frac{\gamma}{r}(\xi\delta v - \eta\delta u) = \mu, \qquad -\frac{\gamma}{r}\left(\xi\frac{\partial\delta x}{\partial\theta} + \eta\frac{\partial\delta y}{\partial\theta}\right) + \xi\delta u + \eta\delta v = \nu$$

on the circle C_r of radius r. The index of the elliptic boundary problem 2 is, by definition, the Fredholm index of the operator which takes $(\delta x, \delta y, \delta u, \delta v)$ to $(\phi, \psi, \alpha, \beta, \kappa, \lambda, \mu, \nu)$. To compute this we make the following change of variables

$$\xi = e^\rho \cos\theta, \qquad \eta = e^\rho \sin\theta,$$

$$s = \frac{\xi\delta x + \eta\delta y}{\xi^2 + \eta^2} \qquad t = \frac{-\eta\delta x + \xi\delta y}{\xi^2 + \eta^2},$$

$$p = \xi\delta u + \eta\delta v, \qquad q = \eta\delta u - \xi\delta v,$$

$$\tilde{\phi} = \cos 2\theta\,\phi + \sin 2\theta\,\psi, \qquad \tilde{\psi} = -\sin 2\theta\,\phi + \cos 2\theta\,\psi,$$

$$\tilde{\alpha} = e^{2\rho}\alpha, \qquad \tilde{\beta} = -e^{2\rho}\beta.$$

It is clear that the operators which take $(\delta x, \delta y, \delta u, \delta v)$ to (s, t, p, q) and $(\phi, \psi, \alpha, \beta)$ to $(\tilde{\phi}, \tilde{\psi}, \tilde{\alpha}, \tilde{\beta})$ are invertible. The index we want is then that of the operator which takes (s, t, p, q) to $(\tilde{\phi}, \tilde{\psi}, \tilde{\alpha}, \tilde{\beta}, \kappa, \lambda, \mu, \nu)$. These variables are related by the equations

$$\frac{\partial s}{\partial\rho} - \frac{\partial t}{\partial\theta} = \tilde{\phi}, \qquad \frac{\partial t}{\partial\rho} + \frac{\partial s}{\partial\theta} = \tilde{\psi},$$

$$\frac{\partial p}{\partial\rho} - \frac{\partial q}{\partial\theta} = \tilde{\alpha}, \qquad \frac{\partial q}{\partial\rho} + \frac{\partial p}{\partial\theta} = \tilde{\beta}$$

for $\log r \le \rho \le 0$,

$$s = \kappa, \qquad -\gamma r\left(\frac{\partial s}{\partial\theta} - t\right) + p = \lambda$$

for $\rho = 0$, and

$$-\frac{\gamma}{r}q = \mu, \qquad -\frac{\gamma}{r}\left(\frac{\partial s}{\partial\theta} - t\right) + p = \nu$$

for $\rho = \log r$. We now decompose all our functions in Fourier series in the θ variable. This gives an uncoupled system of one-dimensional boundary value problems, one for each frequency. It is simple to prove that any one-dimensional problem with as many boundary conditions as unknowns has index zero. It follows that our two-dimensional problem also has index zero. ◻

In fact, a straightforward computation shows that $(\tilde{\phi}, \tilde{\psi}, \tilde{\alpha}, \tilde{\beta})$ is zero precisely when (s, t, p, q) is of the form $(0, \sigma, -\gamma y\sigma, 0)$ and that the equations are solvable precisely when

$$\int_0^{2\pi} \lambda(\theta)\,d\theta = \int_0^{2\pi} \nu(\theta)\,d\theta.$$

The kernel and cokernel are thus one-dimensional. The kernel has a simple interpretation. In terms of the original variables it is

$$\delta x = -\sigma\eta, \quad \delta y = \sigma\xi, \qquad \delta u = -\frac{\gamma r\sigma\xi}{\xi^2 + \eta^2}, \quad \delta v = -\frac{\gamma r\sigma\eta}{\xi^2 + \eta^2}.$$

This just reflects the one-dimensional automorphism group of A. Thus the "obvious" set of solutions is isolated. In principle there might be other connected components of the moduli space of solutions, but this seems unlikely.

Proof of Theorem 2. The inhomogeneous version of Problem 3 for δx, δy, δu, δv may be rewritten as follows. On A, we have

$$(1)\quad (\partial/\partial\xi - i\partial/\partial\eta)(\delta x + i\delta y) = \phi + i\psi, \qquad (\partial/\partial\xi - i\partial/\partial\eta)(\delta u - i\delta v) = \alpha + i\beta.$$

For each $j \in 1, ..., \sigma$, we have on $\partial_j A$ either

$$(2)\qquad \frac{\partial\rho}{\partial x}\delta x + \frac{\partial\rho}{\partial y}\delta y = \kappa_j, \qquad -v\frac{\partial\delta x}{\partial\theta} + u\frac{\partial\delta y}{\partial\theta} + \frac{\partial y}{\partial\theta}\delta u - \frac{\partial x}{\partial\theta}\delta v = \lambda_j,$$

or

$$(3)\qquad u\delta u + v\delta v = \mu_j \qquad -v\frac{\partial\delta x}{\partial\theta} + u\frac{\partial\delta y}{\partial\theta} + \frac{\partial y}{\partial\theta}\delta u - \frac{\partial x}{\partial\theta}\delta v = \nu_j.$$

The conditions (2) and (3) correspond to prescribed and free boundaries of the flow. Denote by i the index of the operator defined by (1), (2), (3). By the index formula of Atiyah-Bott [2], see also [3], we have $i = i_A + \Sigma i_{\partial_j A}$, where i_A depends on the differential operator in A and $i_{\partial_j A}$ are indices associated to the boundary conditions along $\partial_j A$. Their sign changes under a change of the orientation of $\partial_j A$. Now we modify the above boundary operator by replacing (3) with (2) on a component $\partial_k A$ where the former applies. By the above remarks on the index formula, the resulting change of the index equals (up to sign, due to orientation) the index of the centrifuge, which has (2) and (3) on the outer and inner boundary component. But by Proposition 3, the latter vanishes. Repeating this replacement, we see that the operator defined by (1) in A and (2) on every component of ∂A has the same index i. In order to compute i, we change the boundary conditions (2) as follows. Note that since x, y, u, v is a solution of the background Problem 2, we have $(-v, u) = \ell\nabla\rho$ on ∂A for some function ℓ. Then the invertible transformation

$$(\kappa_j, \lambda_j) \to (\kappa_j, \lambda_j + \ell\frac{\partial\kappa_j}{\partial\theta})$$

gives rise to the boundary condition

$$(4)\qquad \frac{\partial\rho}{\partial x}\delta x + \frac{\partial\rho}{\partial y}\delta y = \kappa_j, \qquad -\ell\frac{\partial^2\rho}{\partial x\partial\theta}\delta x - \ell\frac{\partial^2\rho}{\partial y\partial\theta}\delta y + \frac{\partial y}{\partial\theta}\delta u - \frac{\partial x}{\partial\theta}\delta v = \lambda_j.$$

This again gives rise to a Fredholm operator with the same index. We would prefer the boundary value problem

$$(5)\qquad \frac{\partial\rho}{\partial x}\delta x + \frac{\partial\rho}{\partial y}\delta y = \kappa_j, \qquad \frac{\partial y}{\partial\theta}\delta u - \frac{\partial x}{\partial\theta}\delta v = \lambda_j.$$

We therefore introduce the following family of boundary conditions with parameter $\tau \in [0, 1]$:

$$(6)\qquad \frac{\partial\rho}{\partial x}\delta x + \frac{\partial\rho}{\partial y}\delta y = \kappa_j, \qquad -\tau\ell\frac{\partial^2\rho}{\partial x\partial\theta}\delta x - \tau\ell\frac{\partial^2\rho}{\partial y\partial\theta}\delta y + \frac{\partial y}{\partial\theta}\delta u - \frac{\partial x}{\partial\theta}\delta v = \lambda_j.$$

When $\tau = 0$ we have the boundary value problem (5) and when $\tau = 1$ we have (4). For each $\tau \in [0, 1]$, the homogeneous parts of (1), (6) define an elliptic boundary problem of Riemann-Hilbert type see [3], [7] for the holomorphic curve $(\delta x+i\delta y, \delta u-i\delta v)$ in C^2. This is seen from the fact that for every τ, the equations (6) define a totally real plane in $(\delta x+i\delta y, \delta u-i\delta v)$-space for fixed x, y, u, v. In particular by homotopy invariance, the index does not change in τ. Now the homogeneous part of the system (1,5) decouples into two classical Riemann-Hilbert boundary problems: $\overline{\partial}(\delta x+i\delta y)=0$ on A with $\frac{\partial\rho}{\partial x}\delta x+\frac{\partial\rho}{\partial y}\delta y=0$ on ∂A, and $\overline{\partial}(\delta u-i\delta v)=0$ with boundary condition $\frac{\partial y}{\partial\theta}\delta u-\frac{\partial x}{\partial\theta}\delta v=0$. By [9], the index of these is given by $\mu+2-\sigma$. Here, μ is the sum of the winding numbers of $\nabla\rho$ or,equivalently, of $(\frac{\partial y}{\partial\theta}, \frac{\partial x}{\partial\theta})$ over the oriented components of ∂A. Note that $\mu(\frac{\partial y}{\partial\theta}, \frac{\partial x}{\partial\theta})=-\mu(\frac{\partial x}{\partial\theta}, \frac{\partial y}{\partial\theta})$. The index of the system (1,5) with $\tau=0$ is the sum of the indices of the decoupled systems. Since the total winding numbers μ enter with opposite signs, we get $4-2\sigma$ for the index of the system. This completes the proof of Theorem 2. $\square$

REFERENCES

[1] V. Andreev, O. Kaptsov, V. Pukhnachev, and A. Rodionov, *Applications of group-theoretic methods in hydrodynamics*, Nauka, 1994. in russian.

[2] M. Atiyah and R. Bott, *The index problem for manifolds with boundary*, in Differential Analysis, papers presented at the Bombay Colloquium, University Press, Oxford, 1964, pp. 175–186.

[3] B. Booss-Bavnbeck and K. Wojciechowski, *Elliptic boundary problems for Dirac operators*, Birkhäuser, Boston, 1993.

[4] H. Helmholtz, *Über discontinuierliche flüssigkeitsbewegungen*, Monatsberichte der Berl. Akad., 63 (1868), pp. 215–228.

[5] G. Kirchoff, *Zur theorie freier flüssigkeitsstrahlen*, J. Reine Angew. Math., 70 (1869), pp. 289–298.

[6] Y. B. Lopatinski, *On a method of reducing boundary problems for a system of differential equations of elliptic type to regular equations*, Ukraïn. Mat. Zh., 5 (1953)), pp. 123–151.

[7] E. Meister, *Randwertaufgaben der Funktionentheorie, mit Anwendungen auf singuläre Integralgleichungen und Schwingungsgleichungen der mathematischen Physik*, Teubner, Stuttgart, 1983.

[8] F. Noether, *Über eine klasse singulärer integralgleichungen*, Math. Ann., 82 (1921), pp. 42–63.

[9] N. I. Vekua, *Generalized analytic functions*, Pergamon, London, 1962.

[10] E. Wegert, *Nonlinear Boundary Value Problems for Holomorphic Functions and Singular Integral Equations*, Akademie Verlag, Berlin, 1992.