Parallel spinors and connections with skew-symmetric torsion in string theory *

Thomas Friedrich (Berlin) and Stefan Ivanov (Sofia)

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Abstract

We describe all almost contact metric, almost hermitian and G_2-structures admitting a connection with totally skew-symmetric torsion tensor, and prove that there exists at most one such connection. We investigate its torsion form, its Ricci tensor, the Dirac operator and the ∇-parallel spinors. In particular, we obtain solutions of the type II string equations in dimension $n = 5, 6$ and 7.

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1 Introduction

Linear connections preserving a Riemannian metric with totally skew-symmetric torsion recently became a subject of interest in theoretical and mathematical physics. For example, the target space of supersymmetric sigma models with Wess-Zumino term carries a geometry of a metric connection with skew-symmetric torsion [22, 32, 33] (see also [40] and references therein). In supergravity theories, the geometry of the moduli space of a class of black holes is carried out by a metric connection with skew-symmetric torsion [25]. The geometry of NS-5 brane solutions of type II supergravity theories is generated by a metric connection with skew-symmetric torsion [42, 43, 41]. The existence of parallel spinors with respect to a metric connection with skew-symmetric torsion on a Riemannian spin manifold is of importance in string theory, since they are associated with some string solitons (BPS solitons) [41]. Supergravity solutions that preserve some of the supersymmetry of the underlying theory have found many applications in the exploration of perturbative and non-perturbative properties of string theory. An important example is the AdS/CFT correspondence, also known as the Maldacena conjecture, which conjectures duality between the supersymmetric background and a certain superconformal field theory [37, 30, 49].

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In type II string theory one investigates manifolds $N^k \times M^{10-k}$, where N^k is a k-dimensional space-time and M^{10-k} is a Riemannian manifold equipped with an additional structure. Indeed, the basic model is a 5-tuple (M^n, g, H, Φ, Ψ), where g is a Riemannian metric, H is a 3-form, Φ is the socalled dilation function, and Ψ is a spinor field. The string equations, which are a generalization of the Einstein equations, can be written in the following form (see [46]):

$$\mathrm{Ric}^g_{ij} - \frac{1}{4} H_{imn} H_{jmn} + 2 \cdot \nabla^g_i \partial_j \Phi \ = \ 0, \quad \delta(e^{-2\Phi} H) \ = \ 0 .$$

The field equations are supplemented with the so-called Killing spinor equations

$$(\nabla^g_X + \frac{1}{4} X \lrcorner H) \cdot \Psi \ = \ 0, \quad (d\Phi - \frac{1}{6} H) \cdot \Psi \ = \ 0 .$$

Sometimes one requires that the 3-form H is closed, $dH = 0$, and solutions of this type are then called strong. The first of the Killing spinor equations suggests that the 3-form H should be the torsion form of a metric connection ∇ with totally skew-symmetric torsion tensor $T = H$. Then the string equations and the Killing spinor equations can be written as follows (see [34]) :

$$\mathrm{Ric}^\nabla + \frac{1}{2} \delta(T) + 2 \nabla^g d\Phi = 0, \quad \delta(T) = 2 \cdot d\Phi^\# \lrcorner T ,$$

$$\nabla \Psi \ = \ 0, \quad (d\Phi - \frac{1}{6} T) \cdot \Psi \ = \ 0 .$$

Ric^∇ is the Ricci tensor of the metric connection ∇, ∇^g is the Levi-Civita connection of the metric g and $d\Phi^\#$ denotes the vector field dual to the 1-form $d\Phi$. If the dilation is constant, then the string equations are equivalent to the condition that the Ricci tensor of the connection vanishes [34] ,

$$\mathrm{Ric}^\nabla = 0 ,$$

and the Killing spinor equations become

$$\nabla \Psi \ = \ 0, \quad T \cdot \Psi \ = \ 0 .$$

In particular, the spinor field is Riemannian harmonic. More general, the Riemannian Dirac operator D^g acts on a ∇-parallel spinor field via the formula

$$D^g \Psi + \frac{3}{4} T \cdot \Psi \ = \ 0 .$$

The number of preserved supersymmetries depends essentially on the number of ∇-parallel spinors. In this paper we investigate solutions of all these field equations in the case of constant dilation in dimensions $n = 5, 6, 7$. In dimension 7 any solution is strong (Theorem 5.4) and we derive a topological obstruction for the existence of such solutions (Remark 5.5).

A spinor parallel with respect to the Levi-Civita connection on a Riemannian manifold M^n restricts its holonomy group (see [31, 48] and [38]). In a similar way a ∇-parallel spinor field reduces the structure group of the frame bundle. Conversely, let us start with a (non-integrable) G-structure on a Riemannian manifold and ask the question whether or not there exists a G-connection with a totally skew-symmetric torsion and at least one parallel spinor field. In dimension 3 the stabilizer of Spin(3) is trivial and therefore the connection is flat. It is well known (see e.g. [45]) that in this case (M^3, g, ∇, T) carries (locally) a structure of a compact Lie group, g is a biinvariant metric and ∇ is the invariant connection with torsion given by the Lie bracket. In particular, on $SU(2)$ there exists at least one ∇-parallel spinor.

The 4-dimensional case was investigated in earlier papers. The restricted holonomy group of ∇ should be contained in SU(2) (see [41]) and this is equivalent to the local existence of a HKT structure, i.e., a hyperhermitian structure that is parallel with respect to ∇ (see [34]). Surprisingly, the geometry of ∇ depends on the type of the parallel spinor (see [10]). If M^4 is compact, then the holonomy of ∇ is contained in SU(2) if and only if M^4 is either a Calabi-Yau manifold or a Hopf surface (see [34]). We note that there exist Hopf surfaces that do not admit any (global) hyperhermitian structure although the holonomy of ∇ is contained in SU(2) (see [23], [34]). These Hopf surfaces do not admit any ∇-parallel spinors (see [10]), which shows that in the non-simply connected compact case the holonomy condition is not sufficient for the existence of ∇-parallel spinors.

In higher dimensions we fix a subgroup G of $Spin(n)$ preserving a spinor Ψ_0 as well as a geometric G-structure on a Riemannian manifold (M^n, g). Then

a) we describe the set of all G-connections with totally skew-symmetric torsion for any geometric type of G-structures. In particular, we decide whether or not a G-structure admits a connection with totally skew-symmetric torsion T and derive a formula for the torsion;

b) We use the parallel spinor Ψ_0 and the algebraic properties of the Clifford multiplication in special dimensions in order to derive the field equation;

c) We study the space of all ∇-parallel spinors and compare it to the space of ∇-harmonic spinors. In particular, we decide whether or not there exist ∇-parallel spinor fields Ψ such that $T \cdot \Psi = 0$.

In the second and third step we need the Schrödinger-Lichnerowicz-formula for a metric connection with totally skew-symmetric torsion. In particular, we prove this formula in full generality, including the computation of the curvature term.

We will give a complete answer in dimension $n = 7$ and for G_2-structures. In odd dimensions (almost contact metric structures) and in even dimensions (almost hermitian structures) we solve the first problem. However, these geometric structures reduce the structure group of the frame bundle only to the subgroup $U(k)$, which does not coincide with the isotropy group of a spinor. Consequently, the connection ∇ adapted to the geometric structure under consideration does not admit a ∇-parallel spinor automatically and we obtain a further curvature condition for the existence of such spinors. We investigate this additional condition and prove some vanishing theorems for ∇-harmonic spinors.

2 The curvature of connections with totally skew-symmetric torsion

In this section we recall some notions concerning the curvature of a metric connection with totally skew-symmetric torsion from [34]. Let (M^n, g, ∇, T) be an n-dimensional Riemannian manifold with a metric connection ∇ of totally skew-symmetric torsion T. The torsion tensor measures the difference between the connection ∇ and the Levi-Civita connection ∇^g :

$$g(\nabla_X Y, Z) \;=\; g(\nabla^g_X Y, Z) + \frac{1}{2} T(X, Y, Z)\,.$$

Let us fix some notation. The differential of an exterior form α is given by the formula

$$d\alpha \;=\; \sum_{i=1}^{n} e_i \wedge \nabla^g_{e_i} \alpha\,.$$

The codifferential of the form α can be calculated using either the Levi-Civita connection ∇^g or the connection ∇ :

$$\delta^g(\alpha) \;=\; \sum_{i=1}^{n} e_i \lrcorner \nabla^g_{e_i} \alpha, \quad \delta^\nabla(\alpha) \;=\; \sum_{i=1}^{n} e_i \lrcorner \nabla_{e_i} \alpha\,.$$

On the 3-form T, the two codifferentials coincide :

$$\delta^g(T) \;=\; \delta^\nabla(T)\,.$$

This formula is a consequence of the assumption that ∇ has a totally skew-symmetric torsion tensor. Let us introduce the 4-form σ^T

$$\begin{aligned}\sigma^T(X, Y, Z, V) &:= g(T(X,Y), T(Z,V)) + g(T(Y,Z), T(X,V)) + g(T(Z,X), T(Y,V)) \\ &= \frac{1}{2}\sum_{i=1}^{n} (e_i \lrcorner T) \wedge (e_i \lrcorner T)(X, Y, Z, V)\,.\end{aligned}$$

Then the exterior derivative dT of the torsion tensor T is given in terms of ∇ by the following formula (see e.g.[34])

$$dT(X, Y, Z, V) \;=\; \sigma_{XYZ}\{(\nabla_X T)(Y, Z, V)\} - (\nabla_V T)(X, Y, Z) + 2\sigma^T(X, Y, Z, V),$$

where σ_{XYZ} denotes the cyclic sum over X, Y, Z. Moreover, the curvature tensors of the Levi-Civita connection and the connection ∇ are related via the formula

$$\begin{aligned}R^g(X, Y, Z, V) \;&=\; R^\nabla(X, Y, Z, V) - \frac{1}{2}(\nabla_X T)(Y, Z, V) + \frac{1}{2}(\nabla_Y T)(X, Z, V) \\ &\quad -\frac{1}{4} g\Big(T(X,Y), T(Z,V)\Big) - \frac{1}{4}\sigma^T(X, Y, Z, V)\,.\end{aligned}$$

The first Bianchi identity for ∇ can be written in the form

$$\sigma_{XYZ} R^\nabla(X, Y, Z, V) \;=\; dT(X, Y, Z, V) - \sigma^T(X, Y, Z, V) + (\nabla_V T)(X, Y, Z)$$

and the difference of the Ricci tensors involves the codifferential of T :

$$\mathrm{Ric}^g(X, Y) \;=\; \mathrm{Ric}^\nabla(X, Y) + \frac{1}{2} \cdot \delta^g(T)(X, Y) - \frac{1}{4}\sum_{i=1}^{n} g(T(e_i, X), T(Y, e_i))\,.$$

In particular, the skew-symmetric part of the Ricci tensor Ric^∇ of ∇ is given by the codifferential of the torsion tensor only :

$$\mathrm{Ric}^\nabla(X,Y) - \mathrm{Ric}^\nabla(Y,X) \;=\; -\,\delta^g(T)(X,Y)\,.$$

We denote the scalar curvature of ∇ by Scal^∇, i.e.,

$$\mathrm{Scal}^\nabla \;=\; \sum_{i,j=1}^n R^\nabla(e_i,e_j,e_j,e_i)\,.$$

3 The Schrödinger-Lichnerowicz-formula for connections with totally skew-symmetic torsion

Consider an n-dimensional Riemannian spin manifold (M^n,g,∇,T) with a metric connection ∇ of totally skew-symmetric torsion T and denote by ΣM^n the spinor bundle. The Dirac operator D depending on the connection ∇ is defined by

$$D\Psi \;=\; \sum_{i=1}^n e_i\cdot\nabla_{e_i}\Psi, \quad \Psi\in\Sigma M^n,$$

where $e_1\dots e_n$ is an orthonormal basis. The Dirac operator D is a formally selfadjoint operator since the torsion of the connection is totally skew-symmetric (see [20]). In case of a Riemannian manifold the well-known Schrödinger-Lichnerowicz-formula expresses the square of the Dirac operator with respect to the Levi-Civita connection by the spinorial Laplace operator and some curvature term (see [44, 14]). In the articles [5, 1], a generalization of this formula for connections with arbitrary torsion is indicated. For connections with totally skew-symmetric torsion we shall derive the curvature term and prove the following explicit formula :

Theorem 3.1. (S-L-formula) *Let (M^n,g,∇,T) be an n-dimensional Riemannian spin manifold with a metric connection ∇ of totally skew-symmetric torsion T. Then, for any spinor field Ψ, the formula*

$$D^2\Psi = \nabla^*\nabla\Psi + \frac{3}{4}dT\cdot\Psi - \frac{1}{2}\sigma^T\cdot\Psi + \delta^g(T)\cdot\Psi - \sum_{k=1}^n e_k\,\lrcorner\,T\cdot\nabla_{e_k}\Psi + \frac{1}{4}\mathrm{Scal}^\nabla\cdot\Psi$$

holds, where $\nabla^*\nabla$ is the Laplacian of ∇ acting on spinors by

$$\nabla^*\nabla\Psi \;=\; -\sum_{i=1}^n \nabla_{e_i}\nabla_{e_i}\Psi + \nabla_{\nabla^g_{e_i}e_i}\Psi\,.$$

Proof. At a fixed point $p\in M^n$ we choose an orthonormal basis $e_1,\dots,e_n$ such that $(\nabla_{e_i}e_j)_p = 0$ and $[e_i,e_j]_p = -T(e_i,e_j)_p$. Then the vector field e_i is ∇^g-parallel in the direction of e_i at the point p : $(\nabla^g_{e_i}e_i)_p = 0$. We calculate

$$\begin{aligned}D^2\Psi &= \sum_{i,j=1}^n e_i\nabla_{e_i}e_j\nabla_{e_j}\Psi = -\sum_{i=1}^n\nabla_{e_i}\nabla_{e_i}\Psi + \sum_{i,j=1}^n R^\nabla(e_i,e_j)e_i\cdot e_j\cdot\Psi - \sum_{i=1}^n e_i\,\lrcorner\,T\cdot\nabla_{e_i}\Psi\\ &= \nabla^*\nabla\Psi + \frac{1}{2}\sum_{i<j,k<l}R^\nabla(e_i,e_j,e_k,e_l)e_i\cdot e_j\cdot e_k\cdot e_l\cdot\Psi - \sum_{i=1}^n e_i\,\lrcorner\,T\cdot\nabla_{e_i}\Psi\,.\end{aligned}$$

The curvature term in the latter equation can be written in the form

$$\hat{\sigma}(R^{\nabla}) \cdot \Psi - \sum_{j<k}[\mathrm{Ric}^{\nabla}(e_j, e_k) - \mathrm{Ric}^{\nabla}(e_k, e_j)]e_j \cdot e_k \cdot \Psi + \frac{1}{2}\mathrm{Scal}^{\nabla} \cdot \Psi \,,$$

where $\hat{\sigma}(R^{\nabla})$ is the 4-form given by

$$\hat{\sigma}(R^{\nabla})(X,Y,Z,V) \;=\; \sigma_{XYZ}\{R^{\nabla}(X,Y,Z,V) - R^{\nabla}(V,X,Y,Z)\}\,.$$

Using the formulas for dT as well as the formula comparing the curvature tensors R^g and R^{∇} we obtain

$$\hat{\sigma}(R^{\nabla}) \;=\; \frac{3}{2}dT - \sigma^T\,.$$

Inserting the latter formula as well as the formula for the skew-symmetric part of the Ricci tensor into the expression for $D^2\Psi$ yields the desired formula. □

Corollary 3.2. *Let Ψ be a parallel spinor with respect to ∇. Then the following formulas hold:*

$$\frac{3}{4}dT \cdot \Psi - \frac{1}{2}\sigma^T \cdot \Psi + \frac{1}{2}\delta^g(T) \cdot \Psi + \frac{1}{4}\mathrm{Scal}^{\nabla} \cdot \Psi \;=\; 0\,,$$

$$\sum_{i<j<k} \left\{\frac{1}{2}dT(X, e_i, e_j, e_k) + (\nabla_X T)(e_i, e_j, e_k)\right\} e_i \cdot e_j \cdot e_k \cdot \Psi - \mathrm{Ric}^{\nabla}(X) \cdot \Psi \;=\; 0\,.$$

Proof. The first formula follows directly from Theorem 3.1. We prove the second one by contracting the well-known formula

$$0 \;=\; \nabla\nabla\Psi \;=\; \sum_{i,j=1}^{n} R^{\nabla}(e_i, e_j) \cdot e_i \cdot e_j \cdot \Psi$$

and using the formulas relating the symmetrization of the curvature tensor R^{∇} and the derivative of the torsion form T. □

The next formula compares the action of the Dirac operator with the action of the torsion form on spinors.

Theorem 3.3. *Let (M^n, g, ∇, T) be an n-dimensional Riemannian spin manifold with a metric connection ∇ of totally skew-symmetric torsion. Then*

$$DT + TD \;=\; dT + \delta^g(T) - 2\sigma^T - 2\sum_{i=1}^{n} e_i \,\lrcorner\, T \cdot \nabla_{e_i}\,.$$

Proof. The proof is similar to the proof of Theorem 3.1. □

Let us apply Theorem 3.1 and Theorem 3.3 in case of a ∇-harmonic spinor field Ψ on a compact manifold M^n. Since $\delta^g(T)$ is a 2-form, the real part of the hermitian product $(\delta^g(T) \cdot \Psi, \Psi)$ vanishes. Moreover, the Dirac operator D is symmetric. Combining Theorem 3.1 and Theorem 3.3 we obtain the condition

$$\int_{M^n} \Big(\|\nabla\Psi\|^2 + \frac{1}{4}(dT \cdot \Psi, \Psi) + \frac{1}{2}(\sigma^T \cdot \Psi, \Psi) + \frac{1}{4}\mathrm{Scal}^{\nabla} \cdot \|\Psi\|^2\Big) \;=\; 0\,.$$

This formula proves the following vanishing theorem.

Theorem 3.4. *Let (M^n, g, ∇, T) be an n-dimensional Riemannian spin manifold with a metric connection ∇ of totally skew-symmetric torsion T. Suppose, moreover, that the eigenvalues of the endomorphism $dT + 2 \cdot \sigma^T + \mathrm{Scal}^\nabla$ acting on spinors are non-negative. Then any ∇-harmonic spinor field is ∇-parallel. In case the eigenvalues of the endomorphism are positive, there are no ∇-parallel spinor fields.*

4 G_2-connections with totally skew-symmetric torsion

We study under which conditions a fixed G-structure on a Riemannian manifold admits an affine connection preserving the G-structure and having totally skew-symmetric torsion tensor. For this purpose we describe the different geometric types of G-structures from the point of view of gauge theory using a certain 1-form Γ with values in the associated bundle of typical fibres $\mathfrak{so}(n)/\mathfrak{g}$. This approach is completely equivalent to the classification of different geometric G-structures used in differential geometry and mainly based on the decomposition of the covariant derivative of the tensor related with the G-structure. The advantage of our approach is that the method applies even in cases where the G-structure is not defined by a tensor (see [15, 47]). To begin with, let (M^n, g) be an oriented Riemannian manifold and denote by $\mathcal{F}(M^n)$ its frame bundle. The Levi-Civita connection is a 1-form

$$Z^g : T(\mathcal{F}(M^n)) \longrightarrow \mathfrak{so}(n)$$

with values in the Lie algebra $\mathfrak{so}(n)$. Its torsion tensor vanishes. We fix a closed subgroup G of the orthogonal group $SO(n)$. A G-structure on M^n is a G-subbundle $\mathcal{R} \subset \mathcal{F}(M^n)$. We decompose the Lie algebra $\mathfrak{so}(n)$ into the subalgebra $\mathfrak{g}$ and its orthogonal complement $\mathfrak{m}$:

$$\mathfrak{so}(n) \;=\; \mathfrak{g} \oplus \mathfrak{m} \,.$$

In a similar way we decompose the restriction of the 1-form Z^g

$$Z^g|_{T(\mathcal{R})} \;=\; \tilde{Z} \oplus \Gamma \,.$$

$\tilde{Z}$ is a connection in the principal G-bundle $\mathcal{R}$ and Γ is a 1-form with values in the associated bundle $\mathcal{R} \times_G \mathfrak{m}$. The different geometric types of G-structures are defined by the irreducible G-components of the representation $\mathbb{R}^n \otimes \mathfrak{m}$. An arbitrary G-connection Z differs from $\tilde{Z}$ by an 1-form Σ with values in the Lie algebra $\mathfrak{g}$,

$$Z \;=\; Z^g - \Gamma + \Sigma \,,$$

and the corresponding covariant derivative ∇ is given by the formula

$$\nabla_X Y \;=\; \nabla^g_X Y - \Gamma(X)(Y) + \Sigma(X)(Y) \,.$$

Since the Levi-Civita connection ∇^g is torsion free, the torsion tensor of ∇ depends on Γ and Σ :

$$T(X,Y,Z) \;=\; -g(\Gamma(X)(Y), Z) + g(\Gamma(Y)(X), Z) + g(\Sigma(X)(Y), Z) - g(\Sigma(Y)(X), Z) \,.$$

T is a 3-form if and only if

$$g(\Gamma(Y)(X),Z)+g(\Gamma(Z)(X),Y) \;=\; g(\Sigma(Z)(X),Y)+g(\Sigma(Y)(X),Z)$$

holds. Now we introduce the following G-invariant maps:

$$\begin{array}{lll} \Phi: & \mathbb{R}^n\otimes\mathfrak{g}\to\mathbb{R}^n\otimes S^2(\mathbb{R}^n), & \Phi(\Sigma)(X,Y,Z):=g(\Sigma(Z)(X),Y)+g(\Sigma(Y)(X),Z), \\ \Psi: & \mathbb{R}^n\otimes\mathfrak{m}\to\mathbb{R}^n\otimes S^2(\mathbb{R}^n), & \Psi(\Gamma)(X,Y,Z):=g(\Gamma(Y)(X),Z)+g(\Gamma(Z)(X),Y)\,. \end{array}$$

Consequently, we proved the following

Proposition 4.1. *A G-reduction $\mathcal{R}\subset\mathcal{F}(M^n)$ admits a G-connection Z with a totally skew-symmetric torsion tensor T if and only if $\Psi(\Gamma)$ is contained in the image of the homomorphism Φ. In this case the set of all these connections Z is an affine space over the vector space* $\ker(\Phi)$.

We will use representation theory in order to study the diagramme

$$\begin{tikzcd} \mathbb{R}^n\otimes\mathfrak{g} \arrow[r, "\Phi"] & \mathbb{R}^n\otimes S^2(\mathbb{R}^n) \\ \mathbb{R}^n\otimes\mathfrak{m} \arrow[ur, "\Psi"'] & \end{tikzcd}$$

By splitting the G-representation $\mathbb{R}^n\otimes\mathfrak{m}$ into irreducible components we can decide whether or not the image of a certain component is contained in the image of Φ. In this way we characterize the geometric G-structures admitting a G-connection Z with a totally skew-symmetric torsion tensor. We will apply this general method to the subgroup $G_2\subset SO(7)$. Therefore, let us recall some notions of G_2-geometry in dimension seven. The group G_2 is the isotropy group of the 3-form in seven variables

$$\omega^3 \;:=\; e_1\wedge e_2\wedge e_7+e_1\wedge e_3\wedge e_5-e_1\wedge e_4\wedge e_6-e_2\wedge e_3\wedge e_6-e_2\wedge e_4\wedge e_5+e_3\wedge e_4\wedge e_7+e_5\wedge e_6\wedge e_7\,.$$

The 3-form ω^3 corresponds to a real spinor $\Psi_0\in\Delta_7$ and, therefore, G_2 can be defined as the isotropy group of a non-trivial real spinor. We identify the Lie algebra of the group $SO(7)$ with the space of all 2-forms :

$$\mathfrak{so}(7) \;=\; \Lambda^2(\mathbb{R}^7) \;=\; \Big\{\sum_{i<j}\omega_{ij}\cdot e_i\wedge e_j\Big\}\,.$$

The Lie algebra $\mathfrak{g}_2$ of the group G_2 is given by the equations

$$\begin{array}{lll} \omega_{12}+\omega_{34}+\omega_{56}\;=\;0, & -\,\omega_{13}+\omega_{24}-\omega_{67}\;=\;0, & \omega_{14}+\omega_{23}+\omega_{57}\;=\;0, \\ \omega_{16}+\omega_{25}-\omega_{37}\;=\;0, & \omega_{15}-\omega_{26}-\omega_{47}\;=\;0, & \omega_{17}+\omega_{36}+\omega_{45}\;=\;0, \\ & \omega_{27}+\omega_{35}-\omega_{46}\;=\;0\,. & \end{array}$$

The space $\mathbb{R}^7:=\Lambda^1_7$ is an irreducible G_2-representation, the 2-forms $\Lambda^2=\Lambda^2_7\oplus\Lambda^2_{14}$ split into two irreducible G_2-components :

$$\begin{aligned} \Lambda^2_7 &:= \{\alpha^2\in\Lambda^2\;:\;*(\omega^3\wedge\alpha^2)=2\cdot\alpha^2\} \;=\; \{X\,\lrcorner\,\omega^3\;:\;X\in\mathbb{R}^7\}\,, \\ \Lambda^2_{14} &:= \{\alpha^2\in\Lambda^2\;:\;*(\omega^3\wedge\alpha^2)=-\,\alpha^2\} \;=\; \mathfrak{g}_2\,. \end{aligned}$$

The space of 3-forms $\Lambda^3 = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$ decomposes into three irreducible G_2-components :

$$\begin{aligned}
\Lambda^3_1 &:= \{t \cdot \omega^3 \ : \ t \in \mathbb{R}\} \ , \\
\Lambda^3_7 &:= \{*(\omega^3 \wedge \alpha^1) \ : \ \alpha^1 \in \Lambda^1\} \ = \ \{X \lrcorner *\omega^3 \ : \ X \in \mathbb{R}^7\} \ , \\
\Lambda^3_{27} &:= \{\alpha^3 \in \Lambda^3 \ : \ \alpha^3 \wedge \omega^3 = 0, \ \alpha^3 \wedge *\omega^3 = 0\} \ .
\end{aligned}$$

The representation Λ^3_{27} is isomorphic to the representation of G_2 in the space $S^2_0(\mathbb{R}^7)$ of all traceless symmetric bilinear forms.

Proposition 4.2. *The map* $\Phi : \mathbb{R}^7 \otimes \mathfrak{g}_2 \to \mathbb{R}^7 \otimes S^2(\mathbb{R}^7)$ *is injective.*

Corollary 4.3. *Let (M^7, g, ω^3) be an oriented, 7-dimensional Riemannian manifold with a fixed G_2-structure ω^3. Then there exists at most one affine connection ∇ such that $\nabla\omega^3 = 0$ and the torsion tensor T is a 3-form.*

Proof. Given $\Sigma \in \mathbb{R}^7 \otimes \mathfrak{g}_2$, the condition $\Phi(\Sigma) = 0$ is equivalent to

$$Z \lrcorner \Sigma(Y) + Y \lrcorner \Sigma(Z) \ = \ 0$$

for any two vectors $Y, Z \in \mathbb{R}^7$. Using the standard basis $e_\alpha \wedge e_\beta$ of the Lie algebra $\mathfrak{so}(7)$ we decompose the elements $\Sigma(e_i) \in \mathfrak{g}_2 \subset \mathfrak{so}(7)$,

$$\Sigma(e_i) \ := \sum_{1 \leq \alpha, \beta \leq 7} \omega_{i\alpha\beta} \cdot e_\alpha \wedge e_\beta \ .$$

The condition $\Phi(\Sigma) = 0$ implies $\omega_{i\alpha\beta} = -\omega_{\alpha i\beta}$. Consequently, Σ depends on 35 parameters $\omega_{i\alpha\beta}$ $(1 \leq i < \alpha < \beta \leq 7)$. Moreover, $\Sigma(e_i)$ belongs to the Lie-Algebra $\mathfrak{g}_2$ and hence we obtain seven equations for any index $i = 1, 2, \ldots, 7$. Altogether, these are 49 equations for 35 variables and a careful examination of the system yields the result that $\omega_{i\alpha\beta} = 0$ is the only solution. $\square$

The low-dimensional G_2-representations, their highest weights, dimensions etc. are listed in the following table (see [21]):

highest weight	dimension	space
$(0,0)$	1	$\Lambda^0_1 := \Lambda^0(\mathbb{R}^7) = \Lambda^3_1$
$(1,0)$	7	$\Lambda^1_7 := \Lambda^1(\mathbb{R}^7) = \Lambda^2_7 = \Lambda^3_7$
$(0,1)$	14	$\mathfrak{g}_2 = \Lambda^2_{14}$
$(2,0)$	27	$\Lambda^3_{27} = S^2_0(\mathbb{R}^7)$
$(1,1)$	64	Λ_{64}
$(3,0)$	77	Λ_{77}

We now compute the G_2-decomposition of the three representations related with the maps Ψ and Φ.

Proposition 4.4. *The following decompositions into irreducible G_2-representations hold:*

1) $\mathbb{R}^7 \otimes \mathfrak{m} = \Lambda^0_1 \oplus \Lambda^1_7 \oplus \Lambda^2_{14} \oplus \Lambda^3_{27}$;

2) $\mathbb{R}^7 \otimes \mathfrak{g}_2 = \Lambda^1_7 \oplus \Lambda^3_{27} \oplus \Lambda_{64}$;

3) $\mathbb{R}^7 \otimes S^2(\mathbb{R}^7) = 2\Lambda^1_7 \oplus \Lambda^2_{14} \oplus \Lambda^3_{27} \oplus \Lambda_{64} \oplus \Lambda_{77}$.

Proof: The first decomposition is an elementary one, the second and third decomposition can be obtained using a suitable computer programme. □

Since the G_2-map $\Phi : \mathbb{R}^7 \otimes \mathfrak{g}_2 \to \mathbb{R}^7 \otimes S^2(\mathbb{R}^7)$ is injective and the multiplicity of Λ^3_{27} in $\mathbb{R}^7 \otimes S^2(\mathbb{R}^7)$ is one, we obtain

Corollary 4.5.

1) $\Psi(\Lambda^0_1 \oplus \Lambda^3_{27}) \subset \mathrm{Im}(\Phi)$;

2) $\Psi(\Lambda^2_{14}) \cap \mathrm{Im}(\Phi) = 0$.

Finally, we have to decide whether or not the space $\Psi(\Lambda^1_7)$ is contained in the image of Φ. Since the representation Λ^1_7 has multiplicity two in $\mathbb{R}^7 \otimes S^2(\mathbb{R}^7)$, we cannot use a universal argument is before.

Proposition 4.6. $\Psi(\Lambda^1_7)$ *is not contained in* $\mathrm{Im}(\Phi)$.

Proof. First of all we compute $\Psi(\Gamma)$ for a given vector $\Gamma \in \Lambda^1_7$. The element in $\mathbb{R}^7 \otimes \mathbb{R}^7$ related to Γ is the 2-form (skew-symmetric endomorphism) $\Gamma \lrcorner \omega^3$ and therefore we obtain

$$\Psi(\Gamma)(X,Y,Y) \;=\; 2\sum_{i=1}^{7} \omega^3(\Gamma, Y, e_i) \cdot \omega^3(e_i, X, Y)\,.$$

Suppose now that $\Psi(\Lambda^1_7) \subset \mathrm{Im}(\Phi)$. Since Φ is injective, there should exist a map $\Sigma : \Lambda^1_7 \to \mathbb{R}^7 \otimes \mathfrak{g}_2$ such that

$$\Phi(\Sigma(\Gamma)) \;=\; \Psi(\Gamma)\,.$$

The multiplicity of Λ^1_7 in $\mathbb{R}^7 \otimes \mathfrak{g}_2$ equals one. Consequently, $\Sigma(\Gamma)$ is proportional to the map

$$\Sigma(\Gamma)(Y) \;:=\; \mathrm{pr}(\Gamma \wedge \mathrm{Y})\,,$$

where $\mathrm{pr} : \Lambda^2(\mathbb{R}^7) = \mathfrak{so}(7) \to \mathfrak{g}_2$ is the orthogonal projection. The 2-forms

$$\frac{1}{\sqrt{3}} e_1 \lrcorner \omega^3, \quad \ldots \quad , \frac{1}{\sqrt{3}} e_7 \lrcorner \omega^3$$

constitute an orthonormal basis of the subspace $\mathfrak{m} \subset \mathfrak{so}(n)$ and the projection is given by the formula

$$\mathrm{pr}(\alpha^2) \;=\; \alpha^2 - \frac{1}{3}\sum_{i=1}^{7}(e_i \lrcorner \omega^3, \alpha^2) \cdot (e_i \lrcorner \omega^3)\,.$$

Now we compute $\Phi(\Sigma(\Gamma))$:

$$\begin{aligned}\Phi(\Sigma(\Gamma))(X,Y,Y) &= 2g(X \lrcorner \mathrm{pr}(\Gamma \wedge \mathrm{Y}), \mathrm{Y}) \\ &= 2(\Gamma \wedge Y)(X,Y) - \frac{2}{3}\sum_{i=1}^{7}(e_i \lrcorner \omega^3, \Gamma \wedge Y)\cdot \omega^3(e_i, X, Y)\,.\end{aligned}$$

Finally, we obtain

$$\Phi(\Sigma(\Gamma))(X,Y,Y) = 2g(\Gamma, X)\cdot g(Y,Y) - 2g(\Gamma, Y)\cdot g(X,Y) - \frac{1}{3}\Psi(\Gamma)(X,Y,Y)\,,$$

a contradiction. □

Recall that a G_2-structure is called cocalibrated if $\delta^g\omega^3 = 0$. To summarize the previous discussion we proved the following theorem.

Theorem 4.7. *Let (M^7, g, ω^3) be a 7-dimensional Riemannian manifold with a G_2-structure ω^3. The following conditions are equivalent:*

1) The G_2-structure is cocalibrated or, equivalently, the $(\Lambda^2_7 \oplus \Lambda^2_{14})$-component of Γ is zero.

2) There exists an affine connection ∇ with totally skew-symmetric torsion preserving the G_2-structure

$$\nabla\omega^3 \;=\; 0\,.$$

In this case the connection ∇ is unique.

The Riemannian covariant derivative $\nabla^g\omega^3$ of the G_2-structure is given by the formula

$$\nabla^g\omega^3 \;=\; \varrho_3(\Gamma)(\omega^3)\,,$$

where ϱ_3 is the differential of the representation of $SO(7)$ in $\Lambda^3(\mathbb{R}^7)$. The G_2-map of $\mathbb{R}^7 \otimes \mathfrak{m}$ into $\mathbb{R}^7 \otimes \Lambda^3(\mathbb{R}^7)$ given by the formula $X \otimes Y \to X \otimes \varrho_3(Y)\omega^3$ is injective (see [15]). Consequently, the different geometric classes of G_2-structures introduced by Fernandez/Gray (see [11]) using the covariant derivative $\nabla^g\omega^3$ can be defined via the algebraic type of $\Gamma \in \Lambda^1(M^7) \otimes \Lambda^1(M^7)$. The condition $\Gamma \in \Lambda^0_1 \oplus \Lambda^3_{27}$ characterizes the so-called cocalibrated G_2-structures, i.e., the G_2-structures of type $W_1 \oplus W_3$ in the notation of [11]. The case of $\Gamma = \lambda\cdot \mathrm{Id}_{T(M^7)}$ corresponds to nearly parallel G_2-structures and the covariant derivative $\nabla^g\omega^3$ is then given by the formula

$$\nabla^g_X\omega^3 \;=\; -\,3\cdot\lambda\cdot(X \lrcorner *\omega^3)\,.$$

The Riemannian manifold (M^7, g) is Einstein and the parameter λ is related to the scalar curvature τ of M^7 via the formula (see [18])

$$\tau \;=\; 7\cdot 24\cdot\Big(\frac{3}{2}\lambda\Big)^2 \;=\; 378\cdot\lambda^2\,.$$

In this case we have $\Psi(\Gamma) = 0$ and $\Sigma = 0$ defines the unique affine connection preserving the 3-form ω^3 with totally skew-symmetric torsion tensor.

Theorem 4.8. *Let (M^7, g, ω^3) be a 7-dimensional Riemannian manifold with a nearly parallel G_2-structure ($\Gamma = \lambda \cdot Id$). Then there exists a unique affine connection ∇ such that*

$$\nabla\omega^3 \ = \ 0 \quad \textit{and} \quad T \textit{ is a 3-form.}$$

The torsion tensor is given by the formula $T = -\,2{\cdot}\lambda{\cdot}\omega^3$. T is ∇-parallel, $\nabla T = \delta T = 0$.

Corollary 4.9. *Let (M^7, g, ω^3) be a 7-dimensional nearly parallel G_2-manifold. Then the triple $(M^7, g, T^* := 3 \cdot T)$ is a solution of the string equations with constant dilation :*

$$R^g_{ij} - \frac{1}{4} T^*_{imn} T^*_{jmn} \ = \ 0, \quad \delta^g(T^*) \ = \ 0\,.$$

We investigate now the case $\Gamma \in \Lambda^3_{27}$. Then Γ is a 3-form and the covariant derivative $\nabla^g\omega^3$ is given by $\nabla^g\omega^3 = \varrho_3(\Gamma)(\omega^3)$. Using the formula

$$\delta^g(\omega^3) \ = \ \sum_{i=1}^{7} e_i \lrcorner \nabla_{e_i}\omega^3 \ = \ \sum_{i=1}^{7} e_i \lrcorner \Big(\varrho(\Gamma(e_i))(\omega^3)\Big)$$

and the decomposition $\Lambda^2(\mathbb{R}^7) = \Lambda^2_7 \oplus \Lambda^2_{14}$ we conclude that $\delta^g(\omega^3) = 0$. Consequently, the G_2-structure is cocalibrated. In a similar way the differential is given by the formula

$$d\omega^3 \ = \ \sum_{i=1}^{7} e_i \wedge \nabla^g_{e_i}\omega^3 \ = \ \sum_{i=1}^{7} e_i \wedge \varrho_3(\Gamma(e_i))(\omega^3)\,.$$

We consider the 3-form $\Gamma \in \Lambda^3_{27}$ as a 1-form $\Gamma : \mathbb{R}^7 \to \mathfrak{m} = \Lambda^2_7$ via the formula

$$\Gamma(X) \ = \ 3 \cdot \mathrm{pr}_{\mathfrak{m}}(X \lrcorner \Gamma)\ ,$$

where $\mathrm{pr}_{\mathfrak{m}} : \mathfrak{so}(7) \to \mathfrak{m}$ denotes the projection onto $\mathfrak{m}$. Then by

$$\mathrm{pr}_{\mathfrak{m}}(e_i \lrcorner \Gamma) \ = \ \frac{1}{3} \sum_{j=1}^{7} (e_i \lrcorner \Gamma, e_j \lrcorner \omega^3) \cdot (e_j \lrcorner \omega^3)$$

we obtain

$$d\omega^3 \ = \ \sum_{i,j=1}^{7} (e_i \lrcorner \Gamma, e_j \lrcorner \omega^3) \cdot e_i \wedge \rho_3(e_j \lrcorner \omega^3)(\omega^3)\,.$$

We need a formula describing the action $\varrho_3(\alpha^2)$ of a 2-form $\alpha^2 \in \Lambda^2(\mathbb{R}^7) = \mathfrak{so}(7)$ on ω^3. Denote by

$$\mathrm{pr}_{\mathfrak{m}}(\alpha^2) \ = \ Z \lrcorner \omega^3$$

its projection onto the space Λ^2_7. Then the representation $\varrho_3(\alpha^2)$ acts on the 3-form ω^3 by the formula

$$\varrho_3(\alpha^2)(\omega^3) \ = \ -\,3 \cdot (Z \lrcorner *\omega^3)\,.$$

Proposition 4.10. *Let (M^7, g, ω^3) be a G_2-structure of pure type $\Gamma \in \Lambda^3_{27}$. Then the exterior derivative $d\omega^3$ is given by the 3-form Γ:*

$$d\omega^3 \ = \ -3 \cdot \sum_{i,j=1}^{7} (e_i \lrcorner \Gamma, e_j \lrcorner \omega^3) \cdot e_i \wedge (e_j \lrcorner *\omega^3)\ .$$

Conversely, $d\omega^3$ *defines* $\Gamma \in \Lambda^3_{27}$.

The 1-form $\Psi(\Gamma)$ with values in $S^2(\mathbb{R}^7)$ is given by

$$\Psi(\Gamma)(X,Y,Y) \;=\; 2g\Big(\Gamma(Y)(X),Y\Big) \;=\; 6\cdot g\Big(\mathrm{pr}_{\mathfrak{m}}(Y \lrcorner \Gamma)(X),Y\Big)\,.$$

On the other hand, let us introduce the map $\Sigma : \Lambda^3_{27} \to \mathbb{R}^7 \otimes \mathfrak{g}_2$ given by the formula

$$\Sigma(\Gamma)(Y) \;:=\; -\,3\cdot \mathrm{pr}_{\mathfrak{g}_2}(Y \lrcorner \Gamma)\,,$$

where $\mathrm{pr}_{\mathfrak{g}_2} : \Lambda^2(\mathbb{R}^7) \to \mathfrak{g}_2$ is the projection onto $\mathfrak{g}_2$.

Lemma 4.11. *For any 3-form* $\Gamma \in \Lambda^3_{27}$ *we have* $\Phi(\Sigma(\Gamma)) = \Psi(\Gamma)$.

Proof. A direct calculation yields the result:

$$\begin{aligned}
\Phi(\Sigma(\Gamma))(X,Y,Y) &= -6\cdot g(X \lrcorner \mathrm{pr}_{\mathfrak{g}_2}(Y \lrcorner \Gamma),Y) \\
&= -6\cdot (Y \lrcorner \Gamma)(X,Y) + 6\cdot g(X \lrcorner \mathrm{pr}_{\mathfrak{m}}(Y \lrcorner \Gamma),Y) \\
&= \Psi(\Gamma)(X,Y,Y)\ .
\end{aligned}$$

□

Now we can compute the torsion tensor T of the unique affine connection ∇ related to a G_2-structure of type Λ^3_{27}.

Theorem 4.12. *Let* (M^7, g, ω^3) *be a 7-dimensional Riemannian manifold with a* G_2*-structure of type* Λ^3_{27}*. Then there exists a unique affine connection* ∇ *such that*

$$\nabla\omega^3 \;=\; 0 \quad \textit{and} \quad T \textit{ is a 3-form.}$$

The torsion form $T = -\,6\cdot\Gamma$ *is of type* Λ^3_{27} *and related with the exterior derivative of the* G_2*-structure by the formula*

$$d\omega^3 \;=\; \frac{1}{2}\sum_{i,j=1}^{7} (e_i \lrcorner T, e_j \lrcorner \omega^3)\cdot e_i \wedge (e_j \lrcorner *\omega^3) \;=\; -\,*T\,.$$

Proof. Since $\Gamma(X) - \Sigma(\Gamma)(X) = 3\cdot(X \lrcorner \Gamma)$ holds, we obtain that the torsion tensor T is proportional to Γ :

$$\begin{aligned}
T(X,Y,Z) &= -g(\Gamma(X),Y,Z) + g(\Gamma(Y),X,Z) + g(\Sigma(\Gamma)(X)(Y),Z) - g(\Sigma(\Gamma)(Y)(X),Z) \\
&= -\,6\cdot\Gamma(X,Y,Z)\ .
\end{aligned}$$

The map $\Lambda^3_{27} \to \Lambda^4 = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$ given by the formula

$$T \;\longmapsto\; \frac{1}{2}\sum_{i,j=1}^{7} (e_i \lrcorner T, e_j \lrcorner \omega^3)\cdot e_i \wedge (e_j \lrcorner *\omega^3)$$

commutes with the G_2-action and, hence, there exists a constant a such that it coincides with the map $T \mapsto a\cdot(*T)$. Inserting a special 3-form of type Λ^3_{27}, we compute the constant $a = -1$. □

Remark 4.13. We combine the formulas for the torsion tensor proved in Theorem 4.8 and Theorem 4.12 and consider now an arbitrary cocalibrated G_2-structure. Then Γ is given by a function λ and a Γ^3_{27} of type Λ^3_{27},

$$\Gamma \;=\; \lambda \cdot \mathrm{Id} + \Gamma^3_{27}\,,$$

and the differential of the G_2 structure as well as the torsion form T are given by the formulas :

$$d\omega^3 \;=\; -\,12 \cdot \lambda \cdot (*\omega^3) + 6 \cdot (*\Gamma^3_{27})$$

$$T \;=\; -\,2 \cdot \lambda \cdot \omega^3 - 6 \cdot \Gamma^3_{27} \;=\; -\,(*d\omega^3) - 14 \cdot \lambda \cdot \omega^3\ .$$

The torsion form T is of type $\Lambda^3_1 \oplus \Lambda^3_{27}$ and we obtain for any cocalibrated G_2-structure the equation

$$T \wedge \omega^3 \;=\; -\,(*d\omega^3) \wedge \omega^3 \;=\; 0\ .$$

The codifferential of the torsion form is given by the formula

$$d * T \;=\; -\,14 \cdot d\lambda \wedge *\omega^3\ .$$

In particular, T is coclosed if and only if λ is constant. This occurs, for example, if the G_2-structure is of pure type Λ^3_1 or Λ^3_{27}.

5 The ∇-Ricci tensor of a cocalibrated G_2-structure

We consider a 7-dimensional Riemannian manifold (M^7, g, ω^3) with a cocalibrated G_2-structure. Then there exists a unique connection ∇ preserving the G_2-structure having totally skew-symmetric torsion tensor T. It is a 3-form of type $\Lambda^3_1 \oplus \Lambda^3_{27}$. Moreover, the ∇-parallel 3-form ω^3 defines a ∇-parallel spinor field Ψ_0 (see [18]). For any vector field X the covariant derivative $\nabla_X T$ is again a 3-form of type $\Lambda^3_1 \oplus \Lambda^3_{27}$ and, consequently, the Clifford product $(\nabla_X T) \cdot \Psi_0$ depends only on the Λ^3_1-part of $\nabla_X T$:

$$(\nabla_X T) \cdot \Psi_0 \;=\; \frac{1}{7}(\nabla_X T, \omega^3) \cdot \omega^3 \cdot \Psi_0 \;=\; -\,(\nabla_X T, \omega^3) \cdot \Psi_0\,.$$

The second equation of Corollary 3.2 becomes

$$\frac{1}{2}(X \lrcorner\, dT) \cdot \Psi_0 - \frac{1}{2}(\nabla_X T, \omega^3) \cdot \Psi_0 - \mathrm{Ric}^{\nabla}(X) \cdot \Psi_0 \;=\; 0\,.$$

Using once again the fact that the Clifford product of the 3-form $X \lrcorner\, dT$ by the spinor Ψ_0 depends only on the $(\Lambda^3_1 \oplus \Lambda^3_7)$-part

$$\pi^3_1(X \lrcorner\, dT) \;:=\; \frac{1}{7}(X \lrcorner\, dT, \omega^3) \cdot \omega^3, \quad \pi^3_7(X \lrcorner\, dT) \;:=\; \frac{1}{4}\sum_{i=1}^{7}(X \lrcorner\, dT, e_i \lrcorner *\omega^3) \cdot (e_i \lrcorner *\omega^3)\,,$$

as well as the algebraic formula

$$(X \lrcorner *\omega^3) \cdot \Psi_0 \;=\; 4 \cdot X \cdot \Psi_0$$

valid for the special spinor Ψ_0 related to ω^3 we conclude

$$(X \lrcorner\, dT, \omega^3) \;=\; -\,(\nabla_X T, \omega^3), \quad \mathrm{Ric}^{\nabla}(X) \;=\; \frac{1}{2}\sum_{i=1}^{7}(X \lrcorner\, dT, e_i \lrcorner *\omega^3) \cdot e_i\ .$$

The relation between the Ricci tensors

$$\mathrm{Ric}^{\nabla}(X) \;=\; \mathrm{Ric}^g(X) + \frac{1}{4}\sum_{i,j=1}^{n} g(T(e_i,X),T(e_j,e_i))\cdot e_j - \frac{1}{2}\sum_{i=1}^{n}\delta^g(T)(X,e_i)\cdot e_i$$

allows us to compute the Riemannian Ricci tensor. We summarize all the derived results on cocalibrated G_2-structures in one theorem.

Theorem 5.1. *Let (M^7,g,ω^3) be a 7-dimensional Riemannian manifold with a cocalibrated G_2-structure. Then there exists a unique connection ∇ preserving the G_2-structure with a totally skew-symmetric torsion tensor. The torsion form is given by the formula*

$$T = -*\,d\omega^3 - 14\cdot\lambda\cdot\omega^3, \quad \lambda \;=\; -\frac{1}{7\cdot 12}(d\omega^3, *\omega^3)\ .$$

T is a solution of the equation

$$R^g_{ij} - \frac{1}{4}T_{imn}T_{jmn} - \frac{1}{2}(e_i \,\lrcorner\, dT, e_j \,\lrcorner\, *\omega^3) - \frac{1}{2}\delta^g(T)_{ij} \;=\; 0\ ,$$

and satisfies, for any vector X, the condition $(X \,\lrcorner\, dT,\omega^3) = -\,(\nabla_X T,\omega^3)$. The codifferential of the torsion form is given by

$$d*T \;=\; -\,14\cdot d\lambda\wedge *\omega^3\ .$$

There exists a ∇-parallel spinor field Ψ_0 such that the 3-form ω^3 and the Riemannian Dirac operator D^g act on it by

$$\omega^3\cdot\Psi_0 = -\,7\cdot\Psi_0, \quad D^g(\Psi_0) \;=\; -\frac{21}{2}\cdot\lambda\cdot\Psi_0, \quad T\cdot\Psi_0 \;=\; 14\cdot\lambda\cdot\Psi_0\ .$$

Example 5.2. Consider the case of a nearly-parallel G_2-structure ($\Gamma = \lambda\cdot\mathrm{Id}_{T(M^7)}$). The torsion form is proportional to the form of the G_2-structure, $T = -2\lambda\cdot\omega^3$, and its differential is given by $dT = 24\lambda^2(*\omega^3)$. Then we have (see Corollary 4.9)

$$R^g_{ij} \;=\; 54\cdot\lambda^2\cdot\delta_{ij}, \quad \frac{1}{4}T_{imn}T_{jmn} \;=\; 6\cdot\lambda^2\cdot\delta_{ij}, \quad \frac{1}{2}(e_i \,\lrcorner\, dT, e_j \,\lrcorner\, *\omega^3) \;=\; 48\cdot\lambda^2\cdot\delta_{ij}\ .$$

The formula for the Ricci tensor generalizes the well-known fact that a nearly-parallel G_2-manifold is an Einstein space (see [18]). In case M^7 is a compact, cocalibrated G_2-manifold we can apply the estimate for the first eigenvalue of the Riemannian Dirac operator (see [13])

$$\frac{7}{4\cdot 6}\|\Psi_0\|^2\cdot\mathrm{vol}(M^7)\cdot\mathrm{Scal}^g_{min} \leq \int_{M^7}(D^g\Psi_0, D^g\Psi_0)\,,$$

where Scal^g_{min} denotes the minimum of the Riemannian scalar curvature. Using the equation $2\cdot D^g(\Psi_0) = -21\cdot\lambda\cdot\Psi_0$ and the definition of the function λ we obtain an L^2-lower bound for $(d\omega^3, *\omega^3)$.

Theorem 5.3. *For any compact, cocalibrated G_2-manifold the following inequality holds*

$$\frac{4\cdot 14}{3}\cdot\mathrm{vol}(M^7)\cdot\mathrm{Scal}^g_{min} \leq \int_{M^7}(d\omega^3, *\omega^3)^2\ .$$

Equality occurs if and only if the cocalibrated G_2-structure is nearly parallel. If the G_2-structure is of pure type Λ^3_{27}, then the minimum of the scalar curvature is non-positive.

The formula of Theorem 5.1 computes in particular the Ricci tensor and the scalar curvature of the connection ∇ :

$$\mathrm{R}^{\nabla}_{ij} \;=\; \frac{1}{2}(e_i \lrcorner\, dT, e_j \lrcorner *\omega^3), \quad \mathrm{Scal}^{\nabla} \;=\; \frac{1}{2}\sum_{i=1}^{7}(e_i \lrcorner\, dT, e_i \lrcorner *\omega^3)\,.$$

A cocalibrated G_2-structure $(\omega^3, \nabla, \Psi_0)$ together with its canonical connection and spinor field solves all of the three string equations $\mathrm{Ric}^{\nabla} = 0$, $\nabla\Psi_0 = 0$, $T \cdot \Psi_0 = 0$ if and only if the G_2-structure is geometrically flat ($\lambda = 0 = T$). Therefore we study the first of these equations only.

Theorem 5.4. *Let (M^7, g, ω^3) be a 7-dimensional Riemannian manifold with a cocalibrated G_2-structure. The following conditions are equivalent:*

1) *The Ricci tensor Ric^{∇} vanishes.*
2) *The torsion form T is closed and coclosed, $dT = 0 = \delta^g(T)$.*
3) *λ is constant and the G_2-structure ω^3 satisfies the equation*

$$d * d\omega^3 + 14 \cdot \lambda \cdot d\omega^3 \;=\; 0\;.$$

Moreover, in this case we have

$$\Big(* \,d\omega^3 + 14 \cdot \lambda \cdot \omega^3 \Big) \wedge d\omega^3 \;=\; 0\;.$$

If the G_2-structure is of pure type Λ^3_1 or Λ^3_{27} and Ric^{∇} vanishes, then the Riemannian manifold M^7 is a Ricci flat space with holonomy G_2.

Proof. The condition $\mathrm{Ric}^{\nabla} = 0$ means that $X \lrcorner\, dT$ is orthogonal to the subspace $\{Y \lrcorner *\omega^3\} = \Lambda^3_7$. Moreover, $X \lrcorner\, dT$ is orthogonal to ω^3 and therefore it belongs to the subspace Λ^3_{27}. Consequently we obtain for any vector X the conditions

$$(X \lrcorner\, dT) \wedge \omega^3 \;=\; 0 \quad \text{and} \quad (X \lrcorner\, dT) \wedge *\omega^3 \;=\; 0\,.$$

The subspace of all 4-forms satisfying these algebraic equations is a G_2-invariant subspace of Λ^4 and it is not hard to see that this space is trivial, i.e. we conclude that $dT = 0$. Since $\delta^g(T)$ is the antisymmetric part of the Ricci tensor Ric^{∇}, the codifferential of the torsion form must vanish, too. These arguments prove the equivalence of the three conditions in the Theorem. The torsion form is of type $\Lambda^3_1 \oplus \Lambda^3_{27}$ and we differentiate the equation $T \wedge \omega^3 = 0$. Then we obtain the last equation of the Theorem. □

Remark 5.5. The cocalibrated G_2-structure $(M^7, g, \omega^3, \nabla)$ defines a (homogeneous) solution string equations with constant dilation

$$R^g_{ij} - \frac{1}{4}T_{imn}T_{jmn} \;=\; 0, \quad \delta^g(T) \;=\; 0$$

if and only if ω^3 is a solution of the cubic equation

$$d * d\omega^3 + 14 \cdot \lambda \cdot d\omega^3 \;=\; 0\,,$$

and $7 \cdot 12 \cdot \lambda = -\,(d\omega^3, *\omega^3)$ is constant. In this case the torsion form is closed and coclosed. In particular, if M^7 is a non-flat G_2-structure, we obtain a necessary topological condition:

$$H^3(M^7;\mathbb{Z}) \;\neq\; 0\,.$$

We have not succeeded in constructing any cocalibrated G_2-structure satisfying this non-linear equation.

Theorem 5.6. *Let $(M^7, g, \omega^3, \nabla)$ be a 7-dimensional compact nearly parallel G_2-manifold and ∇ be the unique G_2-connection with totally skew-symmetric torsion. Then every ∇-harmonic spinor Ψ is ∇-parallel. Moreover, the space of ∇-parallel spinors is one-dimensional.*

Proof. The Dirac operator D is selfadjoint. Let M^7 be compact and consider a ∇-harmonic spinor, $D\Psi = 0$. Then Theorem 3.3 implies

$$\sum_{i=1}^{7} \int_{M^7} (e_i \lrcorner T \cdot \nabla_{e_i}\Psi, \Psi) \;=\; 0\,,$$

since $dT = 2\sigma^T$ holds in case of a nearly-parallel structure. Using the latter equality as well as the Schrödinger-Lichnerowicz formula we obtain

$$\int_{M^7} \Big(\|\nabla\Psi\|^2 + \frac{1}{2}(dT \cdot \Psi, \Psi) + \frac{1}{4}\mathrm{Scal}^{\nabla}\|\Psi\|^2 \Big) \;=\; 0\,.$$

The 4-form $*\omega^3$ acts on spinors as a symmetric endomorphism with the eigenvalues $+1$ and -7. The result follows now from the estimate

$$\frac{1}{2}(dT \cdot \Psi, \Psi) + \frac{1}{4}\mathrm{Scal}^{\nabla}\|\Psi\|^2 \geq \Big(\frac{1}{2} \cdot 24 \cdot \lambda^2(-7) + \frac{1}{4} \cdot 48 \cdot \lambda^2 \cdot 7\Big)\|\Psi\|^2 \;=\; 0\,. \qquad \Box$$

Remark 5.7. Let us discuss the result from the point of view of the spectrum of the Riemannian Dirac operator. The first eigenvalue of the Riemannian Dirac operator on a compact, simply connected nearly parallel manifold $(M^7, g, \omega^3, \nabla)$ is

$$\mu \;=\; \frac{1}{2}\sqrt{\frac{7 \cdot R}{6}} \;=\; \frac{21 \cdot \lambda}{2}$$

(see [13]). Here we have constructed a ∇-parallel spinor

$$\nabla\Psi_0 \;=\; \nabla^g_X\Psi_0 - \frac{1}{2} \cdot \lambda \cdot (X \lrcorner\, \omega^3) \cdot \Psi_0 \;=\; 0\,.$$

Let us compute the Riemannian Dirac operator :

$$D^g\Psi_0 - \frac{3}{2} \cdot \lambda \cdot \omega^3 \cdot \Psi_0 \;=\; 0\,.$$

The endomorphism ω^3 acts Ψ_0 by multiplication by -7. Therefore we obtain

$$D^g\Psi_0 \;=\; -\frac{21}{2} \cdot \lambda \cdot \Psi_0\,,$$

i.e. the ∇-parallel spinor field on M^7 is the real Killing spinor on M^7. In this sense the ∇-parallel spinors on (non-nearly parallel) calibrated G_2-structures generalize the Killing spinors.

6 Examples

Denote by H(3) the 6-dimensional, simply-connected Heisenberg group and consider the product $M^7 := \mathrm{H}(3) \times \mathbb{R}$. There exists a left invariant metric and an orthonormal frame $e_1, \ldots, e_7$ such that the corresponding 3-form ω^3 defines a cocalibrated G_2-structure of pure type Λ^3_{27} (see [12]). Indeed, the exterior differentials are given by the formulas

$$de_1 \;=\; de_2 \;=\; de_3 \;=\; de_6 \;=\; de_7 \;=\; 0\;,$$

$$de_4 \;=\; e_1 \wedge e_6 + e_3 \wedge e_7, \quad de_5 \;=\; e_1 \wedge e_3 - e_6 \wedge e_7$$

and an easy computation yields the following formula for the differential

$$d\omega^3 \;=\; e_1 \wedge e_2 \wedge e_3 \wedge e_4 + e_2 \wedge e_4 \wedge e_6 \wedge e_7 + e_1 \wedge e_2 \wedge e_5 \wedge e_6 - e_2 \wedge e_3 \wedge e_5 \wedge e_7\;.$$

We see that $d\omega^3 \wedge \omega^3 = 0$, $d\omega^3 \wedge *\omega^3 = 0$, i.e., the G_2-structure is of pure type Λ^3_{27}. The torsion form $T = - * d\omega^3$ equals

$$T \;=\; -\,(e_5 \wedge e_6 \wedge e_7 - e_1 \wedge e_3 \wedge e_5 + e_3 \wedge e_4 \wedge e_7 + e_1 \wedge e_4 \wedge e_6)$$

and its differential is given by

$$dT \;=\; -\,4 \cdot e_1 \wedge e_3 \wedge e_6 \wedge e_7\,.$$

The Ricci tensor $2 \cdot \mathrm{Ric}^\nabla(X,Y) = (X \lrcorner\, dT, Y \lrcorner *\omega^3)$ is a diagonal matrix

$$\mathrm{Ric}^\nabla \;=\; \mathrm{diag}\,(-2,\,0,\,-2,\,0,\,0,\,-2,\,-2)$$

and the scalar curvature becomes negative, $\mathrm{Scal}^\nabla = -8$. The symmetric tensor $T_{imn}T_{jmn}$ is of diagonal form too,

$$T_{imn}T_{jmn} \;=\; \mathrm{diag}\,(4,\,0,\,4,\,4,\,4,\,4,\,4)\,,$$

and thus we obtain the Riemannian Ricci tensor

$$R^g_{ij} \;=\; \frac{1}{4}T_{imn}T_{jmn} + \mathrm{Ric}^\nabla_{ij} \;=\; \mathrm{diag}\,(-1,\,0,\,-1,\,1,\,1,\,-1,\,-1)\,.$$

Now we study the ∇-parallel spinors. First of all we need the 4-forms

$$\frac{1}{4}dT + \frac{1}{2}\sigma^T \;=\; -\,e_1 \wedge e_3 \wedge e_6 \wedge e_7 + (e_3 \wedge e_4 \wedge e_5 \wedge e_6 - e_1 \wedge e_4 \wedge e_5 \wedge e_7 - e_1 \wedge e_3 \wedge e_6 \wedge e_7)\,,$$

$$\frac{3}{4}dT - \frac{1}{2}\sigma^T \;=\; -\,3e_1 \wedge e_3 \wedge e_6 \wedge e_7 - (e_3 \wedge e_4 \wedge e_5 \wedge e_6 - e_1 \wedge e_4 \wedge e_5 \wedge e_7 - e_1 \wedge e_3 \wedge e_6 \wedge e_7)\,.$$

Lemma 6.1.

1) *The 4-form $dT/4+\sigma^T/2$ acts in the spinor bundle as a symmetric endomorphism with eigenvalues* $(2,\,-4,\,2,\,0,\,2,\,0,\,2,\,-4)$.

2) *The 4-form $3\cdot dT/4-\sigma^T/2$ acts in the spinor bundle as a symmetric endomorphism with eigenvalues* $(2,\,0,\,2,\,-4,\,2,\,-4,\,2,\,0)$.

The proof of Lemma 6.1 is an easy computation in the spin representation of the 7-dimensional Clifford algebra.

Corollary 6.2. *There are four ∇-parallel spinor fields on M^7. The torsion form acts trivially on any of these spinors, $T\cdot\Psi = 0$.*

Let G be a discrete group of isometries acting on M^7 and preserving the G_2-structure ω^3. Then M^7/G admits a G_2-structure of type Λ^3_{27}.

Corollary 6.3. *If M^7/G is a compact manifold and Ψ is ∇-harmonic, then*

$$6\cdot\int_{M^7/G}\|\Psi\|^2 \;\geq\; \int_{M^7/G}\|\nabla\Psi\|^2 \;.$$

Proof. Combining the Schrödinger-Lichnerowicz formula and Theorem 3.3 we obtain, in case of a ∇-harmonic spinor, the equation

$$\int_{M^7/G}\Big(\|\nabla\Psi\|^2+\frac{1}{4}(dT\cdot\Psi,\Psi)+\frac{1}{2}(\sigma^T\cdot\Psi,\Psi)+\frac{1}{4}\mathrm{Scal}^\nabla\cdot\|\Psi\|^2\Big) \;=\; 0\;.$$

Since $\mathrm{Scal}^\nabla = -8$, the proof follows directly by Lemma 6.1. □

We now discuss a second example. The product $M^7 = N^6\times\mathbb{R}^1$ of $\mathbb{R}^1$ by a 3-dimensional complex, solvable Lie group N^6 admits a left invariant metric such that the following structure equations hold (see [9]):

$$de_1 \;=\; 0,\quad de_2 \;=\; 0,\quad de_7 \;=\; 0\;,$$

$$de_3 \;=\; e_1\wedge e_3 - e_2\wedge e_4,\quad de_4 \;=\; e_2\wedge e_3+e_1\wedge e_4\;,$$

$$de_5 \;=\; -\,e_1\wedge e_5+e_2\wedge e_6,\quad de_6 \;=\; -\,e_2\wedge e_5-e_1\wedge e_6\;.$$

A computation of the exterior products yields the formulas :

$$d*\omega^3 \;=\; 0,\quad d\omega^3 \;=\; 2\cdot e_1\wedge e_3\wedge e_4\wedge e_7 - 2\cdot e_1\wedge e_5\wedge e_6\wedge e_7\;.$$

In particular, the corresponding G_2-structure is cocalibrated and of pure type Λ^3_{27},

$$*d\omega^3\wedge\omega^3 \;=\; 0,\quad *d\omega^3\wedge *\omega^3 \;=\; 0\;.$$

The torsion tensor T of the connection associated with the G_2-structure is given by

$$T \;=\; 2\cdot e_2\wedge e_5\wedge e_6 - 2\cdot e_2\wedge e_3\wedge e_4,\quad dT \;=\; -4\cdot e_1\wedge e_2\wedge e_5\wedge e_6 - 4\cdot e_1\wedge e_2\wedge e_3\wedge e_4\;.$$

We compute the scalar curvature $\mathrm{Scal}^\nabla = -16$ of the connection ∇ and the 4-forms

$$\frac{3}{4}dT-\frac{1}{2}\sigma^T \;=\; -\,3\cdot e_1\wedge e_2\wedge e_5\wedge e_6 - 3\cdot e_1\wedge e_2\wedge e_3\wedge e_4 + 2\cdot e_3\wedge e_4\wedge e_5\wedge e_6\;,$$

$$\frac{1}{4}dT+\frac{1}{2}\sigma^T \;=\; -\,e_1\wedge e_2\wedge e_5\wedge e_6 - e_1\wedge e_2\wedge e_3\wedge e_4 - 2\cdot e_3\wedge e_4\wedge e_5\wedge e_6\;.$$

Lemma 6.4.

1) *In the spinor bundle the 4-forms $dT/4 + \sigma^T/2$ acts in the spinor bundle as a symmetric endomorphism with eigenvalues* $(4,\,4,\,-2,\,-2,\,-2,\,-2,\,0,\,0)$.

2) *The 4-form $3 \cdot dT/4 - \sigma^T/2$ acts as a symmetric endomorphism with eigenvalues* $(4,\,4,\,2,\,2,\,2,\,2,\,-8,\,-8)$ *in the spinor bundle.*

Corollary 6.5. *There are two ∇-parallel spinor fields on M^7. The torsion form acts trivially on any of these spinors, $T \cdot \Psi = 0$.*

Let G be a discrete group of isometries acting on M^7 and preserving the G_2-structure ω^3. Then M^7/G admits a G_2-structure of type Λ^3_{27}.

Corollary 6.6. *If M^7/G is a compact manifold and Ψ is ∇-harmonic, then*

$$6 \cdot \int_{M^7/G} \|\Psi\|^2 \;\geq\; \int_{M^7/G} \|\nabla\Psi\|^2 \;.$$

We would like to mention that any hypersurface $M^7 \subset \mathbb{R}^8$ admits a cocalibrated G_2-structure (see [11]). This structure is of pure type Λ^3_1 if and only if the hypersurface is umbilic. The pure type Λ^3_{27} occurs if and only if the hypersurface is minimal. The function $\lambda = (d\omega^3, *\omega^3)$ is proportional to the mean curvature of the hypersurface (see [11]). Moreover, it turns out that in the decomposition

$$T \;=\; -\,2 \cdot \lambda \cdot \omega^3 - 6 \cdot \Gamma^3_{27} \;=\; -\,(*d\omega^3) - 14 \cdot \lambda \cdot \omega^3$$

of the torsion tensor T (see Remark 4.13) the 3-form Γ^3_{27} corresponds to the traceless part of the second fundamental form of the hypersurface via the G_2-isomorphism $S^2_0(\mathbb{R}^7) = \Lambda^3_{27}$. The torsion form is coclosed for hypersurfaces of constant mean curvature. In this case we obtain solutions of the equations in Theorem 5.1 such that $\delta^g(T) = 0$.

7 Sasakian manifolds in dimension five

The case of the group G_2 and dimension $n = 7$ discussed in detail fits into a more general approach described in the introduction. We study two further natural geometric structures: almost metric contact structures and almost hermitian structures. To begin with, let us consider the case of 5-dimensional Sasakian manifolds.

Proposition 7.1. *Every Sasakian manifold $(M^{2k+1}, g, \xi, \eta, \varphi)$ admits a unique metric connection with totally skew-symmetric torsion preserving the Sasakian structure :*

$$\nabla\xi \;=\; \nabla\eta \;=\; \nabla\varphi \;=\; 0\;.$$

The connection ∇ is given by

$$g(\nabla_X Y, Z) \;=\; g(\nabla^g_X Y, Z) + \frac{1}{2}(\eta \wedge d\eta)(X, Y, Z), \quad T \;=\; \eta \wedge d\eta\;.$$

The torsion form T is ∇-parallel and henceforth coclosed, $\delta^g(T) = 0$. The 4-form $2 \cdot \sigma^T$ coincides with dT,

$$2 \cdot \sigma^T \;=\; dT \;=\; d\eta \wedge d\eta \ .$$

Proof. The existence of the connection has been noticed in earlier papers (see e.g.[36]). The uniqueness will be proved in a more general context in Theorem 8.2. □

We now consider a 5-dimensional Sasakian manifold M^5 and orient it by the condition that the differential of the contact form is given by

$$d\eta \;=\; 2 \cdot (e_1 \wedge e_2 + e_3 \wedge e_4) \ .$$

Furthermore, we fix a spin structure. The endomorphism $\eta \wedge d\eta = 2 \cdot (e_1 \wedge e_2 + e_3 \wedge e_4) \wedge e_5$ acts in the 5-dimensional spin representation with eigenvalues $(-4, 0, 0, 4)$. Consequently, the spinor bundle splits into two 1-dimensional and one 2-dimensional ∇-parallel subbundles. The Clifford multiplication by ξ preserves this decomposition of the spinor bundle and acts on the 1-dimensional bundles by multiplication by i, on the 2-dimensional bundle by multiplication by $-i$. Suppose that there exists a ∇-parallel spinor Ψ. Then one of the subbundles under consideration admits a ∇-parallel spinor

$$\nabla^g_X \Psi + \frac{1}{4}(X \lrcorner\, \eta \wedge d\eta) \cdot \Psi \;=\; 0$$

and the Riemannian Dirac operator for this spinor is given by the formula

$$D^g \Psi + \frac{3}{4}(\eta \wedge d\eta) \cdot \Psi \;=\; 0 \ .$$

Let us first discuss the case that Ψ belongs to one of the 1-dimensional subbundles defined by the algebraic equation $\xi \cdot \Psi = i \cdot \Psi$. In this case we apply Corollary 3.2

$$\sum_{i<j<k} \left\{ \frac{1}{2} dT(X, e_i, e_j, e_k) \right\} e_i \cdot e_j \cdot e_k \cdot \Psi - \mathrm{Ric}^{\nabla}(X) \cdot \Psi \;=\; 0$$

as well as the following algebraic lemma.

Lemma 7.2. *The spinor $\Psi = (1, 0, 0, 0)$ or $(0, 0, 0, 1)$ belongs to the kernel of the endomorphism*

$$\sum_{1 \leq i<j<k \leq 5} t_{ijk} \cdot e_i \cdot e_j \cdot e_j + \sum_{i=1}^{5} x_i \cdot e_i$$

in the 5-dimensional spin representation if and only if the following equations hold :

$$x_1 \;=\; -\,t_{234}, \quad x_2 \;=\; t_{134}, \quad x_3 \;=\; -\,t_{124}, \quad x_4 \;=\; t_{123}, \quad x_5 \;=\; 0 \ ,$$

$$t_{125} \;=\; -\,t_{345}, \quad t_{235} \;=\; -\,t_{145}, \quad t_{245} \;=\; t_{135} \ .$$

Using these formulas we conclude $\mathrm{Ric}^{\nabla} = \mathrm{diag}(a, a, a, a, 0)$, where $2 \cdot a := dT(e_1, e_2, e_3, e_4) = d\eta \wedge d\eta(e_1, e_2, e_3, e_4) = 8$. In particular, we obtain

$$\mathrm{Ric}^g - \mathrm{diag}(2,\ 2,\ 2,\ 2,\ 4) \;=\; \mathrm{Ric}^g - \frac{1}{4} T_{imn} T_{jmn} \;=\; \mathrm{Ric}^{\nabla} \ ,$$

i.e., $\mathrm{Ric}^g = \mathrm{diag}(6,\,6,\,6,\,6,\,4)$. A proof similar to the proof of the existence of Killing spinors on Einstein-Sasakian manifolds (see [17]) shows that this condition is the only integrability condition for ∇-parallel spinors in the 1-dimensional subbundles. The endomorphisms act in the spinor bundle

$$\frac{3}{4}dT - \frac{1}{2}\sigma^T + \frac{1}{4}\mathrm{Scal}^\nabla \;=\; 4\cdot(e_1\cdot e_2\cdot e_3\cdot e_4 + 1) \;=\; \frac{1}{4}dT + \frac{1}{2}\sigma^T + \frac{1}{4}\mathrm{Scal}^\nabla$$

with eigenvalues 0 and 4. Corollary 3.2 and Theorem 3.3 yield

Theorem 7.3. *Let $(M^5, g, \xi, \eta, \varphi)$ be a simply connected 5-dimensional Sasakian spin manifold and consider the unique linear connection ∇ with totally skew-symmetric torsion preserving the Sasakian structure. There exists a ∇-parallel spinor in the subbundle defined by the algebraic equation $\xi\cdot\Psi = i\cdot\Psi$ if and only if the Riemannian Ricci tensor of M^5 has the eigenvalues $(6,\,6,\,6,\,6,\,4)$. A ∇-parallel spinor of this algebraic type is an eigenspinor of the Riemannian Dirac operator, $D^g\Psi = \pm\,3\cdot\Psi$. In case M^5 is compact, any ∇-harmonic spinor Ψ is ∇-parallel.*

Example 7.4. Sasakian manifolds with the described form of the Ricci tensor can be constructed – for example – as bundles over 4-dimensional Kähler-Einstein manifolds with positive scalar curvature. Indeed, consider a simply connected Kähler-Einstein manifold (N^4, J, g^*) with scalar curvature $\mathrm{Scal}^* = 32$. Then there exists an S^1-bundle $M^5 \to N^4$ as well as a Sasakian structure on M^5 such that the Ricci tensor has the eigenvalues $\mathrm{Ric}^g \;=\; \mathrm{diag}(6,\,6,\,6,\,6,\,4)$ (see [19]). More general, the Tanno deformation of an arbitrary 5-dimensional Einstein-Sasakian structure yields for a special value of deformation parameter examples of Sasakian manifolds satisfying the condition of Theorem 7.3 (see Example 9.3). The Einstein-Sasakian manifolds constructed recently in [6] admit ∇-parallel spinors with respect to a Tanno deformation of the Sasakian structure.

We discuss the case that the ∇-parallel spinor field Ψ belongs to the 2-dimensional subbundle defined by the algebraic equation $d\eta\cdot\Psi = 0$. A spinor field of this type is a Riemannian harmonic spinor. Let us again compute the Ricci tensor Ric^∇:

Lemma 7.5. *The spinor $\Psi = (0,\,1,\,0,\,0)$ belongs to the kernel of the endomorphism*

$$\sum_{1\le i<j<k\le 5} t_{ijk}\cdot e_i\cdot e_j\cdot e_j + \sum_{i=1}^{5} x_i\cdot e_i$$

in the 5-dimensional spin representation if and only if the following equations hold :

$$x_1 \;=\; t_{234},\quad x_2 \;=\; -\,t_{134},\quad x_3 \;=\; t_{124},\quad x_4 \;=\; -\,t_{123},\quad x_5 \;=\; 0\;,$$

$$t_{125} \;=\; t_{345},\quad t_{235} \;=\; t_{145},\quad t_{245} \;=\; -\,t_{135}\;.$$

In this case we obtain

$$\mathrm{Ric}^\nabla \;=\; \mathrm{diag}(-4,\,-4,\,-4,\,-4,\,0),\quad \mathrm{Ric}^g \;=\; \mathrm{diag}(-2,\,-2,\,-2,\,-2,\,4)\;.$$

We compute the endomorphisms acting in the spinor bundle :

$$\frac{3}{4}dT - \frac{1}{2}\sigma^T + \frac{1}{4}\mathrm{Scal}^\nabla \;=\; 4\cdot(e_1\cdot e_2\cdot e_3\cdot e_4 - 1) \;=\; \frac{1}{4}dT + \frac{1}{2}\sigma^T + \frac{1}{4}\mathrm{Scal}^\nabla\;.$$

The Clifford product $e_1 \cdot e_2 \cdot e_3 \cdot e_4$ acts in the 2-dimensional subbundle of the spin bundle as the identity. Corollary 3.2 and Theorem 3.3 yield

Theorem 7.6. *Let $(M^5, g, \xi, \eta, \varphi)$ be a 5-dimensional Sasakian spin manifold and consider the unique linear connection ∇ with totally skew-symmetric torsion preserving the Sasakian structure. If there exists a ∇-parallel spinor in the subbundle defined by the algebraic equation $d\eta \cdot \Psi = 0$, then the Riemannian Ricci tensor of M^5 has the eigenvalues $(-2, -2, -2, -2, 4)$. Any ∇-parallel spinor in this 2-dimensional subbundle satisfies the equations*

$$\nabla^g_\xi \Psi \;=\; 0, \quad \nabla^g_X \Psi \;=\; \frac{1}{2}\,\varphi(X)\cdot\xi\cdot\Psi \;=\; -\frac{i}{2}\,\varphi(X)\cdot\Psi, \quad d\eta\cdot\Psi \;=\; 0\,.$$

In particular, it is harmonic with respect to the Riemannian connection. Any ∇-harmonic spinor Ψ on a compact manifold M^5 satisfying the algebraic condition $d\eta \cdot \Psi = 0$ is ∇-parallel.

Example 7.7. Sasakian manifolds with the described form of the Ricci tensor may be constructed as bundles over 4-dimensional selfdual, Ricci-flat Kähler manifolds. Indeed, suppose that the Sasakian structure on M^5 is regular. Then M^5 is an S^1-bundle over a Ricci-flat Kähler manifold N^4. Moreover, the spinor field Ψ is projectable and induces a parallel spinor field Ψ^* in the negative spinor bundle $\Sigma^-(N^4)$ over N^4 (see [39]). Consequently, N^4 is a selfdual, Ricci-flat Kähler manifold. Vice versa, let N^4 be a $K3$-surface equipped with its selfdual, Ricci-flat metric. The Weitzenböck formula for 2-forms proves that any harmonic, antiselfdual 2-form is parallel. Since $b_2^+ = 3$, the dimension of this space equals three. Moreover, for any 2-form $\Omega \in \Lambda^2_-(N^4)$ the corresponding symmetric form

$$\sum_{k=1}^{4} \Omega_{ki}\cdot\Omega_{kj} \;=\; g_{ij}$$

is a positive multiple of the metric. Fix the parallel (harmonic) form Ω in such a way that its de-Rham cohomolgy class is an integral class. Then there exists a unique S^1-bundle $M^5 \to N^4$ as well as a connection in the S^1-bundle such that Ω is the curvature form of the connection. A Kaluza-Klein construction defines a Sasakian structure on the total space M^5 and the Ricci tensor has the eigenvalues $(-2, -2, -2, -2, 4)$ (see [16]). Moreover, the parallel spinor fields Ψ^* on N^4 pull back to ∇-parallel spinors on M^5.

8 Almost contact connections with totally skew-symmetric torsion

Let us discuss the latter results from a more general point of view and consider an almost contact metric manifold $(M^{2k+1}, g, \xi, \eta, \varphi)$, i.e., a Riemannian manifold equipped with a 1-form η, a (1,1)-tensor φ and a vector field ξ dual to η with respect to the metric g such that the following compatibility conditions are satisfied (see [4]):

$$\eta(\xi) \;=\; 1, \quad \varphi^2 \;=\; -\operatorname{Id}+\eta\otimes\xi, \quad g(\varphi(X),\varphi(Y)) \;=\; g(X,Y)-\eta(X)\cdot\eta(Y), \quad \varphi(\xi) \;=\; 0.$$

Let us introduce the fundamental form $F(X,Y) := g(X,\varphi(Y))$ as well as the Nijenhuis tensor

$$N(X,Y) \ := \ [\varphi(X),\varphi(Y)] + \varphi^2[X,Y] - \varphi[\varphi(X),Y] - \varphi[X,\varphi(Y)] + d\eta(X,Y)\cdot\xi \ ,$$

$$N^2(X,Y) \ := \ d\eta(\varphi(X),Y) + d\eta(X,\varphi(Y)) \ .$$

We recall some notions describing the different types of almost contact metric structures. If $2\cdot F = d\eta$, then we have a *contact metric structure*, if $N=0$, we have a *normal contact structure*. A *K-contact structure* is a contact metric structure such that the vector field ξ is a Killing vector field. If the structure is normal and K-contact, then it is a *Sasaki structure* (a complete classification of almost contact metric structures is presented in [2, 7, 8]). The Nijenhuis tensor of type (0, 3) is given by $N(X,Y,Z) = g(N(X,Y),Z)$. We have the following general identities [4, 7, 8]:

$$\begin{aligned} 2\cdot g((\nabla^g_X\varphi)(Y),Z) &= dF(X,\varphi(Y),\varphi(Z)) - dF(X,Y,Z) + N(Y,Z,\varphi(X)) \\ &\quad +\eta(X)\cdot N^2(Y,Z) + \eta(Z)\cdot d\eta(\varphi(Y),X) + \eta(Y)\cdot d\eta(X,\varphi(Z)) \ , \end{aligned}$$

$$g((\nabla^g_X\varphi)(Y),Z) + g((\nabla^g_X\varphi)(\varphi(Y)),\varphi(Z)) = \eta(Y)\cdot(\nabla^g_X\eta)(\varphi(Z)) - \eta(Z)\cdot(\nabla^g_X\eta)(\varphi(Y)) \ ,$$

$$g((\nabla^g_X\varphi)(\varphi(Y)),\xi) \ = \ (\nabla^g_X\eta)(Y) \ = \ g(\nabla^g_X\xi,Y) \ ,$$

$$\begin{aligned} N(X,Y,Z) &= -N(\varphi(X),\varphi(Y),Z) + \eta(X)\cdot N(\xi,Y,Z) + \eta(Y)\cdot N(X,\xi,Z) \\ &= -N(\varphi(X),Y,\varphi(Z)) + \eta(Z)\cdot N(\xi,X,Y) - \eta(X)\cdot N(\xi,\varphi(Y),\varphi(Z)) \ . \end{aligned}$$

Finally, we introduce the forms

$$\begin{aligned} dF^-(X,Y,Z) &:= dF(X,\varphi(Y),\varphi(Z)) + dF(\varphi(X),Y,\varphi(Z)) \\ &\quad + dF(\varphi(X),\varphi(Y),Z) - dF(X,Y,Z)\,, \\ d^\varphi F(X,Y,Z) &:= -dF(\varphi(X),\varphi(Y),\varphi(Z))\,, \end{aligned}$$

and a direct consequence of the definitions is the following

Proposition 8.1. On any almost contact metric manifold the identities hold:

1) $dF^-(X,Y,Z) = -N(X,Y,\varphi(Z)) - N(Y,Z,\varphi(X)) - N(Z,X,\varphi(Y))$,

2) $N(X,Y) = (\nabla^g_{\varphi(X)}\varphi)(Y) - (\nabla^g_{\varphi(Y)}\varphi)(X) + (\nabla^g_X\varphi)(\varphi(Y)) - (\nabla^g_Y\varphi)(\varphi(X))$
$-\eta(Y)\cdot\nabla^g_X\xi + \eta(X)\cdot\nabla^g_Y\xi\,.$

A linear connection ∇ is said to be an almost contact connection if it preserves the almost contact structure :

$$\nabla g \ = \ \nabla\eta \ = \ \nabla\varphi \ = \ 0\ .$$

Theorem 8.2. *Let $(M^{2k+1},g,\xi,\eta,\varphi)$ be an almost contact metric manifold. The following conditions are equivalent:*

1) *The Nijenhuis tensor N is skew-symmetric and ξ is a Killing vector field.*

2) *There exists an almost contact linear connection ∇ with totally skew-symmetric torsion tensor T.*

Moreover, this connection is unique and determined by

$$g(\nabla_X Y, Z) \; = \; g(\nabla^g_X Y, Z) + \frac{1}{2} T(X, Y, Z) \; ,$$

where ∇^g is the Levi-Civita connection and the torsion T is defined by

$$T \; = \; \eta \wedge d\eta + d^{\varphi} F + N - \eta \wedge (\xi \lrcorner N) \; .$$

Proof. Let us assume that such a connection exists. Then

$$0 \; = \; g(\nabla^g_X \xi, Z) + \frac{1}{2} T(X, \xi, Z)$$

holds and the skew-symmetry of T yields that ξ is a Killing vector field, $d\eta = \xi \lrcorner T$, $\xi \lrcorner d\eta = 0$ and

$$T(\varphi(X), \varphi(Y), Z) - T(X, Y, Z) + T(\varphi(X), Y, \varphi(Z)) + T(X, \varphi(Y), \varphi(Z)) \; = \; -N(X, Y, Z) \, .$$

The latter formula shows that N is skew-symmetric. Since φ is ∇-parallel, we can express the Riemannian covariant derivative of φ by the torsion form:

$$T(X, Y, \varphi(Z)) + T(X, \varphi(Y), Z) \; = \; -2g((\nabla^g_X \varphi)Y, Z) \, .$$

Taking the cyclic sum in the above equality, we obtain

$$\sigma_{X,Y,Z} \, T(X, Y, \varphi(Z)) \; = \; -\sigma_{X,Y,Z} \, g((\nabla^g_X \varphi)Y, Z) \, .$$

We use Proposition 8.1 as well as the identity preceding it to get

$$-\sigma_{X,Y,Z} \, T(X, Y, \varphi(Z)) \; = \; \sigma_{X,Y,Z} \, g((\nabla^g_X \varphi)Y, Z) \; = \; -dF(X, Y, Z) \, .$$

Adding this result to the formula expressing the Nijenhuis tensor N by the torsion T, some calculations yield

$$T(\varphi(X), \varphi(Y), \varphi(Z)) \; = \; dF(X, Y, Z) - N(X, Y, \varphi(Z)) - \eta(Z) \cdot N^2(X, Y).$$

By replacing X, Y, Z by $\varphi(X), \varphi(Y), \varphi(Z)$ and using the symmetry property of the Nijenhuis tensor mentioned before Proposition 8.1, we obtain the formula for the torsion tensor T.
For the converse, suppose that the almost contact structure has the properties 1) and define the connection ∇ by the formulas in 2). Clearly T is skew-symmetric and $\xi \lrcorner T = d\eta = 2\nabla^g \eta$. Since ξ is a Killing vector field, we conclude $\nabla g = \nabla \xi = 0$. Furthermore, using the conditions 1) and Proposition 8.1, we obtain $\xi \lrcorner dF = N^2$. Finally we have to prove that $\nabla \varphi = 0$. This follows by straightforward computations using the relation between $\nabla \varphi$ and the torsion tensor T, Proposition 8.1 as well as the following lemma. □

Lemma 8.3. *Let $(M^{2k+1}, g, \xi, \eta, \varphi)$ be an almost contact metric manifold with a totally skew-symmetric Nijenhuis tensor N. Then the following equalities hold:*

$$\nabla^g_\xi \xi \; = \; \xi \lrcorner d\eta \; = \; 0 \; ,$$

$$(\nabla^g_X\eta)(Y) + (\nabla^g_Y\eta)(X) \;=\; -\,(\nabla^g_{\varphi(X)}\eta)(\varphi(Y)) - (\nabla^g_{\varphi(Y)}\eta)(\varphi(X))\ ,$$

$$N(\varphi(X),Y,\xi) \;=\; N(X,\varphi(Y),\xi) \;=\; N^2(X,Y) \;=\; dF(X,Y,\xi) \;=\; -\,dF(\varphi(X),\varphi(Y),\xi)\ .$$

Proof. The identities in and before Proposition 8.1 imply

$$0 \;=\; N(X,\xi,\xi) \;=\; (\nabla^g_\xi\varphi)(\xi), \quad 0 \;=\; N(\xi,X,X) \;=\; (\nabla^g_X\eta)(X) - (\nabla^g_{\varphi(X)}\eta)(\varphi(X))\ ,$$

$$N(X,Y,\xi) \;=\; d\eta(X,Y) - d\eta(\varphi(X),\varphi(Y))\ .$$

Hence, the first two equalities follow. We take the cyclic sum in the second idendity before Proposition 8.1 and put $X = \xi$ to obtain

$$-\,dF(\xi,Y,Z) + g((\nabla^g_\xi\varphi)(\varphi(Y)),\varphi(Z)) \;=\; -\,(\nabla^g_Y\eta)(\varphi(Z)) + (\nabla^g_Z\eta)(\varphi(Y))\ .$$

On the other hand, using again the general formulas we calculate that

$$-\,dF(\xi,\varphi(Y),\varphi(Z)) \;=\; g((\nabla^g_\xi\varphi)(\varphi(Y)),\varphi(Z)) + (\nabla^g_{\varphi(Y)}\eta)(Z) - (\nabla^g_{\varphi(Z)}\eta)(Y)\ .$$

Summing up the latter two equalities and taking into account the second already proved equality of Lemma 8.3 we obtain the last equality. □

We discuss these results for some special contact structures.

Theorem 8.4. *Let $(M^{2k+1},g,\xi,\eta,\varphi)$ be an almost contact metric manifold with totally skew-symmetric Nijenhuis tensor N. Then the condition $dF = 0$ implies $N = 0$.*

1) *A contact metric structure ($2\cdot F = d\eta$) admits an almost contact connection with totally skew-symmetric torsion if and only if it is Sasakian. In this case, the connection is unique, its torsion is given by*

$$T \;=\; \eta\wedge d\eta$$

and T is parallel, $\nabla T = 0$.

2) *A normal ($N = 0$) almost contact structure admits a unique almost contact connection with totally skew-symmetric torsion if and only if ξ is a Killing vector field. The torsion T is then given by*

$$T \;=\; \eta\wedge d\eta + d^\varphi F.$$

Proof. If $dF = 0$, Lemma 8.3 implies that $N^2 = \xi\lrcorner N = 0$. Then Proposition 8.1 leads to $0 = dF^-(X,Y,Z) = -3\cdot N(\varphi(X),Y,Z)$. The assertion that $\nabla T = 0$ in a Sasakian manifold follows by direct verification. □

9 Almost contact structures, parallel spinors and holonomy group

Let $(M^{2k+1},g,\xi,\eta,\varphi)$ be a $(2k+1)$-dimensional almost contact metric manifold with totally skew-symmetric Nijenhuis tensor N and Killing vector ξ and denote by ∇ the unique almost contact connection with a totally skew-symmetric torsion (Theorem 8.2).

Since $\nabla\xi = 0$ the (restricted) holonomy group Hol^{∇} of ∇ is contained in $U(k)$. This group cannot occur as the isotropy group of any spinor. The spinor bundle Σ of a contact spin manifold decomposes under the action of the fundamental form F into the sum (see [19])

$$\Sigma \ = \ \Sigma_0 \oplus \ldots \Sigma_k, \quad \dim(\Sigma_r) \ = \ \binom{k}{r} \ .$$

The isotropy group of a spinor of type Σ_0 or Σ_k coincides with the subgroup $SU(k) \subset U(k)$. Consequently, there exists locally a ∇-parallel spinor of type Σ_0, Σ_k if and only if Hol^{∇} is contained in $SU(k)$. We shall express this condition in terms of the curvature of ∇. The group Hol^{∇} is contained in $SU(k)$ if and only if the 2-form

$$\varrho^{\nabla}(X,Y) \ := \ \frac{1}{2}\sum_{i=1}^{2k+1} R^{\nabla}(X,Y,e_i,\varphi(e_i)) \ .$$

vanishes, $\varrho^{\nabla} = 0$. Let us introduce the torsion 1-form ω^{∇} as well as the 2-form λ^{∇} by

$$\omega^{\nabla}(X) \ := \ -\frac{1}{2}\sum_{i=1}^{2k+1} T(X,e_i,\varphi(e_i))\,, \quad \lambda^{\nabla}(X,Y) \ := \ \frac{1}{2}\sum_{i=1}^{2k+1} dT(X,Y,e_i,\varphi(e_i)) \ .$$

Proposition 9.1. *Let $(M^{2k+1},g,\xi,\eta,\varphi)$ be a $(2k+1)$-dimensional almost contact metric manifold with totally skew-symmetric Nijenhuis tensor N and Killing vector ξ. Let ∇ be the unique almost contact connection with totally skew-symmetric torsion. Then one has*

$$\varrho^{\nabla}(X,Y) \ = \ \mathrm{Ric}^{\nabla}(X,\varphi(Y)) - (\nabla_X\omega^{\nabla})(Y) + \frac{1}{4}\lambda^{\nabla}(X,Y) \ .$$

Proof. We follow the scheme in [34], Section 3, and use the curvature properties of ∇ from Section 2 to calculate $\lambda^{\nabla}(X,Y)$:

$$-2(\nabla_X\omega)(Y) + 2(\nabla_Y\omega)(X) + 2\sum_{i=1}^{2k+1}\Big(\sigma^T(X,Y,e_i,\varphi(e_i)) - (\nabla_{\varphi(e_i)}T)(X,Y,e_i)\Big) \ .$$

The first Bianchi identity for ∇ together with the latter identity implies

$$4\rho^{\nabla}(X,Y) + 2\mathrm{Ric}^{\nabla}(Y,\varphi(X)) - 2\mathrm{Ric}^{\nabla}(X,\varphi(Y)) \ = \ \lambda^{\nabla}(X,Y) - 2(\nabla_X\omega)(Y) + 2(\nabla_Y\omega)(X) \ .$$

Using the relation between the curvature tensors of ∇ and ∇^g, we obtain

$$\mathrm{Ric}^{\nabla}(Y,\varphi(X)) + \mathrm{Ric}^{\nabla}(X,\varphi(Y)) \ = \ (\nabla_X\omega^{\nabla})(Y) + (\nabla_Y\omega^{\nabla})(X) \ .$$

The last two equalities lead to the desired formula. □

We apply Proposition 9.1 in case of a Sasakian manifold.

Theorem 9.2. *Let $(M^{2k+1},g,\xi,\eta,\varphi,\nabla)$ be a simply connected $(2k+1)$-dimensional Sasakian spin manifold and ∇ be the unique almost contact connection with totally skew-symmetric torsion. Then there exists a ∇-parallel spinor of type Σ_0 or Σ_k if and only if the Ricci tensor is given by the formula :*

$$\mathrm{Ric}^{\nabla} \ = \ 4\cdot(k-1)\cdot(g - \eta\otimes\eta) \ .$$

This condition is equivalent to

$$\mathrm{Ric}^g \;=\; 2\cdot(2k-1)\cdot g - 2\cdot(k-1)\cdot\eta\otimes\eta\ .$$

Proof. On a Sasakian manifold $T = \eta\wedge d\eta = 2\cdot\eta\wedge F$ and $\nabla T = 0$, where $F(X,Y) = g(X,\varphi(Y))$ is the fundamental form of the Sasakian structure. Consequently, we calculate that

$$\nabla(\omega^\nabla) \;=\; 0, \quad \lambda^\nabla \;=\; 16\cdot(1-k)F, \quad \sum_{i=1}^{2k+1} g(T(X,e_i),T(Y,e_i)) \;=\; 8\cdot g + 8\cdot(k-1)\eta\otimes\eta\ ,$$

and the proof follows from Proposition 9.1. □

Remark 9.3. Sasakian manifolds with the prescribed form of the Ricci tensor admit Sasakian quasi-Killing spinors of type $(\pm 1/2, b)$ (see [19], Theorem 6.3). The Tanno deformation of an Einstein-Sasakian manifold defined by the formulas

$$\tilde\varphi \;:=\; \varphi, \quad \tilde\xi \;:=\; a^2\cdot\xi, \quad \tilde\eta \;:=\; a^{-2}\cdot\eta, \quad \tilde g \;:=\; a^{-2}\cdot g + (a^{-4}-a^{-2})\cdot\eta\otimes\eta$$

yields for the parameter $a^2 := 2k/(k+1)$ a Sasakian manifold satisfying the condition of Theorem 9.2 and vice versa (see [19], Lemma 6.7 and Lemma 6.8).

10 Almost hermitian connections with totally skew-symmetric torsion

In this section we study connections with totally skew-symmetric torsion and preserving an almost complex structure. These exist, for example, for nearly Kähler manifolds. In dimension $n = 6$ nearly Kähler manifolds have special properties and they are precisely the 6-dimensional manifolds admitting real Killing spinors (see [28] and [29]).

Theorem 10.1. *Let (M^{2n}, g, J) be a $2n$-dimensional almost complex manifold. Then there exists a linear connection with totally skew-symmetric torsion preserving the hermitian structure (g, J) if and only if the Nijenhuis tensor $N(X,Y,Z) := g(N(X,Y),Z)$ is a 3-form. In this case the connection is unique and is determined by*

$$T(X,Y,Z) \;=\; -\,d\Omega(J(X),J(Y),J(Z)) + N(X,Y,Z)\,,$$

where Ω is the Kähler form.

Proof. The result can be derived from the considerations in [24] or from Proposition 4.1. We sketch a direct proof. Since $\nabla g = \nabla J = 0$, we have

$$T(J(X),J(Y),Z) - T(X,Y,Z) + T(J(X),Y,J(Z)) + T(X,J(Y),J(Z)) \;=\; -\,N(X,Y,Z)\,,$$

which shows that N is a 3-form. The formula for the torsion form follows from the following identities on an almost complex manifold with skew symmetric tensor N.

$$\begin{aligned}
2\cdot g\Big((\nabla^g_X J)Y, Z\Big) \;&=\; d\Omega(X,J(Y),J(Z)) - d\Omega(X,Y,Z) + N(Y,Z,J(X)) \\
4\cdot d\Omega^-(X,Y,Z) \;&:=\; d\Omega(X,J(Y),J(Z)) - d\Omega(X,Y,Z) + d\Omega(J(X),J(Y),Z) \\
&+\; d\Omega(J(X),Y,J(Z)) \;=\; -\,3\cdot N(J(X),Y,Z)\ .
\end{aligned}$$

□

Corollary 10.2. *On an almost Kähler manifold there does not exist a hermitian connection with totally skew-symmetric torsion.*

Corollary 10.3. *On any nearly Kähler manifold the torsion form T is ∇-parallel and henceforth coclosed, $\delta^g(T) = 0$.*

Proof. If (M^{2n}, g, J) is a nearly Kähler manifold, then

$$d\Omega(X,Y,Z) \;=\; -\,d\Omega^-(X,Y,Z) \;=\; 3\cdot N(J(X),Y,Z)$$

and Theorem 10.1 yields that $4\cdot T = N$. Moreover, ∇ is the characteristic connection considered by Gray [26] and T is ∇-parallel, $\nabla T = \nabla N = 0$ (see e.g. [35, 3]). □

We compute the Ricci tensor Ric^∇ for a 6-dimensional nearly Kähler spin manifold.

Proposition 10.4. *On a 6-dimensional nearly Kähler manifold with non-vanishing Nijenhuis tensor $N \neq 0$ we have*

$$T_{imn}T_{jmn} \;=\; 2\cdot a\cdot g_{ij}, \quad \mathrm{Ric}^g \;=\; \frac{5}{2}\cdot a\cdot g, \quad \mathrm{Ric}^\nabla \;=\; 2\cdot a\cdot g\ .$$

The 4-form $2\cdot\sigma^T$ coincides with dT,

$$2\cdot\sigma^T \;=\; dT \;=\; a\cdot(\Omega\wedge\Omega)\ .$$

Proof. We recall (see [26]) that any 6-dimensional nearly Kähler manifold is Einstein and of constant type, i.e.

$$\mathrm{Ric}^g \;=\; \frac{5}{2}\cdot a\cdot g, \quad ||(\nabla^g_X J)Y||^2 \;=\; \frac{1}{2}\cdot a\cdot(||X||^2\cdot||Y||^2 - g^2(X,Y) - g^2(X,J(Y)))$$

where $a := \mathrm{Scal}^g/15$ is a positive constant. Polarizing the latter equality und using the identity $4\cdot J(\nabla^g_X J)Y = N(X,Y) = 4\cdot T(X,Y)$ we get

$$T_{imn}T_{jmn} \;=\; 2\cdot a\cdot g_{ij}, \quad 2\cdot\sigma^T \;=\; a\cdot\Omega\wedge\Omega\ .$$

We calculate $\mathrm{Ric}^\nabla_{ij} = \mathrm{Ric}^g_{ij} - \frac{1}{4}\cdot T_{imn}T_{jmn} = 2\cdot a\cdot g_{ij}$ and the result follows. □

We consider again the general almost complex case. Let (M^{2n}, g, J) be an almost complex manifold with totally skew-symmetric tensor N. Then M^{2n} is of type G_1 according to Gray-Hervella classification (see [27]). Denote by ∇ the unique hermitian connection with totally skew-symmetric torsion T described in Theorem 9.1. The Ricci form of ∇ is defined by

$$\varrho^\nabla(X,Y) \;=\; \frac{1}{2}\sum_{i=1}^{2n} R^\nabla(X,\,Y,\,e_i,\,J(e_i))\ .$$

The holonomy group Hol^∇ of ∇ is contained in $SU(n)$ if and only if $\varrho^\nabla = 0$. We define the Lee form θ and the tensor λ^ω by

$$\theta(X) \;=\; -\frac{1}{2}\sum_{i=1}^{2n} T(J(X),\,e_i,\,J(e_i)), \quad \lambda^\omega(X,Y) \;=\; \sum_{i=1}^{2n} dT(X,\,Y,\,e_i,\,J(e_i))\ .$$

We remark that the formula (3.16) in [34] holds in the general case of a G_1-manifold,

$$\varrho^{\nabla}(X,Y) \;=\; \mathrm{Ric}^{\nabla}(X,J(Y)) + (\nabla_X\theta)J(Y) + \frac{1}{4}\lambda^{\omega}(X,Y)\;.$$

Theorem 10.5. *Let (M^{2n},g,J,∇) be an almost hermitian manifold of type G_1 with its unique linear connection ∇ with totally skew-symmetric torsion. Then $Hol^{\nabla}\subset SU(n)$ if and only if*

$$0 \;=\; \mathrm{Ric}^{\nabla}(X,J(Y)) + (\nabla_X\theta)J(Y) + \frac{1}{4}\lambda^{\omega}(X,Y)\;.$$

Corollary 10.6. *Let (M^{2n},g,J,∇) be a nearly Kähler manifold. Then* Hol^{∇} *is contained in SU(n) if and only if*

$$\mathrm{Ric}^{\nabla}(X,Y) \;=\; \frac{1}{4}\lambda^{\omega}(X,J(Y))\;.$$

If the manifold is not Kähler and the dimension is 6*, then the above condition is always satisfied.*

Proof. In the nearly Kähler case the torsion form T is ∇-parallel and therefore θ is parallel too, $\nabla\theta = 0$. In the 6-dimensional strictly nearly Kähler case the condition of the Corollary 10.6 is a consequence of Proposition 10.4. □

Finally we study in detail the ∇-parallel and ∇-harmonic spinor fields on a 6-dimensional nearly Kähler manifold M^6. The spinor bundle $\Sigma(M^6)$ splits into the 1-dimensional subbundles $E^{\pm}\subset\Sigma^{\pm}(M^6)$ of the spinor bundle defined by the equation

$$J(X)\cdot\Psi^{\pm} \;=\; \mp i\cdot X\cdot\Psi^{\pm}$$

as well as their orthogonal complements. The connection ∇ preserves this decomposition. We discuss the integrability condition (see Corollary 3.2) for the existence of a ∇-parallel spinor. A purely algebraic computation in the 6-dimensional spin representation proves the following

Lemma 10.7. *A spinor $\Psi^{\pm}$ satisfies the equation*

$$\frac{1}{2}(X \lrcorner\, dT)\cdot\Psi^{\pm} - \mathrm{Ric}^{\nabla}(X)\cdot\Psi^{\pm} \;=\; 0$$

for all vectors X if and only if it belongs to $E^{\pm}$. The endomorphism

$$\frac{3}{4}dT - \frac{1}{2}\sigma^T + \frac{1}{4}\mathrm{Scal}^{\nabla} \;=\; \frac{1}{4}dT + \frac{1}{2}\sigma^T + \frac{1}{4}\mathrm{Scal}^{\nabla}$$

$$=\; a\cdot(e_1\cdot e_2\cdot e_3\cdot e_4 + e_1\cdot e_2\cdot e_5\cdot e_6 + e_3\cdot e_4\cdot e_5\cdot e_6 + 3)$$

acts in the spinor modules $\Delta_6^{\pm}$ with eigenvalues $(0,4\,a,4\,a,4\,a)$.

Consequently, Corollary 3.2 and Theorem 3.3 yield the following results :

Theorem 10.8. *Let (M^6,g,J) be a* 6*-dimensional nearly Kähler spin manifold and let ∇ be the unique linear connection with totally skew-symmetric torsion preserving the nearly Kähler structure. Then there exist two ∇-parallel spinors. These spinors are* *sections in the subbundles $E^{\pm}$. If M^6 is compact, then every ∇-harmonic spinor is ∇-parallel.*

Remark 10.9. A 6-dimensional nearly Kähler manifold admits two Killing spinors with respect to the Levi-Civita connection (see [28]) and these spinor fields are the ∇-parallel spinors.

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Thomas Friedrich
Humboldt-Universität zu Berlin
Institut für Mathematik
Sitz: Rudower Chaussee 25
10099 Berlin, Germany
e-mail: friedric@mathematik.hu-berlin.de

Stefan Ivanov
University of Sofia "St. Kl. Ohridski"
Faculty of Mathematics and Informatics
blvd. James Bourchier 5
1164 Sofia, Bulgaria
e-mail: ivanovsp@fmi.uni-sofia.bg