AUTOMORPHISM GROUPS OF NORMAL CR 3-MANIFOLDS

FLORIN ALEXANDRU BELGUN

ABSTRACT. We classify the normal CR structures on S^3 and their automorphism groups. Together with [3], this closes the classification of normal CR structures on contact 3-manifolds. We give a criterion to compare 2 normal CR structures, and we show that the underlying contact structure is, up to homotopy, unique.

1. INTRODUCTION

A pseudoconvex CR manifold is called *normal* if it admits a *Reeb* vector field whose flow preserves the CR structure (in other words, it admits a symmetry transversal to the underlying contact structure). If the manifold has dimension 3, this is equivalent to the fact that it admits a compatible Riemannian *Sasakian* structure, for which the above mentioned Reeb vector field is a unitary Killing vector field.

In a previous paper [3] we have classified the Sasakian structures on compact 3-manifolds and shown that the underlying (therefore normal) CR structures essentially determine the Sasakian structure (in other words, the symmetry group of the CR structure is one-dimensional), in the case where the manifold is not a finite quotient of the 3-sphere. We obtained therefore the classification of the normal CR structures on these manifolds. On the other hand, finite quotients of S^3 admit normal CR structures with bigger automorphism groups (for example the standard *flat* (see next section) CR structure on S^3 admits $PSU(2,1)$ as automorphism group), and in this case the problem of classifying these structures was still open.

The first purpose of this paper is to complete the classification of the normal CR structures on compact 3-manifolds by solving the remaining case of S^3 and its quotients. To do that, we study the automorphisms of the normal CR structures on S^3 and use the classification of the Sasakian structures [3].

It turns out that the only (locally) homogeneous normal CR structure is the standard one, and otherwise the (connected component of the) automorphism group is either a circle or a 2-torus (section 3, Theorem 1). There are other examples of homogeneous CR structures on compact 3-manifolds, but they are not normal: a left-invariant CR structure on S^3 admits no infinitesimal automorphism *transversal* to the contact distribution unless it is also right-invariant, i.e. flat.

Date: October 24th, 2000.
AMS classification : 53A40, 53C25, 53D10.
Supported by the SFB 288 of the DFG.

Remark. A direct consequence of that is the following result, similar to the well-known theorem of Obata and Lelong-Ferrand in conformal geometry: if the automorphism group of a compact CR 3-manifold M is non-compact, then M is isomorphic to S^3 with its canonical CR structure. In the general theory of *rigid geometric structures* — e.g. conformal, projective, pseudo-Riemannian, CR structures — of Gromov [13], [1], the above results are particular (solved) cases of the *vague conjecture*: apart from exceptional cases (here: the flat structure), the automorphism group of a rigid geometric structure on a compact manifold should be compact.

The classification of the normal CR structures on S^3 is then achieved in terms of the *wrapping numbers* of a certain uniquely defined CR Reeb vector field T_0, whose orbits are all closed, and of the metric g^{T_0} induced by projection on the orbit space of T_0 (this space is topologically S^2, possibly with 1 or 2 conical points) — see section 3 and Theorem 3. Note that, *a priori*, the invariant tensor of a CR structure in dimension 3 is the *Tanaka curvature tensor* (an analogue of the Cotton-York tensor from conformal geometry), which, even in the simple case of a normal CR structure, depends on a second order derivative of the curvature of g^T.

Finally, we consider the following question: which contact structures admit compatible normal CR structures? In dimension 3, any contact manifold (M,Q) admits an infinity of compatible CR structures on it : to pick such a complex structure J on Q, all we need is to choose a positive definite metric on Q, as there is no integrability condition for J required. So, the existence of a CR structure in dimension 3 implies no topological condition at all (except orientability), because every 3-manifold admits a contact structure [17]. But if we want to find compatible *normal* CR structures on (M,Q), then we already know that M has to be a finite quotient of a principal circle bundle over a Riemann surface, of non-zero Chern class [3], [10].

Moreover, we want to know what contact structures, up to diffeomorphism, on such a manifold admit compatible normal CR structures. It turns out that there exists at most one, which turns out to be *tight* (Proposition 2).

The paper is organized as follows: in section 2 we recall the classification of Sasakian structures on S^3 and we study in detail the orbits of the occurring CR Reeb vector fields; in section 3 we consider, in general, the differential equation related to the existence of multiple CR Reeb vector fields and describe the structure of the Lie algebra of the CR automorphism group. The proof of the main result (Theorem 1, stated at the beginning of section 3) is contained in section 4 and 5. The classification result (Theorem 3) follows in section 6 from the previous sections and the investigation of the flat structures on S^3. In section 7 we describe the contact structures underlying a normal CR structure in dimension 3.

Acknowledgments: The author benefited from useful discussions with Prof. Friedrich.

2. Preliminaries: Sasakian structures on S^3

In this paper, we call a CR structure (Q,J) on an odd-dimensional oriented manifold M^{2n+1} with a *contact* distribution of hyperplanes $Q \subset TM$

(such that, if η is a non-vanishing *(contact)* 1-form for which $Q = \ker \eta$, then $\eta \wedge (d\eta)^n$ is a non-vanishing volume form on M), endowed with a *compatible* complex structure J, i.e., for η as above, $d\eta|_Q = d\eta(J\cdot, J\cdot)|_Q$.

In this case, $d\eta(J\cdot,\cdot)$ is a bilinear symmetric non-degenerate form on Q. If $2n+1 = 3$, we will always consider *positive* contact forms η, i.e. such that $h := -\frac{1}{2}d\eta(J\cdot,\cdot)$ is a positive definite Hermitian metric on Q (the renormalization factor is useful when considering Sasakian structures — to be defined below). The canonical orientation of the manifold is then, by definition, given by the volume form $\eta \wedge d\eta$.

Remark. With this convention, the canonical orientation of S^3 with its standard CR structure induced from its embedding in $\mathbb{C}^2$ is reverse to the usual one.

In general, in contact geometry one can associate to each contact form η a canonical vector field V, called a *Reeb* vector field, such that $\eta(V) = 1$ and $\mathcal{L}_V\eta = 0$, where $\mathcal{L}$ stands for the Lie derivative. V is called a CR Reeb vector field if, in addition, $\mathcal{L}_V|_Q J = 0$. Here we follow the convention from [3] and consider as a *tensor* on a contact manifold any tensorial combination of elements in the tensorial algebras of TM, Q, TM/Q and their duals; with respect to that, $J \in End(Q)$ is a tensor and the Lie derivative $\mathcal{L}_V$ can be restricted to $End(Q)$ because Q is V-invariant, so the equation $\mathcal{L}_V|_Q J = 0$ makes sense. If V satisfies this equation, it is an infinitesimal automorphism of the CR structure of M, transverse to Q; conversely, any such vector field is a CR Reeb vector field for an appropriate contact form. A Reeb vector field is *positive* if associated to a positive contact form, *negative* otherwise.

Definition 1. *A* CR *structure is called* normal *if it admits a CR Reeb vector field.*

Definition 2. *A* Sasakian *structure (g, T) on a 3-dimensional manifold M is a Riemannian metric g admitting a unitary Killing vector field T such that ∇T is a complex structure J on $Q := T^\perp$.*

Remark. If T is unitary Killing, then ∇T is antisymmetric and vanishes on T; the above condition ensures, in addition, that its square is minus the identity on Q. In higher odd dimensions J has to satisfy some integrability conditions, which are vacuous in dimension 3.

It is easy to see that, if (M, g, T) is Sasakian, then (M, Q, J), with the notations above, is a normal CR structure, because T is a CR Reeb vector field. Conversely, it turns out that for any positive CR Reeb vector field V, the Riemannian metric defined to be equal to $h = -\frac{1}{2}d\eta(J\cdot,\cdot)$ on Q, for $\eta(V) = 1$, $Q = \ker \eta$, and such that V is unitary and orthogonal to Q, is Sasakian [3], [2].

The Sasakian structures on a compact 3-manifold have been completely classified [3] using the theory of *locally conformally Kähler metrics* with parallel Lee form on compact complex surfaces [2], and, if we could tell which Sasakian metrics on M have the same underlying CR structure, we could reduce the problem of classification of the normal CR structures on M to the previous one, which is solved.

It turns out [3], [2], [10], that M is then always a finite quotient of a non-flat (i.e. with non-zero Chern class) circle bundle over a Riemann surface (and endowed with the appropriate orientation), and if this surface has positive genus, the orbits of a Reeb vector field are always the fibers of such a fibration (The corresponding Sasakian metrics are called then *regular*). In this case, it follows [3] that each normal CR structure on M admits, up to multiplication by a constant, exactly one CR Reeb vector field, and, if we fix the length of its orbits to be equal to 2π, the CR structures on M are in 1–1 correspondence to the Sasakian structures satisfying the above normalization condition. That closes the classification of normal CR structures on 3-manifolds which are not covered by S^3.

Locally, every Sasakian structure can be realized as a fibration over a surface, with a suitable connection on it, and we are frequently going to use the terminology from fiber bundles: we will often refer to the vectors in Q as *horizontal*, and to T as *vertical.*

From the classification [3] we retain the following facts concerning Sasakian structures on S^3 (and its quotients) : On S^3 there are regular Sasakian structures (with respect to the Hopf fibration of S^3 over $\mathbb{CP}^1$), but also *quasi-regular* (i.e. all the orbits of the Reeb vector field are closed, but of different lengths) or *irregular* (some orbits of the Reeb vector field are not closed). The structure of the orbits will be discussed below in detail.

The orbits of a quasi-regular CR Reeb vector field are the fibers of a Seifert fibration of S^3 over a simply-connected *orbifold* with 1 conical point, or with 2 conical points with different angles. In general, an *orbifold* is a topological space endowed with a differential structure with singularities, and it is given by an atlas of charts pointing to open sets in $\mathbb{R}^n$ or in quotients of $\mathbb{R}^n$ by finite linear groups. In dimension 2, a *conical* point of *angle* $2\pi/k$ has a neighborhood diffeomorphic to the quotient of the unit disc in $\mathbb{C}$ by the cyclic group generated by multiplication by a k-th root of unity. It is well-known that the orbifolds that occur above are not finite quotients of any smooth surface.

Convention. From now on, the manifold denoted by M will be implicitly considered to be diffeomorphic to S^3 and endowed with a normal CR structure. Except otherwise stated, the symbol S^3, when used to denote a CR manifold, will be reserved to the standard (flat) CR structure on the three-sphere.

As already mentioned, the Sasakian structures on $M \simeq S^3$ have been investigated by considering the Riemannian product $M \times S^1$, endowed naturally with a complex structure (we extend J such that it sends T into a unitary vector field on the S^1 factor); this complex surface is a *primary Hopf* surface *without resonance (or of class 1)* [2], i.e. it is the quotient of $\mathbb{C}^2 \smallsetminus \{0\}$ by an infinite cyclic group generated by a contraction

$$\gamma(x,y) := (\alpha x, \beta y), \text{ where } 0 < |\alpha| \leq |\beta| < 1, \tag{1}$$

and the metric on it is a Hermitian metric with Kähler form ω and with parallel *Lee form* θ (defined by the equation $d\omega = -2\theta \wedge \omega$) equal to (a constant times) ds, the length form on the factor S^1 [2]. The key information that we will use here is that the *Lee vector field*, which is the dual of the

Lee form, and equal to (a constant times) $-JT$ (where T is the Reeb vector field on the Sasakian manifold (M, g, T)), turns out ([2], Proposition 8) to be, up to a constant, identified on $\mathbb{C}^2$ with

$$\log|\alpha| x\partial_x + \log|\beta| y\partial_y,$$

and to get the expression that corresponds to T we have to multiply this with i and, possibly, a real constant.

The condition to get the above product structure on $\mathbb{C}^2 \smallsetminus \{0\}$ (for example, it is necessary, but not sufficient, the orbits of JT to be closed) is that $\alpha = |\alpha|$ and $\beta = |\beta|$, so both are real positive numbers [3]. The regularity of the Sasakian structure (S^3, g, T) is then decided by the ratio $\log\alpha/\log\beta$:

Lemma 1. *T is regular, quasi-regular, resp. irregular if and only if $\log\alpha/\log\beta$ is equal to 1, rational or irrational. This ratio is exactly the ratio of the lengths of the* exceptional *orbits of T contained in the complex lines $\{x = 0\}$, resp. $\{y = 0\}$. If $\log\alpha/\log\beta = p/q$, an irreducible fraction, the lengths of the orbit in $\{x = 0\}$, in $\{y = 0\}$, resp. generic (i.e., different from the 2 exceptional ones) orbit are qc, pc and pqc, where c is a positive constant. If $\log\alpha/\log\beta$ is irrational, then a generic orbit of T is dense in a 2-dimensional torus; outside the exceptional orbits, the rest of S^3 is foliated by these tori, to which are tangent all CR Reeb vector fields (which commute with T) and are represented in $\mathbb{C}^2$ by $iax\partial_x + iby\partial_y$, for $a, b \in \mathbb{R}$ (in particular, this set contains the regular CR Reeb vector field represented by $ix\partial_x + iy\partial_y$).*

Proof. It is enough to note that the orbits of T in M can be identified to the orbits of $i\log\alpha x\partial_x + i\log\beta y\partial_y$ on $\mathbb{C}^2$. □

Remark. Let (M, g, T) be a Sasakian structure such that its product with S^1 leads to the Hopf surface $\mathbb{C}^2 \smallsetminus \{0\}/\langle\gamma\rangle$, where $\gamma(x, y) = \left(a^{1/p}x, a^{1/q}y\right)$, for $p, q \in \mathbb{N}$ mutually prime and $0 < a < 1$ is a constant. Then $T \equiv -px\partial_x - qy\partial_y$ is quasi-regular, and the space B of its orbits in $M \simeq S^3$ (seen as a factor in the Riemannian product $M \times S^1$, and not as the standard 3-sphere in $\mathbb{C}^2$) is the space of the complex orbits of T – seen as a holomorphic vector field – in $\mathbb{C}^2 \smallsetminus \{0\}$. It is topologically $\mathbb{CP}^1$, and the projection from $\mathbb{C}^2 \smallsetminus \{0\}$ is the holomorphic map $(x, y) \mapsto [x^p : y^q]$, but this fails to be a submersion at the *poles* $[1 : 0]$ and $[0 : 1]$.

To recover the orbifold structure of the basis B we proceed as follows: around the exceptional orbit contained in $\mathbb{C} \times \{0\}$ we consider a covering $\mathbb{C}^* \times \mathbb{C} \to \mathbb{C}^* \times \mathbb{C}$ defined by $(x, y) \mapsto (x^q, y)$. The pull-back of T is $\tilde{T} = -pqx\partial_x - qy\partial_y$, the space of its orbits $\tilde{B}$ (in $\mathbb{C}^* \times \mathbb{C}$) is $\mathbb{C}$, seen as the open set in $\mathbb{CP}^1$ outside the point $[0 : 1]$. The corresponding projection $\mathbb{C}^* \times \mathbb{C} \to \mathbb{CP}^1$ is equivalent with $(x, y) \mapsto [x^p : y]$ which is a submersion everywhere. The map from $\tilde{B}$ to B, which assigns to each orbit of $\tilde{T}$ the corresponding orbit of T, is then the ramified covering $[x : y] \mapsto [x^q : y^q]$, which gives the orbifold structure around the pole $[1 : 0]$ — this is thus a conical point of angle $2\pi/q$ (a smooth point has angle 2π). Note that the orbits of $\tilde{T}$ have the same length as the generic orbits of T, over each of which project q different orbits of $\tilde{T}$. Over the exceptional orbit contained in $\mathbb{C} \times \{0\}$ is "wrapped" one orbit of $\tilde{T}$ q times. An analogue statement holds for the other pole: $[0:1]$ is then a conical point of angle $2\pi/p$, there is a ramified covering with p leaves over it, the source smooth manifold being the orbit space of $\tilde{T}$, the lift of T on some p-fold smooth covering of $\mathbb{C}\times\mathbb{C}^*$, these orbits generating a fibration, and the generic orbits of T are covered by p different $\tilde{T}$ orbits, the exceptional one by only one, "wrapped" p times.

The situation where one of p or q is equal to 1 corresponds to the *teardrop* orbifold (a sphere with one conical point), if both are superior to one we get two conical points.

Definition 3. *The ratios between the lengths of a generic orbit and an exceptional one, in case of a quasi-regular CR Reeb vector field T, are called the* wrapping numbers *of T.*

Of course, the wrapping numbers coincide with the multiplicity, at the conical points, of the ramified coverings defined above.

3. Main statement. The structure of the Lie algebra of infinitesimal automorphisms of a normal CR structure on S^3

An important fact, that does not directly follow from the section above, but from [2], [3], is the following:

Proposition 1. *The CR structure of an irregular Sasakian metric (M,g,T) always admits a regular CR Reeb vector field T_0 which commutes with T; on the other hand, if (g_0,T_0) is a regular Sasakian metric on S^3, it admits a CR Reeb vector field commuting with T_0 if and only if the metric induced by g_0 on the basis $\mathbb{CP}^1$ (recall that the orbits of T_0 are the fibers of a S^1 fibration on $\mathbb{CP}^1$) admits a Killing vector field.*

To summarize, there are three types of normal CR structures on S^3: with a regular CR Reeb vector field; with a regular and an irregular CR Reeb vector field, commuting; with a quasi-regular CR Reeb vector field. The last situation can be described, locally around each Reeb orbit, as (a finite quotient of) a regular Sasakian structure (see the construction in the remark above; the lifted vector $\tilde{T}$ is regular). Note that we do not know yet whether these types overlap or not. In the next sections, we are going to prove:

Theorem 1. *If a normal CR structure admits no irregular CR Reeb vector field, then it admits a unique (up to a constant) regular or quasi-regular CR Reeb vector field; the connected component of its CR automorphism group is then S^1.*

If a normal CR structure admits an irregular CR Reeb vector field, then it is either the standard flat one (in which case the automorphism group is $PSU(2,1)$) or it admits an unique regular CR Reeb vector field; in this latter case, the 2 CR Reeb fields span the (Abelian) Lie algebra of the CR automorphism group, whose connected component is a 2-dimensional torus.

To prove this theorem we will investigate the existence, on a Sasakian manifold (S^3,g,T), of other CR Reeb vector fields

$$T' = fT + X_f, \tag{2}$$

where X_f is the component in $Q = T^\perp$ of T', and f, a positive function. It turns out [3] that T' is a Reeb vector field if and only if

$$X_f = \frac{1}{2}J(df|_Q)^\sharp, \tag{3}$$

and that it is a CR Reeb vector field if and only if

$$\square_X f := X.JX.f + JX.X.f - \nabla_X JX.f - \nabla_{JX}X.f = 0,\ \forall X \in Q, \tag{4}$$

where ∇ is the *Tanaka-Webster* connection on (M, Q, J, T), defined as follows:

$$\nabla T = 0;\ \nabla X \in Q\ \forall X \in Q,\ \nabla J = 0,$$

and the following conditions on its torsion τ:

$$\tau(T, X) = 0\ \forall X \in Q,\ \tau(X, JX) = 2T\|X\|^2,$$

where the norm is relative to the Sasakian metric. The Tanaka-Webster connection can easily be related to the Levi-Civita connection of the above Sasakian metric [3], but it is more suitable than this one in the case of a regular CR Reeb vector field T, because it is easily constructed from the fibration (following the orbits of T) data [3]; in particular, the operator $\square_X f$, if f is a function constant along T, can be computed using formally the same expression above, but replacing ∇ with ∇^B, the Levi-Civita connection on the basis B (the space of the orbits of T). Such functions appear when we look for CR Reeb vector fields $T' = fT + X_f$ commuting with T, when $T.f = 0$ and X_f is the horizontal lift of the vector field $\frac{1}{2}Jdf^\sharp$ on B; the following Lemma holds:

Lemma 2. *The horizontal part X_f of a CR Reeb vector field commuting with the (quasi)-regular CR Reeb vector field T on S^3 projects on the basis B as a Killing vector field, whose orbits, with the exception of two, which are points (among which the one or two possible conic points of the orbifold B), are all circles.*

Proof. The regular case has been treated in [3]. The fact that X_f projects on a Killing vector field follows from a local computation, so it holds also in the quasi-regular case. As any isometry of an orbifold has to fix (or interchange) the conical points, it follows that X_f vanishes at the conical points. Now, if a Killing vector field on a (even open) surface has a zero, then it is isolated and has index 1; we can smoothen the conical points and deform a little X_f such as the resulting vector field on S^2 has the same number of zeros, each with the same index as before. They must be thus exactly 2, as the Euler characteristic of S^2; we will call these degenerate orbits of X_f *poles.* The other orbits of X_f are smooth submanifolds, because they coincide with the level sets of f (recall that $X_f = \frac{1}{2}Jdf^\sharp$), and that df is non-degenerate away from the 2 poles). □

Lemma 3. *Let T be an irregular CR Reeb vector field on S^3. Then there is a unique regular CR Reeb vector field T_0 which commutes with T.*

Proof. We know that there exists such a T_0 [3]. Suppose T_0' is another regular CR Reeb vector field such that $[T, T_0'] = 0$. Consider the closure of an orbit of T: this is a torus $\mathcal{T}$. From the standpoint of T_0', T is a commuting CR Reeb vector field: as such, $T = f'T_0' + X_f'$, X_f' is a Killing vector field on B_0', the space of the orbits of T_0', and we know from Lemma 1 that the orbits of T — as an irregular CR Reeb vector field commuting with T_0' — are dense in tori to which T_0' is tangent. So T_0', as well as T_0, is tangent to the closures of T's orbits. On a dense set of such a torus $\mathcal{T}$, T_0' is determined by its value at a point x (as T_0' commutes with T, we transport, through the flow of T, this vector to all the points situated on the orbit of T passing through x, and this set is dense). The same holds for T_0, so if T_0' coincides with T_0 at a point of $\mathcal{T}$, it coincides everywhere. In any case we get that $[T_0, T_0'] = 0$ on $\mathcal{T}$, and, as $\mathcal{T}$ was arbitrarily chosen, T_0 and T_0' commute on S^3. As they are both regular, they coincide (Lemma 1, see also the remark below). $\square$

Remark. Any CR Reeb vector field has at least 2 closed orbits; for example, if we consider a CR Reeb vector field T' commuting with a regular one, T, the orbits of T and T' over the two poles P_1 and P_2 coincide, but they have different lengths; if they both have length 1 for T (measured using the dual contact form η), their lengths for T' (measured using the contact 1-form $\eta' := f^{-1}\eta$ dual to T') are $f(P_1)^{-1}$, resp. $f(P_2)^{-1}$. Unless f is everywhere constant, these are then different, because the values of f at the poles are the only critical points of f, thus its 2 extrema. So there is no other regular CR Reeb vector field commuting with T, and if T admits other commuting CR Reeb fields, one can always find irregular ones among them (we can modify the ratio of the maximum and minimum of f, if f is not constant, by adding to f some positive constant). The last statement also holds if T is a quasi-regular CR Reeb vector field: any solution f of the equation $\Box_X f = 0$, $X \in Q$ still has only 2 critical points (see above), hence they coincide with its extrema.

Of course, the equation $\Box_X f = 0$ being linear, the space of its solutions is a vector space, containing the constants; this space can be identified with the Lie algebra of the CR automorphisms group, the solution corresponding to T is the constant 1, and the solutions corresponding to CR Reeb vector fields form the cone of everywhere-positive solutions. If T is regular, then we can integrate f on the fibers (the orbits of T) using the contact 1-form η dual to T, and get the function $\boldsymbol{I}^T f$, which is constant on the orbits of T, and still verifies the equation $\Box_X \boldsymbol{I}^T f = 0$; it thus corresponds to a CR Reeb vector field (more generally, if f is not everywhere positive, a CR infinitesimal automorphism) which commutes with T. We can consider then $\boldsymbol{I}^T : \mathfrak{g} \to \mathfrak{g}$ as a linear projection on the commutator of T (it is even a projector if we fix, by no loss of generality, the length of the orbits of T to be equal to 1). This commutator is 1-dimensional if B admits no Killing vector fields, 2-dimensional if it admits one, and 4-dimensional if $B \simeq S^2$ has a round metric.

If T is a quasi-regular CR Reeb vector field, whose generic orbits have length pql, and its exceptional orbits C_p and C_q have lengths pl, resp. ql (for $p, q \in \mathbb{N}^*$ mutually prime, and $l > 0$), and if f is a solution to the equation $\Box_X f = 0$, $X \in Q$, we define $\boldsymbol{I}^T f$ on a generic orbit as the integral of f on it, and on an exceptional orbit, say, C_p, as q times the integral of f on it. The resulting function $\boldsymbol{I}^T f$ on B, the space of the T orbits, is smooth in all the smooth points of B, and continuous at the conical points. Moreover, if we lift it to one of the local ramified coverings of one of the poles, say, $\{C_p\}$ (see above), $\tilde{B}_p$, we get a smooth function, which is actually obtained by integration along the orbits of the regular CR Reeb vector field $\tilde{T}$ on some smooth covering of a neighborhood of C_p (see above). It follows that $\boldsymbol{I}^T f$ satisfies the equation for CR infinitesimal automorphisms commuting with T, and $\boldsymbol{I}^T : \mathfrak{g} \to \mathfrak{g}$ is, also in this case, a linear projection (a projector if $pql = 1$) on the commutator of T, which can be 1-dimensional (no Killing vector field on B), or 2-dimensional. As a non-smooth orbifold cannot admit more than 1 linear independent Killing fields, these are the only cases.

Remark. If a quasi-regular CR Reeb field T admits no linearly independent commuting CR Reeb field, and if $\Box_X f = 0,\ X \in Q$, then $\boldsymbol{I}^T f$ is constant everywhere.

We are going to use, in the next section, a criterion that will allow us to conclude, in certain circumstances, that 2 CR Reeb vector fields T and $T' = fT + X_f$ coincide:

Lemma 4. *1. If $f \equiv 1$ in a neighborhood of an orbit of T, then $f \equiv 1$ everywhere.*

2. Suppose that the commutator of T in $\mathfrak{g}$ is 1-dimensional. Then both T and T' are regular or quasi-regular, in the latter case with the same exceptional orbits, and same wrapping numbers. If the lengths of the generic orbits of both T and T' are equal to 1, and if $\boldsymbol{I}^T f = 1$ and $\boldsymbol{I}^{T'} f^{-1} = 1$, then $f \equiv 1$ and $T \equiv T'$.

Proof. In the first case, T and T' coincide in a neighborhood of one of their common generic orbits. If they are irregular, we consider the uniquely (see Lemma 3) associated regular CR Reeb vector fields T_0, resp. T_0', and we easily get, as in Lemma 3, that they commute in a neighborhood of a torus $\mathcal{T}$, which is the closure of a common orbit of T and T'. Then, for a suitable constant k, $T_0 + k[T_0, T_0']$ is a CR Reeb vector field, and it coincides with T_0 on a neighborhood of an orbit of T_0. If we prove that they coincide everywhere, it follows that T_0 and T_0' are two regular CR Reeb vector fields that commute, so (Lemma 3 and the Remark thereafter) they coincide.

We can suppose thus that T and T' are both (quasi-) regular, and they coincide on a neighborhood of some orbit, which we may suppose generic. Then f (the function defining T' starting from T) is constant in such a neighborhood, and $\boldsymbol{I}^T f$ is constant on an open set of B, the orbit space of T. But $\Box_X \boldsymbol{I}^T f = 0, \forall X \in TB$, so $J(d\boldsymbol{I}^T f)^\sharp$ is a Killing vector field, vanishing in an open set, thus everywhere. So $\boldsymbol{I}^T f$ is constant. At this point we can apply [3], Corollary 2, or the method below, to conclude that f is constant.

The method we apply to prove the second claim in the Lemma is the same we used — in the regular case — in [3], Lemma 1; we recall it briefly: Using the Hölder inequality

$$\left(\int_M f^{-2}\frac{1}{2}\eta \wedge d\eta\right)\left(\int_{S^3} f\frac{1}{2}\eta \wedge d\eta\right)^2 \geq \left(\int_M \frac{1}{2}\eta \wedge d\eta\right)^3 = v^3,$$

we get the implication

$$(\boldsymbol{I}^T f \equiv 1 \text{ and } l^T = 1) \Longrightarrow v \geq v',$$

with equality if and only if f is constant. Here, η is the contact form such that $\eta(T) \equiv 1$, v is the volume of M with respect to the Sasakian metric defined by T, and l^T is the length of the orbits of T. The hypothesis of the Lemma ensures that the parenthesis of the above implication, as of the following one, are true, so we get $v = v'$ and $f = cst.$:

$$(\boldsymbol{I}^{T'} f^{-1} \equiv 1 \text{ and } l^{T'} = 1) \Longrightarrow v' \geq v.$$

If T is quasi-regular, we denote by l^T the length of the generic orbits of T ; besides, all the integrals above will be taken over the complement of the exceptional orbits in S^3. Note that the exceptional orbits, as well as the wrapping numbers of T and T' coincide (otherwise a linear combination of them would be irregular — see the Remark at the end of Lemma 3 —, and this implies that $\mathfrak{g}$ is even-dimensional (see Corollary 1 below), but if the commutator of T is 1-dimensional, then $\mathfrak{g}$ is odd-dimensional, contradiction). The proof follows as in the regular case (we get equality in the Hölder inequality, thus f is a constant). □

The Lemma above is useful because we can describe Sasakian metrics [3], but have *a priori* no simple criterion to check if two different Sasakian structures are CR-isomorphic, or if they admit other CR Reeb vector fields (except for the commuting ones); as our goal is to prove, roughly, that there are very few cases when $\dim G > 1$ — where G is the group of CR automorphisms of M —, we will investigate the orbits of G and search for geometric facts that would imply the technical hypotheses in Lemma 4.

Lemma 5. *Let $\mathfrak{g}$ be the Lie algebra of infinitesimal automorphisms of a CR structure on $M \simeq S^3$, and let $T \in \mathfrak{g}$ be a quasi-regular CR Reeb vector field. Then the Lie bracket with T is $\mathrm{ad}_T : \mathfrak{g} \to \mathfrak{g}$, and the integration along the orbits of T yields a projection $\boldsymbol{I}^T : \mathfrak{g} \to \mathfrak{g}$. Let $K := K^T := \ker \mathrm{ad}_T$ and $W := W^T := \ker \boldsymbol{I}^T$.*

Then $\mathrm{ad}_T(\mathfrak{g}) = \mathrm{ad}_T(W) = W$ and $\mathrm{ad}_T|_W$ is an endomorphism of W whose eigenvalues are all pure imaginary (non-zero). In particular, $\mathfrak{g} = K^T \oplus W^T$, and W^T is even-dimensional.

Proof. The exponential of ad_T is the adjoint action of the flow of T on $\mathfrak{g}$; as this flow is periodic, it follows that the eigenvalues of ad_T are imaginary. The image of ad_T lies in W, because, for an infinitesimal CR automorphism $T' = fT + X_f$, $\mathrm{ad}_T(T') = f'T + X_{f'}$, where $f' = T.f$, and, of course, the integral of f' along the orbits of T is zero.

It remains to prove that if $T' = fT + X_f \in W \cap K$, then $f \equiv 0$. $T' \in K$ implies that f is constant on the orbits of T, and $T' \in W$ implies that this constant is 0. □

Corollary 1. *If (M, Q, J) admits an irregular CR Reeb vector field, then $\dim \mathfrak{g}$ is even; if it admits a (quasi-regular) CR Reeb vector field whose commutator has dimension 1, then $\dim \mathfrak{g}$ is odd.*

Proof. It follows from the previous Lemma that $\dim W^T$ is always even. The second claim readily follows. For the first claim, we take T to be the regular CR Reeb vector field which commutes with an irregular one, and then $\dim K^T$ is 2 or 4 (see above). □

We consider in the next sections the cases when the commutator of a CR Reeb vector field is 1 or 2-dimensional (the remaining case, where the commutator is 4-dimensional, is the flat CR structure on S^3; a detailed study of this structure is contained in Section 5).

4. Proof of Theorem 1: case with an irregular CR Reeb vector field

Suppose we have a normal CR structure on $M \simeq S^3$ with an irregular CR Reeb vector field T'; denote by T the (unique) regular one that commutes to it. We suppose that the CR structure is not flat; then $\dim K^T = 2$ and $\dim \mathfrak{g} \geq 2$ is even. We have to prove that $\dim \mathfrak{g} = 2$.

If $\dim \mathfrak{g} \geq 4$, then all the open orbits of G in S^3 are CR flat. Indeed, E. Cartan proved that the automorphism group of a non-flat 3-dimensional CR manifold has dimension 3 [5]. If the union of these orbits is a dense open set in S^3, the CR structure is flat. The orbits of G already contain the closures of the orbits of T', which are tori, and 2 exceptional orbits. If the CR structure is not flat, then, in a neighborhood of such a torus $\mathcal{T}$, these tori need to coincide with the orbits of G. As $\dim \mathfrak{g} > \dim K^T = 2$, there is an irregular CR Reeb vector field $\tilde{T}'$ which does not commute with T or T'; denote by $\tilde{T}$ the corresponding regular CR Reeb vector field, such that $[\tilde{T}, \tilde{T}'] = 0$. Let the function f be such that $\tilde{T} = fT + X_f$. We will prove that, on a neighborhood of $\mathcal{T}$, f is constant on the above mentioned tori. From this it will follow that $[T, \tilde{T}]$ vanishes on a neighborhood of an orbit of T, thus, from Lemma 4, everywhere.

Suppose f is not constant on $\mathcal{T}$. As $\tilde{T} = fT + X_f$ is tangent to $\mathcal{T}$, it follows that $df(Y) = 2g(JX_f, Y) = 0$, where $Y \in Q \cap T\mathcal{T}$; so f is constant on the horizontal curves that project on circles on B, the space of orbits of T. If x is a regular point of f in $\mathcal{T}$, then the horizontal curve $C_x \subset \mathcal{T}$ containing x is included in a level set of f, thus it is a circle, "wrapping" p times over the circle on B on which $\mathcal{T}$ projects. For neighboring points $x' \in \mathcal{T}$, the same thing holds, and the length of $C_{x'}$, in the Sasakian metric determined by T, is the same as the length of C_x. If we look at C_x and $C_{x'}$ as the horizontal curves for the Sasakian metric associated to $\tilde{T}$, they should still have the same length in this other Sasakian metric; but the horizontal part of this latter one is multiplied by f^{-1}, which is not locally constant, contradiction.

So f is constant on all the 2-dimensional orbits of G; as we supposed that their union contains a non-empty neighborhood of $\mathcal{T}$, then T and $\tilde{T}$ commute on this open set, thus (see Lemma 4, first point) everywhere.

The Lie algebra $\mathfrak{g}$ is thus 2-dimensional and Abelian, so G is a quotient of $\mathbb{R}^2$. In order to prove that G is actually a torus, we proceed as follows: let T be the regular CR Reeb vector field, whose orbits are of length 1, and let f be a function on the orbit space B of T, such that its differential is the dual of a Killing vector field on B, and such that the image of f is the interval $[0, 1]$. Then any infinitesimal CR automorphism of M is of the type $T_{a,b} := (a + (b - a)f)T + X_{a+(b-a)f}$, where a and b are real numbers, positive if and only if $T_{a,b}$ is a CR Reeb vector field. For example $T = T_{1,1}$, and $T_{2,1}$

and $T_{1,2}$ are both quasi-regular CR Reeb vector fields with one exceptional orbit of length $1/2$, all the others being of length 1; the exponentials of these elements of $\mathfrak{g}$ act trivially on M, thus they are equal to $1 \in G$. From this it follows that $\exp(T_{p,q}) = 1 \in G$, $\forall p, q \in \mathbb{Z}$, thus $G \simeq \mathfrak{g}/Z^2$ is a torus.

This proves the Theorem in case where there exists an irregular CR Reeb vector field, and the CR structure is not flat.

5. Proof of Theorem 1: case without irregular CR Reeb vector fields

Suppose every CR Reeb vector field on (M, Q, J) is quasi-regular, and denote by $p := p^T, q := q^T$ the wrapping numbers of such a field T. Suppose that the exceptional orbits C_1^T, C_2^T have length length p, resp. q (if p or q, or if both of them are equal to 1, there is only one, resp. no exceptional orbit), and all other orbits have length pq, with respect to the dual 1-form η^T. It is then clear that, if p^T or/and q^T are greater that 1, then the exceptional orbits $C_1^{\tilde{T}}$ or/and $C_2^{\tilde{T}}$, and the numbers $p^{\tilde{T}}, q^{\tilde{T}}$ have to be the same for all CR Reeb vector fields $\tilde{T}$ close to T: this is because the quotient of the lengths of 2 orbits passing through 2 fixed points a, b is always a rational number, and it must vary continuously when deforming T to $\tilde{T}$, it is thus constant; hence $p^{\tilde{T}} = p^T$ and $q^{\tilde{T}} = q^T$ and the orbits of $\tilde{T}$ passing through points in the open set $M \setminus (C_1^T \cup C_2^T)$ have all the same length, which proves that the exceptional orbits of $\tilde{T}$ coincide with the ones of T, as claimed.

We study now the orbits of the connected component of 1 in G, the Lie group of CR automorphisms. The following Lemmas can be proven using Lemmas 4 and 5, by the same method as in [3]:

Lemma 6. *There are no 2-dimensional orbits of G.*

Lemma 7. *There is at most a finite set of 1-dimensional orbits of G*

We have thus one open, dense orbit, and a finite number of circular orbits of G, in $M \simeq S^3$. G is odd-dimensional (see Corollary 1) and, if its dimension is greater that 3, the open dense orbit (hence the whole manifold) is CR flat [5], case which we exclude (it admits irregular CR Reeb fields). So, if we suppose that there are at least 2 linearly independent CR Reeb vector fields, $\dim G = 3$, and its Lie algebra is generated by T (that we fix from now on), $V := fT + X_f$ and $V' := f'T + X_{f'}$, where $f : M \to \mathbb{R}$, $\boldsymbol{I}^T f = 0$ and $f' = T.f$. Moreover, we have $f'' = af$, where $a < 0$, thus f, restricted to the orbits of T, is a sinusoid, and its critical points are its maximum and minimum. In fact, it has $2pn$ extrema on C_1 (of length p), $2qn$ extrema on C_2, and $2pqn$ extrema on a generic orbit of T ($n \in \mathbb{N}^*$).

Lemma 8. *If $\dim G = 3$, then $n = 1$, and at least one of p or q is equal to 1.*

As the above Lemmas, this follows as in [3], from Lemmas 4, 5, and the following:

Lemma 9. *If $\dim G = 3$, then $\mathfrak{g} \simeq \mathfrak{sl}_2(\mathbb{R})$.*

The proof of that follows as in [3], from Lemma 5, and the fact that, if U and $V = [T, U]$ span W^T, then their bracket commutes with T and is non-zero, thus is a constant times T; we can then renormalize T, U, V such that they correspond, through an isomorphism from $\mathfrak{g}$ to $\mathfrak{sl}_2(\mathbb{R})$, to the following matrices:

$$T \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \; U \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; V \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5}$$

We have thus excluded all but the following situation: $\mathfrak{g} \simeq \mathfrak{sl}_2(\mathbb{R})$, and the orbits of G are: one circle C and one open orbit O, on which we have a non-flat CR structure. This structure comes then from a left-invariant CR structure on $G_0 := \tilde{SL}_2(\mathbb{R})$ which extends smoothly on a compactification (by adding C) of the open orbit O. We will prove that this is only possible if the CR structure on G_0 is flat.

G is a quotient of $G_0 = \tilde{SL}(2,\mathbb{R})$ by some central subgroup. The centrum of G_0 is an infinite cyclic group generated by e, and thus $G = G_p := G_0/\langle e^p\rangle$, where p is a prime number. For example, $G_1 = PSL(2,\mathbb{R})$ and $G_2 = SL(2,\mathbb{R})$ (in the latter, e projects on minus the identity). The orbit O is thus isomorphic to a quotient of G_0 by a discrete subgroup Γ. As O can be retracted to a circle, $\Gamma \simeq \mathbb{Z}$; as the orbits of T in O are closed, Γ must contain a subgroup of the center of G_0 (the orbit of T in G_0 contains all the center). So O is a finite quotient of some G_p, $p \in \mathbb{N}^*$.

Important example. There is a standard action of $SL(2,\mathbb{R})$ on S^3 which has one open orbit and a 1-dimensional, circular orbit: consider S^3 (with the canonical CR structure) as embedded in $\mathbb{C}^2$, and let the action of $SL(2,\mathbb{R})$ be generated by the following holomorphic vector fields in $\mathbb{C}^2$:

$$\begin{array}{cclcl} T & \rightsquigarrow & -2ix\partial_x & - & iy\partial_y \\ U & \rightsquigarrow & (1-x^2)\partial_x & - & xy\partial_y \\ V & \rightsquigarrow & i(1+x^2)\partial_x & + & ixy\partial_y \end{array} \tag{6}$$

Where T, U, V are the generators of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ (5), and satisfy the commutation relations

$$[T,U] = -2V, \;\; [T,V] = 2U, \;\; [U,V] = 2T. \tag{7}$$

We get these vectors by looking for holomorphic vector fields in $\mathbb{C}^2$, tangent to S^3 (or, equivalently, elements in $\mathfrak{su}(2,1)$), whose flows preserve the circle $C_0 := \{y = 0\}$; we get a 4-dimensional Lie algebra $\mathfrak{k}$, generated by the above vector fields and $ix\partial_x + iy\partial_y$ (the standard regular CR Reeb vector field on S^3), which commutes with T, and then we keep only the three above, which generate $[\mathfrak{k},\mathfrak{k}]$; note that the CR Reeb vector fields contained in this Lie algebra are all quasi-regular, and all the generic orbits are twice as long as the unique exceptional orbit C_0. For the moment, we only have a Lie algebra action of $\mathfrak{sl}_2(\mathbb{R})$ on S^3. We easily check that it produces a Lie group action of $SL(2,\mathbb{R})$ on S^3, and that the local diffeomorphism $\psi : SL(2,\mathbb{R}) \to O_0$, where $O_0 := S^3 \setminus C_0$, defined by $\psi(\mu) := \mu.(0,1) \in O_0$, $\mu \in SL(2,\mathbb{R})$, is bijective and has the following expression:

$$\psi(\mu) := \psi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(i\frac{a-d+i(b+c)}{c-b+i(a+d)}, \frac{2i}{c-b+i(a+d)} \right). \tag{8}$$

If we denote by x_1, resp. x_2, the numerator and the denominator of the first fraction appearing in the above equation, we get:

$$\psi(\mu) = (x,y) = \left(i\frac{x_1}{x_2}, \frac{2i}{x_2} \right),$$

and $d\psi_\mu = \left(i\frac{dx_1}{x_2} - x\frac{dx_2}{x_2}, -y\frac{dx_2}{x_2}\right)$, from which we conclude that the right-invariant vector fields on $SL(2,\mathbb{R})$, whose value in μ is $T\mu, U\mu, V\mu$, respectively, are sent through ψ in the vector fields from (6), as expected, and the left-invariant vector fields equal to U, V in the identity on $SL(2,\mathbb{R})$ are sent in:

$$\begin{array}{lcrcl} \mu U & \rightsquigarrow & y^2\partial_x & - & y^2\overline{\left(\frac{x}{y}\right)}\partial_y \\ \mu V & \rightsquigarrow & iy^2\partial_x & - & iy^2\overline{\left(\frac{x}{y}\right)}\partial_y \end{array} \tag{9}$$

The fact that the vector field corresponding to μV is exactly i times the one corresponding to μU is due to the fact that the left-invariant CR structure J_1 induced from S^3 on $SL(2,\mathbb{R})$ by ψ is such that U and V generate the contact plane Q, and J_1 sends U into V.

Remark. The adjoint action of T on $\mathfrak{sl}_2(\mathbb{R})$ induces (up to multiplication by the constant $-1/2$) *the same* CR structure on the tangent space of the identity in $SL(2,\mathbb{R})$. Actually, J_1 is the standard left-invariant CR structure on $SL(2,\mathbb{R}) = G_2$ (and on its universal covering G_0); any other left-invariant CR structure $J_q, q \neq 0$, is given — up to an inner automorphism of the group — by the same Q (generated by μU and μV), and such that

$$J_q(\mu U) := \frac{1}{q^2}\mu V.$$

Indeed, any element in $\mathfrak{sl}_2(\mathbb{R})$ whose flow on a (hence on any) $G_p,\ p > 0$, is periodic, is equivalent, via an inner automorphism, to T; any plane in $\mathfrak{sl}_2(\mathbb{R})$ transverse to T and ad_T-invariant can be brought by an inner automorphism of $\mathfrak{sl}_2(\mathbb{R})$ to Q, and, for the Sasakian metrics induced by T in the new and in the standard CR structures on G_0, the 2 metrics on Q have a common, left-invariant, orthogonal basis. By an inner automorphism again, we may bring this basis to $\mu U, \mu V$, as claimed.

We are looking for a left-invariant CR structure on G, thus on its universal covering G_0, which, after a quotient by a discrete group, can be extended smoothly from O to the compact S^3. In particular, if $\tilde{T}$ is a fixed quasi-regular CR Reeb vector field, the metric on the space of its orbits must have bounded curvature (because this curvature is determined by the curvature of the Tanaka-Webster connection corresponding to T, which should be well-defined on the whole compact manifold S^3, see [3]). With no loss of generality (see the Remark above), we suppose that $\tilde{T}$ is identified with the right-invariant vector field on G_2 $T\mu$, where T is the matrix (5) in $\mathfrak{sl}_2(\mathbb{R})$, and that the left-invariant CR structure on G is given on $G_2 = SL(2,\mathbb{R})$ by (Q, J_q) as above. We note that any left-invariant structure on G goes down to any $G_p,\ p > 0$, because we take the quotient only by central elements; it is more convenient to make the computations on $SL(2,\mathbb{R})$, because of the example given above; moreover, the space of the orbits of T in G_0 or in any G_p, $p > 0$, is the same.

First, we know that μU and μV are orthogonal, that the ratio of their norms is 1 in the standard (CR flat) Sasakian metric associated to $T\mu$ and J_1, in general this ratio is $1/q^2$ (for the CR structure (Q, J_q)). The product of their norms is half of the coefficient of $T\mu$ in their commutator, written in the basis $T\mu, \mu U, \mu V$ (note that they are always linearly independent):

$$[\mu U, \mu V] = -2iy\partial_y = 2\alpha^2 T\mu + \ldots (\textit{terms in } Q),$$

where

$$\alpha^2 := \frac{2-|y|^2}{|y|^2}. \tag{10}$$

We have used the notations from the example above, who realizes $SL(2,\mathbb{R})$ as an open set of $S^3 \subset \mathbb{C}^2$. In this case, the projection on the space of orbits of $T\mu$ is

$$\begin{array}{ccccc} S^3 \setminus \{y=0\} & \to & \mathbb{CP}^1 \setminus \{[1:0]\} & \tilde{\to} & \mathbb{C} \\ (x,y) & \mapsto & [x:y^2] & \tilde{\mapsto} & \frac{x}{y^2}. \end{array}$$

By straightforward computation we get the projections on $\mathbb{C}$ of the $T\mu$-invariant vector fields μU and μV to be $\alpha^2\partial_z$, resp. $i\alpha^2\partial_z$, where $z = u+iv$ is the complex parameter on $\mathbb{C}$ and $\alpha^2 := \sqrt{A} := \sqrt{1+4|z|^2}$ is the same as before (10).

For the CR structure given by J_q, the vectors

$$X := X_q := qf\partial_u; \; Y := Y_q := \frac{1}{q}f\partial_v, \; f := \alpha^3 = A^{\frac{3}{4}} \tag{11}$$

form an orthonormal frame for the induced metric on the space $\mathbb{C}$ of the orbits of $T\mu$. We compute

$$[X,Y] = -\frac{1}{q}\partial_v f X + q\partial_x f Y,$$

then the Levi-Civita connection of the metric for which X, Y is an orthonormal frame:

$$\begin{array}{rclcrcl} \nabla_X X & = & \frac{1}{q}\partial_v f Y & \qquad & \nabla_X Y & = & -\frac{1}{q}\partial_v f X \\ \nabla_Y X & = & -q\partial_u f Y & & \nabla_X Y & = & q\partial_u f X \end{array}$$

and its curvature:

$$\begin{array}{rcl} \langle R_{X,Y}Y, X\rangle & = & q^2(f{\partial_u}^2 f - (\partial_u f)^2) + \frac{1}{q^2}(f{\partial_v}^2 f - (\partial_v f)^2) \\ & = & q^2(f\Delta f - |df|^2) + \left(\frac{1}{q^2} - q^2\right)(f{\partial_v}^2 f - (\partial_v f)^2) \\ & = & q^2 \cdot 12A^{-1/2} + \left(\frac{1}{q^2} - q^2\right)(6A^{1/2} - 48v^2A^{-1/2}). \end{array} \tag{12}$$

The first term in the last line is bounded, the second is not (for example, on the Ou axis), unless $q=1$, which corresponds to the standard, flat, CR structure on $SL(2,\mathbb{R})$.

We have proven that a $\tilde{SL}(2,\mathbb{R})$ action on $O = M \setminus C \simeq S^3 \setminus \{\text{circle}\}$, preserving a normal CR structure on $M \simeq S^3$, extends on the whole sphere if and only if the CR structure is flat. As this was the only case where $\dim G$ could be greater that 1, it follows that, if there are no irregular CR Reeb vector fields, then $\mathfrak{g} = \mathbb{R}$ and the connected component of G is a circle. The proof of Theorem 1 is complete.

Remark. Even in the flat case we can classify the above $\tilde{SL}(2,\mathbb{R})$ actions on S^3: as we will see in the next section, all regular CR Reeb fields are equivalent (modulo an inner automorphism of $PSU(2,1)$ — the group of CR automorphisms of the canonical structure of S^3), and all quasi-regular

CR Reeb vector fields commute with exactly one regular one. If we look for the subgroup of $PSU(2,1)$ preserving a circle in S^3, we can suppose that this circle is C_0 as above, and then get a 4-dimensional Lie algebra, which contains $\mathfrak{sl}_2(\mathbb{R})$ as its commutator. We obtain therefore the action described above, which is an action of $SL(2,\mathbb{R})$. Note that this is the only possible action of a group of type G_p, $p > 0$ prime, satisfying the required conditions.

6. CR flat Sasakian structures on S^3

It is known that a CR flat Sasakian structure on a 3-manifold which is not covered by S^3 is — up to a finite covering — regular, and comes from an S^1 fibration over a Riemann surface with constant curvature k [3]. This is an easy consequence of the fact that, if a regular Sasakian structure is CR flat, then the *Tanaka curvature tensor* $\Phi : S^2Q \to TM/Q$ vanishes. This tensor is trace-free, and is computed (in the case of a regular Sasakian structure) from sectional curvature k of the orbit space B of the Reeb vector field:

$$\Phi(X,X)(T) = -\frac{1}{2}\Box_X k, \ \forall X \in Q,$$

and it vanishes if and only if $J(dk)^\sharp$ is a Killing vector field on B. If the genus of B is greater than 0, this implies that k is constant.

If B is a sphere, there are functions f such that $J(df)^\sharp$ is a non-zero Killing vector field. We are going to investigate the case where this function is the curvature itself.

Let g be a metric on $B \simeq S^2$ with a Killing vector field Y, which vanishes at the *poles* P_1 and P_2. Its orbits are then *parallels* and we suppose that they have period 2π. The orbits of JY are then *meridians*: geodesics starting from P_1 and pointing towards P_2 (we label the poles as such). Define X to be the unitary vector field on $S^2 \setminus \{P_1, P_2\}$ which is collinear with Y and points in the same sense. There is a function r such that $Y = rX$. This function will be viewed as a function on the real line as follows: it is enough to evaluate it on a meridian, for which we use the arc length parameterization, the origin of $\mathbb{R}$ corresponds to the pole P_1 (the starting point of the meridian), and τ, the length of a meridian, corresponds to the pole P_2; we extend then $r : \mathbb{R} \to \mathbb{R}$ to be an odd function, periodic with period 2τ. The fact that this function is smooth can be easily seen as follows: we complete a meridian to a closed geodesic; on half of it, X will be a parallel unitary vector field, and if we extend it to $\tilde{X}$ to be parallel on the whole closed geodesic (of length 2τ), then $Y(x) = r(x)\tilde{X}(x)$ on half of it, and $Y(x) = -r(x)\tilde{X}(x)$ on the other half. The function $r : \mathbb{R} \to \mathbb{R}$ defined above is just the scalar product between $\tilde{X}$ and Y, in the arc length parameterization of a geodesic. It is smooth.

Remark. If the metric on S^2 has conical points at the poles (we have an orbifold – this happens if the CR Reeb vector field on S^3 is quasi-regular), we can still "glue together" r to get a smooth function: the argument above, applied to a local finite ramified covering around one pole, identifies that "glued" function with a scalar product restricted to a geodesic. Note that we cannot complete canonically a meridian in this case, but the function r

is constant on the parallels anyway, so any other such completion, done in a local ramified covering, yields the desired result.

The fact that the Killing vector field $Y = rX$ is given by the differential of the curvature k of the metric implies that

$$k' = lr, \tag{13}$$

where we have "glued" k as above, and considered as a periodic function on $\mathbb{R}$ (with period 2τ – the length of a closed geodesic extending a meridian), the derivative is taken with respect to the arc length parameterization (i.e. $k' \equiv JX.k$), and l is a positive real constant (if l were negative, we could change X in $-X$ and so on).

On the other hand, k can be retrieved, by computation, from r: We use that $\nabla_{JX}JX = \nabla_{JX}X = 0$, as JX is unitary and is tangent to the meridians, which are geodesics, and $[JX, Y] = [JX, rX] = 0$, as Y is Killing and thus preserves the meridians, and get:

$$[X, JX] = \frac{r'}{r}X$$

everywhere except in the poles, thus

$$R_{X,JX}JX = \left(-\frac{r''r - (r')^2}{r^2} - \frac{(r')^2}{r^2}\right)X = -\frac{r''}{r}X,$$

on the above dense open set in S^2. The derivatives are taken with respect to the arc length parameterization, hence correspond to derivations along the vector field JX. We get thus the following equation:

$$r'' = -kr, \tag{14}$$

which, together with (13), yields a second order non-linear differential equation on $\mathbb{R}$, for which we seek periodic solutions:

$$k'' = -\frac{1}{2}k^2 + c,\ c \in \mathbb{R}. \tag{15}$$

This equation cannot be integrated, in general (using elementary functions), but it is equivalent to a first order system of ordinary differential equations, given by the following vector field on $\mathbb{R}^2$ (in coordinates x, y):

$$Z := \left(-\frac{1}{2}x^2 + c\right)\partial_y + y\partial_x. \tag{16}$$

(For any solution k of (15), the curve (k, k') is an integral curve of Z, and conversely, the first component of any integral curve of Z is a solution of (15).)

Remark. Any solution to the equation (15) is a solution of an equation of the type

$$(z')^2 = z^3 + pz + q,\ p, q \in \mathbb{R}, \tag{17}$$

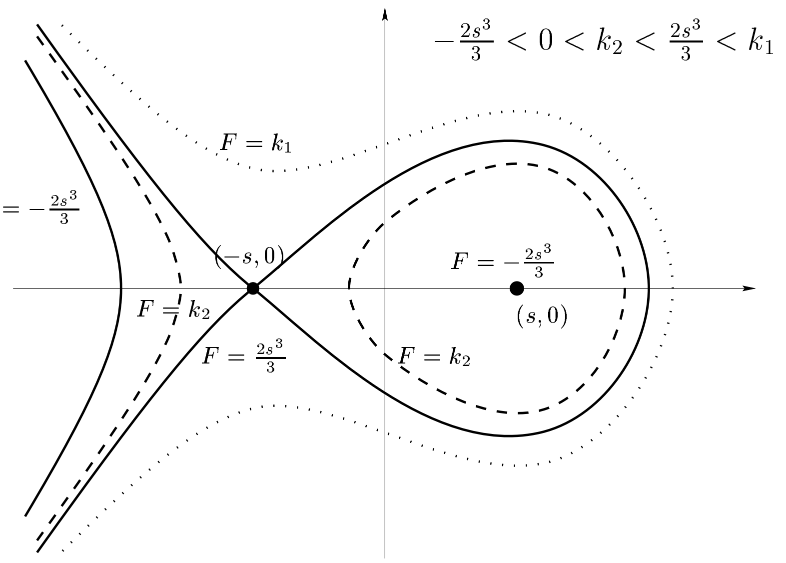
which comes from the fact that, if z solves (15), then (z, z') satisfy an order 3 algebraic equation (see below) that leads to (17). The general solution to this equation is an elliptic function and cannot be expressed, in general, in terms of elementary functions.

Remark. A simple way to find a solution of (15) could be, *a priori*, the following: A regular CR Reeb vector field T on S^3 equipped with the standard CR structure is still a Killing vector field for the Sasakian structure given by another CR Reeb vector field $\tilde{T}$, provided that $[T, \tilde{T}] = 0$; the metrics induced, by projection, on the space B of orbits of T, from the Sasakian metric on S^3 determined by T or by $\tilde{T}$ are different, and their curvatures $k, \tilde{k}$ both satisfy the equation $\square_X \tilde{k} = \square_X k = 0, \forall X \in TB$; it is, however, easy to check that $\tilde{k}$ is, as well as k, constant; not only they are conformally equivalent (through the identity of B), but also isometric up to a global homothety. So we don't get this way any non-trivial solution to (15).

We seek for periodic solutions of (15), hence for closed integral curves of Z (16). First we note that, for negative c, the component along ∂_y of Z is always negative, fact which excludes any periodic orbit. For $c = 0$ the only periodic orbit is the fixed point $(0,0)$. In general, the orbits of Z are contained in the level sets of the function $F : \mathbb{R}^2 \to \mathbb{R}$,

$$F(x,y) := y^2 + \frac{1}{3}x^3 - 2cx,$$

which are cubics in $\mathbb{R}^2$:



For values of F contained between $-\frac{2s^3}{3}$ and $\frac{2s^3}{3}$, where $s := \sqrt{2c}$, its level sets are cubics with 2 connected components in $\mathbb{R}^2$, one of which is an embedded line, and the other an embedded circle — each of which is an orbit of Z, because Z does not vanish on these curves —; the level set $F = -\frac{2s^3}{3}$ has 2 components, one of which is the fixed point $(s,0)$ and the other an embedded line — which is an orbit of Z —; the level set $F = \frac{2s^3}{3}$ has only 1 connected component — a cubic with a simple double point in $(-s,0)$ —, and contains one stationary orbit $\{(0,0)\}$ and 3 different, non-periodic orbits of Z, which are exactly the connected components of the above cubic after removal of the fixed point $(-s,0)$ of Z. For other values of F its level sets consist, each of them, of one connected cubic in $\mathbb{R}^2$, diffeomorphic to $\mathbb{R}$, and containing one (non-periodic) orbit of Z. (see figure above)

The only case which is interesting for the equation (15), for which we seek periodic solutions, is $c = \frac{1}{2}s^2$, $s > 0$ (thus $c > 0$), and the solutions whose orbits coincide with the embedded circles contained in the level sets of F, for values of F between $-\frac{2ls^3}{3}$ and $\frac{2s^3}{3}$. Actually all the solutions obtained this way satisfy the equation (15), and yield metrics on $S^2 \setminus \{P_1, P_2\}$ with the desired property (J times the differential of the curvature is dual to a Killing vector field); it remains to check the smoothness at the poles: the only condition is that the constructed metric gives an angle of 2π around each pole. This angle can be measured by integration around a pole (that is, multiplication by 2π — the period of the Killing vector field $Y = rX$) of the differential of the length of Y. We need thus

$$|r'| = 1 \text{ in } P_1, P_2, \tag{18}$$

which is equivalent to

$$\left| -\frac{1}{2}s_i^2 + \frac{1}{2}s^2 \right| = l, \ i = 1, 2,$$

where s_i are the solutions, between $-s$ and s, of the equation $F(x, 0) = 0$, or equivalently, $(s_i, 0)$ are the points where the considered orbit of Z crosses the Ox axis — these points correspond to P_1 and P_2, which are the extrema of k, thus where k' vanishes. Because $s_1 \in (-s, s)$ and $s_2 \in (s, 2s)$ (the orbit "turns around" the fixed point $(s, 0)$), the equation (18) is equivalent to:

$$s^2 - s_1^2 = 2l \text{ and } s_2^2 - s^2 = 2l.$$

On the other hand, the fact that $F(s_1, 0) = F(s_2, 0)$ (and $s_1 \neq s_2$) is equivalent to

$$s_1^2 + s_1 s_2 + s_2^2 = 3s^2. \tag{19}$$

It follows then that $s_1^2 + s_2^2 = 2s^2$, which leads to $\sqrt{s^4 - 4l^2} = s^2$, contradiction.

So there is no metric on S^2 such that J times the curvature is dual to a Killing vector field, except the constant curvature metric (in which case the corresponding Killing vector field is trivial).

If we are looking for a metric with conical points at the poles P_1, P_2, with wrapping (natural) numbers q_1, resp. q_2, the reasoning above remains unchanged, except for the part referring to the angles given by the metric at the poles; we need to replace equation (18) by the following system:

$$|r'| = \frac{1}{q_1} \text{ in } P_1 \text{ and } |r'| = \frac{1}{q_2} \text{ in } P_2, \tag{20}$$

which is equivalent to

$$s^2 - s_1^2 = \frac{2l}{q_1} \text{ and } s_2^2 - s^2 = \frac{2l}{q_2}.$$

From here, we get the expressions of s_1, s_2, then replace them in (19) and finally get:

$$\left(s^2 - \frac{2l}{q_1}\right)\left(s^2 + \frac{2l}{q_2}\right) = \left(s^2 - \frac{2l}{q_2} + \frac{2l}{q_1}\right)^2,$$

which implies

$$s = \sqrt{l \frac{(2/q_2)^3 + (2/q_1)^3}{3((2/q_2)^2 - (2/q_1)^2)}}, \text{ for } q_1 > q_2.$$

Thus, for any pair of mutually prime numbers $q_1 > q_2$ (the last one may be equal to 1, in which case we get a teardrop orbifold metric on S^2), and for each fixed $l > 0$, there is exactly one value of the parameter s, and exactly one orbit of Z, thus exactly one solution of (15), which is equivalent to

$$k'' = \frac{1}{2}(-k^2 + s^2);$$

from the point of view of the metric that we obtain on $B \simeq S^2$, the change in the parameter l means a homothety (i.e., if we fix the total volume of B, then l is unique).

We have proven:

Theorem 2. *For any regular CR Reeb vector field on a flat CR structure on S^3, the orbit space has constant curvature. The metric on the orbit space of a quasi-regular CR Reeb vector field T is, up to homothety, determined by the wrapping numbers of T.*

Remark. We obtain, in particular, another proof of the unicity of the CR flat structure on S^3, under the additional supposition that it admits global CR Reeb vector fields. Of course, this fact holds in general, as a consequence of the theory of Cartan connections [19].

Remark. As mentioned before, the orbifold metrics on S^2 for which the curvature function k satisfies $\Box_X k = 0, \forall X \in TS^2$ cannot be expressed in terms of elementary functions *in the arc length parameterization of the meridians.* In fact, these metrics, an their curvatures, are easy to compute, as the corresponding Sasakian metrics on S^3, because they are determined by quasi-regular CR Reeb vector fields on S^3, all of which can be explicitely computed. The metrics on their orbit spaces can be explicitly given in terms of elementary functions. But this does not lead to any contradiction, because it is the *arc length parameterization of the meridians* — as a solution to a second degree differential equation — which cannot be expressed in terms of elementary functions.

Theorem 2 has interesting geometric implications on the automorphism group of the standard CR structure on S^3, namely $PSU(2,1)$:

Corollary 2. *The commutator of any regular CR Reeb vector field in $\mathfrak{psu}(2,1)$ is 4-dimensional. For any 2 quasi-regular CR Reeb vector fields T_1, T_2 with the same wrapping numbers there is an element in $PSU(2,1)$ mapping T_1 on a constant multiple of T_2.*

Another consequence of this is that we can check if two CR structures are isomorphic, just looking at the CR Reeb vector field and at the metric induced on the orbit space (recall that any irregular CR Reeb vector field admits exactly one commuting regular one — it is easy to find it just by looking at the lengths of the exceptional orbits of the irregular CR Reeb vector field — see section 3), thus reducing the problem of comparing 2 CR

structures to the elementary problem of comparing 2 metrics on a Riemann surface — more generally, an orbifold:

Theorem 3. *If 2 Sasakian metrics on a compact 3-manifold M admit the same underlying CR structure, then:*

(1) *they are both regular and are* 0-type deformation *of each other, if M is not a quotient of S^3;*
(2) *if $M = S^3$ or a finite quotient, first we replace (if necessary) the irregular Sasakian metrics with the (uniquely defined) regular Sasakian metrics by* basic first type deformations*, so we only need to compare (quasi-) regular Sasakian structures. If the Sasakian structures are both quasi-regular with the same wrapping numbers they are 0-type deformations of each other; If they have different wrapping numbers the underlying CR structure is flat, and the metric induced on each one's orbit space is determined, following the procedure described in the proof of Theorem 2, by the wrapping numbers.*

By definition [3], a *0-type deformation* of a Sasakian metric is a rescaling of the Reeb vector field (multiplication by a constant), and a *first type deformation* is given by a change in the CR Reeb vector field (2); it is called *basic* if we use in (2) a function constant along the orbits of the Reeb vector field. A 0-type deformation is, thus, a particular basic first type deformation, given by an everywhere constant function. In [2] and [3], a *second type deformation* is defined by keeping the same CR Reeb vector field, the same operator J — previously extended to TM by acting trivially on the Reeb vector field —, and by changing the metric on the orbit space — as a consequence, the contact structure will be modified.

Corollary 3. *A non-trivial second type deformation of a Sasakian structure always changes the underlying CR structure.*

Proof. If the CR Reeb vector field is quasi-regular, this follows from the theorems above; if it is irregular, then any second type deformation of it necessarily commutes with the first type deformation that replaces the irregular CR Reeb vector field with the unique (commuting) regular one, and we can apply the conclusion above. □

7. Contact compact 3-manifolds admitting normal CR structures

Proposition 2. *Let M be a compact 3-manifold. Then, up to isomorphism, there exists at most one contact structure on M underlying a normal CR structure. Moreover, this contact structure is* tight.

By definition, an *over twisted* contact structure is one such that it exists an embedded disc $D \subset M$, transverse to Q everywhere except in a point P in the interior of D, and such that the border of D is tangent to Q. A CR structure is called *tight* if it is not overtwisted. Overtwisted contact structures exist on any orientable 3-manifold [17], and the isotopy classes of overtwisted contact structures on M coincide with their homotopy classes in the category of 2-plane fields in M [7]. On the other hand, tight structures are more rigid: there is only one isotopy class of tight contact structures on

S^3 [8]; recently, there have been found examples of 3-manifolds not admitting tight contact structures [9].

Proof. We know, from [3], that, if M admits a normal CR structure, then it is a finite quotient of a circle bundle M_0 over a Riemann surface B with non-zero Chern class, and, in most of the cases (see the exceptions below), this CR structure is S^1-invariant, where the S^1-action is free and transverse to the underlying contact structure. The latter is thus given by a connection on the S^1-bundle $M_0 \to B$ (where B is the orbit space of S^1), such that its curvature is a given volume form on B (from here follows that the Chern class should be non-zero). On M_0, we have thus a Γ-invariant Sasakian structure, where Γ is the covering group of the finite covering $M_0 \to M$. On M_0, all (Γ-invariant) contact structures which are invariant and transverse to the free action of S^1 are homotopic within (Γ-invariant) contact structures [16], and this (Γ-equivariant) homotopy is followed by a (Γ-equivariant) isotopy [18], [12].

On the other hand, all free S^1 actions on M (and, implicitly, the diffeomorphism type of the quotient space B) are isomorphic, because the only invariants are $b_1(B)$ and the Chern class of the fibration $M \to B$, and these are determined by the cohomology of M.

The exceptional cases are the normal CR structures on S^3 (or finite quotients) which are S^1-invariant, but the S^1 action is not free; it is given by a (unique up to a constant factor) quasi-regular CR Reeb vector field. We can, nevertheless, deform the metric on the orbifold B such that it admits a Killing vector field; the corresponding (second type) deformation of the CR structures is such that the underlying contact structures are all isotopic [18]; but the new CR structure admits at least 2 linearly independent CR Reeb vector fields, therefore it is invariant to a free S^1 action, which is the case treated above.

On the other hand, the S^1 invariant and transversal contact structures on the total space of a circle fiber bundle as above are tight [11]. $\square$

References

[1] G. D'Ambra. M. Gromov, *Lectures on transformation groups: Geometry and dynamics*, Proc. Conf., Cambridge/MA (USA) 1990, Surv. Differ. Geom., J. Differ. Geom., **Suppl. 1** (1991), 19-111.

[2] F.A. Belgun, *On the metric structure of non-Kähler complex surfaces*, Math. Ann. **317** (2000), 1-40.

[3] F.A. Belgun, *Normal CR structures on compact 3-manifolds*, to appear in Math. Zeit.

[4] D. Burns, Jr., S. Shnider, *Spherical Hypersurfaces in Complex Manifolds*, Inv. Math. **33** (1976), 223-246.

[5] E. Cartan, *Sur la géométrie pseudo-conforme des hypersurfaces de l'éspace des deux variables complexes*, Ann. Math. Pura Appl. **4** (1932), 17-90 (or Oeuvres II, 2, 1231-1304).

[6] S. Dragomir, *On pseudohermitian immersions between strictly pseudoconvex CR manifolds*, Am. J. Math. **117** (1995), 169-202.

[7] Y. Eliashberg, Classification of overtwisted contact structures on 3-manifolds, Inv. Math. **98** (1989), 623-637.

[8] Y. Eliashberg, Contact 3-manifolds twenty years after Martinet's work, Ann. Instit. Fourier **42** (1992), 165-192.

[9] J.B. Etnyre, K. Honda, On the non-existence of tight contact structures, arXiv: math.GT/9910115 (1999), 17 pages.
[10] H. Geiges, *Normal contact structures on 3-manifolds*, Tôhoku Math. J. **49** (1997), 415-422.
[11] E. Giroux, *Structures de contact sur les variétés fibrées en cercles au-dessus d'une surface*, 1999, arXiv: math.GT/9911235, 38 pages.
[12] J.W. Gray, Some global properties of contact structures, Ann. of Math. **2** (1959), 421-450.
[13] M. Gromov, *Rigid transformations groups*, in Géométrie différentielle, Colloq. Geom. Phys., Paris/Fr. 1986, Trav. Cours **33** (1988), 65-139.
[14] Y. Kamishima, T. Tsuboi, *CR structures on Seifert manifolds*, Inv. Math. **104** (1991), 149-163.
[15] S. Kobayashi, *Transformation groups in Differential Geometry*, Springer-Verlag, 1972.
[16] R. Lutz, *Structures de contact sur les fibre's principaux en cercles de dimension trois*, Ann. Inst. Fourier **27** (1977), 1-15.
[17] J. Martinet, *Formes de contact sur les variétés de dimension 3*, Lect. Notes in Math. **209** (1971), 142-163.
[18] J. Martinet, *Sur les singularités des formes différentielles*, Ann. Inst. Fourier **20** (1970), 95-178.
[19] N. Tanaka, *On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections*, Japan. J. Math. **2** (1976), 131-190.

Institut für Mathematik
Humboldt-Universität zu Berlin
Sitz: Rudower Chaussee 25
D-10099 Berlin
Germany
e-mail: `belgun@mathematik.hu-berlin.de`