Constructing Infinite Particle Spectra

O.A. Castro-Alvaredo and A. Fring
Institut für Theoretische Physik, Freie Universität Berlin, Arnimallee 14, D-14195 Berlin, Germany
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We propose a general construction principle which allows to include an infinite number of resonance states into a scattering matrix of hyperbolic type. As a concrete realization of this mechanism we provide new S-matrices generalizing a class of hyperbolic ones, which are related to a pair of simple Lie algebras, to the elliptic case. For specific choices of the algebras we propose elliptic generalizations of affine Toda field theories and the homogeneous sine-Gordon models. For the generalization of the sinh-Gordon model we compute explicitly renormalization group scaling functions by means of the c-theorem and the thermodynamic Bethe ansatz. In particular we identify the Virasoro central charges of the corresponding ultraviolet conformal field theories.

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I. INTRODUCTION

Treating quantum field theories in 1+1 dimensions as a test laboratory for realistic theories in higher dimensions, this paper is concerned with the general question of how to enlarge a given finite particle spectrum of a theory to an infinite one.

In general the bootstrap [1], which is the construction principle for the scattering matrix, is assumed to close after a finite number of steps, which means it involves a finite number of particles. However, from a physical as well as from a mathematical point of view, it appears to be natural to extend the construction in such a way that it would involve an infinite number of particles. The physical motivation for this are string theories, which admit an infinite particle spectrum. Mathematically the infinite bootstrap would be an analogy to infinite dimensional groups, in the sense that two entries of the S-matrix are combined into a third, which is again a member of the same infinite set. It appears to us that it is impossible to construct an infinite bootstrap system involving asymptotic states and find the mathematical analogue to infinite groups in this sense (see also footnote 2 and the paragraph after figure 1). However, it is possible to introduce an infinite number of unstable particles into the spectrum. Scattering matrices which would allow such type of interpretation have occurred in the literature [3–5], although only in the latter paper a reference to unstable particles has been made. In [4,5] these matrices were found to be expressible in terms of elliptic functions, a feature very common in the context of lattice models, e.g. [7]. The main purpose of this paper is to suggest a general construction principle for such type of S-matrices starting from some known theory with a finite particle spectrum of a special, albeit quite generic, form. As particular examples we provide elliptic generalizations of scattering matrices related to a pair of Lie algebras [6], which contain the affine Toda S-matrices [8] and homogeneous sine-Gordon S-matrices [9] for particular choices of the algebras.

Our paper is organized as follows: In section II we provide a general principle for the construction of scattering matrices which involve an infinite number of unstable particles and present some explicit examples. In section III we construct renormalization group (RG) scaling functions by means of the c-theorem and the thermodynamic Bethe ansatz (TBA) for the generalization of the sinh-Gordon model. Our conclusions and an outlook towards open problems are stated in section IV.

II. CONSTRUCTION PRINCIPLE

Let us consider the huge class of two-particle S-matrices, which describe the scattering between particles of type a and b as a function of the rapidity difference θ, of the general form[1]

$$S_{ab}(\theta) = S_{ab}^{\min}(\theta) S_{ab}^{\mathrm{CDD}}(\theta)\,. \tag{1}$$

Here $S_{ab}^{\min}(\theta)$ denotes the so-called minimal S-matrix which satisfies the consistency relations [1], namely unitarity, crossing and the fusing bootstrap equations and possibly possess poles on the imaginary axis in the sheet $0 \leq \operatorname{Im}\theta \leq \pi$, which is physical for asymptotic states. The CDD-factor [2], referred to as $S_{ab}^{\mathrm{CDD}}(\theta)$, also satisfies these equations, but has its poles in the sheet $-\pi \leq \operatorname{Im}\theta \leq 0$, which is the "physical one" for resonance states. $S_{ab}^{\mathrm{CDD}}(\theta)$ might depend on additional constants like the effective coupling constant or a resonance parameter. A simple prescription to introduce now an infinite number of resonance poles is to replace the CDD-factor in (1) by

$$\hat{S}_{ab}^{\mathrm{CDD}}(\theta, N) = \prod_{n=-N}^{N} S_{ab}^{\mathrm{CDD}}(\theta + n\omega)\,, \tag{2}$$

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[1]Exceptions to this factorization are for instance the scattering matrices of affine Toda field theories related to non-simply laced Lie algebras, which was first noted in [10].

where ω is taken to be real. By construction the new S-matrix, $\hat{S}_{ab}(\theta, N) = S_{ab}^{\min}(\theta)\hat{S}_{ab}^{\rm CDD}(\theta, N)$ satisfies the bootstrap consistency equations and possible poles in the sheet $-\pi \leq \operatorname{Im}\theta \leq 0$ have now been duplicated $2N$ times within this sheet, such that they admit an interpretation as unstable particles. Therefore, when $N \to \infty$ we have an infinite number of resonance poles. Since, as a consequence of crossing and unitarity, the S-matrix is known to be a $2\pi i$-periodic function, a property shared individually by $S_{ab}^{\rm CDD}(\theta, N)$, we expect to recover a double periodic function in the limit $N \to \infty$

$$\lim_{N\to\infty} \hat{S}_{ab}^{\rm CDD}(\theta, N) = \lim_{N\to\infty} \hat{S}_{ab}^{\rm CDD}(\theta + \mu 2\pi i + \nu\omega, N) \quad (3)$$

for $\mu, \nu \in \mathbb{Z}$. At this stage it is not clear whether the prescription (2) is meaningful at all, in the sense that it leads to meaningful quantum field theories, and in particular one has to be concerned about the convergence of the infinite product in (3). Since $\lim_{N\to\infty} \hat{S}_{ab}^{\rm CDD}(\theta, N)$ is a double periodic function we expect that it is somehow related to elliptic functions (see e.g. [11] for their properties). Let us therefore now look concretely at the building blocks which can be used to make up the entire scattering matrix in the non-elliptic case, when backscattering is absent. In that case the S-matrices are diagonal and known [12] to be of the general form

$$S_{ab}(\theta) = \prod_{x\in\mathcal{A}} \{x\}_\theta^\sigma = \prod_{x\in\mathcal{A}} \frac{\tanh(\theta - i\pi x + \sigma)/2}{\tanh(\theta + i\pi x + \sigma)/2}, \quad (4)$$

with $x \in \mathbb{Q}$ and $\sigma \in \mathbb{R}$. A specific theory is then characterized by the finite set $\mathcal{A}$[2]. This means, if we demonstrate that the prescription (3) is meaningful for each individual building block $\{x\}_\theta^\sigma$ as defined in (4), in particular we need to demonstrate the convergence of the infinite product, we have established that it is sensible for the entire scattering matrix. For this purpose we note the identity

$$\{x\}_{\theta,\ell}^\sigma := \prod_{n=-\infty}^{\infty} \{x\}_{\theta+n\omega}^\sigma = \frac{\operatorname{sc}\theta_- \operatorname{dn}\theta_+}{\operatorname{sc}\theta_+ \operatorname{dn}\theta_-}. \quad (5)$$

Here we abbreviated $\theta_\pm = (\theta \pm i\pi x + \sigma)iK_\ell/\pi$ and used the Jacobian elliptic functions in the standard notation $\operatorname{pq}(z)$ with p,q $\in$ {s,c,d,n} (see e.g. [11]). The quarter periods K_ℓ depending on the parameter $\ell \in [0,1]$ are defined in the usual way through the complete elliptic integral

$$K_\ell = \int_0^{\pi/2} (1 - \ell\sin^2\theta)^{-1/2} d\theta\,. \quad (6)$$

[2]The fact that $x \in \mathbb{Q}$ together with the bootstrap leads to a finite set $\mathcal{A}$ and therefore a finite number of asymptotic states. Taking instead $x \in \mathbb{R}$ could possibly lead to an infinite number, but a consistent closure of the bootstrap is not known up to now.

The period of $\{x\}_{\theta,\ell}^\sigma$ is chosen to be $\omega = \pi K_{(1-\ell)}/K_\ell$. The last identity in (5) is easily derived from the infinite product representations of the elliptic functions which can be found in various places as for instance in [11]

$$\operatorname{sc} x = k\tan\frac{\pi x}{2K_\ell} \prod_{n=1}^{\infty} \frac{1 - 2q^{2n}\cos(\pi x/K_\ell) + q^{4n}}{1 + 2q^{2n}\cos(\pi x/K_\ell) + q^{4n}}, \quad (7)$$

$$\operatorname{dn} x = k^{-1} \prod_{n=1}^{\infty} \frac{1 + 2q^{2n-1}\cos(\pi x/K_\ell) + q^{4n-2}}{1 - 2q^{2n-1}\cos(\pi x/K_\ell) + q^{4n-2}}\,, \quad (8)$$

with $k = (1-\ell)^{-1/4}$ and $q = \exp(-\omega)$. Recalling the well known limits $\lim_{\ell\to 0} K_\ell = \pi/2$ and $\lim_{\ell\to 0} K_{(1-\ell)} = \infty$ we obtain

$$\lim_{\ell\to 0} \{x\}_{\theta,\ell}^\sigma = \{x\}_\theta^\sigma\,. \quad (9)$$

This means in the limit $\ell \to 0$ the elliptic S-matrix $\hat{S}_{ab}(\theta)$ collapses to the hyperbolic one, that is $S_{ab}(\theta)$. Notice that due to the general identity $\operatorname{pr}(x)/\operatorname{qr}(x) = \operatorname{pq}(x)$, we could also write (5) in terms of various other combinations of elliptic functions. For instance replacing sc by sn / cn is probably most intuitive, since it allows an alternative prescription to (3) for the construction of elliptic scattering matrices: Replace sinh $\to$ sn, cosh $\to$ cn and correct the consistency equations by a factor dn, which reduces always to 1 in the hyperbolic limit, in such a way that no resonance poles are left inside the physical sheet. Defining now the function

$$\theta_{\mu,\nu}(x, \sigma, \omega) := 2\pi i(\nu + x/2) + 2\omega\mu - \sigma \quad (10)$$

the singularities of $\{x\}_{\theta,\ell}^\sigma$ are easily identified as

$$\text{zeros}: \quad \theta_{\mu,\nu}(x, \sigma, \omega), \qquad \theta_{\mu,\nu}(1 - x, \sigma, \omega), \quad (11)$$

$$\text{poles}: \quad \theta_{\mu,\nu}(-x, \sigma, \omega), \quad \theta_{\mu,\nu}(x - 1, \sigma, \omega)\,. \quad (12)$$

Note that when taking $0 \leq x \leq 1$ the poles are situated in the non-physical sheet.

This brings us to the question of how to interpret these poles and how can we characterize the physical properties of the related particles? Considering the S-matrix $S_{ab}(\theta)$, which describes the scattering of two particles of type a and b with masses m_a and m_b, we assume that there is a resonance pole situated at $\theta_R = \sigma - i\bar{\sigma}$. According to the Breit-Wigner formula [17] (see also e.g. [18]) the mass $M_{\tilde{c}}$ and the decay width $\Gamma_{\tilde{c}}$ of an unstable particle of type $\tilde{c}$ can be conveniently expressed as

$$2M_{\tilde{c}}^2 = \sqrt{\gamma^2 + \tilde{\gamma}^2} + \gamma, \quad \Gamma_{\tilde{c}}^2/2 = \sqrt{\gamma^2 + \tilde{\gamma}^2} - \gamma, \quad (13)$$

where

$$\gamma = m_a^2 + m_b^2 + 2m_a m_b\cosh\sigma\cos\bar{\sigma}, \quad (14)$$

$$\tilde{\gamma} = 2m_a m_b \sinh|\sigma|\sin\bar{\sigma}\,. \quad (15)$$

We keep here in mind that this description, although frequently used e.g. [9,5,13], is not entirely rigorous and requires additional investigation. This caution is based on various facts. First, the relations (13) are simply derived by carrying over a prescription from usual quantum mechanics to quantum field theory, i.e. complexifying the mass of a stable particle. Second, solving the Breit-Wigner formula for the quantities $M_{\tilde{c}}$, $\Gamma_{\tilde{c}}$ and treating them literally as mass and decay width is somewhat problematic since this is in conflict with Heisenberg's uncertainty principle, because apparently we know simultaneously the energy and the time. Third, the Breit-Wigner relations presume an exponential decay in momentum space, which is in fact incompatible with the general principles of quantum field theory and therefore might possibly be a problem in this context [19]. Nonetheless, we employ these quantities and try to find evidence to support that they are indeed meaningful. When taking $M_{\tilde{c}}$ to be the mass of the unstable particle there should be a threshold for energetic reasons of the type

$$M_{\tilde{c}} \geq m_a + m_b \,, \qquad (16)$$

with the consequence that the decay width is bounded by

$$\Gamma_{\tilde{c}}^2 \geq 8 m_a m_b (1 - \cosh\sigma \cos\bar{\sigma}) \;. \qquad (17)$$

So far evidence for these thresholds has not been found in the literature. One reason for this is that the unstable particles enter the bootstrap principle in a more passive way than the stable particles, whose properties are directly used in the construction procedure. Hence one expects that signs for these thresholds will emerge in a more indirect way.

A summary of our statements about the pole structure of $\hat{S}(\theta)$ is depicted in figure 1.

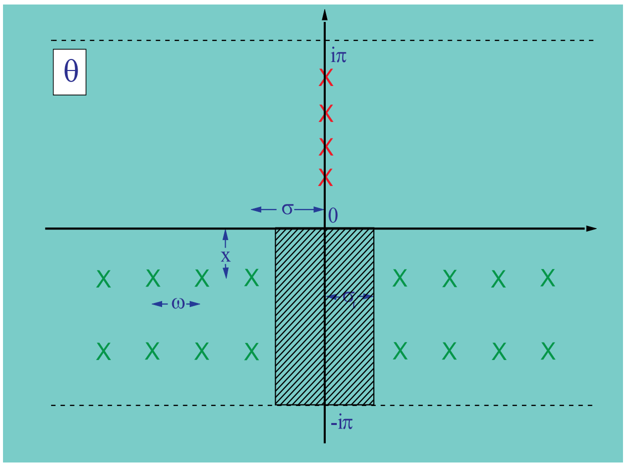

Figure 1: The poles of the blocks $\{x\}_{\theta,\ell}^{\sigma}$ are the crosses in the sheet $-\pi \leq \operatorname{Im}\theta \leq 0$. The crosses on the positive part of the imaginary axis are associated, as usual, with stable particles. For equal masses of the stable particles the threshold (16) is $\sigma_t = \operatorname{arccosh}[(3 - \cos\pi x)/(1 + \cos\pi x)]$.

Since the poles inside the sheet $0 \leq \operatorname{Im}\theta \leq \pi$ are associated to $S_{ab}^{\min}(\theta)$, it is also obvious from figure 1 why the prescription (2) may not be employed for this part of the S-matrix, since it would lead to a pole structure which is, according to (13), non physical for $M_{\tilde{c}}$ and $\Gamma_{\tilde{c}}$.

A. Examples

It is clear by construction that our prescription includes all affine Toda field theories related to simply laced Lie algebras, since they all factorize as (1) and may be represented in the form (4), see e.g. [8]. Taking the resonance parameter σ to be zero and the set $\mathcal{A} = \{t\}$ for $0 \leq t \leq 1$, we recover as a special case the elliptic version of the sinh-Gordon model proposed in [5]. Reintroducing σ, its scattering matrix reads

$$\hat{S}(\theta) = \prod_{n=-\infty}^{\infty} \frac{\tanh(\theta - i\pi x + n\omega + \sigma)/2}{\tanh(\theta + i\pi x + n\omega + \sigma)/2} \,. \qquad (18)$$

According to (13) the masses and decay width of the unstable particles are

$$M_{\mu,\nu}^{\sigma,n\omega} = m\sqrt{2}\cosh\frac{\theta_{\mu,\nu}(y,\sigma,n\omega)}{2}, \quad y = -x, x-1 \quad (19)$$

$$\Gamma_{\mu,\nu}^{\sigma,n\omega} = m2\sqrt{2}\sinh\frac{\theta_{\mu,\nu}(y,\sigma,n\omega)}{2}, \quad y = -x, x-1 \quad (20)$$

where m denotes the mass of the stable particle. The thresholds (16) and (17) translate in this case into

$$\cosh(n\omega + \sigma) \geq \frac{3 \mp \cos\pi x}{1 \pm \cos\pi x}, \Gamma \geq 4m\frac{\sin^2\frac{\pi(2x+1\pm1)}{4}}{\cos\frac{\pi(2x+1\pm1)}{4}}. \qquad (21)$$

As a further example we consider the elliptic generalization of the $A_1|A_{N-1}$-theory ($\equiv SU(N)_2$-homogeneous sine-Gordon model). The two-particle S-matrix describing the scattering of two stable particles of type a and b, with $1 \leq a, b \leq N-1$, related to the non-elliptic version of this model was proposed in [9]. In our notation it may be written as

$$S_{ab}(\theta, \sigma_{ab}) = (-1)^{\delta_{ab}} \left[c_a \sqrt{\{1/2\}_{\theta}^{\sigma_{ab}}} \right]^{I_{ab}} . \qquad (22)$$

Here I denotes the incidence matrix of the $SU(N)$-Dynkin diagram, the resonance parameters have the property $\sigma_{ab} = -\sigma_{ba}$ and $c_a = \pm 1$ depending on whether a is even or odd. According to our prescription outlined in the previous paragraph, the elliptic generalization of (22) is

$$\hat{S}_{ab}(\theta, \sigma_{ab}, \ell) = (-1)^{\delta_{ab}} \left[c_a \sqrt{\{1/2\}_{\theta,\ell}^{\sigma_{ab}}} \right]^{I_{ab}} . \qquad (23)$$

Note that despite the appearance of the square root, S as well as $\hat{S}$ are still meromorphic functions in θ.

III. RG-SCALING FUNCTIONS

Having established that our prescription leads to sensible solutions of the bootstrap consistency equations, we would also like to know what kind of quantum field theories these scattering matrices correspond to. Up to now all known solutions to the on-shell consistency equations have led to sensible QFT's, albeit a rigorous proof which would establish that indeed *all* solutions are well-defined local QFT's is still an outstanding issue. Some crucial characteristics of the theory are contained in the renormalization group scaling functions, which we now want to determine. In particular, we want to identify in the extreme ultraviolet limit the Virasoro central charges of the corresponding conformal field theories.

A. The c-theorem

We carry out this task by evaluating the c-theorem [14] in the version presented in [13]

$$c(r) = 3\sum_{n=1}^{\infty}\sum_{\mu_1\ldots\mu_n}\int_{-\infty}^{\infty}\frac{d\theta_1\ldots d\theta_n}{n!(2\pi)^n}e^{-rE} \tag{24}$$
$$\times\left|F_n^{\Theta|\mu_1\ldots\mu_n}(\theta_1,\ldots,\theta_n)\right|^2\frac{(6+6rE+3r^2E^2+r^3E^3)}{2E^4}\,.$$

The sum of the on-shell energies is here denoted by $E=\sum_{i=1}^{n}m_{\mu_i}\cosh\theta_i$, with m_{μ_i} being the masses of the theory and the correlation function for the trace of the energy momentum tensor Θ has been expanded in terms of n-particle form factors $F_n^{\Theta|\mu_1\ldots\mu_n}(\theta_1,\ldots,\theta_n)$ (see [20] for general properties and [16] for explicit sinh-Gordon formulae). We normalized Θ and m_{μ_i} by an overall mass scale, such that E as well as the renormalization group parameter r become dimensionless. In particular $\lim_{r\to0}c(r)$ is the ultraviolet Virasoro central charge.

Let us now start with the evaluation of $c(r)$ as defined in (24) for the elliptic version of the sinh-Gordon model. As the input for this we need to know the n-particle form factors. Since so far it is not known how to compute the sum in n analytically, we have to resort to a numerical treatment and it is clear that we have to terminate the series at a certain value of n. Fortunately, it was observed explicitly in [16], that in fact the expression for $n=2$ is already very close to the exact answer for the sinh-Gordon model. We assume here that the convergence behaviour is still true when we generalize the scattering matrix to (18). Note that in general one has to be careful with this approximation, since the higher particle contributions are crucial in some models in order to obtain a good approximation to $c(r)$ [21,13,22]. In the two-particle approximation, indicated by the superscript, one can perform one of the integrations analytically and (24) acquires the simple form

$$\lim_{r\to0}c^{(2)}(r) = \frac{3}{2}\int_0^{\infty}d\theta\frac{|F_2^{\Theta}(2\theta)|^2}{\cosh^4\theta}\,. \tag{25}$$

It is here crucial to note that besides the formulation of $\hat{S}$ in terms of elliptic functions for $N\to\infty$, it can also be expressed equivalently in terms of the usual sinh-Gordon S-matrix (5). When trying to solve now the form factor consistency equations [20], we can exploit this observation. Since for the model at hand there is neither a kinematic nor a bound state pole in $F_2^{\Theta}(\theta)$, the only equations to be solved are Watson's equations. The two particle form factor is then easily obtained to be

$$\hat{F}_2^{\Theta}(\theta,N) = 2\pi\prod_{n=-N}^{N}\frac{F_{\min}(\theta+n\omega)}{F_{\min}(i\pi+n\omega)}, \tag{26}$$

where $F_{\min}(\theta)$ is the minimal form factor of the sinh-Gordon model obtained in [16]

$$F_{\min}(\theta) = \exp\left[4\int_0^{\infty}\frac{dt}{t}\left(\cos\left(\frac{t\theta}{\pi}\right)\coth t + i\sin\left(\frac{t\theta}{\pi}\right)\right)\right.$$
$$\left.\times\frac{\sinh(\frac{t(x-1)}{2})\sinh(\frac{tx}{2})\sinh(\frac{t}{2})}{\sinh(t)}\right]\,. \tag{27}$$

Using the infinite product representation for $F_{\min}(\theta)$ [16] the solution (26) for $N\to\infty$ coincides with the equation (5.3) in [5]. Proceeding now to the evaluation of (24), we require $|\hat{F}_2^{\Theta}(\theta,N)|^2$, whose characteristics are captured in figure 2.

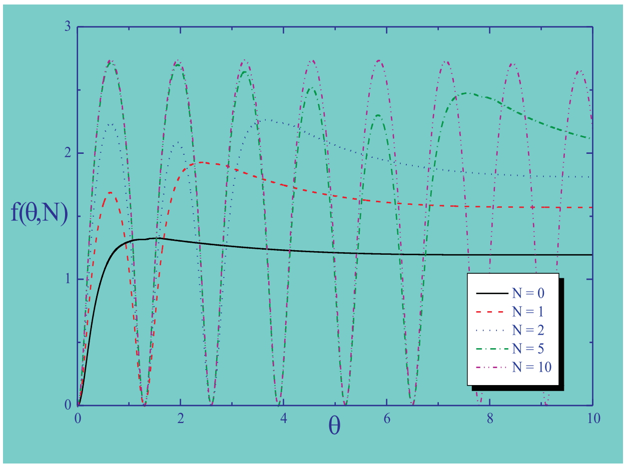

Figure 2: Absolute value squared of the two particle form factors $\mathrm{f}(\theta,N) = |\hat{F}_2^{\Theta}(\theta,N)/2\pi|^2$ as functions of the rapidity for different values of N for $\omega=1.3$ and $x=0.1$.

We observe that for a certain value of N the function starts to converge, which is of course important from a technical point of view when we want to compute the limiting case $N\to\infty$. The other observation we make in figure 2, see also figure 5 in [16], is that $\mathrm{f}(\theta,N=0)$

always has a distinct maximum, which we refer to as θ_m. From (27) follows that it is determined by the solution of

$$\frac{4\theta_m}{\pi}\cosh\theta_m \sin\pi x + \cosh 2x\pi \coth\frac{\theta_m}{2}$$
$$= \frac{\cosh\frac{3\theta_m}{2}}{\sinh\frac{\theta_m}{2}} + 2(2x-1)\cos\pi x \sinh\theta_m . \qquad (28)$$

Solving this equation for various values of x, we find that θ_m is always slightly greater than the smallest threshold bound obtained from (21). For instance for $x = 0.1$ we obtain $\theta_m \simeq 1.439$ and $n\omega + \sigma > 0.315$ and for $x = 0.5$ we have $\theta_m \simeq 2.040$ and $n\omega + \sigma > 1.763$. We interpret this as an indication that the form factors "know" about the thresholds (21). We support this now by considering $\lim_{N\to\infty} \mathrm{f}(\theta, N)$ for various values of ω.

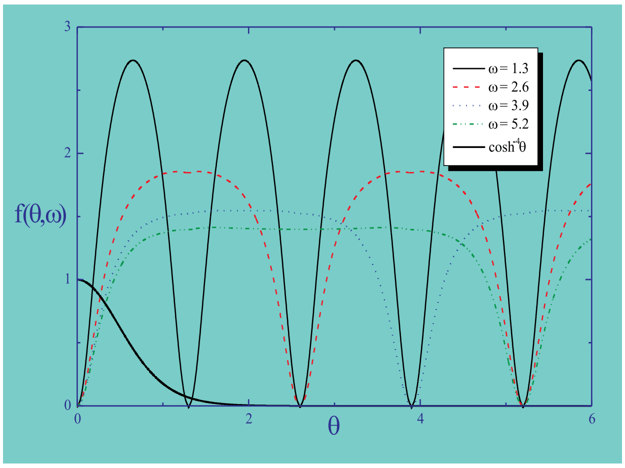

Figure 3: Absolute value squared of minimal form factors $\mathrm{g}(\theta,\omega) = \lim_{N\to\infty} |\hat{F}_{\min}(\theta, N)|^2$ as functions of the rapidity for different values of ω and $x = 0.1$.

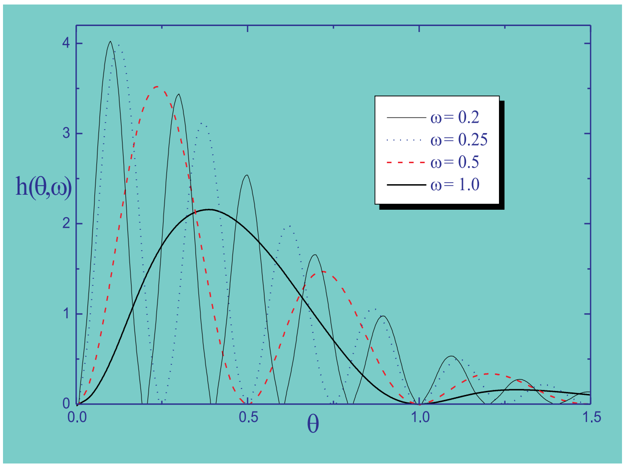

Figure 4: Integrand of equation (25), that is $\mathrm{h}(\theta,\omega) = \lim_{N\to\infty} |\hat{F}_{\min}(\theta, N)|^2/\cosh^4\theta$ as a function of the rapidity for different values of ω and $x = 0.1$.

We observe that in the region in which the factor $1/\cosh^4(\theta)$, emerging in (25), is still non vanishing the integrals $\lim_{N\to\infty} \int d\theta |\hat{F}_{\min}(\theta, N)|^2$ are decreasing functions of ω. This behaviour is changed once we take $\omega < \theta_m$ as we can explicitly extract from figure 4.

Naturally this features are also reflected in the scaling functions. Presuming that for each value of N we have a consistent theory, we would like to know which ultraviolet central charges these models possess and in addition we want to identify a value of N for which the related model constitutes reasonably good approximation for the elliptic models. That such an identification is possible is exhibited in figure 5. In addition we observe, that for fixed ω and x the scaling function is a monotonically increasing when N is varied.

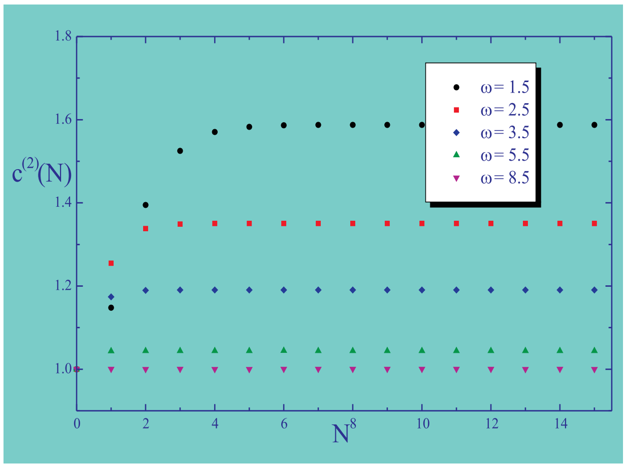

Figure 5: Ultraviolet Virasoro central charge $c^{(2)}$ as a function of N.

Focussing now on the elliptic case, that is we select a large enough N such that this case is well approximated, we compute the scaling function in dependence of ω for various values of r. Our results are depicted in figure 6.

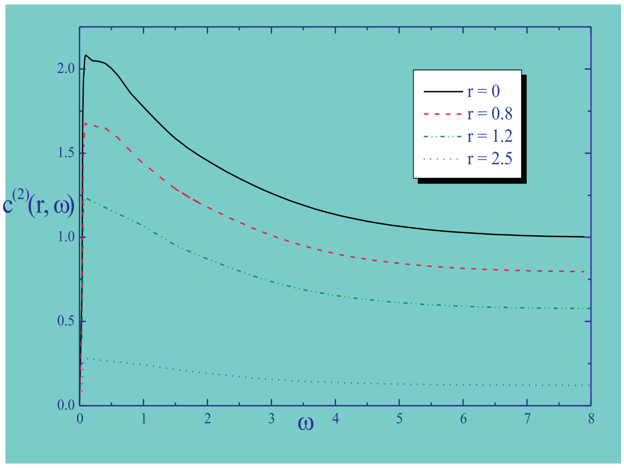

Figure 6: Ultraviolet Virasoro central charge $c^{(2)}$ as a function of ω.

In the extreme limits we obtain $\lim_{\omega\to 0} c(r,\omega) = 0$ and for large ω we recover the values of the sinh-Gordon model, $\lim_{\omega\to\infty} c(r,\omega) = c_{\rm SG}(r)$. The latter limit follows from (9) and is in addition compatible with $\lim_{\theta\to\infty} F_{\rm min}(\theta) = 1$. The values for the extremal points were already quoted in [5], however, we also observe that the function is not monotonically increasing between these points as claimed in there. In fact, in the physical region, that is for values of $\omega > 0.315$, the function is monotonically decreasing and does not take on values between 0 and 1. Remarkably, the threshold is quite clearly exhibited by a drastic change in the behaviour of c, that is the onset of a small plateau as is visible in figure 6. We performed the same computation for different values of x and observed that this onset moves in the direction predicted by equation (21).

B. The thermodynamic Bethe ansatz

Let us now compare the results of the previous section with an alternative method, namely the thermodynamic Bethe ansatz [15]. For this we first have to solve the TBA-equations

$$r\,\hat{m}_i \cosh\theta + \ln(1 - e^{-L_i(\theta)}) = \sum_j \varphi_{ij} * L_j(\theta) \qquad (29)$$

for the function $L_i(\theta)$. The information of the scattering matrix is captured in the kernel $\varphi_{ij}(\theta) = -id \ln S_{ij}(\theta)/d\theta$ of the rapidity convolution, which is denoted as usual by $f * g(\theta) := \int d\theta'/2\pi\, f(\theta - \theta')g(\theta')$. The dimensionless parameter $r = m_1 T^{-1}$ is the inverse temperature T times the overall mass scale of the lightest particle m_1. Also all masses have been normalized in this way, i.e. $\hat{m}_i = m_i/m_1$. Having determined the $L_i(\theta)$-functions, we may compute the scaling function by means of

$$c'(r) = \frac{3\,r}{\pi^2} \sum_i \hat{m}_i \int_0^\infty d\theta \cosh\theta\, L_i(\theta) \;. \qquad (30)$$

Once again $\lim_{r\to 0} c'(r)$ is the ultraviolet Virasoro central charge. We would like to recall here that the scaling functions $c(r)$ and $c'(r)$ are not identical, but contain qualitatively the same information in the RG sense.

In order to carry out this analysis we need to know in (29) the kernel $\varphi(\theta)$ as input. For the model under consideration we can exploit the factorization property (18) for a finite product and trivially obtain

$$\varphi_N(\theta) = \sum_{n=-N}^{N} \varphi_{\rm SG}(\theta + n\omega + \sigma)\,, \qquad (31)$$

where $\varphi_{\rm SG}(\theta)$ is the sinh-Gordon kernel, e.g. [23]

$$\varphi_{\rm SG}(\theta) = \frac{4\sin(\pi x)\cosh\theta}{\cosh(2\theta) - \cos(2\pi x)}\,. \qquad (32)$$

Using alternatively the representation of the S-matrix (18) in terms of elliptic functions, we compute the kernel directly to

$$\lim_{N\to\infty} \varphi_N(\theta) = \frac{K_\ell}{\pi} \sum_{k=-,+} \left[\frac{{\rm dc}\,\theta_k}{{\rm sn}\,\theta_k} + \ell(1-\ell)\frac{{\rm sn}\,\theta_k}{{\rm dc}\,\theta_k} \right] \;. \qquad (33)$$

With these expression we carry out our numerical analysis, that is we solve iteratively the equation (29) and evaluate (30) thereafter. The results of this investigations are presented in figure 7.

Unfortunately for very small values of ω and r our numerical iteration procedure does not converge reliably. However, we will be content at this stage with the data obtained so far, since they already support qualitative our c-theorem analysis. They confirm that above threshold the scaling function is monotonically decreasing as a function of ω and also that values greater than 1 may be reached, even for finite values of r.

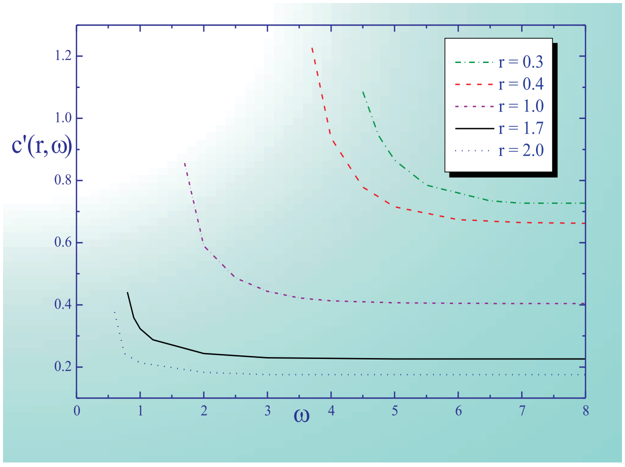

Figure 7: TBA scaling function.

IV. CONCLUSIONS

Starting from a given scattering matrix of hyperbolic type, we have demonstrated that it is possible to include consistently an arbitrary number of unstable particles into the spectrum of the theory. In particular when this number becomes infinite the S-matrix may be expressed in terms of elliptic functions.

For the generalization of the sinh-Gordon model we computed RG scaling functions. Within these analysis we found clear evidence for the thresholds which constrain the masses of the unstable particles. Above threshold, the values the ultraviolet Virasoro central charges may take are between 1 and 2 (possibly slightly greater than 2) and not between 0 and 1 as suggested in [5]. The theories are consistent for each finite value of N. For fixed resonance parameters ω and σ the scaling functions are non-decreasing for increasing N.

Concerning the investigation of the c-theorem, it would be desirable to refine the analysis. In particular one should include higher n-particle form factors into the expansion. For the elliptic version some of them were already presented in [5], but in general it remains a challenge to find closed expressions for arbitrary particle numbers. At present the TBA analysis is the least conclusive exploration and deserves further consideration in future. In particular the regions of ω and r, which were not accessible to us, should be explored and might possibly lead to a further more concrete indication of the thresholds also in this context. In regard to this, it will be useful develop existence criteria for the solution of the TBA equations analogue to the one derived in [23]. The one presented in there can not be taken over directly, since it makes use of the fact that $\int d\theta |\varphi(\theta)|$ equals 2π, whereas for the model investigated here this is $2\pi N$. It would be desirable to develop analytic approximations for the TBA solutions in the extreme ultraviolet limit, i.e. $r = 0$, similar to the ones already existing for theories with different characteristic features.

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