Eigenvalue estimates of the Dirac operator depending on the Ricci tensor. *

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Abstract

We prove a new lower bound for the first eigenvalue of the Dirac operator on a compact Riemannian spin manifold by refined Weitzenböck techniques. It applies to manifolds with harmonic curvature tensor and depends on the Ricci tensor. Examples show how it behaves compared to other known bounds.

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0 Introduction

If M^n is a compact Riemannian spin manifold with positive scalar curvature R, then each eigenvalue λ of the Dirac operator D satisfies the inequality

$$\lambda^2 \geq \frac{nR_0}{4(n-1)} \ , \tag{1}$$

where R_0 is the minimum of R on M^n. The estimate (1) is sharp in the sense that there exist manifolds for which (1) is an equality for the first eigenvalue λ_1 of D. If this is the case, then each eigenspinor ψ corresponding to λ_1 is a Killing spinor with the Killing number λ_1/n, i.e., ψ is a solution of the field equation

$$\nabla_X\psi + \frac{\lambda_1}{n} X \cdot \psi = 0 \tag{2}$$

and M^n must be an Einstein space (see [7]). A generalization of this inequality was proved in the paper [10], where a conformal lower bound for the spectrum of the Dirac operator occured. Moreover, for special Riemannian manifolds better estimates for the eigenvalues of the Dirac operator are known, see [11], [12]. However, all these estimates of the spectrum of the Dirac operators depend only on the scalar curvature of the underlying

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manifold. Therefore it is a natural question whether or not one may relate the spectrum of the Dirac operator to more refined curvature data.

In this paper we shall prove an estimate depending on the Ricci tensor for the eigenvalues of the Dirac operator on compact Riemannian manifolds with harmonic curvature tensor. The main idea is the investigation of the differential operators

$$Q^t : \Gamma(S) \to \Gamma(TM^n \otimes S)$$

depending on a real parameter $t \in \mathbb{R}$ and defined by

$$Q^t_X \psi := \nabla_X D\psi + \tfrac{1}{n} X \cdot D^2\psi + t \cdot (\mathrm{Ric} - \tfrac{R}{n})(X) \cdot \psi,$$

where Ric denotes the Ricci tensor. Under the assumption that the curvature tensor is harmonic we prove a formula expressing the length $|Q^t\psi|^2$ by the Dirac operator $D\psi$, the covariant derivatives $\nabla D\psi$ and $\nabla\psi$ as well as by some curvature terms (Theorem 1.6). Integrating this formula we obtain, for any $t \geq 0$, an inequality for the eigenvalues of the Dirac operator depending on the scalar curvature, the minimum of the eigenvalues of the Ricci tensor and its length. An optimal choice of the parameter t bounds the spectrum of the Dirac operator from below. For example, we prove the inequality

$$\lambda^2 > \frac{1}{4} \cdot \frac{|\mathrm{Ric}|_0^2}{|\mathrm{Ric}|_0 \sqrt{\frac{n-1}{n}} + |\kappa_0|}$$

for compact Riemannian spin manifolds with harmonic curvature tensor and vanishing scalar curvature, where κ_0 and $|\mathrm{Ric}|_0$ denote the minimum of the eigenvalues and the length of the Ricci tensor, respectively.

1 The Weitzenböck formula for the operator Q^t

First of all let us fix some notations. In the following $(X_1, \ldots, X_n)$ is always any local frame of vector fields, and $(X^1, \ldots, X^n)$ is the associated frame defined by $X^k := g^{kl} X_l$, where the g^{kl} denote the components of the inverse of the Riemannian metric $(g_{kl}) := (g(X_k, X_l))$. Using the twistor operator (see [1], Section 1.4)

$$\mathcal{D} : \Gamma(S) \longrightarrow \Gamma(TM^n \otimes S)$$

locally given by $\mathcal{D}\psi := X^k \otimes \mathcal{D}_{X_k}\psi$ and $\mathcal{D}_X\psi \; := \; \nabla_X\psi + \frac{1}{n} X \cdot D\psi$, we may rewrite the operator Q^t as

$$Q^t\psi \;=\; \mathcal{D}D\psi + t \cdot X^k \otimes (\mathrm{Ric} - \tfrac{R}{n})(X_k) \cdot \psi. \tag{3}$$

The image of the twistor operator $\mathcal{D}$ is contained in the kernel of the Clifford multiplication $\mu : TM^n \otimes S \to S$, i.e.,

$$\mu(\mathcal{D}\psi) \;=\; X^k \cdot \mathcal{D}_{X_k}\psi = 0\ . \tag{4}$$

As endomorphisms acting on the spinor bundle the following identities are well known:

$$X^k \cdot \mathrm{Ric}(X_k) \;=\; \mathrm{Ric}(X_k) \cdot X^k \;=\; -R, \quad X^k \cdot (\mathrm{Ric} - \tfrac{R}{n})(X_k) \;=\; 0\ . \tag{5}$$

In particular, we see that the image of the operators Q^t is contained in the kernel of the Clifford multiplication. By definition, a spinor field ψ belongs to the kernel of the operator Q^t if and only if it satisfies the equation

$$\nabla_X D\psi + \tfrac{1}{n} X \cdot D^2\psi + t \cdot (\mathrm{Ric} - \tfrac{R}{n})(X) \cdot \psi \;=\; 0 \tag{6}$$

for each vector field X. In the following we shall use the Weitzenböck formula

$$|\mathcal{D}\psi|^2 \;=\; |\nabla\psi|^2 - \tfrac{1}{n}|D\psi|^2 \tag{7}$$

for the twistor operator $\mathcal{D}$.

Lemma 1.1: *For any spinor field $\psi \in \Gamma(S)$, the following formula holds:*

$$\begin{aligned} |Q^t\psi|^2 \;&=\; |\nabla D\psi|^2 - \tfrac{1}{n}|D^2\psi|^2 + 2t \cdot \tfrac{R}{n} \cdot \mathrm{Re}(\langle D^2\psi, \psi\rangle) + t^2 \cdot |\mathrm{Ric} - \tfrac{R}{n}|^2 \cdot |\psi|^2 \\ &\quad - 2t \cdot \mathrm{Re}\Big(\langle \mathrm{Ric}(X^k) \cdot \nabla_{X_k} D\psi,\, \psi\rangle\Big) \end{aligned} \tag{8}$$

Proof: Using the formulas (3), (4) and (7) we have

$$\begin{aligned} |Q^t\psi|^2 \;&=\; \langle Q^t_{X_k}\psi, Q^t_{X^k}\psi\rangle \\ &\overset{(3)}{=} \Big\langle \mathcal{D}_{X_k} D\psi + t(\mathrm{Ric} - \tfrac{R}{n})(X_k) \cdot \psi, \mathcal{D}_{X^k} D\psi + t(\mathrm{Ric} - \tfrac{R}{n})(X^k) \cdot \psi \Big\rangle \\ &= |\mathcal{D}D\psi|^2 - 2t \cdot \mathrm{Re}\left(\Big\langle (\mathrm{Ric} - \tfrac{R}{n})(X^k)\mathcal{D}_{X_k} D\psi, \psi\Big\rangle\right) + t^2 \cdot |\mathrm{Ric} - \tfrac{R}{n}|^2 \cdot |\psi|^2 \\ &\overset{(4)}{=} |\mathcal{D}D\psi|^2 + t^2 \cdot |\mathrm{Ric} - \tfrac{R}{n}|^2 \cdot |\psi|^2 - 2t \cdot \mathrm{Re}\left(\langle \mathrm{Ric}(X^k) \cdot \mathcal{D}_{X_k} D\psi, \psi\rangle\right) \\ &\overset{(7)}{=} |\nabla D\psi|^2 - \tfrac{1}{n}|D^2\psi|^2 + t^2 \cdot \left|\mathrm{Ric} - \tfrac{R}{n}\right|^2 \cdot |\psi|^2 + 2t \cdot \tfrac{R}{n} \cdot \mathrm{Re}\left(\langle D^2\psi, \psi\rangle\right) \\ &\quad -2t \cdot \mathrm{Re}\left(\langle \mathrm{Ric}(X^k) \cdot \nabla_{X_k} D\psi, \psi\rangle\right). \end{aligned}$$

□

Equation (8) is a preliminary version of the Weitzenböck formula, which we will apply in the proof of our main result. Our next aim is to express the uncontrollable last term on the right-hand side by terms that are controllable. For this purpose we need a condition on the covariant derivative of the Ricci tensor. For vector fields X, Y, we use the notation

$$\nabla_{X,Y} := \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$$

for the corresponding tensorial derivatives of second order in TM^n as well as in S. By K we denote the Riemannian curvature tensor and by C the curvature tensor in the spinor bundle S. Then, for all $X, Y, Z \in \Gamma(TM^n)$ and all $\psi \in \Gamma(S)$, we have

$$K(X,Y)(Z) \;=\; \nabla_{X,Y} Z - \nabla_{Y,X} Z \quad , \quad C(X,Y)\psi \;=\; \nabla_{X,Y}\psi - \nabla_{Y,X}\psi$$

as well as the well known relation between the two curvatures

$$C(X,Y) \cdot \psi \;=\; \tfrac{1}{4} X^k \cdot K(X,Y)(X_k) \cdot \psi \;=\; \tfrac{1}{4} g(K(X,Y)(X^k), X^l) X_k \cdot X_l \cdot \psi\;. \tag{9}$$

Considering C as a map from $\Gamma(S)$ to $\Gamma(TM^n \otimes TM^n \otimes S)$ locally defined by

$$C\psi \;:=\; X^k \otimes X^l \otimes C(X_k, X_l)\psi,$$

the length $|C\psi|^2$ is just the scalar product

$$|C\psi|^2 = \langle C(X_k, X_l)\psi, C(X^k, X^l)\psi\rangle.$$

Moreover, for two spinor fields ψ, φ we introduce a complex vector field $\langle C\psi, \nabla\varphi\rangle$ defined by the formula

$$\langle C\psi, \nabla\varphi\rangle \;:=\; \langle C(X^k, X^l)\psi, \nabla_{X_l}\varphi\rangle \cdot X_k \; .$$

Lemma 1.2: *For any $\psi \in \Gamma(S)$, we have the equation*

$$\begin{aligned} \Big\langle C(X^k, X^l)\nabla_{X_k}\psi, \nabla_{X_l}\psi\Big\rangle \;&=\; \mathrm{div}\Big\langle C\psi, \nabla\psi\Big\rangle - \tfrac{1}{2}\Big|C\psi\Big|^2 \\ &+ \tfrac{1}{4}\cdot\Big\langle \psi, \Big((\nabla_{X_k}\mathrm{Ric})(X^l)\cdot X^k - X^k\cdot(\nabla_{X_k}\mathrm{Ric})(X^l)\Big)\nabla_{X_l}\psi\Big\rangle. \end{aligned} \tag{10}$$

Proof: Let $x \in M^n$ be any point and let $(X_1, \ldots, X_n)$ be any orthonormal frame in a neighbourhood of the point such that $(\nabla X_k)_x = 0$ holds for $k = 1, \ldots, n$. Then we have at $x \in M^n$ that

$$\begin{aligned} &\langle C(X^k, X^l)\nabla_{X_k}\psi, \nabla_{X_l}\psi\rangle \;=\; -\langle \nabla_{X_k}\psi, C(X^k, X^l)\nabla_{X_l}\psi\rangle \\ &= -X_k(\langle\psi, C(X^k, X^l)\nabla_{X_l}\psi\rangle) + \langle\psi, (\nabla_{X_k}C)(X^k, X^l)\nabla_{X_l}\psi\rangle + \langle\psi, C(X^k, X^l)\nabla_{X_k}\nabla_{X_l}\psi\rangle \\ &= X_k(\langle C(X^k, X^l)\psi, \nabla_{X_l}\psi\rangle) + \langle\psi, (\nabla_{X_k}C)(X^k, X^l)\nabla_{X_l}\psi\rangle + \tfrac{1}{2}\langle\psi, C(X^k, X^l)C(X_k, X_l)\psi\rangle \\ &= \;\mathrm{div}\langle C\psi, \nabla\psi\rangle + \langle\psi, (\nabla_{X_k}C)(X^k, X^l)\nabla_{X_l}\psi\rangle - \tfrac{1}{2}|C\psi|^2 \;, \end{aligned}$$

and we obtain the following formula for the left-hand side of the expression (10)

$$\langle C(X^k, X^l)\nabla_{X^k}\psi, \nabla_{X_l}\psi\rangle = \mathrm{div}\langle C\psi, \nabla\psi\rangle - \tfrac{1}{2}|C\psi|^2 + \langle\psi, (\nabla_{X_k}C)(X^k, X^l)\nabla_{X_l}\psi\rangle.$$

On the other hand, from (9) we obtain $(\nabla_Z C)(X, Y) = \frac{1}{4}\cdot X^k\cdot(\nabla_Z K)(X, Y)(X_k)$ and

$$(\nabla_{X_k}C)(X^k, X^l) \;=\; \frac{1}{4}\cdot g\Big((\nabla_{X_k}K)(X^k, X^l)(X^i), X^j\Big)X_i\cdot X_j \; .$$

The Bianchi identity

$$(\nabla_X K)(Y, Z) + (\nabla_Y K)(Z, X) + (\nabla_Z K)(X, Y) \;=\; 0$$

implies the relation

$$g((\nabla_{X_k}K)(X^k, X)(Y), Z) \;=\; g((\nabla_Z\mathrm{Ric})(Y) - (\nabla_Y\mathrm{Ric})(Z), X) \; .$$

The latter two equations yield

$$\begin{aligned} (\nabla_{X_k}C)(X^k, X^l) \;&=\; \tfrac{1}{4}g\Big((\nabla_{X^j}\mathrm{Ric})(X^i) - (\nabla_{X^i}\mathrm{Ric})(X^j), X^l\Big)X_i\cdot X_j \\ &=\; \tfrac{1}{4}\Big(g\Big(X^i, (\nabla_{X^j}\mathrm{Ric})(X^l)\Big) - g\Big(X^j, (\nabla_{X_i}\mathrm{Ric})(X^l)\Big)\Big)X_i\cdot X_j \\ &=\; \tfrac{1}{4}\Big((\nabla_{X_j}\mathrm{Ric})(X^l)\cdot X^j - X^i\cdot(\nabla_{X_i}\mathrm{Ric})(X^l)\Big). \end{aligned}$$

Inserting this formula we obtain (10). □

In the following we use the Schrödinger-Lichnerowicz formula

$$\nabla^*\nabla \;=\; D^2 - \tfrac{1}{4}R\ . \tag{11}$$

The local expression of the Bochner Laplacian $\nabla^*\nabla$ is

$$\nabla^*\nabla \;=\; -\nabla_{X_k,X^k} \;=\; -\nabla_{X_k}\nabla_{X^k} + {\Gamma_k}^{kl}\nabla_{X_l}\ , \tag{12}$$

where the Christoffel symbols ${\Gamma_{ij}}^k$ are defined by $\nabla_{X_i}X_j = {\Gamma_{ij}}^k X_k$. In the proof of the following lemma we also use the well known general formulas

$$X^k\cdot\nabla_{X,X_k}\psi \;=\; \nabla_X D\psi\ , \tag{13}$$

$$X^k\cdot\nabla_{X_k,X}\psi \;=\; D\nabla_X\psi - X^k\cdot\nabla_{\nabla_{X_k}X}\psi \;=\; \nabla_X D\psi + \tfrac{1}{2}\mathrm{Ric}(X)\cdot\psi. \tag{14}$$

Moreover, for $\psi,\varphi\in\Gamma(S)$, let $\psi\varphi$ and $\langle\psi,\nabla\varphi\rangle$ be the complex vector fields on M^n locally given by

$$\psi\varphi \;:=\; i\cdot\langle\psi, X^k\cdot\varphi\rangle\cdot X_k \quad , \quad \langle\psi,\nabla\varphi\rangle \;:=\; \langle\psi,\nabla_{X^k}\varphi\rangle\cdot X_k\ .$$

The vector field satisfies the relation

$$i\cdot\mathrm{div}(\psi\varphi) \;=\; \langle D\psi,\varphi\rangle - \langle\psi, D\varphi\rangle. \tag{15}$$

Lemma 1.3: *Let ψ be any spinor field. Then there is the identity*

$$\begin{aligned}
&\Big\langle C(X^k,X^l)\nabla_{X^k}\psi,\nabla_{X_l}\psi\Big\rangle = -\mathrm{Re}\left(\Big\langle \mathrm{Ric}(X^k)\nabla_{X_k}D\psi,\psi\Big\rangle\right) + \Big\langle\nabla_{\mathrm{Ric}(X^k)}\psi,\nabla_{X^k}\psi\Big\rangle\\
&+\tfrac{1}{4}\Big|\mathrm{Ric}\Big|^2\cdot\Big|\psi\Big|^2 - \tfrac{1}{2}\Big|C\psi\Big|^2 + \Big|\nabla D\psi\Big|^2 - \tfrac{R}{4}\cdot\Big|\nabla\psi\Big|^2 - \Big|(D^2-\tfrac{R}{4})\psi\Big|^2\\
&+\mathrm{div}\Big(i\left(\nabla_{X_k}D\psi + \tfrac{1}{2}\mathrm{Ric}(X_k)\cdot\psi\right)(\nabla_{X_k}\psi) + \Big\langle C\psi,\nabla\psi\Big\rangle - \Big\langle (D^2-\tfrac{R}{4})\psi,\nabla\psi\Big\rangle\Big).
\end{aligned}\tag{16}$$

Proof: Let $x\in M^n$ be any point and let $(X_1,\ldots,X_n)$ be any orthonormal frame in a neighbourhood of x such that $(\nabla X_k)_x = 0$ for $k=1,\ldots,n$. We use the notations

$$R_{ijkl} \;:=\; g(K(X_i,X_j)(X_k),X_l),\quad R_{ij} \;:=\; g(\mathrm{Ric}(X_i),X_j) = {R_{ik}}^k{}_j\ .$$

Then, we have ${\Gamma_{ij}}^k = 0$ at the point x and

$$R_{ijk}{}^l = X_i({\Gamma_{jk}}^l) - X_j({\Gamma_{ik}}^l),\quad R_i{}^j = X_i({\Gamma_k}^{kj}) - X_k({\Gamma_i}^{kj})\ . \tag{$*$}$$

Using this we calculate

$$\begin{aligned}
\Big\langle C(X^k,X^l)\nabla_{X_k}\psi,\nabla_{X_l}\psi\Big\rangle \;&=\; \Big\langle\nabla_{X^k}\nabla_{X^l}\nabla_{X_k}\psi - \nabla_{X^l}\nabla_{X^k}\nabla_{X_k}\psi,\nabla_{X_l}\psi\Big\rangle\\
=\;&\Big\langle\nabla_{X^k}\nabla_{X^l}\nabla_{X_k}\psi,\nabla_{X_l}\psi\Big\rangle + \Big\langle\nabla_{X^l}(-\nabla_{X^k}\nabla_{X_k}\psi),\nabla_{X_l}\psi\Big\rangle\\
=\;&\Big\langle\nabla_{X^k}(\nabla_{X^l,X_k}\psi + {\Gamma_{lk}}^p\nabla_{X_p}\psi),\nabla_{X^l}\psi\Big\rangle + \Big\langle\nabla_{X_l}(-\nabla_{X^k,X_k}\psi - {\Gamma_k}^{kp}\nabla_{X_p}\psi),\nabla_{X^l}\psi\Big\rangle\\
\overset{(11)(12)}{=}\;&\Big\langle\nabla_{X^k}(C(X_l,X_k)\psi + \nabla_{X_k,X_l}\psi),\nabla_{X^l}\psi\Big\rangle + X_k({\Gamma_{lk}}^p)\Big\langle\nabla_{X_p}\psi,\nabla_{X^l}\psi\Big\rangle\\
&+\Big\langle\nabla_{X_l}((D^2-\tfrac{R}{4})\psi),\nabla_{X^l}\psi\Big\rangle - X_l({\Gamma_k}^{kp})\Big\langle\nabla_{X_p}\psi,\nabla_{X^l}\psi\Big\rangle\\
=\;&\Big\langle\nabla_{X_k}(C(X^l,X^k)\psi),\nabla_{X_l}\psi\Big\rangle + \Big\langle\nabla_{X^k}\nabla_{X_k,X_l}\psi,\nabla_{X^l}\psi\Big\rangle + (X_k({\Gamma_l}^{kp}) - X_l({\Gamma_k}^{kp}))\cdot
\end{aligned}$$

$$
\begin{aligned}
&\Big\langle\nabla_{X_p}\psi,\nabla_{X^l}\psi\Big\rangle+X_l\Big(\Big\langle(D^2-\tfrac{R}{4})\psi,\nabla_{X^l}\psi\Big\rangle\Big)-\Big\langle(D^2-\tfrac{R}{4})\psi,\nabla_{X_l}\nabla_{X^l}\psi\Big\rangle\\
\overset{(*)}{=}\ & X_k\Big(\Big\langle C(X^l,X^k)\psi,\nabla_{X_l}\psi\Big\rangle\Big)-\Big\langle C(X^l,X^k)\psi,\nabla_{X_k}\nabla_{X_l}\psi\Big\rangle\\
&+\Big\langle\nabla_{X^k}(\nabla_{X_k}\nabla_{X_l}\psi-\Gamma_{kl}{}^p\nabla_{X_p}\psi),\nabla_{X^l}\psi\Big\rangle-R_l{}^p\Big\langle\nabla_{X_p}\psi,\nabla_{X^l}\psi\Big\rangle\\
&+\operatorname{div}\Big\langle(D^2-\tfrac{R}{4})\psi,\nabla\psi\Big\rangle+\Big|(D^2-\tfrac{R}{4})\psi\Big|^2\\
=\ &-\operatorname{div}\Big\langle C\psi,\nabla\psi\Big\rangle+\tfrac12|C\psi|^2-\Big\langle(D^2-\tfrac{R}{4})\nabla_{X_k}\psi,\nabla_{X^k}\psi\Big\rangle-X^k(\Gamma_{kl}{}^p)\Big\langle\nabla_{X_p}\psi,\nabla_{X^l}\psi\Big\rangle\\
&-\Big\langle\nabla_{\mathrm{Ric}(X^k)}\psi,\nabla_{X_k}\psi\Big\rangle+\operatorname{div}\Big\langle(D^2-\tfrac{R}{4})\psi,\nabla\psi\Big\rangle+\Big|(D^2-\tfrac{R}{4})\psi\Big|^2.
\end{aligned}
$$

Hence, it holds that

$$
\begin{aligned}
&\Big\langle C(X^k,X^l)\nabla_{X_k}\psi,\nabla_{X_l}\psi\Big\rangle\ =\ \Big|(D^2-\tfrac{R}{4})\psi\Big|^2+\tfrac12\Big|C\psi\Big|^2-\Big\langle\nabla_{\mathrm{Ric}(X^k)}\psi,\nabla_{X_k}\psi\Big\rangle\\
&+X_k(\Gamma^{kpl})\cdot\Big\langle\nabla_{X_p}\psi,\nabla_{X_l}\psi\Big\rangle+\tfrac{R}{4}\cdot\Big|\nabla\psi\Big|^2-\Big\langle D^2\nabla_{X_l}\psi,\nabla_{X^l}\psi\Big\rangle\\
&+\operatorname{div}\Big(\Big\langle(D^2-\tfrac{R}{4})\psi,\nabla\psi\Big\rangle-\Big\langle C\psi,\nabla\psi\Big\rangle\Big)\ .
\end{aligned}
\tag{2*}
$$

Further, we have

$$
\begin{aligned}
&\Big\langle D^2\nabla_{X_l}\psi,\nabla_{X^l}\psi\Big\rangle\ =\ \Big\langle D(D\nabla_{X_l}\psi),\nabla_{X^l}\psi\Big\rangle\\
\overset{(14)}{=}\ &\Big\langle D(\nabla_{X_l}D\psi+\tfrac12\mathrm{Ric}(X_l)\cdot\psi+\Gamma_{kl}{}^pX^k\cdot\nabla_{X_p}\psi),\nabla_{X^l}\psi\Big\rangle\\
=\ &\Big\langle D(\nabla_{X_l}D\psi+\tfrac12\mathrm{Ric}(X_l)\cdot\psi),\nabla_{X^l}\psi\Big\rangle+X_q(\Gamma_{kl}{}^p)\Big\langle X^q\cdot X^k\cdot\nabla_{X_p}\psi,\nabla_{X^l}\psi\Big\rangle\\
\overset{(15)}{=}\ & i\cdot\operatorname{div}\Big((\nabla_{X_l}D\psi+\tfrac12\mathrm{Ric}(X_l)\cdot\psi)(\nabla_{X^l}\psi)\Big)+\Big\langle\nabla_{X_l}D\psi+\tfrac12\mathrm{Ric}(X_l)\cdot\psi,D\nabla_{X_l}\psi\Big\rangle\\
&+\Big(\tfrac12(X_q(\Gamma_{kl}{}^p)-X_k(\Gamma_{ql}{}^p))+\tfrac12(X_q(\Gamma_{kl}{}^p)+X_k(\Gamma_{ql}{}^k))\Big)\Big\langle X^q\cdot X^k\cdot\nabla_{X_p}\psi,\nabla_{X^l}\psi\Big\rangle\\
\overset{(15),(*)}{=}\ & i\cdot\operatorname{div}\Big((\nabla_{X_l}D\psi+\tfrac12\mathrm{Ric}(X_l)\cdot\psi)(\nabla_{X^l}\psi)\Big)\\
&+\Big\langle\nabla_{X_l}D\psi+\tfrac12\mathrm{Ric}(X_l)\cdot\psi,\nabla_{X^l}D\psi+\tfrac12\mathrm{Ric}(X^l)\cdot\psi\Big\rangle\\
&+\tfrac12R_{qkl}{}^p\langle X^q\cdot X^k\cdot\nabla_{X_p}\psi,\nabla_{X^l}\psi\rangle-X_k(\Gamma^{klp})\langle\nabla_{X_p}\psi,\nabla_{X_l}\psi\rangle\\
=\ & i\cdot\operatorname{div}\Big((\nabla_{X_l}D\psi+\tfrac12\mathrm{Ric}(X_l)\cdot\psi)(\nabla_{X^l}\psi)\Big)+|\nabla D\psi|^2-\mathrm{Re}\Big(\langle\mathrm{Ric}(X^l)\nabla_{X_l}D\psi,\psi\rangle\Big)\\
&+\tfrac14|\mathrm{Ric}|^2|\psi|^2-\tfrac12R^{plqk}\langle X_q\cdot X_k\cdot\nabla_{X_p}\psi,\nabla_{X^l}\psi\rangle+X_k(\Gamma^{kpl})\langle\nabla_{X_p}\psi,\nabla_{X_l}\psi\rangle\\
\overset{(9)}{=}\ & i\cdot\operatorname{div}\Big((\nabla_{X_l}D\psi+\tfrac12\mathrm{Ric}(X_l)\cdot\psi)(\nabla_{X^l}\psi)\Big)+|\nabla D\psi|^2+\tfrac14|\mathrm{Ric}|^2\cdot|\psi|^2\\
&-\mathrm{Re}\Big(\langle\mathrm{Ric}(X^l)\nabla_{X_l}D\psi,\psi\rangle\Big)+X_k(\Gamma^{kpl})\langle\nabla_{X_p}\psi,\nabla_{X_l}\psi\rangle-2\Big\langle C(X^p,X^l)\nabla_{X_p}\psi,\nabla_{X_l}\psi\Big\rangle\ .
\end{aligned}
$$

Inserting the latter equation into (2∗) we obtain (16). □

Comparing the equations (10) and (16) we obtain immediately

Lemma 1.4: *For any spinor field ψ, we have the identity*

$$\begin{aligned} & \mathrm{Re}\Big(\Big\langle \mathrm{Ric}(X^k)\nabla_{X_k} D\psi, \psi\Big\rangle\Big) + \tfrac{1}{4}\Big\langle \psi, \Big((\nabla_{X_k}\mathrm{Ric})(X^l)X^k - X^k(\nabla_{X_k}\mathrm{Ric})(X^l)\Big)\nabla_{X_l}\psi\Big\rangle \\ (17) \quad & = \Big|\nabla D\psi\Big|^2 - \Big|(D^2 - \tfrac{R}{4})\psi\Big|^2 - \tfrac{R}{4}\Big|\nabla\psi\Big|^2 + \tfrac{1}{4}\Big|\mathrm{Ric}\Big|^2\Big|\psi\Big|^2 + \Big\langle \nabla_{\mathrm{Ric}(X^k)}\psi, \nabla_{X_k}\psi\Big\rangle \\ & \quad + \mathrm{div}\Big(i(\nabla_{X_k} D\psi + \tfrac{1}{2}\mathrm{Ric}(X_k)\psi)(\nabla_{X^k}\psi) - \Big\langle (D^2 - \tfrac{R}{4})\psi, \nabla\psi\Big\rangle\Big) \ . \end{aligned}$$

The following purely algebraic condition on the covariant derivative of the Ricci tensor implies that the second term in formula (17) vanishes. The proof is an easy computation using the relations in the Clifford algebra. A thorough geometric discussion of this condition will be provided in Section 2.

Lemma 1.5: *If the covariant derivative of the Ricci tensor satisfies*

$$(\nabla_X \mathrm{Ric})(Y) \;=\; (\nabla_Y \mathrm{Ric})(X) \ ,$$

then, for any spinor field ψ and any vector field Y, the Clifford product

$$\Big((\nabla_{X_k}\mathrm{Ric})(Y)\cdot X^k - X^k\cdot(\nabla_{X_k}\mathrm{Ric})(Y)\Big)\cdot\psi \;=\; 0$$

vanishes.

We thus obtain the following Weitzenböck formula for the length $|Q^t\psi|^2$, which is fundamental for all our further considerations.

Theorem 1.6: *Let M^n be a Riemannian spin manifold and suppose that*

$$(\nabla_X \mathrm{Ric})(Y) \;=\; (\nabla_Y \mathrm{Ric})(X) \ .$$

Then, for any spinor field ψ, there exists a vector field $X_\psi \in \Gamma(TM^n)$ such that

$$\begin{aligned} \Big|Q^t\psi\Big|^2 \;&=\; \Big|\nabla D\psi\Big|^2 - \tfrac{1}{n}\cdot\Big|D^2\psi\Big|^2 + t^2\cdot\Big|\mathrm{Ric} - \tfrac{R}{n}\Big|^2\cdot\Big|\psi\Big|^2 + 2t\cdot\Big(\tfrac{R}{n}\cdot\mathrm{Re}(\langle D^2\psi,\psi\rangle) \\ (18) \quad & \quad + \Big|(D^2 - \tfrac{R}{4})\psi\Big|^2 + \tfrac{R}{4}\cdot\Big|\nabla\psi\Big|^2 - \Big|\nabla D\psi\Big|^2 - \tfrac{1}{4}\cdot\Big|\mathrm{Ric}\Big|^2\Big|\psi\Big|^2 \\ & \quad - \Big\langle \nabla_{\mathrm{Ric}(X^k)}\psi, \nabla_{X_k}\psi\Big\rangle + \mathrm{div}(X_\psi)\Big) \ . \end{aligned}$$

Proof: The formula follows from (8) and (17) if one defines X_ψ locally by

$$X_\psi \;:=\; \mathrm{Re}\Big(\Big\langle (D^2 - \tfrac{R}{4})\psi, \nabla\psi\Big\rangle - i\Big(\nabla_{X_k}D\psi + \tfrac{1}{2}\mathrm{Ric}(X_k)\cdot\psi\Big)\Big(\nabla_{X^k}\psi\Big)\Big). \qquad \square$$

2 A mini-max principle for the estimate of the eigenvalues

In this section we assume that M^n is compact, connected and that the Ricci tensor satisfies the condition

$$(\nabla_X \mathrm{Ric})(Y) \;=\; (\nabla_Y \mathrm{Ric})(X) \ .$$

It is an easy consequence of the Bianchi idendity that the scalar curvature of the manifold must be constant. Then the tensor

$$T(X) \;:=\; \frac{1}{n-2}\Big(\frac{R}{2(n-1)}\cdot X - \mathrm{Ric}(X)\Big)$$

has the same properties as the Ricci tensor. In dimension $n = 3$ the manifold is conformally flat. If $n \geq 4$, we obtain the identity

$$(\nabla_{X_k} W)(X, Y, X^k) \;=\; (n-3)\cdot((\nabla_X T)(Y) - (\nabla_Y T)(X))$$

by computing the divergence of the Weyl tensor W (see [16]). Therefore, the manifold satisfies the mentioned condition for the Ricci tensor if and only if it has constant scalar curvature and a harmonic Weyl tensor. Moreover, these two properties are equivalent to the condition that the curvature tensor is harmonic (see Chapter 16 in [2]). The following examples are known:

1. Local products of Einstein manifolds;
2. conformally flat manifolds with constant scalar curvature;
3. warped products $S^1 \times_{f^2} N^{n-1}$ of an Einstein manifold with positive scalar curvature $R = 4(n-1)/n$ by S^1 (see [4], [5], [6]), where the function $F := f^{n/2}$ is a positive, periodic solution of the differential equation

$$F'' \;-\; F^{1-\frac{4}{n}} \;=\; -F\;;$$

4. warped products over Riemann surfaces.

We denote by $\kappa_1(x) \leq \kappa_2(x) \leq \ldots \leq \kappa_n(x)$ the eigenvalues of the Ricci tensor at the point $x \in M^n$ and by κ_0 the minimum of κ_1. If $D\psi = \lambda\psi$ is an eigenspinor, then (18) yields the inequality

$$\int_{M^n}\Big(\big(\frac{n-1}{n}\lambda^4 - \frac{R}{4}\lambda^2 + 2t(\frac{R}{n}\lambda^2 - \frac{1}{4}|\mathrm{Ric}|^2) + t^2|\mathrm{Ric} - \frac{R}{n}|^2\big)|\psi|^2 - 2t\big\langle\nabla_{\mathrm{Ric}(X^k)}\psi, \nabla_{X_k}\psi\big\rangle\Big) \;\geq\; 0\,.$$

In case of an Einstein manifold we get back the inequality (1). In general, the Schrödinger-Lichnerowicz formula and the estimation

$$\kappa_0|\nabla\psi|^2 \;\leq\; \langle\nabla_{\mathrm{Ric}(X^k)}\psi, \nabla_{X_k}\psi\rangle$$

imply the inequality

$$\kappa_0\cdot\int_{M^n}(\lambda^2 - \frac{R}{4})|\psi|^2 \;\leq\; \int_{M^n}\langle\nabla_{\mathrm{Ric}(X^k)}\psi, \nabla_{X_k}\psi\rangle$$

and, finally, for any $t \geq 0$ we obtain the condition

$$\lambda^2\Big(\lambda^2 - \frac{nR}{4(n-1)}\Big) + 2t\frac{n}{n-1}\Big(\frac{R}{n} - \kappa_0\Big)\Big(\lambda^2 - \frac{R}{4}\Big) + \frac{n}{n-1}\max_{x\in M^n}\Big[\Big(t^2 - \frac{t}{2}\Big)\Big|\mathrm{Ric} - \frac{R}{n}\Big|^2\Big] \;\geq\; 0\,.$$

This is a min-max principle and can be used in order to estimate the eigenvalues of the Dirac operator from below. Of course, only parameters between $0 \leq t \leq 1/2$ are interesting. A similar result involving only the scalar curvature was proved in [9]. For $\lambda = 0$ we immediately obtain the following result

Theorem 2.1: *Let M^n be a compact Riemannian spin manifold with harmonic curvature tensor. If κ_0 and $|\mathrm{Ric}|_0^2$ denote the minimum of the eigenvalues and the length of the Ricci tensor, respectively, and if*

$$|\mathrm{Ric}|_0^2 \ > \ R \cdot \kappa_0$$

holds, then there are no harmonic spinors.

If the scalar curvature is positive, we know that $\lambda^2 \geq nR/4(n-1)$ and the mini-max principle yields a better estimate only in case that the left-hand side is negative for $\lambda^2 = nR/4(n-1)$ and some $t > 0$. This condition is equivalent to

$$|\mathrm{Ric}|_0^2 \ > \ \frac{R}{n-1}(R - \kappa_0) \ ,$$

where κ_0 and $|\mathrm{Ric}|_0$ are the minimum of the eigenvalues and the length of the Ricci tensor, respectively

Example 1: The warped product $S^1 \times_{f^2} N^{n-1}$ of an Einstein manifold N^{n-1} with positive scalar curvature $R = 4(n-1)/n$ by S^1 never satisfies the latter condition. Solving the differential equation $F'' - F^{1-\frac{4}{n}} = -F$ for $n = 5$, $F'(0) = 0$ and $F(0) = 0.1$ we obtain, for example, $\kappa_0 = -8.5$ and $|\mathrm{Ric}|_0^2 = 2$. One series of examples which we can apply our inequality to consist of products $(S^1 \times_{f^2} N^{n-1}) \times \ldots \times (S^1 \times_{f^2} N^{n-1})$ with a sufficiently large number of factors. A second series are products $\Sigma^k \times (S^1 \times_{f^2} N^{n-1})$ by an Einstein manifold Σ^k with sufficiently large scalar curvature. We describe the case of a two-dimensional sphere Σ^2 and a 4-dimensional Einstein spin manifold N^4 with scalar curvature $R_N = 16/5$ in greater delail. Consider the positive, periodic solution $F = f^{5/2}$ of the differential equation $F'' - F^{1/5} = -F$ with initial values $F(0) = 0.1$ and $F'(0) = 0$. The Ricci tensor of the manifold $S^1 \times_{f^2} N^4$ has two eigenvalues

$$\kappa_1 \ = \ \frac{24}{25}\Big(\frac{F'}{F}\Big)^2 + \frac{8}{5}\Big(1 - F^{-\frac{4}{5}}\Big), \quad \kappa_2 \ = \ \frac{1}{4}\Big(\frac{16}{5} - \kappa_1\Big) \ .$$

The multiplicity of κ_1 is one, the multiplicity of κ_2 is four, the scalar curvature of the warped product equals $16/5$. Denote by R_Σ the scalar curvature of the sphere Σ^2 and consider the manifold $M^7 := \Sigma^2 \times (S^1 \times_{f^2} N^4)$. Then we have

$$|\mathrm{Ric}_M|_0^2 \ = \ \frac{R_\Sigma^2}{2} + |\mathrm{Ric}_{(S^1 \times_{f^2} N^4)}|_0^2 \ = \ \frac{R_\Sigma^2}{2} + 2, \quad R_M \ = \ R_\Sigma + \frac{16}{5}, \quad \kappa_0 \ = \ -8.5 \ .$$

For the optimal parameter $t = 0.212$ we obtain the estimate

$$\lambda^2 \ \geq \ -1.74873 + 0.1105 \cdot R_\Sigma + 0.235194 \cdot \sqrt{120.053 + 12.8828 \cdot R_\Sigma + R_\Sigma^2} \ \approx \ 0.3457 \cdot R_\Sigma \ ,$$

whereas the inequality (1) yields the estimate $\lambda^2 \geq \frac{7}{24} R_\Sigma + \frac{14}{15} \approx 0.29 \cdot R_\Sigma$. □

The discussion of the limiting case yields a spinor field ψ in the kernel of one of the operators Q^t with $t \geq 0$. Moreover, at every point we have

$$\kappa_1 |\nabla\psi|^2 \ = \ \langle \nabla_{\mathrm{Ric}(X^k)}\psi, \nabla_{X_k}\psi \rangle \ ,$$

i.e., the derivative of ψ vanish in all directions Y that are orthogonal to the κ_1-eigenspace of the Ricci tensor, $\nabla_Y\psi = 0$. The equation $Q^t\psi = 0$ means that the eigenspinor ψ satisfies the equation

$$\nabla_X\psi + \frac{\lambda}{n}X\cdot\psi + \frac{t}{\lambda}(\mathrm{Ric} - \frac{R}{n})(X)\cdot\psi \;=\; 0$$

for each vector field X. In particular, the length of the spinor field is constant. If $t = 0$, then ψ is a Killing spinor. In case $t > 0$, we consider the largest eigenvalue κ_n at a minimum $x_0 \in M^n$ of κ_1 and insert an eigenvector. Then we obtain

$$\lambda^2 + t\cdot(n\cdot\kappa_n(x_0) - R) \;=\; 0\;.$$

But $n\cdot\kappa_n(x_0) - R$ is positive, a contradiction. Thus the limiting case in the inequality cannot occur except that M^n is an Einstein manifold with a Killing spinor.

First we consider the case that the scalar curvature $R = 0$ vanishes. Then κ_0 is negative and for any positive t we have

$$\frac{n-1}{n}\lambda^4 - 2t\cdot\kappa_0\cdot\lambda^2 + \max_{x\in M^n}\Big[\Big(t^2 - \frac{t}{2}\Big)|\mathrm{Ric}|^2\Big] \;>\; 0\;.$$

An elementary discussion yields the proof of the following theorem.

Theorem 2.2: *Let M^n be a compact, non-Ricci flat Riemannian spin manifold with harmonic curvature tensor and vanishing scalar curvature. If κ_0 and $|\mathrm{Ric}|_0^2$ denote the minimum of the eigenvalues and the length of the Ricci tensor, respectively, then the eigenvalues of the Dirac operator are bounded by*

$$\lambda^2 > \frac{1}{4}\cdot\frac{|\mathrm{Ric}|_0^2}{|\mathrm{Ric}|_0\sqrt{\frac{n-1}{n}} + |\kappa_0|}\;.$$

Remark: The Schrödinger-Lichnerowicz formula implies the well known fact that a compact, non Ricci-flat Riemannian spin manifold with $R \equiv 0$ does not admit harmonic spinors. The estimate in Theorem 2.2 is a quantitative improvement of this fact for manifolds with harmonic curvature tensor and vanishing scalar curvature.

Example 2: Let $\Gamma \subset \mathrm{Conf}(S^n)$ be a geometrically finite Kleinian group of compact type and denote by $\Lambda(\Gamma)$ its limit set. Then $X^n(\Gamma) := (S^n - \Lambda(\Gamma))/\Gamma$ is a closed manifold equipped with a flat conformal structure. If the Hausdorff dimension of the limit set equals $(n-2)/2$, then there exists a Riemannian metric in the conformal class with vanishing scalar curvature (see [14]). S. Nayatani constructed this metric explicitly and studied its Ricci tensor ([13]).

Example 3: Let us continue Example 1. If Σ^2 is a compact surface with scalar curvature $R_\Sigma = -\frac{16}{5}$, then $M^7 := \Sigma^2 \times (S^1 \times_{f^2} N^4)$ has a harmonic curvature tensor and vanishing scalar curvature. Theorem 2.2 proves the estimate

$$\lambda^2 \;\geq\; 0.17833\;.$$

Example 4: If Σ^2 is a compact surface with scalar curvature $R_\Sigma = -4$, then $M^7 := \Sigma^2 \times (S^1 \times_{f^2} N^4)$ has a harmonic curvature tensor and *negative* scalar curvature. We apply the mini-max principle and obtain

$$\lambda^2 \;\geq\; 0.052 \;.$$

In particular, M^7 has no harmonic spinors.

3 An estimate of the eigenvalues

Let us introduce the short cuts

$$a \;:=\; \tfrac{nR}{8(n-1)}, \quad b \;:=\; \tfrac{n}{n-1}(\tfrac{R}{n} - \kappa_0), \quad c \;:=\; |\mathrm{Ric} - \tfrac{R}{n}|_0\sqrt{\tfrac{n}{n-1}}, \quad A \;:=\; \tfrac{c^2}{4} + 2\tfrac{n-1}{n}ab$$

as well as the new parameter $s := t/\lambda^2$. Then, by definition, b and c are non-negative and the condition $|\mathrm{Ric}|_0^2 \;>\; R \cdot \kappa_0$ of Theorem 2.1 is equivalent to $A > 0$. The mini-max principle yields immediately

$$\lambda^2\Big(\lambda^2 - 2a + 2bs(\lambda^2 - \tfrac{R}{4}) + (\lambda^2 s^2 - \tfrac{s}{2})c^2\Big) \;\geq\; 0$$

and then

$$\lambda^2 \;\geq\; \tfrac{2(a+As)}{1+2bs+c^2s^2} \;=:\; f(s)$$

for any $s \geq 0$. The function $f(s)$ attains its maximum at the point

$$s_0 \;:=\; \frac{A-2ab}{ac^2+c\sqrt{a^2c^2+A(A-2ab)}} \;.$$

Hence, in case $R \leq 0$ ($a \leq 0$), the parameter s_0 is automatically positive. In case $R > 0$ ($a > 0$) the parameter $s_0 > 0$ is positive if and only if $A - 2ab > 0$ or, equivalently, if

$$|\mathrm{Ric}|_0^2 \;>\; \tfrac{R}{n-1}(R - \kappa_0) \;,$$

holds. We summarize the main result.

Theorem 3.1: *Let M^n be a compact Riemannian spin manifold with harmonic curvature tensor such that the condition*

$$\Big|\mathrm{Ric} - \frac{R}{n}\Big|_0^2 \;>\; \Big(\frac{R}{n} - \kappa_0\Big)\max\Big\{\frac{R}{n-1}, -R\Big\} \tag{19}$$

is satisfied. Then every eigenvalue λ of the Dirac operator satisfies

$$\lambda^2 \;>\; \frac{A^2}{bA - ac^2 + c\sqrt{a^2c^2 + A(A-2ab)}} \;>\; 0 \;. \tag{20}$$

Proof: In case $R \leq 0$, the condition (19) is equivalent to $|\mathrm{Ric}|_0^2 \;>\; R\kappa_0$ and the case of a positive scalar curvature we already discussed. Then we obtain $\lambda^2 \geq f(s_0)$ and equality cannot occur for an eigenvalue λ of D. □

We remark that the compact, conformally flat 3-manifolds with constant scalar curvature and constant length of the Ricci tensor are the 3-dimensional space forms and the product of a 2-dimensional space form M^2 by S^1 (see [3]). These manifolds do not satisfy the condition (19).

In order to express the lower bound of the eigenvalue estimate in a convenient way we introduce the new variables α, β by the formulas

$$\alpha \ := \ ac^2 + (A-2ab)b, \quad \beta \ := \ (c^2-b^2)(A-2ab)^2 \ .$$

They are polynomials of degree three and six, depending on the eigenvalues of the Ricci tensor. The inequality (20) can be reformulated in the following form.

Corollary 3.2: *Let M^n be a compact Riemannian spin manifold with harmonic curvature tensor and positive scalar curvature $R > 0$. Suppose, moreover, that*

$$\left|\mathrm{Ric}\right|_0^2 \ > \ \frac{R}{n-1}\Big(R - \kappa_0\Big)$$

holds. Then each eigenvalue of the Dirac operator satisfies the estimate

$$\lambda^2 \ > \ \frac{nR}{4(n-1)} + \frac{(A-2ab)^2}{\alpha + \sqrt{\alpha^2+\beta}} \ > \ \frac{nR}{4(n-1)} \ .$$

We remark that the condition in Corollary 3.2 is satisfied in case that the scalar curvature is positive and at least one eigenvalue of the Ricci tensor is negative. Consequently, we obtain

Corollary 3.3: *Let M^n be a compact Riemannian spin manifold with parallel Ricci tensor, positive scalar curvature and at least one negative eigenvalue of the Ricci tensor. Then the estimate of Corollary* 3.2 *holds.*

Example 1: Let us consider the product manifold $M^4 = T^2 \times S^2$ equipped with the Riemannian metric induced by the metric of the flat torus T^2 and the metric of the standard sphere $S^2 \subset \mathbb{R}^3$. Then the Ricci tensor of M^4 is parallel and the set of eigenvalues of Ric is given by $(\kappa_1, \ldots, \kappa_4) = (0,0,1,1)$. Hence, we have $\kappa_1 = 0$, $R = 2 = |\mathrm{Ric}|^2$. The condition (19) is satisfied since

$$|\mathrm{Ric} - \tfrac{R}{4}|^2 \ = \ 1 \ > \ \tfrac{1}{3} \ = \ \tfrac{1}{2}\max\left\{\tfrac{2}{3}, -2\right\} \ .$$

Moreover, we find $a = \frac{1}{3}$, $b = \frac{2}{3}$, $c = \sqrt{\frac{4}{3}}$, $A = \frac{2}{3}$, $bA - ac^2 = 0$. Inserting this into (20) we obtain

$$(*) \qquad \lambda^2 > \tfrac{1}{2}\sqrt{2} \ .$$

The Riemannian estimate (1) yields the inequality $\lambda^2 \geq \frac{2}{3}$ and the Kähler estimate (see [11]) gives the lower bound $\lambda^2 \geq 1$. Since $\frac{2}{3} < \frac{1}{2}\sqrt{2} < 1$, the estimation of the first eigenvalue of the Dirac operator on the product considered as a Kähler manifold is the best one. We remark that $\lambda_1 = 1$ becomes an equality for the first eigenvalue (see [8]).

Example 2: Let N^2 be any compact Riemannian surface of constant Gaussian curvature -1 and let $S^2(r) \subset \mathbb{R}^3$ be the standard sphere of radius $r > 0$. Then the Ricci tensor of the Riemannian product $M^4(r) := S^2(r) \times N^2$ is parallel and $(\kappa_1, \ldots, \kappa_4) = (-1, -1, r^{-2}, r^{-2})$ is the corresponding set of eigenvalues of Ric. Thus, here we have

$$\kappa \;=\; -1, \quad R \;=\; -2\cdot(1-\tfrac{1}{r^2}), \quad |\mathrm{Ric}|^2 \;=\; 2\cdot(1+\tfrac{1}{r^4}), \quad |\mathrm{Ric}-\tfrac{R}{4}|^2 \;=\; (1+\tfrac{1}{r^2})^2 \;,$$

and

$$(\tfrac{R}{4}-\kappa)\cdot\max\left\{\tfrac{R}{3},\; -R\right\} \;=\; \max\left\{\tfrac{1}{3}(\tfrac{1}{r^4}-1), 1-\tfrac{1}{r^4}\right\}.$$

This shows that the condition (19) is satisfied and we find

$$a \;=\; \tfrac{1}{3}(-1+\tfrac{1}{r^2}) \quad , \quad b \;=\; \tfrac{2}{3}(1+\tfrac{1}{r^2}) \quad , \quad c \;=\; (1+\tfrac{1}{r^2})\sqrt{\tfrac{4}{3}},$$

$$A \;=\; \tfrac{2}{3}\tfrac{1}{r^2}(1+\tfrac{1}{r^2}) \quad , \quad c^2-b^2 \;=\; \tfrac{8}{9}(1+\tfrac{1}{r^2})^2, \quad bA-ac^2 \;=\; \tfrac{4}{9}(1+\tfrac{1}{r^2})^2 \;.$$

Inserting this into the inequality (20) we obtain the estimation

$$(*) \qquad \lambda^2 \;>\; \tfrac{1}{2}(\sqrt{1+\tfrac{2}{r^4}}-1) \;>\; 0 \;.$$

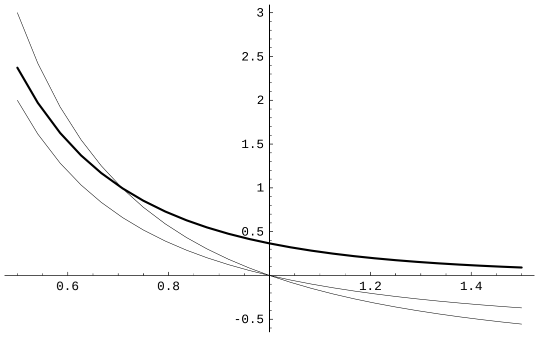
Hence, we see that, on the product $M^4(r)$, the Dirac operator has a trivial kernel even in case the scalar curvature is negative. In case the scalar curvature is positive, we can compare the new estimation $(*)$ with the estimation (1) and with the estimation for Kähler manifolds (see [11]), respectively. For positive scalar curvature ($r < 1$), the lower bound $(*)$ is obviously better than the Riemannian estimate (1),

$$\tfrac{2}{3}(-1+\tfrac{1}{r^2}) \;<\; \tfrac{1}{2}\left(\sqrt{1+\tfrac{2}{r^4}}-1\right) .$$

If we compare the new lower bound $(*)$ and the lower bound $-1+\frac{1}{r^2}$ in the Kähler case then, in the region $1/\sqrt{2} < r < 1$ ($0 < R < 2$), the inequality $(*)$ is the better one (see the figure):

$$\tfrac{1}{2}\left(\sqrt{1+\tfrac{2}{r^4}}-1\right) \;>\; -1+\tfrac{1}{r^2} \quad \text{if} \quad 1/\sqrt{2} < r < 1 \;.$$

This example shows that in certain cases with positive scalar curvature the estimate given by Theorem 3.1 is even better than the bound in [11] for Kähler manifolds.



The preceding two examples are special cases of a more general situation. Consider compact Einstein manifolds with spin structures $M_1^{n_1}, \ldots, M_k^{n_k}$ of dimensions $n_1, \ldots, n_k$ (in the case of $n_i = 2$, we assume that M^{n_i} is a surface of constant Gaussian curvature). Then the Riemannian product $M^n := M_1^{n_1} \times \ldots \times M_k^{n_k}$ is a compact Riemannian spin manifold with parallel Ricci tensor. Let R_i be the scalar curvature of $M_i^{n_i}$. Then the scalar curvature R as well as the length of the Ricci tensor of M^n are given by

$$R \;=\; \sum_{i=1}^{k} R_i, \qquad |\mathrm{Ric}|^2 \;=\; \sum_{i=1}^{k} \frac{R_i^2}{n_i}\,.$$

Moreover, let us assume that the smallest eigenvalue of the Ricci tensor is $\kappa_1 = \frac{R_1}{n_1}$. Then the conditions under which we can apply our estimate are equivalent to

$$\sum_{i=1}^{k} \frac{R_i^2}{n_i} \;>\; \frac{R_1}{n_1}\Big(\sum_{i=1}^{k} R_i\Big) \quad (R \;\leq\; 0), \quad \text{and} \quad \sum_{i=1}^{k} \frac{R_i^2}{n_i} \;>\; \frac{1}{n-1}\Big(\sum_{i=1}^{k} R_i\Big)\Big(\sum_{i=2}^{k} R_i\Big) \quad (R \geq 0)$$

respectively. Remark that, by the theorem of de Rham-Wu [15], any compact, simply connected Riemannian manifold with parallel Ricci tensor splits into a Riemannian product of Einstein manifolds, i.e., the product situation is the general one for a parallel Ricci tensor.

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