Invariant nearly Kähler f-structures on homogeneous spaces

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Abstract

We introduce a concept of nearly Kahler f-structures naturally generalizing well-known nearly Kähler structures in Hermitian geometry. A remarkable collection of invariant structures of such a kind is presented. It is based on the canonical f-structures on homogeneous Φ-spaces, i.e. homogeneous spaces defined by Lie group automorphisms. As a result, the analogy with homogeneous nearly Kähler manifolds is obtained. Some general and particular examples including the 6-dimensional generalized Heisenberg group are considered in more detail.

Contents

1 Introduction

Homogeneous spaces of Lie groups are of great importance in differential geometry. In particular, homogeneous spaces defined by Lie group automorphisms Φ provided the remarkable examples for main classes in Hermitian geometry [WG]. The most important role belongs to Riemannian 3-symmetric spaces (i.e. $\Phi^3 = id$) that in naturally reductive case are just homogeneous nearly Kähler manifolds. The key point is that homogeneous 3-symmetric spaces possess the canonical almost complex structure constructed by means of Φ (see [S2], [WG], [G2]). It should be mentioned that this structure became an effective tool and a remarkable example in some deep constructions of differential geometry and global analysis such as homogeneous structures ([TV], [Sat], [Ki4], [GV], [LV], [AG] etc.), Einstein metrics ([SW], [SY]), holomorphic and minimal submanifolds [Sal1,2], real Killing spinors ([Gru], [BFGK]).

It turns out that the situation can be essentially extended. On the one hand, an almost complex structure is the particular case of f-structures by K.Yano [Y]. On the other hand, homogeneous Φ-spaces possess a remarkable collection of canonical structures of classical type including f-structures [BS1,2]. It gives the opportunity to introduce a class of f-structures such that it both generalizes classical nearly Kähler structures and contains many homogeneous examples on the base of canonical f-structures.

Let (M, g) be a (pseudo-)Riemannian manifold, ∇ the Levi-Civita connection of the metric g, J an almost complex structure on M. Recall that (g, J) is an almost Hermitian structure if $g(JX, JY) = g(X, Y)$ for any $X, Y \in \mathfrak{X}(M)$. If, in addition,

$$\nabla_X(J)X = 0,$$

then J is called *a nearly Kähler structure* (breafly, *NK-structure*) [G1]. Note that invariant NK-structures were intensively investigated in [WG], [G2], [Ki1] and other papers.

Let us now turn to f-structures on (M, g). f is *a metric f-structure* if

$$g(fX, Y) + g(X, fY) = 0$$

(see [Ki2]). Further, a metric f-structure on (M, g) is said to be *a Killing f-structure* if f is a Killing tensor that is equivalent to the condition

$$\nabla_X(f)X = 0$$

(see [Gr1,2]). Obviously, this notion coincides with that of a nearly Kähler structure if $Ker\ f = 0$. But this is actually a problem to find essential examples of invariant Killing f-structures (see [B2]).

Now we introduce the following more general concept. A metric f-structure on (M, g) is called *a nearly Kähler f-structure* (breafly, *NKf-structure*) if

$$\nabla_{fX}(f)fX = 0$$

for all $X \in \mathfrak{X}(M)$ [B1].

It should be noted that both notions above are included in a large class of generalized G_1-manifolds in the general concept of *generalized Hermitian geometry* [Ki2].

In the paper we concentrate on classes and remarkable examples of invariant NKf-structures. In particular, it is proved that the canonical f-structures on naturally reductive homogeneous Φ-spaces of orders 4 and 5 are NKf-structures.

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2 Invariant NKf-structures on naturally reductive homogeneous spaces

Let G be a connected Lie group, H its closed subgroup, g an invariant (pseudo-) Riemannian metric on the homogeneous space G/H. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras corresponding to G and H respectively. Suppose that G/H is a reductive homogeneous space, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ the reductive decomposition of the Lie algebra $\mathfrak{g}$. As usual, we identify $\mathfrak{m}$ with the tangent space $T_o(G/H)$ at the point $o = H$. Then the invariant metric g is completely defined by its value at the point o. For convenience we denote by the same manner both any invariant structure on G/H and its value at o throughout the rest of the paper.

Further, denote by $\mathfrak{m}^{(\lambda,\mu)}$ the linear span of all vectors $[\lambda X, \mu X]$, where λ, μ are linear operators on $\mathfrak{m}$, $X \in \mathfrak{m}$. Note that for arbitrary regular Φ-space all subspaces $\mathfrak{m}^{(\lambda,\mu)}$ carry a remarkable property [B1].

Recall that $(G/H, g)$ is *naturally reductive* [KN] if

$$g([X,Y]_{\mathfrak{m}}, Z) = g(X, [Y,Z]_{\mathfrak{m}})$$

for all $X, Y, Z \in \mathfrak{m}$. Here the subscript $\mathfrak{m}$ denotes the projection of $\mathfrak{g}$ onto $\mathfrak{m}$ with respect to the reductive decomposition.

Now we can characterize invariant Killing and nearly Kähler f-structures in terms of special subspaces $\mathfrak{m}^{(\lambda,\mu)}$.

Theorem 1 . *Let $(G/H, g)$ be a naturally reductive homogeneous space, (g, f) an invariant metric f-structure, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ the corresponding reductive decomposition.*

(i) f is a Killing f-structure iff $\mathfrak{m}^{(1,f)} \subset \mathfrak{h}$.

(ii) f is a nearly Kähler f-structure iff $\mathfrak{m}^{(f,f^2)} \subset \mathfrak{h}$.

Proof. The technique is traditional for invariant structures on reductive homogeneous spaces [N]. For instance, we prove (ii).

The condition $\nabla_{fX}(f)fX = 0$ means that

$$\nabla_{fX} f^2 X = f \nabla_{fX} fX$$

for all vector fields X on G/H. Denote by α the Nomizu function of an invariant affine connection∇ [N]. Then, using special vector fields in a neighborhood of the point o [N] it can be shown that the previous condition is equivalent to that of

$$\alpha(fX, f^2X) = f\ \alpha(fX, fX),\ X \in \mathfrak{m}.$$

Since the metric g is naturally reductive,

$$\alpha(X,Y) = \frac{1}{2}[X,Y]_{\mathfrak{m}},\ X, Y \in \mathfrak{m}.$$

Hence we obtain $[fX, f^2X] \in \mathfrak{h}$ for any $X \in \mathfrak{m}$. This is nothing but the condition $\mathfrak{m}^{(f,f^2)} \subset \mathfrak{h}$.

Statement (i) can be proved in exactly the same way. $\square$

Remark 1 . Consider the particular case $f = J$ $(J^2 = -1)$ of Theorem 1. Then both the conditions are reduced to that of $\mathfrak{m}^{(1,J)} \subset \mathfrak{h}$, i.e. $[X, JX] \in \mathfrak{h}$ for all $X \in \mathfrak{m}$. This is just the well-known criterion for nearly Kähler structures in the case ([WG], p.138).

Any invariant metric f-structure on G/H determines the orthogonal decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ such that $\mathfrak{m}_1 = Im\, f$, $\mathfrak{m}_2 = Ker\, f$.

Theorem 2 . *Let $(G/H, g, f)$ be a naturally reductive homogeneous space with a metric f-structure, $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ the decomposition with respect to f. The following conditions are equivalent:*

(i) (g, f) is a Killing f-structure;

(ii) (g, f) is NKf-structure and $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{h}$.

Proof. Taking any $X = X_1 + X_2$, $X_1 \in \mathfrak{m}_1$, $X_2 \in \mathfrak{m}_2$, we get

$$[X, fX] = [X_1, fX_1] + [X_2, fX_1]. \tag{1}$$

If (g, f) is a Killing f-structure, then by Theorem 1(i) the vectors $[X, fX]$ and $[X_1, fX_1]$ belong to $\mathfrak{h}$. From (1) it follows immediately that $[X_2, fX_1] \in \mathfrak{h}$. Since f is non-singular on $\mathfrak{m}_1$, we obtain $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{h}$. So, (i) implies (ii).

Conversely, suppose (g, f) is NKf-structure and $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{h}$. By Theorem 1(ii) we have

$$[X_1, fX_1] = [fY, f^2Y] \in \mathfrak{h},$$

where $Y \in \mathfrak{m}$. Since $[X_2, fX_1] \in \mathfrak{h}$ by assumption it follows from (1) that $[X, fX] \in \mathfrak{h}$. Hence by Theorem 1(i) (g, f) is a Killing f-structure. This completes the proof. □

3 Canonical structures on regular Φ-spaces

Here we breafly collect some basic definitions and results in the theory of homogeneous Φ-spaces and canonical structures on them. These structures are of fundamental importance for the paper subject. As to details and related objects, we refer to [BS2], [WG], [Ko], [F], [S1,2].

Let G be a connected Lie group, Φ its (analytic) automorphism. Denote by G^Φ the subgroup of all fixed points of Φ and G^Φ_o the identity component of G^Φ. Suppose a closed subgroup H of G satisfies the condition

$$G^\Phi_o \subset H \subset G^\Phi.$$

Then G/H is called *a homogeneous Φ-space.*

Let G/H be a homogeneous Φ-space, $\mathfrak{g}$ and $\mathfrak{h}$ the corresponding Lie algebras for G and H, $\varphi = d\Phi_e$ the automorphism of $\mathfrak{g}$. Consider the linear operator $A = \varphi - id$ and the Fitting decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with respect to A, where $\mathfrak{g}_0$ and $\mathfrak{g}_1$ denote $0-$ and 1-component of the decomposition respectively. It is clear that $\mathfrak{h} = Ker\, A$ and $\mathfrak{h} \subset \mathfrak{g}_0$.

Definition 1 ([S1], [BS2], [F]) . A homogeneous Φ-space G/H is called *regular* if one of the following equivalent conditions is satisfied:

1. $\mathfrak{h} = \mathfrak{g}_0$;

2. $\mathfrak{g} = \mathfrak{h} \oplus A\mathfrak{g}$;

3. The restriction of the operator A to $A\mathfrak{g}$ is non-singular;

4. $A^2X = 0 \Longrightarrow AX = 0$ for all $X \in \mathfrak{g}$.

We recall two basic facts:

Theorem 3 ([S1]).

- *Any homogeneous Φ-space of order k ($\Phi^k = id$) is a regular Φ-space.*
- *Any regular Φ-space is reductive. More exactly, the Fitting decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \ \ \mathfrak{m} = A\mathfrak{g}$$

is a reductive one. It is called the canonical reductive decomposition *corresponding to G/H.*

Furthermore, denote by θ the restriction of φ to $\mathfrak{m}$. Now we turn to *affinor structures* on G/H, i.e. tensor fields of type $(1,1)$.

Definition 2 ([BS1,2]) . An invariant affinor structure F on a regular Φ-space G/H is called *canonical* if its value at the point $o = H$ is a polynomial in θ.

Denote by $\mathcal{A}(\theta)$ the set of all canonical affinor structures on a regular Φ-space G/H. It is easy to see that $\mathcal{A}(\theta)$ is a commutative subalgebra of the algebra $\mathcal{A}$ of all invariant affinor structures on G/H. Moreover,

$$dim\ \mathcal{A}(\theta) = deg\ \nu\ \leq\ dim\ G/H,$$

where ν is the minimal polynomial of the operator θ. Note that the algebra $\mathcal{A}(\theta)$ is trivial for any symmetric Φ-space ($\Phi^2 = id$), more exactly, it consists of scalar structures.

No doubt, almost product structures P ($P^2 = 1$), almost complex structures J ($J^2 = -1$), and, more generally, f-structures ($f^3 + f = 0$) belong to the most important affinor structures. We shall say that P, J, f are structures of *classical type.* All the canonical structures of classical type on regular Φ-spaces were completely described [BS1,2]. In particular, for homogeneous Φ-spaces of arbitrary finite order k, precise computational formulae were indicated.

For future reference we select here some results about canonical structures P, J, f on homogeneous Φ-spaces of orders $k = 3, 4, 5$ only. Note that we indicate the structures considered up to sign.

Theorem 4 ([BS1,2]).

1. *There exists a unique canonical structure of classical type on any homogeneous Φ-space of order 3. This is the well-known canonical almost complex structure*

$$J = \frac{1}{\sqrt{3}}(\theta - \theta^2)$$

(see [S2], [WG], [G2]*).*

2. *Let G/H be a homogeneous Φ-space of order 4. There exist a unique canonical almost product structure*
$$P = \theta^2,$$
and a unique canonical f-structure
$$f = \frac{1}{2}(\theta - \theta^3).$$
In addition, the following conditions are equivalent:
   (a) *the structure P is trivial;*
   (b) *the above-mentioned f-structure is an almost complex structure;*
   (c) $-1 \notin$ *spec* θ.

3. *There exist only the following canonical structures of classical type on any homogeneous Φ-space of order 5:*
$$P = \frac{1}{\sqrt{5}}(\theta - \theta^2 - \theta^3 + \theta^4);$$
$$J_1 = \alpha(\theta - \theta^4) - \beta(\theta^2 - \theta^3); \quad J_2 = \beta(\theta - \theta^4) + \alpha(\theta^2 - \theta^3);$$
$$f_1 = \gamma(\theta - \theta^4) + \delta(\theta^2 - \theta^3); \quad f_2 = \delta(\theta - \theta^4) - \gamma(\theta^2 - \theta^3);$$
where $\alpha = \frac{\sqrt{5+2\sqrt{5}}}{5}$; $\beta = \frac{\sqrt{5-2\sqrt{5}}}{5}$; $\gamma = \frac{\sqrt{10+2\sqrt{5}}}{10}$; $\delta = \frac{\sqrt{10-2\sqrt{5}}}{10}$.
In addition, the following conditions are equivalent:
   (a) *spec θ consists of two elements;*
   (b) *The structure P is trivial;*
   (c) *the structures J_1 and J_2 coincide (up to sign);*
   (d) *one of the structures f_1 and f_2 is trivial, while the other is an almost complex structure coinciding with one of the structures J_1 and J_2.*

4 Canonical NKf-structures on regular Φ-spaces

The main purpose of this Section is to show that naturally reductive regular Φ-spaces with some canonical f-structures provide a wide class of homogeneous NKf-manifolds.

Theorem 5 . *Let G/H be a regular Φ-space, g a naturally reductive metric on G/H with respect to the canonical reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, (g, f) a metric canonical f-structure on G/H. Suppose the f-structure satisfies the condition*
$$f^2 = \pm\theta\, f. \qquad (2)$$
Then $(G/H, g, f)$ is a nearly Kähler f-manifold.

Proof. By Theorem 1(ii) it suffices to prove that $[fX, f^2X] \in \mathfrak{h}$ for any $X \in \mathfrak{m}$. Using (2), we have

$$\varphi[fX, f^2X] = [\theta fX, \theta f^2X] = [f^2X, f^3X] = [f^2X, -fX] = [fX, f^2X].$$

It follows that $[fX, f^2X] \in \mathfrak{h}$. □

Remark 2 . Assumptions for a metric g in Theorem 5 are satisfied in the following important case. Let G/H be a regular Φ-space of a semi-simple Lie group G. The invariant metric g determined on G/H by means of the Killing form of the Lie algebra $\mathfrak{g}$ is usually called *standard*. It is known (see [S1]) that the standard metric on any regular Φ-space of such a kind is naturally reductive. Moreover, this metric is compatible with any canonical f-structure on a homogeneous Φ-space of order k, i.e. (g, f) is a metric f-structure (see [B1]).

Corollary 1 . *Let $(G/H, g)$ be a naturally reductive homogeneous Φ-space of order $k = 4n$, $n \geq 1$. If $\{i, -i\} \subset$ spec θ, then there exists a non-trivial canonical NKf-structure on G/H.*

Proof. The procedure of describing canonical structures on homogeneous Φ-spaces of finite order is constructive (see [BS2], §4 and §5). We use it to present a canonical f-structure required. Consider the decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ such that $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are determined by the eigenvalues $\{i, -i\}$ and all the others of the operator θ respectively. Now we put f to be the complex structure on $\mathfrak{m}_1$ and vanish on $\mathfrak{m}_2$. In other words, consider the collection

$$(\zeta_1, ..., \zeta_l, ..., \zeta_u), \ \zeta_j \in \{-1, 0, 1\}$$

for constructing canonical f-structures (see [BS2], Theorem 5.3), where ζ_l corresponds to the eigenvalue i of θ. Then we put

$$\zeta_l = 1, \ \zeta_m = 0, \ m \in \{1, 2, ..., u\}, \ m \neq l.$$

It is not difficult to see that this f-structure satisfies the condition $f^2 = \theta f$. It remains to apply Theorem 5. □

We formulate the following evident but important particular case of Corollary 1:

Corollary 2 . *Any naturally reductive homogeneous Φ-space $(G/H, g)$ of order 4 is NKf-manifold with respect to the canonical f-structure $f = \frac{1}{2}(\theta - \theta^3)$.*

Indeed, in this case the only canonical f-structure on G/H (see Theorem 4(2)) satisfies condition (2). Notice that if $Ker\ f = 0$, then G/H is a Hermitian symmetric space (see [WG], p.136).

It should be noted that there exist canonical NKf-structures on regular Φ-spaces such that condition (2) is not satisfied. We consider in this respect canonical f-structures on homogeneous Φ-spaces of order 5. First we need some special facts on these spaces.

Lemma 1 ([BC],[C1,2],[BS2]). *Let G/H be a homogeneous Φ-space of order 5, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ the canonical reductive decomposition of $\mathfrak{g}$, P the canonical almost product structure. Consider the decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$, where $\mathfrak{m}_1$ and $\mathfrak{m}_2$ correspond to the eigenvalues $+1$ and -1 of the operator P respectively.*

1. *The following relations hold:*

$$[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{h} \oplus \mathfrak{m}_2, \quad [\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{h} \oplus \mathfrak{m}_1, \quad [\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}.$$

2. *The canonical structures J_1, J_2, f_1, f_2 can be represented in the form*

$$J_1 = (I, J), \quad J_2 = (I, -J), \quad f_1 = (I, 0), \quad f_2 = (0, J),$$

where I and J are special complex structures on $\mathfrak{m}_1$ and $\mathfrak{m}_2$ respectively.

3. *For all $X_1, Y_1 \in \mathfrak{m}_1$, $X_2, Y_2 \in \mathfrak{m}_2$ we have*

$$I\,[X_2, Y_2]_1 = -[X_2, J\,Y_2]_1; \;\; J\,[X_1, Y_1]_2 = [X_1, I\,Y_1]_2;$$
$$I\,[X_1, Y_2]_1 = -[I\,X_1, Y_2]_1 = [X_1, J\,Y_2]_1;$$
$$I\,[X_1, Y_2]_2 = -[I\,X_1, Y_2]_2 = -[X_1, J\,Y_2]_2.$$

Here subscripts 1 and 2 mean projections of vectors $Z \in \mathfrak{g}$ with respect to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$.

Theorem 6 . *Let G/H be a homogeneous Φ-space of order 5, g a naturally reductive metric on G/H with respect to the canonical reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Then both the canonical structures f_1 and f_2 on G/H are NKf-structures.*

Proof. Consider the canonical f-structure $f_1 = (I, 0)$. Then for any $X \in \mathfrak{m}$ we have

$$[f_1 X, f_1^2 X] = [Y_1, f_1\,Y_1] = [Y_1, I\,Y_1],$$

where $Y_1 \in \mathfrak{m}_1$. By Lemma 1(3), we obtain

$$[Y_1, I\,Y_1]_2 = J\,[Y_1, Y_1]_2 = 0.$$

Besides, from Lemma 1(1) it follows that

$$[Y_1, I\,Y_1] \in [\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{h} \oplus \mathfrak{m}_2.$$

Now we can conclude that $[f_1 X, f_1^2 X] \in \mathfrak{h}$. By Theorem 1(ii) it means that f_1 is NKf-structure. The analogous statement for the structure $f_2 = (0, J)$ can be proved in the same manner. $\square$

5 Examples

In conclusion, we indicate some general and particular examples of homogeneous NKf-manifolds.

5.1 Riemannian 4- and 5-symmetric spaces

Riemannian homogeneous Φ-spaces of order 4 of classical compact Lie groups were classified and geometrically described in [J]. The similar problem for homogeneous Φ-spaces of order 5 was solved in [TX]. By Corollary 2 and Theorem 6, it presents a collection of homogeneous NKf-manifolds. Note that the canonical f-structures under consideration are generally non-integrable.

5.2 The sphere S^5

It is well-known (see [Sa],[Ko]) that the sphere

$$S^5 \cong SU(3)/SU(2)$$

admits the structure of Riemannian homogeneous Φ-space of order 4. Its canonical structure $f = \frac{1}{2}(\theta - \theta^3)$ was indicated in [BS2]. Clearly, this is NKf-structure. It can be easily shown that f is not a Killing f-structure.

It should be mentioned that nevertheless the sphere S^5 allows a Killing f-structure. This is a so-called weakly cosymplectic structure (see [Bl],[Ki3]), however this f-structure is not invariant. Moreover, from the results in [Gr1],[Gr2] it follows that S^5 does not admit any invariant Killing f-structure.

5.3 Homogeneous Φ-spaces of order 8

The canonical NKf-structure described in Corollary 1 can be effectively constructed for homogeneous Φ-spaces of any order $k = 4n$. As an example, consider the case $k = 8$. Applying the general formula (see [BS2], Theorem 5.3) for the situation, we obtain that all canonical f-structures can be represented in the form

$$f = \frac{1}{4}\sum_{m=1}^{3}(\sum_{j=1}^{3}\zeta_j \sin\frac{\pi m j}{4})(\theta^m - \theta^{8-m}),$$

where $\zeta_j \in \{-1; 0; 1\}$. Put $\zeta_2 = 1$, $\zeta_1 = \zeta_3 = 0$, then we obtain

$$f = \frac{1}{4}(\theta - \theta^3 + \theta^5 - \theta^7). \tag{3}$$

It is easy to verify that this f-structure satisfies condition (2). So, we obtain

Theorem 7 . *Let $(G/H, g)$ be a naturally reductive homogeneous Φ-space of order 8. If $\{i, -i\} \subset spec\,\theta$, then the canonical f-structure (3) is a non-trivial NKf-structure on G/H.*

5.4 The 6-dimensional generalized Heisenberg group

We breafly formulate some notions and results related to the 6-dimensional generalized Heisenberg group (N, g). As to details, we refer to [Ka1], [Ka2], [TV].

The Lie algebra $\mathfrak{n}$ of the 6-dimensional generalized Heisenberg group can be defined as follows. Consider the linear space

$$\mathfrak{n} := V \oplus Z = L(x_1, x_2, x_3, x_4) \oplus L(a_1, a_2)$$

with the brackets:

$$\begin{cases} [x_1,x_2]=a_1, \quad [x_1,x_3]=a_2, \\ [x_2,x_4]=-a_2, \quad [x_3,x_4]=a_1, \\ \text{all the other brackets are zero.} \end{cases}$$

It is a 2-step nilpotent Lie algebra with center Z. The simply connected, connected Lie group with the Lie algebra $\mathfrak{n}$ is exactly the 6-dimentional generalized Heisenberg group N. Besides, the Lie group N has a left invariant metric g induced by an inner product on $\mathfrak{n}$.

It is known (see [TV], p.112) that (N,g) is a Riemannian homogeneous Φ-space of order 4. More exactly, the automorphism Φ is determined by means of the isometric automorphism φ of the Lie algebra $\mathfrak{n}$ such that $\varphi^4 = id$. For convenience, we consider φ written in the form

$$\varphi: (x_1,x_2,x_3,x_4,a_1,a_2) \to (-x_4,-x_3,x_2,x_1,-a_1,-a_2).$$

By our notations we have $\theta = \varphi$. Directly calculating the canonical f-structure $f=\frac{1}{2}(\theta-\theta^3)$ on (N,g), we obtain

$$f \; : \; (x_1,x_2,x_3,x_4,a_1,a_2) \;\to\; (-x_4,-x_3,x_2,x_1,0,0).$$

It follows

$$f|_V = \varphi, \;\; f|_Z = 0,$$

hence

$$\mathfrak{m}_1 = Im\, f = V, \;\; \mathfrak{m}_2 = Ker\, f = \; Z.$$

It means that f has defect 2. It is easy to show that (g,f) is a metric f-structure on (N,g).

It is important to note that (N,g) is not a naturally reductive homogeneous space (see [Ka2], [TV],p.96). Therefore we cannot use Corollary 2 above and have to directly calculate. The Levi-Civita connection ∇ of the metric g was indicated in [Ka1]:

$$\begin{cases} \nabla_x y = \frac{1}{2}[x,y], \\ \nabla_a x = \nabla_x a = -\frac{1}{2} j(a)x \quad, \\ \nabla_a b = 0, \end{cases} \tag{4}$$

where $a,b\in Z,\; x,y\in V$ and $x_2=j(a_1)x_1,\; x_3=j(a_2)x_1,\; x_4=j(a_1)j(a_2)x_1$. By α denote as usual the Nomizu function of ∇.

From easy calculations it follows that $[x,\varphi\, x]=0$ for all $x\in V$. Now put $y=fX$ for all $X\in\mathfrak{n}$. Then if we recall (4), we can get

$\alpha(fX,fX)=\frac{1}{2}[y,y]=0,\;\alpha(fX,f^2X)=\frac{1}{2}[y,\varphi\, y]=0.$

It follows that the canonical f-structure satisfies the condition $\nabla_{fX}(f)fX = 0$, i.e. it is NKf-structure. However this f-structure is not Killing.

Besides, using the results of [BD] we can show that f is non-integrable.

To sum up, we have

Theorem 8 . *The 6-dimensional generalized Heisenberg group (N,g) is a nearly Kähler f-manifold with respect to the canonical f-structure $f=\frac{1}{2}(\theta-\theta^3)$ of a homogeneous Φ-space of order 4. This f-structure is not integrable on (N,g).*

Remark 3 . Recall (see [G2]) that a (pseudo-)Riemannian homogeneous Φ-space $(G/H, g)$ of order 3 with the canonical almost complex structure J is nearly Kähler iff the metric g is naturally reductive. As to (pseudo-)Riemannian homogeneous Φ-spaces $(G/H, g)$ of order 4 with the canonical f-structure, the situation is not completely similar. More precisely, by Corollary 2 any naturally reductive space of such a kind is a nearly Kähler f-manifold. However the above example illustrates that the inverse is not true. This is just the difference between the cases $(G/H, g, J)$ for $k = 3$ and $(G/H, g, f)$ for $k = 4$.

5.5 The group of hyperbolic motions of the plane $\mathbb{R}^2$

Finally, we consider the well-known example of a 3-dimensional Riemannian homogeneous Φ-space of order 4 (see [Ko],p.18).

Let

$$G = \left\{ \left(\begin{array}{ccc} e^{-c} & 0 & a \\ 0 & e^{c} & b \\ 0 & 0 & 1 \end{array} \right) \middle| \, a, b, c \in \mathbb{R} \right\}$$

be the Lie group of hyperbolic motions of the plane $\mathbb{R}^2$. This is a solvable Lie group diffeomorphic to $\mathbb{R}^3$. The Riemannian metric g on G determined by the formula

$$ds^2 \;=\; e^{2c}da^2 + e^{-2c}db^2 + \lambda^2 dc^2, \;\; \lambda > 0$$

is invariant with respect to G for any λ indicated. The automorphism Φ of the Lie group G given by the formula

$$\Phi : \left(\begin{array}{ccc} e^{-c} & 0 & a \\ 0 & e^{c} & b \\ 0 & 0 & 1 \end{array} \right) \longrightarrow \left(\begin{array}{ccc} e^{c} & 0 & -b \\ 0 & e^{-c} & a \\ 0 & 0 & 1 \end{array} \right)$$

is an isometry of order 4 with the only fixed point. Hence $(G = \mathbb{R}^3(a, b, c), g)$ is a Riemannian Φ-space of order 4.

Now using the results of [Ko] and calculating as above, we obtain

Theorem 9 . *The canonical f-structure $f = \frac{1}{2}(\theta - \theta^3)$ on the Riemannian homogeneous Φ-space $(\mathbb{R}^3(a, b, c), g)$ of order 4 is a metric f-structure but f is neither integrable nor nearly Kähler.*

References

[AG] Abbena E., Garbiero S., *Almost Hermitian homogeneous manifolds and Lie groups*, Nihonkai Math. J., 1993, V.4, 1-15.

[B1] Balashchenko V.V., *Riemannian geometry of canonical structures on regular Φ-spaces*, Preprint No.174/1994, Fakultät für Mathematik der Ruhr-Universität Bochum, 1994, 1-19.

[B2] Balashchenko V.V., *Naturally reductive Killing f-manifolds*, Russian Math. Surveys, 1999, V.54, no.3, 623-625.

[Bl] Blair D.E., *Contact manifolds in Riemannian geometry.* LN in Math. V.509. Springer-Verlag. 1976.

[BC] Balashchenko V.V., Churbanov Yu.D., *Invariant structures on homogeneous* Φ*-spaces of order* 5, Russian Math. Surveys, 1990, V.45, no.1, 195-197.

[BD] Balashchenko V.V., Dashevich O.V., *Geometry of canonical structures on homogeneous* Φ*-spaces of order* 4, Russian Math. Surveys, 1994, V.49, no.4, 149-150.

[BFGK] Baum H., Friedrich T., Grunewald R., Kath I., *Twistor and Killing Spinors on Riemannian Manifolds*, Teubner-Texte zur Mathematik, V.124, Teubner-Verlag, Stuttgart/Leipzig, 1991.

[BS1] Balashchenko V.V., Stepanov N.A., *Canonical affinor structures on regular* Φ*-spaces*, Russian Math. Surveys, 1991, V.46, no.1, 247-248.

[BS2] Balashchenko V.V., Stepanov N.A., *Canonical affinor structures of classical type on regular* Φ*-spaces*, Sbornik: Mathematics, 1995, V.186, no.11, 1551-1580.

[C1] Churbanov Yu.D., *On some classes of homogeneous* Φ*-spaces of order 5*, Izv. Vyssh. Uchebn. Zaved. Mat., 1992, no.2, 88-90; English transl. in Soviet Math. (Iz. VUZ), 1992, V.36, no.2.

[C2] Churbanov Yu.D., *Canonical* f*-structures of homogeneous* Φ*-spaces of order* 5, Vestnik BGU. Ser.1: Fiz,Mat,Mech., 1994, no.1, 51-54 (in Russian).

[F] Fedenko A.S., *Spaces with symmetries*, Belarusian State University, Minsk, 1977 (in Russian).

[G1] Gray A., *Nearly Kähler manifolds*, J. Diff. Geom., 1970, V.4, no.3, 283-309.

[G2] Gray A., *Riemannian manifolds with geodesic symmetries of order* 3, J. Diff. Geom., 1972, V.7, no.3-4, 343-369.

[Gr1] Gritsans A.S., *Geometry of Killing* f*-manifolds*, Russian Math. Surveys, 1990, V.45, no.4, 168-169.

[Gr2] Gritsans A.S., *On construction of Killing* f*-manifolds*, Izv. Vyssh. Uchebn. Zaved. Mat., 1992, no.6, 49-57; English transl. in Soviet Math. (Iz. VUZ), 1992, V.36, no.6.

[Gru] Grunewald R., *Six-dimensional Riemannian manifolds with a real Killing spinor*, Ann. Global Anal. Geom., 1990, V.8, no.1, 43-59.

[GV] Garbiero S., Vanhecke L., *A characterization of locally* 3*-symmetric spaces*, Riv. Mat. Univ. Parma (5), 1993, V.2, 331-335.

[J] Jimenez J.A., *Riemannian* 4*-symmetric spaces*, Trans. Amer. Math. Soc., 1988, V.306, no.2, 715-734.

[Ka1] Kaplan A., *Riemannian nilmanifolds attached to Clifford modules*, Geom. Dedicata, 1981, V.11, 127-136.

[Ka2] Kaplan A., *On the geometry of groups of Heisenberg type*, Bull. London Math. Soc., 1983, V.15, 35-42.

[Ki1] Kirichenko V.F., *On geometry of homogeneous K-spaces*, Mat. Zametki, 1981, V.30, no.4, 569-582; English transl. in Math. Notes, 1981, V.30.

[Ki2] Kirichenko V.F., *Methods of generalized Hermitian geometry in the theory of almost contact manifolds*, Itogi Nauki i Tekhniki: Probl. Geom. V.18, VINITI, Moscow, 1986, 25-71; English transl. in J. Soviet Math. 1988, V.42, no.5.

[Ki3] Kirichenko V.F., *Sur la geometrie des varietes approximativement cosymplectiques*, C. R. Acad. Sci. Paris, Ser.1, 1982, V.295, no.12, 673-676.

[Ki4] Kirichenko V.F., *Hermitian-homogeneous generalized almost Hermitian manifolds*, Soviet Math. Dokl., 1984, V.30, 267-271.

[KN] Kobayashi S., Nomizu K., *Foundations of differential geometry*, V.2, Intersc. Publ. J.Wiley & Sons, New York - London, 1969.

[Ko] Kowalski O., *Generalized symmetric spaces*, LN in Math., V.805, Berlin, Heidelberg, New York: Springer-Verlag, 1980.

[LV] Ledger A.J., Vanhecke L., *On a theorem of Kiričenko relating to* 3-*symmetric spaces*, Riv. Mat. Univ. Parma (4), 1987, V.13, 367-372.

[N] Nomizu K., *Invariant affine connections on homogeneous spaces*, Amer. J. Math., 1954, V.76, no.1, 33-65.

[Sal1] Salamon S., *Harmonic and holomorphic maps*, Geometry Seminar "Luigi Bianchi" II – 1984, LN in Math., V.1164, Springer–Verlag, 1985, 161-224.

[Sal2] Salamon S., *Minimal surfaces and symmetric spaces*, Diff. Geometry. Proc. Colloq. Santiago de Compostela, 1985, 103-114.

[Sa] Sanchez C.U., *Regular s-structure on spheres*, Indiana Univ. Math. J., 1988, V.37, no.1, 165-180.

[Sat] Sato T., *Riemannian* 3-*symmetric spaces and homogeneous K-spaces*, Memoirs of the Faculty of Technology, Kanazawa Univ. 1979, V.12, no.2, 137-143.

[S1] Stepanov N.A., *Basic facts of the theory of φ-spaces*, Izv. Vyssh. Uchebn. Zaved. Mat., 1967, no.3, 88-95; English transl. in Soviet Math. (Iz. VUZ), 1967, V.11, no.3.

[S2] Stepanov N.A., *Homogeneous* 3-*cyclic spaces*, Izv. Vyssh. Uchebn. Zaved. Mat., 1967, no.12, 65-74; English transl. in Soviet Math. (Iz. VUZ), 1967, V.11, no.12.

[SW] Sekigawa K., Watanabe J., *On some compact Riemannian* 3-*symmetric spaces*, Sci. Reports of Niigata Univ. Ser. A., no.19, 1983, 1-17.

[SY] Sekigawa K., Yoshida H., *Riemannian* 3-*symmetric spaces defined by some outer automorphisms of compact Lie groups*, Tensor, 1983, V.40, no.3, 261-268.

[TV] Tricerri F., Vanhecke L., *Homogeneous structures on Riemannian manifolds*, London Math. Soc., LN Ser., no.83, 1983.

[TX] Tsagas Gr., Xenos Ph., *Homogeneous spaces which are defined by diffeomorphisms of order* 5, Bull. Math. Soc. Sci. Math. RSR, 1987, V.31, no.1, 57-77.

[WG] Wolf J.A., Gray A., *Homogeneous spaces defined by Lie group automorphisms*, J. Diff. Geom., 1968, V.2, no.1-2, 77-159.

[Y] Yano K., *On a structure defined by a tensor field f of type $(1,1)$ satisfying* $f^3+f=0$, Tensor, 1963, V.14, 99-109.

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