THE SPIN^c DIRAC OPERATOR ON HIGH TENSOR POWERS OF A LINE BUNDLE

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Abstract. We study the asymptotic of the spectrum of the spin^c Dirac operator on high tensor powers of a line bundle. As application, we get a simple proof of the main result of Guillemin–Uribe [13, Theorem 2], which was originally proved by using the analysis of Toeplitz operators of Boutet de Monvel and Guillemin [10].

1. Introduction

Let (X,ω) be a compact symplectic manifold of real dimension $2n$. Assume that there exists a hermitian line bundle L over X endowed with a hermitian connection ∇^L with the property that $\frac{\sqrt{-1}}{2\pi}R^L=\omega$, where $R^L=(\nabla^L)^2$ is the curvature of (L,∇^L). Let E be a hermitian vector bundle E on X.

Let g^{TX} be a riemannian metric on X. Let $J_0: TX\longrightarrow TX$ be the skew–adjoint linear map which satisfies the relation $\omega(u,v)=g^{TX}(J_0u,v)$. Let J be an almost complex structure which is compatible with g^{TX} and ω, and which commutes with J_0. Then one can construct canonically a spin^c Dirac operator D_k acting on $\Omega^{0,\bullet}(X,L^k\otimes E)=\bigoplus_{q=0}^n\Omega^{0,q}(X,L^k\otimes E)$, the direct sum of spaces of $(0,q)$–forms with values in $L^k\otimes E$. Let $\lambda=\inf\limits_{u\in T_x^{(1,0)}X,\, x\in X} R^L_x(u,\overline{u})/|u|^2_{g^{TX}}>0$. One of our main results is as follows:

Theorem 1.1. *There exists $C>0$ such that for $k\in\mathbb{N}$, the spectrum of D_k^2 is contained in the set $\{0\}\cup(2k\lambda-C,+\infty)$. Set $D_k^-=D_k\restriction_{\Omega^{0,odd}}$, then for k large enough, we have*

$$\operatorname{Ker} D_k^-=\{0\}. \tag{1.1}$$

We recover with (1.1) the vanishing result of [6, Theorem 2.3], [11, Theorem 3.2]. Another interesting application is to describe the asymptotics of the spectrum of the metric Laplacian $\Delta_k=(\nabla^{L^k\otimes E})^*\nabla^{L^k\otimes E}$ acting on $\mathcal{C}^\infty(X,L^k\otimes E)$. Define the smooth function $\tau(x)=\sum_j R^L(w_j,\overline{w}_j)>0$, $x\in X$, where $\{w_j\}_{j=1}^n$ is an orthonormal basis of $T_x^{(1,0)}X$.

Corollary 1.2. *The spectrum of the Schrödinger operator $\Delta_k^\#=\Delta_k-k\tau$ is contained in the union $(-a,a)\cup(2k\lambda-b,+\infty)$, where a and b are positive constants independent of k. For k large enough, the number d_k of eigenvalues on the interval $(-a,a)$ satisfies $d_k=\langle \operatorname{ch}(L^k\otimes E)\operatorname{Td}(X),[X]\rangle$. In particular $d_k\sim k^n(\operatorname{rank}E)\operatorname{vol}_\omega(X)$.*

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In the case E is a trivial line bundle, Corollary 1.2 is the main result of Guillemin and Uribe [13, Theorem 2] [1]. The idea in [6], [11], [12], [13] is that one first reduces the problem to a problem on the unitary circle bundle of L^*, then one applies Melin inequality [14, Theorem 22.3.2] to show that $\Delta_k^\#$ is semi–bounded from below. In order to prove [13, Theorem 2], they apply the analysis of Toeplitz structures of Boutet de Monvel–Guillemin [10]. For the interesting applications of [13, Theorem 2], we refer the reader to Borthwick and Uribe [6], [8],[9]. Our proof is based on a direct application of Lichnerowicz formula.

This paper is organized as follows. In Section 2, we recall the construction of the spin^c Dirac operator and prove our main technical result, Theorem 2.5. In Section 3, we prove Theorem 1.1 and Corollary 1.2. In Section 4, we generalize our result to the L^2 case. In particular, we obtain a new proof of [12, Theorem 2.6].

2. The Lichnerowicz formula

Let (X, ω) be a compact symplectic manifold. Let (L, h^L), (E, h^E) be two hermitian vector bundles endowed with hermitian connections ∇^L and ∇^E respectively. Let R^L and R^E be their curvatures. We assume $\operatorname{rank} L = 1$ and $R^L = -2\pi\sqrt{-1}\omega$. Let g^{TX} be an arbitrary riemmanian metric on TX. Let J be an almost complex structure which is compatible with g^{TX} and ω. Then J defines canonically an orientation of X. Let $J_0 : TX \longrightarrow TX$ be the skew–adjoint linear map defined by

$$\omega(u, v) = g^{TX}(J_0 u, v), \quad \text{for} \quad u, v \in TX. \tag{2.1}$$

We assume J commutes with J_0 [16, p.61].

Let $TX^c = TX \otimes_{\mathbb{R}} \mathbb{C}$ denote the complexification of the tangent bundle. The almost complex structure J induces a splitting $TX^c = T^{(1,0)}X \oplus T^{(0,1)}X$, where $T^{(1,0)}X$ and $T^{(0,1)}X$ are the eigenbundles of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. Accordingly, we have a decomposition of the complexified cotangent bundle: $T^*X^c = T^{(1,0)*}X \oplus T^{(0,1)*}X$. The exterior algebra bundle decomposes as $\Lambda T^*X^c = \oplus_{p,q}\Lambda^{p,q}$, where $\Lambda^{p,q} := \Lambda^{p,q}T^*X^c = \Lambda^p(T^{(1,0)*}X) \otimes \Lambda^q(T^{(0,1)*}X)$.

Let ∇^{TX} be the Levi–Civita connection of the metric g^{TX}, and let $\nabla^{1,0}$ and $\nabla^{0,1}$ be the canonical hermitian connections on $T^{(1,0)}X$ and $T^{(0,1)}X$ respectively:

$$\begin{aligned}
\nabla^{1,0} &= \tfrac{1}{4}(1 - \sqrt{-1}J)\, \nabla^{TX}\, (1 - \sqrt{-1}J)\,, \\
\nabla^{0,1} &= \tfrac{1}{4}(1 + \sqrt{-1}J)\, \nabla^{TX}\, (1 + \sqrt{-1}J)\,.
\end{aligned}$$

Set $A_2 = \nabla^{TX} - \left(\nabla^{1,0} \oplus \nabla^{0,1}\right) \in T^*X \otimes \operatorname{End}(TX)$ which satisfies $J\, A_2 = -A_2\, J$.

Let us recall some basic facts about the spin^c Dirac operator on an almost complex manifold [15, Appendix D]. The fundamental $\mathbb{Z}_2$ spinor bundle induced by J is given by $\Lambda^{0,\bullet} = \Lambda^{\text{even}}(T^{(0,1)*}X) \oplus \Lambda^{\text{odd}}(T^{(0,1)*}X)$. For any $v \in TX$ with decomposition $v =$

[1] In [13], they only knew $d_k \sim k^n \operatorname{vol}_\omega(X)$. When $J_0 = J$, Borthwick and Uribe [6, p. 854] got the precise value d_k, for large enough k, in this case.

$v_{1,0}+v_{0,1}\in T^{(1,0)}X\oplus T^{(0,1)}X$, let $\overline{v}^*_{1,0}\in T^{(0,1)*}X$ be the metric dual of $v_{1,0}$. Then $\mathbf{c}(v)=\sqrt{2}(\overline{v}^*_{1,0}\wedge -i_{v_{0,1}})$ defines the Clifford action of v on $\Lambda^{0,\bullet}$, where $\wedge$ and i denote the exterior and interior product respectively.

Formally, we may think

$$\Lambda^{0,\bullet}=S\left(TX\right)\otimes\left(\det T^{(1,0)}X\right)^{1/2},$$

where $S\left(TX\right)$ is the spinor bundle of the possibly non–existent spin structure on TX, and $\left(\det T^{(1,0)}X\right)^{1/2}$ is the possibly non–existent square root of $\det T^{(1,0)}X$.

Moreover, by [15, pp.397–398], ∇^{TX} induces canonically a Clifford connection on $\Lambda^{0,\bullet}$. Formally, let $\nabla^{S(TX)}$ be the Clifford connection on $S(TX)$ induced by ∇^{TX}, and let $\nabla^{\det}$ be the connection on $(\det T^{1,0}X)^{1/2}$ induced by $\nabla^{1,0}$. Then

$$\nabla^{\text{Cliff}}=\nabla^{S(TX)}\otimes\mathrm{Id}+\mathrm{Id}\otimes\nabla^{\det}.$$

Let $\{w_j\}_{j=1}^n$ be a local orthonormal frame of $T^{(1,0)}X$. Then

$$e_{2j}=\tfrac{1}{\sqrt{2}}(w_j+\overline{w}_j)\quad\text{and}\quad e_{2j-1}=\tfrac{\sqrt{-1}}{\sqrt{2}}(w_j-\overline{w}_j)\,,\quad j=1,\dots,n\,,\tag{2.2}$$

form an orthonormal frame of TX. Let $\{w^j\}_{j=1}^n$ be the dual frame of $\{w_j\}_{j=1}^n$. Let Γ be the connection form of $\nabla^{1,0}\oplus\nabla^{0,1}$ in local coordinates. Then $\nabla^{TX}=d+\Gamma+A_2$. By [15, Theorem 4.14, p.110], the Clifford connection ∇^{Cliff} on $\Lambda^{0,\bullet}$ has the following local form:

$$\begin{aligned}\nabla^{\text{Cliff}}&=d+\tfrac14\sum_{i,j}\big\langle(\Gamma+A_2)e_i,e_j\big\rangle\mathbf{c}(e_i)\mathbf{c}(e_j)\\&=d+\sum_{l,m}\Big\{\big\langle\Gamma w_l,\overline{w}_m\big\rangle\,\overline{w}^l\wedge\,i_{\overline{w}_m}+\\&\qquad\tfrac12\big\langle A_2w_l,w_m\big\rangle\,i_{\overline{w}_l}\,i_{\overline{w}_m}+\tfrac12\big\langle A_2\overline{w}_l,\overline{w}_m\big\rangle\,\overline{w}^l\wedge\,\overline{w}^m\wedge\Big\}.\end{aligned}\tag{2.3}$$

Let $\nabla^{L^k\otimes E}$ be the connection on $L^k\otimes E$ induced by ∇^L,∇^E. Let $\nabla^{\Lambda^{0,\bullet}\otimes L^k\otimes E}$ be the connection on $\Lambda^{0,\bullet}\otimes L^k\otimes E$,

$$\nabla^{\Lambda^{0,\bullet}\otimes L^k\otimes E}=\nabla^{\text{Cliff}}\otimes\mathrm{Id}+\mathrm{Id}\otimes\nabla^{L^k\otimes E}.\tag{2.4}$$

Along the fibers of $\Lambda^{0,\bullet}\otimes L^k\otimes E$, we consider the pointwise scalar product $\langle\cdot,\cdot\rangle$ induced by g^{TX}, h^L and h^E. Let dv_X be the riemannian volume form of (TX,g^{TX}). The L^2 scalar product on $\Omega^{0,\bullet}(X,L^k\otimes E)$, the space of smooth sections of $\Lambda^{0,\bullet}\otimes L^k\otimes E$, is given by

$$(s_1,s_2)=\int_X\langle s_1(x),s_2(x)\rangle\,dv_X(x)\,.\tag{2.5}$$

We denote the corresponding norm with $\|\cdot\|$.

Definition 2.1. The spin^c Dirac operator D_k is defined by

$$D_k=\sum_{j=1}^{2n}\mathbf{c}(e_j)\nabla^{\Lambda^{0,\bullet}\otimes L^k\otimes E}_{e_j}:\Omega^{0,\bullet}(X,L^k\otimes E)\longrightarrow\Omega^{0,\bullet}(X,L^k\otimes E)\,,\tag{2.6}$$

D_k is a formally self–adjoint, first order elliptic differential operator on $\Omega^{0,\bullet}(X, L^k\otimes E)$, which interchanges $\Omega^{0,\mathrm{even}}(X, L^k\otimes E)$ and $\Omega^{0,\mathrm{odd}}(X, L^k\otimes E)$. We denote

$$D_k^+ = D_k \restriction_{\Omega^{0,\mathrm{even}}}, \quad D_k^- = D_k \restriction_{\Omega^{0,\mathrm{odd}}} . \tag{2.7}$$

Let $R^{T^{(1,0)}X}$ be the curvature of $\left(T^{(1,0)}X, \nabla^{1,0}\right)$. Let

$$\begin{aligned} \omega_d &= -\sum_{l,m} R^L(w_l, \overline{w}_m)\, \overline{w}^m \wedge i_{\overline{w}_l}\, , \\ \tau(x) &= \sum_j R^L(w_j, \overline{w}_j)\, . \end{aligned} \tag{2.8}$$

Remark that by (2.1), at $x\in X$, there exists $\{w_i\}_{i=1}^n$ an orthogonal basis of $T^{1,0}X$, such that $J_0 = \sqrt{-1}\ \mathrm{diag}(a_1(x),\cdots, a_n(x))\in \mathrm{End}(T^{(1,0)}X)$, and $a_i(x)>0$ for $i\in\{1,\cdots,n\}$. So

$$\begin{aligned} \omega_d &= -2\pi \sum_l a_l(x)\, \overline{w}^l \wedge i_{\overline{w}_l}\, , \\ \tau(x) &= 2\pi \sum_l a_l(x)\, . \end{aligned} \tag{2.9}$$

The following Lichnerowicz formula is crucial for us.

Theorem 2.2. *The square of the Dirac operator satisfies the equation:*

$$D_k^2 = \left(\nabla^{\Lambda^{0,\bullet}\otimes L^k\otimes E}\right)^* \nabla^{\Lambda^{0,\bullet}\otimes L^k\otimes E} - 2k\omega_d - k\tau + \tfrac{1}{4}K + \mathbf{c}(R), \tag{2.10}$$

where K is the scalar curvature of (TX, g^{TX}), and

$$\mathbf{c}(R) = \sum_{l<m}\left(R^E + \tfrac{1}{2}\operatorname{Tr}\left[R^{T^{(1,0)}X}\right]\right)(e_l, e_m)\,\mathbf{c}(e_l)\,\mathbf{c}(e_m)\, .$$

Proof. By Lichnerowicz formula [3, Theorem 3.52], we know that

$$D_k^2 = \left(\nabla^{\Lambda^{0,\bullet}\otimes L^k\otimes E}\right)^* \nabla^{\Lambda^{0,\bullet}\otimes L^k\otimes E} + \tfrac{1}{4}K + \mathbf{c}(R) + k\sum_{l<m} R^L(e_l, e_m)\,\mathbf{c}(e_l)\,\mathbf{c}(e_m)\, . \tag{2.11}$$

Now, we identify R^L with a purely imaginary antisymmetric matrix $-2\pi\sqrt{-1}J_0 \in \mathrm{End}(TX)$ by (2.1). As $J_0\in \mathrm{End}(T^{1,0}X)$, by [3, Lemma 3.29], we get (2.10). □

Remark 2.3. Let $\mathcal{E} = \mathcal{E}^+\oplus\mathcal{E}^-$ be a Clifford module. Then it was observed by Braverman [11, §9] that, with the same proof of [3, Proposition 3.35], there exists W a vector bundle on X such that $\mathcal{E} = \Lambda^{0,\bullet}\otimes W$ as a $\mathbb{Z}_2$–graded Clifford module.

As a simple consequence of Theorem 2.2, we recover the statement on the drift of spectrum of the metric Laplacian first proved by Guillemin–Uribe [13, Theorem 1], (see also [6, Theorem 2.1], [11, Theorem 4.4]), by passing to the circle bundle of L^* and applying Melin's inequality [14, Theorems 22.3.2–3]. (In [6, Theorem 2.1], [11, Theorem 4.4], one assumes that R^L is everywhere non–vanishing.)

Corollary 2.4. *There exists $C > 0$ such that for $k \in \mathbb{N}$, the metric Laplacian $\Delta_k = \big(\nabla^{L^k\otimes E}\big)^* \nabla^{L^k\otimes E}$ on $\mathcal{C}^\infty(X, L^k \otimes E)$ satisfies :*

$$\Delta_k - k\tau \geqslant -C \tag{2.12}$$

Proof. By (2.10), $s \in \mathcal{C}^\infty(X, L^k \otimes E)$,

$$\|D_k s\|^2 = \|\nabla^{\Lambda^{0,\bullet}\otimes L^k\otimes E} s\|^2 - k(\tau(x)s, s) + \left(\left(\tfrac{1}{4}K + \mathbf{c}(R)\right) s, s\right) . \tag{2.13}$$

From (2.3), we infer that

$$\big\|\nabla^{\Lambda^{0,\bullet}\otimes L^k\otimes E} s\big\|^2 = \big\|\nabla^{L^k\otimes E} s\big\|^2 + \Big\|\sum_{l,m} \langle A_2 \overline{w}_l, \overline{w}_m\rangle\, \overline{w}^l \wedge \overline{w}^m \wedge s\Big\|^2 .$$

and therefore there exists a constant $C > 0$ not depending on k such that

$$0 \leqslant \|D_k s\|^2 \leqslant \big\|\nabla^{L^k\otimes E} s\big\|^2 - k(\tau(x)s, s) + C\|s\|^2 = \big((\Delta_k - k\tau(x))s, s\big) + C\|s\|^2 .$$

□

The following is our main technical result.

Theorem 2.5. *There exists $C > 0$ such that for any $k \in \mathbb{N}$ and any $s \in \Omega^{>0}(X, L^k \otimes E) = \bigoplus_{q\geqslant 1} \Omega^{0,q}(X, L^k \otimes E)$,*

$$\|D_k s\|^2 \geqslant (2k\lambda - C)\|s\|^2 , \tag{2.14}$$

Proof. By (2.10), for $s \in \Omega^{0,\bullet}(X, L^k \otimes E)$,

$$\|D_k s\|^2 = \{\|\nabla^{\Lambda^{0,\bullet}\otimes L^k\otimes E} s\|^2 - k(\tau(x)s, s)\} - 2k(\omega_d s, s) + \left(\left(\tfrac{1}{4}K + \mathbf{c}(R)\right) s, s\right) \tag{2.15}$$

We consider now $s \in \mathcal{C}^\infty(X, L^k \otimes E')$, where $E' = E \otimes \Lambda^{0,\bullet}$. Estimate (2.12) becomes

$$\big\|\nabla^{L^k\otimes E'} s\big\|^2 - k(\tau(x)s, s) \geqslant -C\|s\|^2 , \tag{2.16}$$

If $s \in \Omega^{>0}(X, L^k \otimes E)$, the second term of (2.15), $-2k(\omega_d s, s)$ is bounded below by $2k\lambda\|s\|^2$. While the third term of (2.15) is $O(\|s\|^2)$. The proof of (2.14) is completed. □

3. Applications of Theorem 2.5

Proof of Theorem 1.1. By (2.14), we get immediately (1.1). For the rest, we use the trick of the proof of Mckean–Singer formula.

Let $\mathcal{H}_\mu$ be the spectral space of D_k^2 corresponding to the interval $(0, \mu)$. Let $\mathcal{H}_\mu^+$, $\mathcal{H}_\mu^-$ be the intersections of $\mathcal{H}_\mu$ with the spaces of forms of even and odd degree respectively. Then $\mathcal{H}_\mu = \mathcal{H}_\mu^+ \oplus \mathcal{H}_\mu^-$. Since D_k^+ commutes with the spectral projection, we have a well defined operator $D_k^+ : \mathcal{H}_\mu^+ \longrightarrow \mathcal{H}_\mu^-$ which is obviously injective. But estimate (2.14) implies that $\mathcal{H}_\mu^- = 0$ for every $\mu < 2k\lambda - C$, hence also $\mathcal{H}_\mu^+ = 0$, for this range of μ. Thus $\mathcal{H}_\mu = 0$, for $0 < \mu < 2k\lambda - C$. The proof of our theorem is completed. □

Proof of Corollary 1.2. Let $P_k : \Omega^{0,\bullet}(X, L^k \otimes E) \longrightarrow \mathcal{C}^\infty(X, L^k \otimes E)$ be the orthogonal projection. For $s \in \Omega^{0,\bullet}(X, L^k \otimes E)$, we will denote $s_0 = P_k s$ its 0 – degree component. We will estimate $\Delta^{\#}_k$ on $P_k(\operatorname{Ker} D^+_k)$ and $(\operatorname{Ker} D^+_k)^\perp \cap \mathcal{C}^\infty(X, L^k \otimes E)$.

In the sequel we denote with C all positive constants independent of k, although there may be different constants for different estimates. From (2.13), there exists $C > 0$ such that for $s \in \mathcal{C}^\infty(X, L^k \otimes E)$,

$$\big| \|D_k s\|^2 - (\Delta^{\#}_k s, s) \big| \leqslant C \|s\|^2 . \tag{3.1}$$

Theorem 1.1 and (3.1) show that there exists $b > 0$ such that for $k \in \mathbb{N}$,

$$(\Delta^{\#}_k s, s) \geqslant (2k\lambda - b)\|s\|^2 , \quad \text{for } s \in \mathcal{C}^\infty(X, L^k \otimes E) \cap (\operatorname{Ker} D^+_k)^\perp. \tag{3.2}$$

We focus now on elements from $P_k(\operatorname{Ker} D^+_k)$, and assume $s \in \operatorname{Ker} D^+_k$. Set $s' = s - s_0 \in \Omega^{>0}(X, L^k \otimes E)$. By (2.15), (2.16),

$$-2k(\omega_d s, s) \leqslant C\|s\|^2 . \tag{3.3}$$

We obtain thus [6, Theorem 2.3] (see also [7], [11, Theorem 3.13]) for $k \gg 1$,

$$\|s'\| \leqslant C k^{-1/2} \|s_0\| . \tag{3.4}$$

(from (3.4), they got $\operatorname{Ker} D^-_k = 0$ for $k \gg 1$). In view of (2.15) and (3.4),

$$\|\nabla^{\Lambda^{0,\bullet} \otimes L^k \otimes E} s\|^2 - k(\tau(x)s_0, s_0) \leqslant C\|s_0\|^2 . \tag{3.5}$$

By (2.3),

$$\nabla^{\Lambda^{0,\bullet} \otimes L^k \otimes E} s = \nabla^{L^k \otimes E} s_0 + A'_2 s_2 + \alpha , \tag{3.6}$$

where s_2 is the component of degree 2 of s, A'_2 is a contraction operator comming from the middle term of (2.3), and $\alpha \in \Omega^{>0}(X, L^k \otimes E)$. By (3.5), (3.6), we know

$$\big\|\nabla^{L^k \otimes E} s_0 + A'_2 s_2\big\|^2 - k(\tau(x)s_0, s_0) \leqslant C\|s_0\|^2 , \tag{3.7}$$

and by (3.4), (3.7),

$$\big\|\nabla^{L^k \otimes E} s_0\big\|^2 \leqslant Ck\|s_0\|^2 , \tag{3.8}$$

By (3.4), (3.7) and (3.8), we get

$$\begin{aligned} \big\|\nabla^{L^k \otimes E} s_0 + A'_2 s_2\big\|^2 &\geqslant \big\|\nabla^{L^k \otimes E} s_0\big\|^2 - 2\big\|\nabla^{L^k \otimes E} s_0\big\|\big\|A'_2 s_2\big\| \\ &\geqslant \big\|\nabla^{L^k \otimes E} s_0\big\|^2 - C\|s_0\|^2. \end{aligned} \tag{3.9}$$

Thus, (3.7) and (3.9) yield

$$\big\|\nabla^{L^k \otimes E} s_0\big\|^2 - k(\tau(x)s_0, s_0) \leqslant C\|s_0\|^2 . \tag{3.10}$$

By (2.12) and (3.10), there exists a constant $a > 0$ such that

$$\big|\big(\Delta^{\#}_k s, s\big)\big| \leqslant a\|s\|^2 , \quad s \in P_k(\operatorname{Ker} D^+_k) . \tag{3.11}$$

By (3.4), we know that for $k \gg 1$, $P_k : \operatorname{Ker} D_k^+ \longrightarrow P_k(\operatorname{Ker} D_k^+)$ is bijective, and

$$\mathcal{C}^\infty(X, L^k \otimes E) = P_k(\operatorname{Ker} D_k^+) \oplus (\operatorname{Ker} D_k^+)^\perp \cap \mathcal{C}^\infty(X, L^k \otimes E)\,. \tag{3.12}$$

The proof is now reduced to a direct application of the minimax principle for the operator $\Delta_k^\#$. It is clear that (3.2) and (3.11) still hold for elements in the Sobolev space $W^1(X, L^k \otimes E)$, which is the domain of the quadratic form $Q_k(f) = \left\|\nabla^{L^k\otimes E} f\right\|^2 - k(\tau(x)f, f)$ associated to $\Delta_k^\#$. Let $\mu_1^k \leqslant \mu_2^k \leqslant \cdots \leqslant \mu_j^k \leqslant \cdots$ ($j \in \mathbb{N}$) be the eigenvalues of $\Delta_k^\#$. Then, by the minimax principle [17, pp.76–78],

$$\mu_j^k = \min_{F \subset \operatorname{Dom} Q_k} \max_{f \in F\,,\, \|f\|=1} Q_k(f)\,. \tag{3.13}$$

where F runs over the subspaces of dimension j of $\operatorname{Dom} Q_k$.

By (3.11) and (3.13), we know $\mu_j^k \leqslant a$, for $j \leqslant \dim \operatorname{Ker} D_k^+$. Moreover, any subspace $F \subset \operatorname{Dom} Q_k$ with $\dim F \geqslant \dim \operatorname{Ker} D_k^+ + 1$ contains an element $0 \neq f \in F \cap (\operatorname{Ker} D_k^+)^\perp$. By (3.2), (3.13), we obtain $\mu_j^k \geqslant 2k\lambda - b$, for $j \geqslant \dim \operatorname{Ker} D_k^+ + 1$.

By Theorem 1.1 and Atiyah–Singer theorem [2],

$$\dim \operatorname{Ker} D_k^+ = \operatorname{index} D_k^+ = \langle \operatorname{ch}(L^k \otimes E)\operatorname{Td}(X), [X]\rangle \tag{3.14}$$

where $\operatorname{Td}(X)$ is the Todd class of an almost complex structure compatible with ω. The index is a polynomial in k of degree n and of leading term $k^n(\operatorname{rank} E)\operatorname{vol}_\omega(X)$, where $\operatorname{vol}_\omega(X)$ is the symplectic volume of X.

The proof of our corollary is completed. □

Remark 3.1. If (X, ω) is Kähler and if L, E are holomorphic vector bundles, then $D_k = \sqrt{2}\big(\overline{\partial} + \overline{\partial}^*\big)$ where $\overline{\partial} = \overline{\partial}^{L^k \otimes E}$. D_k^2 preserves the $\mathbb{Z}$–grading of $\Omega^{0,\bullet}$. By using the Bochner–Kodaira–Nakano formula, Bismut and Vasserot [4, Theorem 1.1] proved Theorem 2.5. As $\overline{\partial} : (\operatorname{Ker} D_k^+)^\perp \cap \mathcal{C}^\infty(X, L^k \otimes E) \longrightarrow \Omega^{0,1}(X, L^k \otimes E)$ is injective, we infer

$$2\left\|\overline{\partial} s\right\|^2 \geqslant (2k\lambda - C)\|s\|^2, \text{ for } s \in (\operatorname{Ker} D_k^+)^\perp \cap \mathcal{C}^\infty(X, L^k \otimes E). \tag{3.15}$$

By Lichnerowicz formula [4, (21)], $2\overline{\partial}^*\overline{\partial} = \Delta_k^\# + \frac{1}{4}K + \mathbf{c}(R)$ on $\mathcal{C}^\infty(X, L^k \otimes E)$, and Corollary 1.2 follows immediately. This observation motivated our work.

Remark 3.2. As in [5], we assume that (L, h^L, ∇^L) is a positive Hermitian vector bundle, i.e. the curvature R^L is an $\operatorname{End}(L)$–valued $(1,1)$–form, and for any $u \in T^{(1,0)}X \smallsetminus \{0\}$, $s \in L$, $\langle R^L(s, \overline{s})u, \overline{u}\rangle > 0$. Let $S^k(L)$ be the k^{th} symmetric tensor power of L. Then if we replace L^k in Sections 2, 3 by $S^k(L)$, or by the irreducible representations of L, which are associated with the weight ka (where a is a given weight), when k tends to $+\infty$, the techniques used in our paper still apply.

4. Covering manifolds

We extend in this section our results to covering manifolds.

4.1. **Covering manifolds, von Neumann dimension.** We present here some generalities about elliptic operators on covering manifolds and Γ–dimension. For details, the reader is referred to [1, §4], [18, §1,§3].

Let $\widetilde{X}$ be a paracompact smooth manifold, such that there is a discrete group Γ acting freely on $\widetilde{X}$ having a compact quotient $X = \widetilde{X}/\Gamma$. Let $g^{T\widetilde{X}}$ be a Γ–invariant metric on $T\widetilde{X}$. Let $p : \widetilde{X} \longrightarrow X$ be the projection.

For a Γ–invariant hermitian vector bundle $(\widetilde{F}, h^{\widetilde{F}})$, we denote by $\mathcal{C}^\infty_c(\widetilde{X}, \widetilde{F})$ the space of compactly supported sections. Then $g^{T\widetilde{X}}$, $h^{\widetilde{F}}$ define an L_2–scalar product on $\mathcal{C}^\infty_c(\widetilde{X}, \widetilde{F})$ as in (2.5). The corresponding L_2 space is denoted by $L_2(\widetilde{X}, \widetilde{F})$.

We have a decomposition $L_2(\widetilde{X}, \widetilde{F}) \cong L_2\Gamma \otimes \mathcal{H}$ where $\mathcal{H} = L_2(U, \widetilde{F})$ is the L_2 space over the relatively compact fundamental domain U of the Γ action. This makes $L_2(\widetilde{X}, \widetilde{F})$ into a free Hilbert Γ–module. Since Γ acts by left translations l_γ on $L_2\Gamma$, we obtain a unitary action of Γ on $L_2(\widetilde{X}, \widetilde{F})$ by left translations $L_\gamma = l_\gamma \otimes \mathrm{Id}$. We will consider in the sequel closed Γ–invariant subspaces of $L_2(\widetilde{X}, \widetilde{F})$ for this action, called (projective) Γ–modules.

Let $\mathcal{A}_\Gamma$ be the von Neumann algebra which consists of all bounded linear operators in $L^2\Gamma \otimes \mathcal{H}$ which commute to the action of Γ. Let $\mathcal{R}_\Gamma$ be the von Neumann algebra of all bounded operators on $L^2\Gamma$ which commute with all l_γ. Then $\mathcal{R}_\Gamma$ is generated by all right translations. Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded operators on $\mathcal{H}$. Then $\mathcal{A}_\Gamma = \mathcal{R}_\Gamma \otimes \mathcal{B}(\mathcal{H})$.

If we consider the orthonormal basis $(\delta_\gamma)_\gamma$ in $L^2\Gamma$, where δ_γ is the Dirac delta function at $\gamma \in \Gamma$, then the matrix of any operator $A \in \mathcal{R}_\Gamma$ has the property that all its diagonal elements are equal. Therefore we define a natural trace on $\mathcal{R}_\Gamma$ as the diagonal element, that is, $\mathrm{tr}_\Gamma A = (A\delta_e, \delta_e)$ where e is the neutral element. Let Tr be the usual trace on $\mathcal{B}(\mathcal{H})$, then we define a trace on $\mathcal{A}_\Gamma$ by $\mathrm{Tr}_\Gamma = \mathrm{tr}_\Gamma \otimes \mathrm{Tr}$.

For any closed Γ–invariant space $V \subset L^2\Gamma \otimes \mathcal{H}$ i.e. for any Γ–module, the projection $P_V \in \mathcal{A}_\Gamma$ and we define $\dim_\Gamma V = \mathrm{Tr}_\Gamma P_V$. In general the Γ–dimension is an element of $[0, \infty]$. We also need the following fact [19, p.398].

Proposition 4.1. *Let $A : V_1 \longrightarrow V_2$ be a bounded linear operator between two Γ–modules, commuting with the action of* Γ. *Then* $\operatorname{Ker} A = 0$ *implies* $\dim_\Gamma V_1 \leqslant \dim_\Gamma V_2$.

Consider an elliptic, Γ–invariant, formally self–adjoint differential operator $\widetilde{P}$ defined in the first instance on $\mathcal{C}^\infty_c(\widetilde{X}, \widetilde{F})$. By a theorem of Atiyah [1, Proposition 3.1], the minimal extension of $\widetilde{P}$ (i.e. the operator closure of $\widetilde{P}$) and the maximal extension of $\widetilde{P}$ (i.e. $\widetilde{P}^*$) coincide. Hence

Lemma 4.2 (Atiyah)**.** *$\widetilde{P}$ defined on $\mathcal{C}^\infty_c(\widetilde{X}, \widetilde{F})$ is essentially self–adjoint.*

Therefore $\widetilde{P}$ has a unique self–adjoint extension, namely its closure. From now on, we always work with this extension of $\widetilde{P}$, which we will denote with the same symbol.

Then the self–adjoint extension $\widetilde{P}$, as well as its spectral projections commute with the action of Γ. In particular, the spectral spaces are Γ–modules. For a Borel set $B \subset \mathbb{R}$, we denote by $E(B,\widetilde{P})$ the spectral projection corresponding to the subset B, and for $\mu\in\mathbb{R}$, set $E_\mu(\widetilde{P})=E\big((-\infty,\mu],\widetilde{P}\big)$. We introduce now a quantitative characteristic of the spectrum, namely the von Neumann spectrum distribution function. For $\mu\in\mathbb{R}$, set

$$N_\Gamma(\mu,\widetilde{P}):=\operatorname{Tr}_\Gamma E_\mu(\widetilde{P})=\dim_\Gamma \operatorname{Range} E_\mu(\widetilde{P})\,.$$

It is non–decreasing and the spectrum of $\widetilde{P}$ coincides with the points of growth of $N_\Gamma(\mu,\widetilde{P})$. If $\widetilde{P}$ is semi–bounded from below, we have $\operatorname{Range} E_\mu(\widetilde{P})\subset \operatorname{Dom}\widetilde{P}^m$ for $m\in\mathbb{N}$. Using the uniform Sobolev spaces [18, pp. 511–2], it is easily seen that $\operatorname{Range} E_\mu(\widetilde{P})\subset\mathcal{C}^\infty(\widetilde{X},\widetilde{F})$, so that $E_\mu(\widetilde{P}):L_2(\widetilde{X},\widetilde{F})\longrightarrow\mathcal{C}^\infty(\widetilde{X},\widetilde{F})$ is linear continuous. Let $K_\mu(\widetilde{x},\widetilde{y})$ be the kernel of $E_\mu(\widetilde{P})$ with respect to the riemannian volume $dv_{\widetilde{X}}$ of $g^{T\widetilde{X}}$. By Schwartz kernel theorem, $K_\mu(\widetilde{x},\widetilde{y})$ is smooth. By [1, Lemma 4.16],

$$N_\Gamma(\mu,\widetilde{P})=\operatorname{Tr}_\Gamma E_\mu(\widetilde{P})=\int_U \operatorname{Tr} K_\mu(\widetilde{x},\widetilde{x})\,dv_{\widetilde{X}}<+\infty.$$

4.2. **The spin^c Dirac operator on a covering manifold.** Assume that there exists a Γ–invariant pre–quantum line bundle $\widetilde{L}$ on $\widetilde{X}$ and a Γ–invariant connection $\nabla^{\widetilde{L}}$ such that $\widetilde{\omega}=\frac{\sqrt{-1}}{2\pi}(\nabla^{\widetilde{L}})^2$ is non–degenerate. Let $(\widetilde{E},h^{\widetilde{E}})$ be a Γ–invariant hermitian vector bundle. Let $\nabla^{\widetilde{E}}$ be an Γ-invariant hermitian connection on $\widetilde{E}$. Let $\widetilde{J}$ be an Γ-invariant almost complex structure on $T\widetilde{X}$ such that $\widetilde{J}$ is compatible with $\widetilde{\omega}$ and $g^{T\widetilde{X}}$. Let $\widetilde{J}_0\in\operatorname{End}(TX)$ be defined by

$$\widetilde{\omega}(u,v)=g^{T\widetilde{X}}(\widetilde{J}_0u,v),\quad\text{for}\quad u,v\in T\widetilde{X}.$$

We assume that $\widetilde{J}$ commutes with $\widetilde{J}_0$. Then, $\widetilde{J}_0,g^{T\widetilde{X}},\widetilde{\omega},\widetilde{J}$ are the pull-back of the corresponding objects in Section 2 for $p:\widetilde{X}\to X$.

We use in the sequel the same notation as in Section 2 for the corresponding objects on X. Following Section 2, we introduce the Γ–invariant spin^c Dirac operator $\widetilde{D}_k$ on $\Omega^{0,\bullet}(\widetilde{X},\widetilde{L}^k\otimes\widetilde{E})$ and the Γ–invariant Laplacian $\widetilde{\Delta}_k=\big(\nabla^{\widetilde{L}^k\otimes\widetilde{E}}\big)^*\nabla^{\widetilde{L}^k\otimes\widetilde{E}}$ on $\mathcal{C}^\infty(\widetilde{X},\widetilde{L}^k\otimes\widetilde{E})$. Let $\widetilde{D}_k^+$ and $\widetilde{D}_k^-$ be the restrictions of $\widetilde{D}_k$ to $L_2^{0,\,\mathrm{even}}(\widetilde{X},\widetilde{L}^k\otimes\widetilde{E})$ and $L_2^{0,\,\mathrm{odd}}(\widetilde{X},\widetilde{L}^k\otimes\widetilde{E})$, respectively.

Proposition 4.3. *There exists $C>0$ such that for $k\in\mathbb{N}$, $\widetilde{\Delta}_k-k\cdot\tau\circ p\geqslant -C$ on $L_2(\widetilde{X},\widetilde{L}^k\otimes\widetilde{E})$.*

Proof. By applying Lichnerowicz formula (2.10) for $s\in\mathcal{C}^\infty_c(\widetilde{X},\widetilde{L}^k\otimes\widetilde{E})$, we obtain as in the proof of Corollary 2.4, that there exists $C>0$ such that $\big((\widetilde{\Delta}_k-k\cdot\tau\circ p)s,s\big)\geqslant -C\|s\|^2$. By Lemma 4.2, this is valid for any $s\in\operatorname{Dom}(\widetilde{\Delta}_k-k\cdot\tau\circ p)$. □

In the same vein, we can generalize Theorem 2.5.

Theorem 4.4. *There exists $C>0$ such that for $k\in\mathbb{N}$ and any $s\in\operatorname{Dom}(\widetilde{D}_k)$ with vanishing degree zero component,*

$$\|\widetilde{D}_ks\|^2\geqslant(2k\lambda-C)\,\|s\|^2\,. \tag{4.1}$$

As an immediate application of the estimate (4.1) for the Dirac operator, we get the following asymptotic vanishing theorem which is the main result in [12, Theorem 2.6].

Corollary 4.5. $\operatorname{Ker}\widetilde{D}_k^- = \{0\}$ *for large enough* k.

We have also an analogue of Theorem 1.1.

Corollary 4.6. *There exists* $C>0$ *such that for* $k\in\mathbb{N}$, *the spectrum of* $\widetilde{D}_k^2$ *is contained in the set* $\{0\}\cup(2k\lambda-C,+\infty)$.

Proof. The proof of Theorem 1.1 does not use the fact that the spectrum is discrete. Therefore it applies in this context, too. ☐

We study now the spectrum of the invariant Schrödinger operator $\widetilde{\Delta}_k - k\cdot\tau\circ p$.

Corollary 4.7. *The spectrum of the Schrödinger operator* $\widetilde{\Delta}_k^{\#}=\widetilde{\Delta}_k-k\cdot\tau\circ p$ *is contained in the union* $(-a,a)\cup(2k\lambda-b,+\infty)$, *where* a *and* b *are positive constants independent of* k. *For large enough* k, *the* Γ*–dimension* d_k *of the spectral space* $E\big((-a,a),\widetilde{\Delta}_k^{\#}\big)$ *corresponding to* $(-a,a)$ *satisfies* $d_k=\langle\operatorname{ch}(L^k\otimes E)\operatorname{Td}(X),[X]\rangle$. *In particular* $d_k\sim k^n(\operatorname{rank}E)\operatorname{vol}_\omega(X)$.

Proof. By repeating the proof of Corollary 1.2, we get estimates (3.2) and (3.11) for smooth elements with compact support. Lemma 4.2 yields then

$$\big|\big(\widetilde{\Delta}_k^{\#}s,s\big)\big|\leqslant a\|s_0\|^2\,,\quad s\in\operatorname{Dom}(\widetilde{\Delta}_k^{\#})\cap P_k(\operatorname{Ker}\widetilde{D}_k^+)\,. \tag{4.2a}$$

$$(\widetilde{\Delta}_k^{\#}s,s)\geqslant(2k\lambda-b)\|s\|^2\,,\quad s\in\operatorname{Dom}(\widetilde{\Delta}_k^{\#})\cap(\operatorname{Ker}\widetilde{D}_k^+)^{\perp}\,, \tag{4.2b}$$

Recall that P_k represents the projection $L_2^{0,\bullet}(\widetilde{X},\widetilde{L}^k\otimes\widetilde{E})\longrightarrow L_2^{0,0}(\widetilde{X},\widetilde{L}^k\otimes\widetilde{E})$. Since the curvatures of all our bundles are Γ–invariant, estimate (3.4) extends to the covering context with the same proof. In particular, $P_k:\operatorname{Ker}\widetilde{D}_k^+\longrightarrow P_k(\operatorname{Ker}\widetilde{D}_k^+)$ is bijective, $P_k\restriction_{\operatorname{Ker}\widetilde{D}_k^+}$ and its inverse are bounded. So $P_k(\operatorname{Ker}\widetilde{D}_k^+)$ is closed. By Proposition 4.1,

$$\dim_\Gamma\operatorname{Ker}\widetilde{D}_k^+=\dim_\Gamma P_k(\operatorname{Ker}\widetilde{D}_k^+)\,. \tag{4.3}$$

As in (3.12), we have

$$\operatorname{Dom}(\widetilde{\Delta}_k^{\#})=P_k(\operatorname{Ker}\widetilde{D}_k^+)\oplus(\operatorname{Ker}\widetilde{D}_k^+)^{\perp}\cap\operatorname{Dom}(\widetilde{\Delta}_k^{\#})\,. \tag{4.4}$$

We use now a suitable form of the minimax principle from [19, Lemma 2.4]:

$$N_\Gamma(\mu,\widetilde{\Delta}_k^{\#})=\sup\{\dim_\Gamma V\,:\,V\subset\operatorname{Dom}\widetilde{\Delta}_k^{\#}\,;\ \big(\widetilde{\Delta}_k^{\#}f,f\big)\leqslant\mu\|f\|^2\,,\forall\,f\in V\} \tag{4.5}$$

where V runs over the Γ–modules of $L_2(\widetilde{X},\widetilde{L}^k\otimes\widetilde{E})$.
By (4.1), (4.2a) and (4.5), we get

$$N_\Gamma(a,\widetilde{\Delta}_k^{\#})\geqslant\dim_\Gamma\operatorname{Ker}\widetilde{D}_k^+\,. \tag{4.6}$$

Let us consider $\nu<2k\lambda-b$. We prove that

$$N_\Gamma(\nu,\widetilde{\Delta}_k^{\#})\leqslant\dim_\Gamma\operatorname{Ker}\widetilde{D}_k^+\,. \tag{4.7}$$

Let $V \subset \mathrm{Dom}(\widetilde{\Delta}_k^{\#})$ be an arbitrary Γ–module with $\big(\widetilde{\Delta}_k^{\#}u, u\big) \leqslant \nu\|u\|^2$. If $\dim_\Gamma V > \dim_\Gamma \operatorname{Ker} \widetilde{D}_k^+$, by Proposition 4.1 and (4.4), there exists $0 \neq v \in V \cap (\operatorname{Ker} D_k^+)^\perp$, which in view of (4.2b) is a contradiction. Therefore $\dim_\Gamma V \leqslant \dim_\Gamma \operatorname{Ker} \widetilde{D}_k^+$. By (4.5), we get (4.7).

By (4.6) and (4.7), we know that the function $N_\Gamma(\nu, \widetilde{\Delta}_k^{\#})$ is constant in the interval $\nu \in [a, 2k\lambda - b)$ and equal to $\dim_\Gamma \operatorname{Ker} \widetilde{D}_k^+$. Enlarging a bit a if necessary, we see that the spectrum of $\widetilde{\Delta}_k^{\#}$ is indeed contained in $(-a,a) \cap (2k\lambda - b, +\infty)$, and the Γ–dimension d_k of the spectral space $E\big((-a,a), \widetilde{\Delta}_k^{\#}\big)$ equals $\dim_\Gamma \operatorname{Ker} \widetilde{D}_k^+$.

By Corollary 4.5, $\dim_\Gamma \operatorname{Ker} \widetilde{D}_k^+ = \operatorname{index}_\Gamma \widetilde{D}_k^+$. Moreover, Atiyah's L_2 index theorem [1, Theorem 3.8] shows that $\operatorname{index}_\Gamma \widetilde{D}_k^+ = \operatorname{index} D_k^+$.

By (3.14), the proof is achieved. □

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