ON HOMOGENEOUS SYMMETRIES FOR EVOLUTION SYSTEMS WITH CONSTRAINTS

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ABSTRACT. The sufficient conditions of time independence and commutativity for local and nonlocal homogeneous symmetries of a large class of (1+1)-dimensional evolution systems are obtained. In contrast with the majority of known results, the verification of our conditions does not require the existence of master symmetry or hereditary recursion operator for the system in question. We also give simple sufficient conditions for the existence of infinite sets of time-independent homogeneous symmetries for (1+1)-dimensional evolution systems within the master symmetry approach.

1. INTRODUCTION

Most of known today integrable systems are homogeneous with respect to some scaling. In this case one looses no generality in assuming the homogeneity of symmetries, master symmetries, recursion operators, etc. This considerably simplifies finding and investigation of such systems, see e.g. [1]–[10]. Moreover, many inhomogeneous systems possess rich sets of homogeneous symmetries.

In this paper we present some new results on the structure of *time-dependent* (cf. e.g. [6, 11, 12, 13, 14] for the time-independent case) formal symmetries for a natural generalization of the systems, considered in [11, 12, 15], namely for (1+1)-dimensional nondegenerate weakly diagonalizable (NWD) evolution systems with constraints. These results turn out to be particularly useful for the study of *homogeneous* (with respect to some scaling) symmetries of such systems. In particular, we have found simple sufficient conditions for the commutativity and time-independence of homogeneous higher order symmetries and for the existence of infinite number of such symmetries for NWD systems with constraints. Note that the overwhelming majority of well known [8, 10, 12] and recently found, see e.g. [7, 16, 17], integrable evolution systems in (1+1) dimensions fits into this class. Most of them are homogeneous, but there is a number of inhomogeneous systems having big sets of homogeneous symmetries as well. Let us mention that our results, unlike the majority of already known ones, can be applied to the systems with time-dependent coefficients, cf. e.g. [18], and are not restricted to scalar evolution equations.

Both the proofs and the application of the results of this paper on commutativity and time independence of homogeneous symmetries do *not* rely on the existence of a master symmetry or e.g. (hereditary) recursion operator. They involve just an easy verification of a few weight-related conditions. In this way we can get rid (cf. [19]) of the tedious direct checks that time-independent symmetries of sufficiently high order commute and that the symmetries with time-independent leading term in fact are time-independent themselves. In addition, our Proposition 5 provides a convenient tool for the proof that a 'candidate' for master symmetry indeed is a nontrivial master symmetry producing the hierarchy of symmetries of growing orders. Let us also stress that the check whether the system under study satisfies the conditions of our propositions is almost entirely algorithmic and thus can be performed using the modern computer algebra software. This makes our results particularly helpful in the study of new integrable systems, for which only a few higher order symmetries

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and (sometimes) a 'candidate' for the master symmetry are known, but no recursion operator is yet found. Moreover, our results, with the exception for Proposition 5, are obviously applicable to non-integrable systems as well.

The paper is organized as follows. In Section 2 we describe the extension of some well-known definitions and facts [11, 12, 15] to the case of explicitly time-dependent evolution systems with constraints. In Section 3 we present the sufficient conditions of well-definiteness of the Lie bracket of two symmetries for the general evolution systems with constraints. Section 4 contains some results on the structure of symmetries and formal symmetries of NWD systems. In Section 5 we present the sufficient conditions of commutativity and time-independence for homogeneous higher order symmetries and for the existence of infinite hierarchies of time-independent homogeneous higher order symmetries for NWD systems with constraints along with the examples illustrating the application of these results.

2. Basic definitions and facts

Consider an evolution system with constraints (cf. [15])

$$\partial \mathbf{u}/\partial t = \mathbf{F}(x, t, \mathbf{u}, \dots, \mathbf{u}_{n'}, \vec{\omega}) \tag{1}$$

for the vector function $\mathbf{u} = (u^1, \dots, u^s)^T$. Here $\mathbf{u}_j = \partial^j \mathbf{u}/\partial x^j$,$\mathbf{u}_0 \equiv \mathbf{u}$ and $\mathbf{F} = (F^1, \dots, F^s)^T$; $\vec{\omega} = (\omega_1, \dots, \omega_c)^T$; ^{T} denotes the matrix transposition. The quantities ω_α, which are usually interpreted as nonlocal variables, are defined here by means of the relations [15, 20]

$$\partial \omega_\alpha/\partial x = X_\alpha(x, t, \mathbf{u}, \mathbf{u}_1, \dots, \mathbf{u}_h, \vec{\omega}), \tag{2}$$

$$\partial \omega_\alpha/\partial t = T_\alpha(x, t, \mathbf{u}, \mathbf{u}_1, \dots, \mathbf{u}_h, \vec{\omega}). \tag{3}$$

We shall denote by Ω the set of nonlocal variables ω_γ, $\gamma = 1, \dots, c$.

Denote by $\mathcal{A}_{j,k}(\Omega)$ the algebra of all locally analytic scalar functions of $x, t, \mathbf{u}, \mathbf{u}_1, \dots, \mathbf{u}_j$, $\omega_1, \dots, \omega_k$ with respect to the standard multiplication. Let $\mathcal{A} \equiv \mathcal{A}(\Omega) = \bigcup_{k=1}^{c} \bigcup_{j=0}^{\infty} \mathcal{A}_{j,k}(\Omega)$, and $\mathcal{A}_{\text{loc}} = \{f \in \mathcal{A} \mid \partial f/\partial \vec{\omega} = 0\}$ be the subalgebra of *local* functions in $\mathcal{A}$. Note that we do not exclude the case $c = \infty$.

The operators of total x- and t-derivatives on $\mathcal{A}$ have the form

$$D \equiv D_x = \frac{\partial}{\partial x} + \sum_{i=0}^{\infty} \mathbf{u}_{i+1} \frac{\partial}{\partial \mathbf{u}_i} + \sum_{\alpha=1}^{c} X_\alpha \frac{\partial}{\partial \omega_\alpha},$$
$$D_t = \frac{\partial}{\partial t} + \sum_{i=0}^{\infty} D^i(\mathbf{F}) \frac{\partial}{\partial \mathbf{u}_i} + \sum_{\alpha=1}^{c} T_\alpha \frac{\partial}{\partial \omega_\alpha}.$$

As in [15, 20], we require that $[D_x, D_t] = 0$ or, equivalently, $D_t(X_\alpha) = D_x(T_\alpha)$ for $\alpha = 1, \dots, c$. We assume that $X_\gamma, T_\gamma \in \mathcal{A}$ for all $\gamma = 1, \dots, c$. We shall denote by $\operatorname{Im} D$ the image of $\mathcal{A}$ under D. Below in this paper (with the exception for Section 3) we make a *blanket assumption* that the kernel of D in $\mathcal{A}$ consists solely of functions of t.

Let $\operatorname{Mat}_p(\mathcal{A})[\![D^{-1}]\!]$ stand for the set of *formal series* in powers of D of the form $\mathfrak{H} = \sum_{j=-\infty}^{q} h_j D^j$, where h_j are $p \times p$ matrices with entries from $\mathcal{A}$, cf. e.g. [11, 12]. For the sake of brevity we shall use the notation $\mathcal{A}[\![D^{-1}]\!]$ instead of $\operatorname{Mat}_1(\mathcal{A})[\![D^{-1}]\!]$.

Recall that the *degree* of formal series $\mathfrak{H} \in \operatorname{Mat}_p(\mathcal{A})[\![D^{-1}]\!]$ is the greatest $m \in \mathbb{Z}$ such that $h_m \neq 0$. It is denoted as $m = \deg \mathfrak{H}$. Following the literature (see e.g. [1]), we assume that $\deg 0 = -\infty$. The formal series $\mathfrak{H}$ of degree m is called *nondegenerate* [12], if $\det h_m \neq 0$. For $\mathfrak{H} = \sum_{j=-\infty}^{m} h_j D^j \in \mathcal{A}[\![D^{-1}]\!]$, $h_m \neq 0$, its *residue* and *logarithmic residue* are defined as $\operatorname{res} \mathfrak{H} = h_{-1}$ and $\operatorname{res} \ln \mathfrak{H} = h_{m-1}/h_m$ [11, 12].

The multiplication law (see e.g. [1])

$$a D^i \circ b D^j = a \sum_{q=0}^{\infty} \frac{i(i-1)\cdots(i-q+1)}{q!} D^q(b) D^{i+j-q}$$

for monomials aD^i, bD^j, $a, b \in \operatorname{Mat}_p(\mathcal{A})$, extended by linearity to the whole $\operatorname{Mat}_p(\mathcal{A})[\![D^{-1}]\!]$, endows the set $\operatorname{Mat}_p(\mathcal{A})[\![D^{-1}]\!]$ by the structure of an associative algebra. The commutator

$[\mathfrak{A},\mathfrak{B}]=\mathfrak{A}\circ\mathfrak{B}-\mathfrak{B}\circ\mathfrak{A}$ makes $\mathrm{Mat}_p(\mathcal{A})[\![D^{-1}]\!]$ into a Lie algebra. Below we omit $\circ$ if this is not confusing.

Following [1, 2, 3], let us say that a vector function $\mathbf{G}\in\mathcal{A}^s$ is a *symmetry* for (1)–(3), if

$$\partial\mathbf{G}/\partial t+[\mathbf{F},\mathbf{G}]=0, \tag{4}$$

where $[\cdot,\cdot]$ is the Lie bracket $[\mathbf{K},\mathbf{H}]=\mathbf{H}'[\mathbf{K}]-\mathbf{K}'[\mathbf{H}]$. Recall that for any (smooth) function $f\in\mathcal{A}^q$ its directional derivative along $\mathbf{H}\in\mathcal{A}^s$ is defined as $f'[\mathbf{H}]=(df(x,t,\mathbf{u}+\epsilon\mathbf{H},\mathbf{u}_1+\epsilon D_x(\mathbf{H}),\dots)/d\epsilon)_{\epsilon=0}$. Note that it is possible to generalize the result of [15] and to show that for any $\mathbf{K}\in\mathcal{A}^s$ we have $\mathbf{K}'\in\mathrm{Mat}_s(\mathcal{A})[\![D^{-1}]\!]$.

In analogy with the notion of order for local functions (cf. e.g. [1, 12]), we shall define the *formal order* of $\vec{f}\in\mathcal{A}^q$ as $\mathrm{ford}\,\vec{f}=\deg\vec{f}'$.

Let $S_F(\mathcal{A})$ denote the set of all symmetries $\mathbf{G}\in\mathcal{A}^s$ for (1)–(3), $S_F^{(k)}(\mathcal{A})=\{\mathbf{G}\in S_F(\mathcal{A})\mid \mathrm{ford}\,\mathbf{G}\le k\}$, $\mathrm{Ann}_F(\mathcal{A})=\{\mathbf{G}\in S_F(\mathcal{A})\mid \partial\mathbf{G}/\partial t=0\}$. In general, unless $\mathcal{A}=\mathcal{A}_{\mathrm{loc}}$, neither $\mathcal{A}^s$ nor $S_F(\mathcal{A})$ are closed under the Lie bracket. However, if $[\mathbf{P},\mathbf{Q}]\in\mathcal{A}^s$ for some $\mathbf{P},\mathbf{Q}\in S_F(\mathcal{A})$, then it is straightforward to verify that $[\mathbf{P},\mathbf{Q}]\in S_F(\mathcal{A})$.

Following [1, 11, 15], we shall call a formal series $\mathfrak{R}=\sum_{j=-\infty}^{r}\eta_j D^j\in\mathrm{Mat}_s(\mathcal{A})[\![D^{-1}]\!]$ the *formal symmetry* of rank m for (1)–(3), if

$$\deg(D_t(\mathfrak{R})-[\mathbf{F}',\mathfrak{R}])\le\deg\mathbf{F}'+\deg\mathfrak{R}-m. \tag{5}$$

Here $D_t(\mathfrak{R})\equiv\sum_{j=-\infty}^{r}D_t(\eta_j)D^j$.

Let $FS_F^{(r)}(\mathcal{A})$ denote the set of all formal symmetries of system (1)–(3) having rank not lower than r. This set is a Lie algebra, because for any $\mathfrak{P}\in FS_F^{(q)}(\mathcal{A})$ and $\mathfrak{Q}\in FS_F^{(q)}(\mathcal{A})$ we obviously have $[\mathfrak{P},\mathfrak{Q}]\in FS_F^{(r)}(\mathcal{A})$ for $r=\min(p,q)$, cf. [12] for local case.

It is well known that (4) is a compatibility condition for (1) and $\partial\mathbf{u}/\partial\sigma=\mathbf{G}$. On the other hand, if $\mathbf{G}\in\mathcal{A}^s$, then $\partial(\partial\mathbf{u}/\partial\sigma)\partial t=D_t(\mathbf{G})$ and $\partial(\partial\mathbf{u}/\partial t)\partial\sigma=\mathbf{F}'[\mathbf{G}]$. Hence, we can rewrite (4) as $D_t(\mathbf{G})=\mathbf{F}'[\mathbf{G}]$.

Let $\mathbf{F}'\equiv\sum\limits_{i=-\infty}^{n}\phi_i D^i$ and $n_0=\begin{cases}1-j, \text{ if } \phi_i=\phi_i(x,t), i=n-j,\dots,n,\\ 2 \text{ otherwise.}\end{cases}$

As $D_t(\mathbf{G})=\mathbf{F}'[\mathbf{G}]$ implies $D_t(\mathbf{G}')-[\mathbf{F}',\mathbf{G}']-\mathbf{F}''[\mathbf{G}]=0$, and $\deg\mathbf{F}''[\mathbf{G}]\le\deg\mathbf{F}'+n_0-2$, we find that $\mathbf{G}'\in FS_F^{(\mathrm{ford}\,\mathbf{G}-n_0+2)}(\mathcal{A})$.

3. On well-definiteness of the Lie bracket

Let $\mathbf{P},\mathbf{Q}\in\mathcal{A}^s$. In general we cannot guarantee (see above) that $[\mathbf{P},\mathbf{Q}]\in\mathcal{A}^s$ or, more broadly, that $\mathbf{Q}_i=\mathrm{ad}^i_{\mathbf{P}}(\mathbf{Q})=[\mathbf{P},\mathbf{Q}_{i-1}]\in\mathcal{A}^s$, $i=1,2,\dots$. In some cases it is possible to make the conditions $\mathrm{ad}^i_{\mathbf{P}}(\mathbf{Q})\in\mathcal{A}^s$ hold, if we introduce new nonlocal variables $\tilde{\omega}_\kappa$ and thus replace $\mathcal{A}$ by a larger algebra $\tilde{\mathcal{A}}$. But in order that $[\mathbf{P},\mathbf{Q}]\in\mathcal{A}^s$ for $\mathbf{P},\mathbf{Q}\in\mathcal{A}^s$ it obviously suffices to require that $\omega'_\mu[\mathbf{P}]\in\mathcal{A}$ for those ω_μ on which $\mathbf{Q}$ actually depends and $\omega'_\nu[\mathbf{Q}]\in\mathcal{A}$ for those ω_ν on which $\mathbf{P}$ actually depends, cf. Ch. 6 in [20].

Moreover, we have

Proposition 1. *Let $\mathbf{P},\mathbf{Q}\in\mathcal{A}^s$, $\omega'_\gamma[\mathbf{Q}]\in\mathcal{A}$ and $\omega'_\gamma[\mathbf{P}]\in\mathcal{A}$ for $\gamma=1,\dots,c$. Then $\mathbf{Q}_l=\mathrm{ad}^l_{\mathbf{P}}(\mathbf{Q})\in\mathcal{A}^s$ for all $l=1,2,\dots$.*

Proof. Let us use induction. To prove that $[\mathbf{P},\mathbf{Q}_l]\in\mathcal{A}^s$, if $\mathbf{Q}_l=[\mathbf{P},\mathbf{Q}_{l-1}]\in\mathcal{A}^s$, it suffices to prove that $\omega'_\nu([\mathbf{P},\mathbf{Q}_{l-1}])\in\mathcal{A}$ for all ω_ν which $\mathbf{P}$ depends on and that $\omega'_\delta[\mathbf{P}]\in\mathcal{A}$ for all ω_δ which $[\mathbf{P},\mathbf{Q}_{l-1}]$ depends on. We have $\omega'_\nu([\mathbf{P},\mathbf{Q}_{l-1}])=(\omega'_\nu[\mathbf{Q}_{l-1}])'[\mathbf{P}]-(\omega'_\nu[\mathbf{P}])'[\mathbf{Q}_{l-1}]$. Hence, it remains to show that $\omega'_\gamma[\mathbf{P}]\in\mathcal{A}$ for all ω_γ which $[\mathbf{P},\mathbf{Q}_{l-1}]$ and $\omega'_\nu[\mathbf{Q}_{l-1}]$ depend on and that $\omega'_\kappa[\mathbf{Q}_{l-1}]\in\mathcal{A}$ for all ω_κ which $\mathbf{P}$ and $\omega'_\nu[\mathbf{P}]$ depend on. For $l=1$ this is obvious (we assume that $\mathbf{Q}_0\equiv\mathbf{Q}$), while for $l=2,3,\dots$ this holds by virtue of inductive hypothesis. Thus, $[\mathbf{P},\mathbf{Q}_l]\in\mathcal{A}^s$, and the result follows. □

If $\mathbf{P}$ is a master symmetry and $\mathbf{Q}$ is a symmetry for (1)–(3), then the above result enables us to ensure that the symmetries $\mathbf{Q}_i$ belong to $\mathcal{A}^s$. Apparently, the conditions of Proposition 1 are satisfied for nearly all known master symmetries of integrable systems (1)–(3) for a suitably chosen set Ω of nonlocal variables ω_γ, so in the majority of cases the action of master symmetries indeed yields the symmetries from $\mathcal{A}^s$. In particular, if $\partial \mathbf{F}/\partial\vec{\omega} = 0$, then the results from Ch. 6 of [20] imply that any $\mathbf{P}, \mathbf{Q} \in S_F(\mathcal{A}(\Omega_{\mathrm{UAC},F}))$ meet the requirements of Proposition 1. Here $\Omega_{\mathrm{UAC},F}$ denotes the set of all nonlocal variables associated with the universal Abelian covering (see [20] for its definition) over (1).

Note that Proposition 1 is obviously valid for more general systems of PDEs with constraints than (1)–(3), if we suitably redefine for them the Lie bracket, the directional derivative and the algebra $\mathcal{A}$.

4. The structure of formal symmetries for NWD systems

Let us consider evolution systems with constraints (1)–(3) having $n \equiv \operatorname{ford}\mathbf{F} \geq 2$ and such that the leading coefficient Φ of the formal series $\mathbf{F}'$ (that is, $\mathbf{F}' \equiv \Phi D^n + \dots$) can be diagonalized by means of a matrix $\Gamma = \Gamma(x, t, \mathbf{u}, \dots, \mathbf{u}_{n'}, \vec{\omega})$ and has s distinct eigenvalues λ_i, i.e., the matrix $\Lambda = \Gamma\Phi\Gamma^{-1}$ is diagonal, cf. [11, 12], $\Lambda \equiv \operatorname{diag}(\lambda_1, \dots, \lambda_s)$, and $\lambda_i \neq \lambda_j$, if $i \neq j$. For systems (1)–(3) with these properties there exists a unique formal series $\mathfrak{T} = \Gamma + \Gamma\sum_{j=1}^{\infty}\Gamma_j D^{-j} \in \mathrm{Mat}_s(\mathcal{A})[\![D^{-1}]\!]$ such that all coefficients of the formal series $\mathfrak{V} = \mathfrak{T}\mathbf{F}'\mathfrak{T}^{-1} + (D_t(\mathfrak{T}))\mathfrak{T}^{-1}$ are diagonal matrices and the diagonal entries of matrices $\Gamma_j, j = 1, 2, \dots,$ are equal to zero. This result is a straightforward generalization of Proposition 3.1 from [11] to the case of evolution systems with constraints. We shall call the systems with constraints (1)–(3) having the above properties and such that $\det\Phi \neq 0$ *nondegenerate weakly diagonalizable (NWD).* If $\mathbf{u}$ is a scalar, i.e. $s = 1$, then any system (1)–(3) with $n \equiv \operatorname{ford}\mathbf{F} \geq 2$ obviously is an NWD system with constraints, and we have $\mathfrak{T} = 1$ and $\mathfrak{V} = \mathbf{F}'$.

Below in this section we assume that (1)–(3) is an NWD system with constraints.

Let (1)–(3) be a NWD system with constraints, and $\mathfrak{R} \in FS_F(\mathcal{A})$ be its formal symmetry of rank m and degree r. The determining equation (5) for $\mathfrak{R}$ under the “gauge” transformation

$$\mathfrak{R} \to \tilde{\mathfrak{R}} = \mathfrak{T}\mathfrak{R}\mathfrak{T}^{-1}, \mathbf{F}' \to \mathfrak{V}, \tag{6}$$

where the formal series $\mathfrak{T}$ and $\mathfrak{V}$ are defined above, goes into

$$\deg(D_t(\tilde{\mathfrak{R}}) - [\mathfrak{V}, \tilde{\mathfrak{R}}]) \leq \deg\mathfrak{V} + \deg\tilde{\mathfrak{R}} - m.$$

This equation is far more convenient for further analysis.

Thus, let us consider the equation

$$\deg(D_t(\mathfrak{P}) - [\mathfrak{V}, \mathfrak{P}]) \leq \deg\mathfrak{V} + \deg\tilde{\mathfrak{P}} - m. \tag{7}$$

Assume that $\mathfrak{P} \equiv \sum\limits_{j=-\infty}^{p}\eta_j D^j$ is a solution for (7). It is clear that for *any* formal series $\mathfrak{P}$ of degree p we have $\deg(D_t(\mathfrak{P}) - [\mathfrak{V}, \mathfrak{P}]) \leq n + p$. If $m > 0$, then equating to zero the coefficient at D^{p+n} in (7) yields

$$[\eta_p, \Lambda] = 0. \tag{8}$$

Since Λ is a diagonal matrix and $\lambda_i \neq \lambda_j$, if $i \neq j$, (8) implies that the matrix η_p is diagonal as well.

Next, if $p > 1$, then, equating to zero the coefficient at D^{p+n-1}, we obtain

$$n\Lambda D(\eta_p) - r\eta_p D(\Lambda) - [\eta_{p-1}, \Lambda] = 0.$$

Taking diagonal and antidiagonal parts of this equation yields

$$n\Lambda D(\eta_p) - p\eta_p D(\Lambda) = 0, \tag{9}$$

$$[\eta_{p-1}, \Lambda] = 0. \tag{10}$$

Let $\eta_p \equiv \operatorname{diag}(\eta_{p,1}, \dots, \eta_{p,s})$. Then (9) reduces to the set of scalar equations of the form

$$n\lambda_I D(\eta_{p,I}) - p\eta_{p,I} D(\lambda_I) = 0, I = 1, \dots, s,$$

where λ_I denote the eigenvalues of the matrix Φ (see above).

The substitution $\eta_{p,I} = h_{p,I}\lambda_I^{p/n}$ further simplifies these equations to $D(h_{p,I}) = 0$. By assumption, $\ker D$ in $\mathcal{A}$ is exhausted by functions of t, so $h_{p,I} = h_{p,I}(t)$ are arbitrary functions of t, and $\eta_{p,I} = h_{p,I}(t)\lambda_I^{p/n}$, $I = 1,\dots,s$. We shall write this in the matrix form as $\eta_p = c_p(t)\Lambda^{p/n}$, where $c_p(t) = \mathrm{diag}(h_{p,1},\dots,h_{p,s})$. Thus, we can represent $\mathfrak{P}$ in the form

$$\mathfrak{P} = c_p(t)\mathfrak{V}^{p/n} + \mathfrak{K}, \tag{11}$$

where $\mathfrak{K}$ is a formal series of degree not higher than $p-1$. Here and below we define the fractional powers of $\mathfrak{V} \equiv \mathrm{diag}(\mathfrak{V}_1,\dots,\mathfrak{V}_s)$ as $\mathfrak{V}^{j/n} = \mathrm{diag}(\mathfrak{V}_1^{j/n},\dots,\mathfrak{V}_s^{j/n})$, and the fractional powers of $\mathfrak{V}_i \in \mathcal{A}[\![D^{-1}]\!]$ are defined like for the local case, cf. e.g. [12]. Let us mention that $[\mathfrak{V}_k^{i/n},\mathfrak{V}_k^{j/n}] = 0$, and hence $[\mathfrak{V}^{i/n},\mathfrak{V}^{j/n}] = 0$ for all integer i,j.

Let us plug (11) into (5). We have $c_p(t)\mathfrak{V}^{p/n} = \mathrm{diag}(h_{p,1}(t)(\mathfrak{V}_1)^{p/n},\dots,h_{p,s}(t)(\mathfrak{V}_s)^{p/n})$, and by virtue of the above we obtain $[c_p(t)\mathfrak{V}^{p/n},\mathfrak{V}] = 0$. Therefore, $\mathfrak{K}$ satisfies the condition $\deg(D_t(\mathfrak{K}) - [\mathfrak{V},\mathfrak{K}]) \le \max(p+n-m, \deg(D_t(c_p(t)\mathfrak{V}^{p/n})))$, whence $\deg(D_t(\mathfrak{K}) - [\mathfrak{V},\mathfrak{K}]) \le p + \max(n-m,0)$. If $m > 2$ and hence $p + \max(n-m,0) < p+n-2$, we can equate to zero the coefficients at D^{p+n-1} and at D^{p+n-2} at the expression $D_t(\mathfrak{K}) - [\mathfrak{V},\mathfrak{K}]$ and repeat the above reasoning.

Iterating these steps, we conclude that if a formal series $\mathfrak{P}$ of degree p satisfies (7) for $m > 1$, then we have

$$\mathfrak{P} = \tilde{\mathfrak{N}} + \sum_{j=j_0}^{p} c_j(t)\mathfrak{V}^{j/n}, \tag{12}$$

where $c_j(t)$ are diagonal $s\times s$ matrices, $\tilde{\mathfrak{N}} \equiv \zeta_{j_0-1}D^{j_0-1} + \zeta_{j_0-2}D^{j_0-2} + \dots$ is some formal series, ζ_{j_0-1} is a diagonal $s\times s$ matrix, $j_0 = p - \min(m,n) + 2$.

Using (6), replacing p by $r = \deg\mathfrak{R}$ and setting $\mathfrak{N} = \mathfrak{T}^{-1}\tilde{\mathfrak{N}}\mathfrak{T}$, we readily see that any formal symmetry $\mathfrak{R} \in FS_F^{(m)}(\mathcal{A})$ of degree r for $m = 2,\dots,n$, can be represented in the form

$$\mathfrak{R} = \mathfrak{T}^{-1}\left(\sum_{j=r-m+2}^{r} c_j(t)\mathfrak{V}^{j/n}\right)\mathfrak{T} + \mathfrak{N},\ \deg\mathfrak{N} < r-m+2. \tag{13}$$

For $m > n$ we can further improve the above results. Namely, equating to zero the coefficient at D^p in $D_t(\mathfrak{P}) - [\mathfrak{V},\mathfrak{P}]$ and taking diagonal and antidiagonal parts of thus obtained equation, we find

$$n\Lambda D(\zeta_{p-n+1}) - (p-n+1)\zeta_{p-n+1}D(\Lambda) = D_t(c_p(t)\Lambda^{p/n}), \tag{14}$$

$$[\Lambda,\zeta_{p-n}] = 0. \tag{15}$$

Making the substitution $\zeta_{p-n+1} = h\Lambda^{\frac{p-n+1}{n}}$, where h is a diagonal $s\times s$ matrix, we obtain from (14) the following equation for h:

$$nD(h) = \dot{c}_p(t)\Lambda^{-1/n} - pc_p(t)D_t(\Lambda^{-1/n}),$$

whence

$$h = c_{p-n+1}(t) + (1/n)D^{-1}(\dot{c}_p(t)\Lambda^{-1/n} - pc_p(t)D_t(\Lambda^{-1/n})).$$

Here $c_{p-n+1}(t)$ is a diagonal $s\times s$ matrix. Its entries are arbitrary functions of t, because $\ker D$ in $\mathcal{A}$ is exhausted by functions of t.

Taking into account (15) and the fact that $\det\Phi \neq 0$ implies $\det\Lambda \neq 0$, we readily see that if a formal series $\mathfrak{B}$ of degree r satisfies (7) with $m > n$, then it can be represented in the form

$$\mathfrak{B} = \tilde{\mathfrak{N}} + \sum_{j=r-n+1}^{r} c_j(t)\mathfrak{V}^{j/n} + (1/n)\left(D^{-1}\left(\dot{c}_r(t)\Lambda^{-1/n} - rc_r(t)D_t(\Lambda^{-1/n})\right)\right)\mathfrak{V}^{\frac{r-n+1}{n}}, \tag{16}$$

where $c_j(t)$ are diagonal $s\times s$ matrices, $\tilde{\mathfrak{N}} \equiv \beta_{r-n}D^{r-n} + \beta_{r-n-1}D^{r-n-1} + \dots$ is some formal series, β_{r-n} is a diagonal $s\times s$ matrix, and dot stands for the *partial* derivative with respect to t.

Setting $\mathfrak{R} = \mathfrak{T}^{-1}\mathfrak{B}\mathfrak{T}$, $\mathfrak{N} = \mathfrak{T}^{-1}\tilde{\mathfrak{N}}\mathfrak{T}$, we readily find that any formal symmetry $\mathfrak{R} \in FS_F^{(n+1)}(\mathcal{A})$ of degree r can be represented in the form

$$\begin{aligned}\mathfrak{R} = \mathfrak{T}^{-1}\left(\sum_{j=r-n+1}^{r} c_j(t)\mathfrak{V}^{j/n}\right)\mathfrak{T} + \frac{1}{n}\mathfrak{T}^{-1}\bigl(D^{-1}\bigl(\dot{c}_r(t)\Lambda^{-1/n} \\ -rc_r(t)D_t(\Lambda^{-1/n})\bigr)\bigr)\mathfrak{V}^{\frac{r-n+1}{n}}\mathfrak{T} + \mathfrak{N},\ \deg\mathfrak{N} < r-n+1.\end{aligned} \tag{17}$$

Note that in the above two formulas we assume that any function of the form $\tilde{h} + a(t)$, where $a(t)$ is an arbitrary function of t, can be taken for $D^{-1}(h)$, if $h = D(\tilde{h})$ and $h, \tilde{h} \in \mathcal{A}$.

Let us mention that (13) and (17) represent a general solution of (5) for $m = 2, \dots, n$ and $m = n+1$, respectively, for *any* NWD system with constraints (1)–(3). Therefore, if at least one entry of the matrix $(\dot{c}_r(t)\Lambda^{-1/n} - rc_r(t)D_t(\Lambda^{-1/n}))$ does not belong to $\operatorname{Im} D$, then the system (1)–(3) has no formal symmetries from $FS_F^{(n+1)}(\mathcal{A})$ for a given matrix $c_r(t)$.

Let $\mathfrak{P}, \mathfrak{Q} \in FS_F^{(n+1)}(\mathcal{A})$. Then $\mathfrak{P} \equiv \mathfrak{T}^{-1}c_p(t)\mathfrak{V}^{p/n}\mathfrak{T} + \cdots$, $\mathfrak{Q} \equiv \mathfrak{T}^{-1}d_q(t)\mathfrak{V}^{q/n}\mathfrak{T} + \cdots$, and the straightforward computation with usage of representations (17) for $\mathfrak{P}$ and $\mathfrak{Q}$ shows that

$$[\mathfrak{P}, \mathfrak{Q}] = \mathfrak{T}^{-1}(1/n)(pc_p(t)\dot{d}_q(t) - qd_q(t)\dot{c}_p(t))\mathfrak{V}^{\frac{p+q-n}{n}}\mathfrak{T} + \mathfrak{K}, \tag{18}$$

where $\mathfrak{K} \in \mathrm{Mat}_s(\mathcal{A})[\![D^{-1}]\!]$ is some formal series, $\deg\mathfrak{K} < p+q-n$.

Let $\mathbf{P}, \mathbf{Q} \in \mathcal{A}^s$, $\mathbf{R} \equiv [\mathbf{P}, \mathbf{Q}]$. Then $\mathbf{R}' = \mathbf{Q}''[\mathbf{P}] - \mathbf{P}''[\mathbf{Q}] - [\mathbf{P}', \mathbf{Q}']$. If $\mathbf{P}, \mathbf{Q} \in S_F(\mathcal{A})$, then (5) and (17) for $\mathfrak{R} = \mathbf{P}'$ and $\mathfrak{R} = \mathbf{Q}'$ imply $\deg\mathbf{P}''[\mathbf{Q}] < p+q-n$ and $\deg\mathbf{Q}''[\mathbf{P}] < p+q-n$ for $p, q > \max(n+n_0-2, 0)$, $p \equiv \operatorname{ford}\mathbf{P}$, $q \equiv \operatorname{ford}\mathbf{Q}$. Hence (18) for $\mathfrak{P} = \mathbf{P}'$, $\mathfrak{Q} = \mathbf{Q}'$ yields

$$[\mathbf{P}, \mathbf{Q}]' = -\mathfrak{T}^{-1}(1/n)(pc_p(t)\dot{d}_q(t) - qd_q(t)\dot{c}_p(t))\mathfrak{V}^{\frac{p+q-n}{n}}\mathfrak{T} + \tilde{\mathfrak{K}}, \tag{19}$$

where $\tilde{\mathfrak{K}} \in \mathrm{Mat}_s(\mathcal{A})[\![D^{-1}]\!]$ is some formal series, $\deg\tilde{\mathfrak{K}} < p+q-n$.

Thus, for $\mathbf{P}, \mathbf{Q} \in S_F(\mathcal{A})$ with $p, q > \max(n+n_0-2, 0)$ we have $\operatorname{ford}\mathbf{R} \le p+q-n$. Clearly, if $\mathbf{R} \in \mathcal{A}^s$, then $\mathbf{R} \in S_F^{(p+q-n)}(\mathcal{A})$ as well. Moreover, $\mathbf{R} \in S_F^{(p+q-n-1)}(\mathcal{A})$, provided $pc_p(t)\dot{d}_q(t) - qd_q(t)\dot{c}_p(t) = 0$.

Let us mention that the formula (19) enables us to compute the leading term of the commutator of two symmetries and thus provides the complete solution to the problem of 'evaluation from the top' for $S_F(\mathcal{A})$, posed by A.M. Vinogradov et al., see e.g. [21].

Let an NWD system (1)–(3) have a nondegenerate formal symmetry $\mathfrak{R} \in \mathrm{Mat}_s(\mathcal{A})[\![D^{-1}]\!]$, $r \equiv \deg\mathfrak{R} \neq 0$, of rank $q > n$. Set $\rho_0^a = \operatorname{res}\ln((\mathfrak{T}\mathfrak{R}\mathfrak{T}^{-1})^{1/r})_{aa}$ and $\rho_j^a = \operatorname{res}((\mathfrak{T}\mathfrak{R}\mathfrak{T}^{-1})^{j/r})_{aa}$ for $j \neq 0$. Then $D_t(\rho_j^a) \in \operatorname{Im} D$ for $a = 1, \dots, s$ and $j = -1, 0, \dots, q-n-2$, i.e., ρ_j^a are conserved densities, cf. [11]. For $n_0 < 2$ we have $\rho_j^a \in \operatorname{Im} D$ (i.e., the densities ρ_j^a are trivial) for all $a = 1, \dots, s$ and $j = -1, 0, \dots, -n_0$.

Proposition 2. *Suppose that an NWD system with constraints (1)–(3) has a nondegenerate formal symmetry* $\mathfrak{R} \in \mathrm{Mat}_s(\mathcal{A})[\![D^{-1}]\!]$, $r \equiv \deg\mathfrak{R} \neq 0$, $q \equiv \operatorname{rank}\mathfrak{R} > n$. *Let for* $a = 1, \dots, s$ *there exist* $m_a \in \{-1, 1, 2, \dots, \min(n-2, q-n-2)\}$ *such that* $m_a \neq 0$, $\rho_{m_a}^a \notin \operatorname{Im} D$ *and* $\rho_j^a \in \operatorname{Im} D$ *for* $j = -1, 1\dots, m_a-1$, $j \neq 0$. *Then for each* $\mathfrak{P} \in FS_F^{(m+n+2)}(\mathcal{A})$, *where* $m = \max\limits_a m_a$, *there exists a constant* $s \times s$ *diagonal matrix* c *such that* $\mathfrak{P} = \mathfrak{T}^{-1}c\mathfrak{R}^{p/r}\mathfrak{T} + \cdots$, $p \equiv \deg\mathfrak{P}$.

Proof. As $\mathfrak{R} \in FS_F^{(n+1)}(\mathcal{A})$, by virtue of (17) we have $\mathfrak{R} = \mathfrak{T}^{-1}h(t)\mathfrak{V}^{r/n}\mathfrak{T} + \cdots$. Furthermore, in complete analogy with (17), for any $\mathfrak{P} \in FS_F^{(n+1)}(\mathcal{A})$ we have (cf. [6, 14])

$$\tilde{\mathfrak{P}} \equiv \mathfrak{T}\mathfrak{P}\mathfrak{T}^{-1} = \sum_{j=p-n+1}^{p} c_j(t)\tilde{\mathfrak{R}}^{j/r} + \frac{1}{n}\bigl(D^{-1}\bigl(\dot{c}_p(t)(h(t))^{n/r}\rho_{-1}\bigr)\bigr)\tilde{\mathfrak{R}}^{\frac{p-n+1}{r}} + \tilde{\mathfrak{N}}. \tag{20}$$

Here $\tilde{\mathfrak{N}} \equiv \sum_{j=-\infty}^{p-n} \tilde{b}_j D^j \in \mathrm{Mat}_s(\mathcal{A})[\![D^{-1}]\!]$, $c_j(t)$, $h(t)$, $\tilde{b}_{p-n}$ are diagonal $s \times s$ matrices, $\rho_{-1} \equiv \operatorname{diag}(\rho_{-1}^1, \dots, \rho_{-1}^s)$, $\tilde{\mathfrak{R}} = \mathfrak{T}\mathfrak{R}\mathfrak{T}^{-1}$. The fractional powers $\tilde{\mathfrak{R}}^{j/r}$ are defined here so that their first r coefficients are diagonal. This is always possible, cf. [11, 12].

For $\mathfrak{P} \in FS_F^{(d)}(\mathcal{A})$ we have $\deg(D_t(\tilde{\mathfrak{P}}) - [\mathfrak{V}, \tilde{\mathfrak{P}}]) \le n + p - d$, and thus $\deg(D_t(\tilde{\mathfrak{P}}_i) - [\mathfrak{V}, \tilde{\mathfrak{P}}_i]) \le n+p+i-\min(q,d)$ for $\tilde{\mathfrak{P}}_i \equiv \tilde{\mathfrak{P}}\tilde{\mathfrak{R}}^{i/r}$. Hence, for $-p-2 < i < \min(q,d)-n-p-1$ we have $\mathrm{res}(D_t(\tilde{\mathfrak{P}}_i) - [\mathfrak{V}, \tilde{\mathfrak{P}}_i]) = 0$.

Let $-p-2 < i < \min(q,d,2n)-n-p-1$. Plug (20) into the equality $\mathrm{res}(D_t(\tilde{\mathfrak{P}}_i)-[\mathfrak{V}, \tilde{\mathfrak{P}}_i]) = 0$ and break it into s scalar equations. Since $\mathrm{res}([\mathfrak{V}, \tilde{\mathfrak{P}}_i])_{aa} \in \mathrm{Im}\, D$ by virtue of Adler's formula (see e.g. [12]), and $D_t(\rho^a_{j+i}) \in \mathrm{Im}\, D$ by assumption, for any $\mathfrak{P} \in FS_F^{(m+n+2)}(\mathcal{A})$ we have $(\dot{c}_p(t))_{aa}\rho^a_{m_a} = 0$ modulo the terms from $\mathrm{Im}\, D$ for all $a = 1, \dots, s$. As $\rho^a_{m_a} \notin \mathrm{Im}\, D$, this implies $\dot{c}_p(t) = 0$, and the result follows. □

Corollary 1. *Under the assumptions of Proposition 2, for any* $\mathbf{G} \in S_F(\mathcal{A})$*,* $k \equiv \mathrm{ford}\, \mathbf{G} \ge m + n + n_0$*, we have* $\mathbf{G}' = \mathfrak{T}^{-1}c\mathfrak{R}^{k/r}\mathfrak{T} + \cdots$*, where* c *is a constant* $s \times s$ *diagonal matrix.*

5. Symmetries of homogeneous NWD systems

Given a scaling vector field $\mathbf{D} = \alpha t\mathbf{F} + x\mathbf{u}_1 + \beta\mathbf{u}$, where $\beta = \mathrm{diag}(\beta_1, \dots, \beta_s)$ is a diagonal matrix, $\alpha, \beta_j = \mathrm{const}$, we can assign the weight $-\alpha$ to the variable t, the weight -1 to x and the weight β_i to u^i, $i = 1, \dots, s$. Assume that the determining equations (2), (3) for ω_γ, $\gamma = 1, \dots, c$, are homogeneous with respect to this weighting, i.e., we can assign some weights μ_γ to all variables ω_γ, $\gamma = 1, \dots, c$, so that (2), (3) are homogeneous. If a formal vector field $\mathbf{G}\partial/\partial\mathbf{u}$ is homogeneous of weight ν with respect to this weighting, then we shall say for short that $\mathbf{G} \in \mathcal{A}^s$ itself is ($\mathbf{D}$-)homogeneous of weight ν and write $\mathrm{wt}(\mathbf{G}) = \nu$. Note that if $\mathbf{G} \in S_F(\mathcal{A})$, then we have $[\mathbf{D}, \mathbf{G}] = \nu\mathbf{G}$.

The commutator of two $\mathbf{D}$-homogeneous symmetries from $S_F(\mathcal{A})$ clearly is a $\mathbf{D}$-homogeneous symmetry from $S_F(\mathcal{A})$ as well, provided it belongs to $\mathcal{A}^s$, so we have the following obvious lemma.

Lemma 1. *Let (1)–(3) be an evolution system with constraints such that (2), (3) are homogeneous with respect to the weighting induced by a scaling* $\mathbf{D} = \alpha t\mathbf{F} + x\mathbf{u}_1 + \beta\mathbf{u}$*, and let* $\mathbf{D}$*-homogeneous* $\mathbf{P}, \mathbf{Q} \in S_F(\mathcal{A})$ *be such that* $[\mathbf{P}, \mathbf{Q}] \in \mathcal{M}$*, where* $\mathcal{M}$ *is a subspace of* $\mathcal{A}^s$*. Suppose that* $\mathrm{wt}(\mathbf{G}) \ne \mathrm{wt}([\mathbf{P}, \mathbf{Q}]) = \mathrm{wt}(\mathbf{P}) + \mathrm{wt}(\mathbf{Q})$ *for all* $\mathbf{D}$*-homogeneous* $\mathbf{G} \in S_F^{(p+q)}(\mathcal{A}) \cap \mathcal{M}$*,* $p \equiv \mathrm{ford}\, \mathbf{P}$*,* $q \equiv \mathrm{ford}\, \mathbf{Q}$*. Then* $[\mathbf{P}, \mathbf{Q}] = 0$.

This result, like other results below, makes it possible to prove the commutativity not only for two given symmetries, but also for large *families* of symmetries. In practice (see examples below) we usually can choose the subspaces like $\mathcal{M}$ sufficiently "large", i.e., such that one can check the condition $[\mathbf{P}, \mathbf{Q}] \in \mathcal{M}$ for all symmetries in the family without actually computing $[\mathbf{P}, \mathbf{Q}]$. On the other hand, the proper choice of these subspaces can considerably reduce the number of weight-related conditions to be verified, and thus make the application of our results truly efficient.

Below in this section we assume that (1)–(3) is an NWD system with constraints and $\mathbf{P}, \mathbf{Q} \in S_F(\mathcal{A})$ are its $\mathbf{D}$-*homogeneous* symmetries for some scaling $\mathbf{D}$ of the form $\mathbf{D} = \alpha t\mathbf{F} + x\mathbf{u}_1 + \beta\mathbf{u}$. We also assume that the determining equations (2), (3) for ω_γ, $\gamma = 1, \dots, c$, are $\mathbf{D}$-homogeneous. Let $p \equiv \mathrm{ford}\, \mathbf{P}$, $q \equiv \mathrm{ford}\, \mathbf{Q}$. Note that if $p, q > \max(n + n_0 - 2, 0)$, then by (19) we should verify the conditions of Lemma 1 only for $\mathbf{G} \in S_F^{(p+q-n)}(\mathcal{A}) \cap \mathcal{M}$ (for $\mathbf{G} \in S_F^{(p+q-n-1)}(\mathcal{A}) \cap \mathcal{M}$, if in addition $pc_p(t)\dot{d}_q(t) - qd_q(t)\dot{c}_p(t) = 0$).

Proposition 3. *Let* $\partial\Phi/\partial t = 0$*,* $\partial X_\gamma/\partial t = \partial T_\gamma/\partial t = 0$*,* $\gamma = 1, \dots, c$*, and all entries of* Φ *be* $\mathbf{D}$*-homogeneous for some* $\mathbf{D} = \alpha t\mathbf{F} + x\mathbf{u}_1 + \beta\mathbf{u}$ *with* $\alpha \ne 0$*. Let* $\mathbf{D}$*-homogeneous* $\mathbf{P}, \mathbf{Q} \in \mathrm{Ann}_F(\mathcal{A})$ *be such that* $[\mathbf{P}, \mathbf{Q}] \in \mathcal{L}$*, where* $\mathcal{L}$ *is a subspace of* $\mathcal{A}^s$*, and* $p, q \ge b_F \equiv \max(n_0, 1)$*,* $p \equiv \mathrm{ford}\, \mathbf{P}$*,* $q \equiv \mathrm{ford}\, \mathbf{Q}$*. Suppose that* $\mathrm{wt}(\mathbf{G}) \ne (p+q)\alpha/n$ *for all* $\mathbf{D}$*-homogeneous* $\mathbf{G} \in S_F^{(n_0-1)}(\mathcal{A}) \cap \mathrm{Ann}_F(\mathcal{A}) \cap \mathcal{L}$*.*

Then $[\mathbf{P}, \mathbf{Q}] = 0$.

Proof. Suppose that $\mathbf{P}, \mathbf{Q} \in \mathrm{Ann}_F(\mathcal{A})$, $[\mathbf{P}, \mathbf{Q}] \in \mathcal{A}^s$, $p, q \ge b_F$. Since $p, q \ge b_F$, using (13) or (17), we easily find that $\deg[\mathbf{P}', \mathbf{Q}'] \le p + q - 1$. It is also easy to show that $\deg \mathbf{P}''[\mathbf{Q}] \le p+q-1$ and $\deg \mathbf{Q}''[\mathbf{P}] \le p+q-1$ for $p, q \ge b_F$. Thus, $\deg[\mathbf{P}, \mathbf{Q}] = \mathrm{ford}[\mathbf{P}, \mathbf{Q}]' =$

$\operatorname{ford}(\mathbf{Q}''[\mathbf{P}]-\mathbf{P}''[\mathbf{Q}]-[\mathbf{P}',\mathbf{Q}'])\leq p+q-1$, and $[\mathbf{P},\mathbf{Q}]\in\mathcal{N}\equiv S_F^{(p+q-1)}(\mathcal{A})\cap\operatorname{Ann}_F(\mathcal{A})$. Eq. (13) or (17) for $\mathfrak{R}=\mathbf{G}'$ imply $\operatorname{wt}(\mathbf{G})=k\alpha/n\neq\operatorname{wt}([\mathbf{P},\mathbf{Q}])=(p+q)\alpha/n$ for all homogeneous $\mathbf{G}\in\mathcal{N}$ with $k\equiv\operatorname{ford}\mathbf{G}\geq n_0$. Hence, $\operatorname{wt}(\mathbf{G})\neq(p+q)\alpha/n$ for all homogeneous $\mathbf{G}\in\mathcal{N}\cap\mathcal{L}\equiv\mathcal{M}$, and thus by Lemma 1 $[\mathbf{P},\mathbf{Q}]=0$. □

It is important to stress that if $\mathbf{D}\in S_F(\mathcal{A})$, then it is often possible to prove that in $S_F(\mathcal{A})\cap\mathcal{L}$ or $\operatorname{Ann}_F(\mathcal{A})\cap\mathcal{L}$ there exists a *basis* made of $\mathbf{D}$-homogeneous symmetries. In this case the usage of Proposition 3 may even allow to prove the commutativity of *all* symmetries of sufficiently high formal order (considered modulo low order ones) from $\operatorname{Ann}_F(\mathcal{A})\cap\mathcal{L}$.

As an example, consider integrable [22] equation $u_t=D^2(u_1^{-1/2})+u_1^{3/2}\equiv K$. It has $n_0=2$, and K is $\mathbf{D}$-homogeneous with $\mathbf{D}=3tK/2+xu_1$. Let $\mathcal{L}=S_K(\mathcal{A}_{\mathrm{loc}})$. The space $S_K^{(1)}(\mathcal{A}_{\mathrm{loc}})\cap\operatorname{Ann}_K(\mathcal{A}_{\mathrm{loc}})$ is spanned by 1 and u_1, and $\operatorname{wt}(1),\operatorname{wt}(u_1)\leq 1<\alpha(p+q)/n=(p+q)/2$ for $p,q\geq b_K=2$. Hence, by Proposition 3 all $\mathbf{D}$-homogeneous time-independent local generalized symmetries of formal order $p>1$ for this equation commute. Furthermore, if we take for $\mathcal{L}'$ the set of x,t,u-independent symmetries of $u_t=K$, then all elements of $\mathcal{L}'$ commute with K,u_1 and 1 by construction. All $\mathbf{D}$-homogeneous elements of $\mathcal{L}'$ having formal order greater than 1 commute by virtue of the above, because $\mathcal{L}'\subset\operatorname{Ann}_K(\mathcal{A}_{\mathrm{loc}})$. As $[u_1,1]=0$, and $\mathcal{L}'$ obviously possesses a basis made of $\mathbf{D}$-homogeneous symmetries only, we conclude that *any* two (not necessarily homogeneous) elements of $\mathcal{L}'$ commute.

Let $n_0\leq 0$, $\partial\Phi/\partial t=0$, $\partial X_\gamma/\partial t=\partial T_\gamma/\partial t=0$, $\gamma=1,\dots,c$, and all entries of Φ are $\mathbf{D}$-homogeneous for some $\mathbf{D}=\alpha t\mathbf{F}+x\mathbf{u}_1+\beta\mathbf{u}$ with $\alpha\neq 0$. Let $\mathbf{D}$-homogeneous $\mathbf{P},\mathbf{Q}\in\operatorname{Ann}_F(\mathcal{A})$ be such that $[\mathbf{P},\mathbf{Q}]\in S_F(\mathcal{A}_{\mathrm{loc}})$, $[\mathbf{P},\mathbf{Q}]$ is x,t-independent and $\operatorname{wt}([\mathbf{P},\mathbf{Q}])\neq 0$. Then $[\mathbf{P},\mathbf{Q}]=0$ by virtue of Proposition 3. Indeed, in this case the weight-related conditions are automatically satisfied, as the only x,t-independent symmetries in $S_F^{(n_0-1)}(\mathcal{A}_{\mathrm{loc}})$ are constant ones, and their weight is zero. E.g., if an NWD system of the form $\mathbf{u}_t=\Phi(x)\mathbf{u}_n+\Psi(x,t)\mathbf{u}_{n-1}+\mathbf{f}(x,t,\mathbf{u},\dots,\mathbf{u}_{n-2})$, where Φ, Ψ are $s\times s$ matrices, is such that all entries of Φ are $\mathbf{D}$-homogeneous for some $\mathbf{D}=\alpha t\mathbf{F}+x\mathbf{u}_1+\beta\mathbf{u}$ with $\alpha\neq 0$, then *all* its $\mathbf{D}$-homogeneous x,t-independent local generalized symmetries of formal order $k>0$ commute.

Proposition 4. *Let (1)–(3) be an NWD system with constraints possessing a symmetry of the form* $\mathbf{D}=\alpha t\mathbf{F}+x\mathbf{u}_1+\beta\mathbf{u}$ *with* $\alpha\neq 0$, *and* $\partial\mathbf{F}/\partial t=0$, $\partial X_\gamma/\partial t=\partial T_\gamma/\partial t=0$, $\gamma=1,\dots,c$*; let* $\mathbf{D}$*-homogeneous* $\mathbf{P}\in S_F(\mathcal{A})$ *be such that* $p\equiv\operatorname{ford}\mathbf{P}\geq n_0$, $\operatorname{ford}\partial\mathbf{P}/\partial t<p$ *and* $[\mathbf{P},\mathbf{F}]\in\mathcal{L}$*, where* $\mathcal{L}$ *is a subspace of* $\mathcal{A}^s$*. Suppose that for all homogeneous* $\mathbf{G}\in S_F^{(p-1)}(\mathcal{A})\cap\mathcal{L}$ *we have* $\operatorname{wt}(\mathbf{G})\neq(p+n)\alpha/n$*. Then* $[\mathbf{P},\mathbf{F}]=0$*, and thus* $\partial\mathbf{P}/\partial t=0$ *and* $\mathbf{P}\in\operatorname{Ann}_F(\mathcal{A})$*.*

Proof. As $\operatorname{ford}\partial\mathbf{P}/\partial t<p$, we have $\partial\mathbf{P}/\partial t=[\mathbf{P},\mathbf{F}]\in S_F^{(p-1)}(\mathcal{A})\cap\mathcal{L}\equiv\mathcal{M}$. The conditions $\operatorname{ford}\partial\mathbf{P}/\partial t<p$ and $p\geq n_0$ by virtue of (13) or (17) for $\mathfrak{R}=\mathbf{P}'$ readily imply $\operatorname{wt}(\mathbf{P})=p\alpha/n$. Hence $\operatorname{wt}([\mathbf{P},\mathbf{F}])=(p+n)\alpha/n$, and thus by Lemma 1 $[\mathbf{P},\mathbf{F}]=0$. □

Suppose that (1)–(3) is an NWD system with constraints having $n_0\leq 0$ and possessing a scaling symmetry $\mathbf{D}=\alpha t\mathbf{F}+x\mathbf{u}_1+\beta\mathbf{u}$ with $\alpha>0$, and $\partial\mathbf{F}/\partial t=0$, $\partial X_\gamma/\partial t=\partial T_\gamma/\partial t=0$, $\gamma=1,\dots,c$. Then by Proposition 4 *any* homogeneous symmetry $\mathbf{K}\in S_F^{(n_0-1)}(\mathcal{A}_{\mathrm{loc}})$ of formal order greater than $k>0$ being polynomial in t and x and such that $\partial^2\mathbf{K}/\partial\mathbf{u}_k\partial t=0$ is in fact time-independent, i.e., $\partial\mathbf{K}/\partial t=0$, and by Proposition 3 any two such symmetries commute. Indeed, if $n_0\leq 0$, then $S_F^{(n_0-1)}(\mathcal{A}_{\mathrm{loc}})$ contains only the symmetries $\mathbf{G}=\mathbf{G}(x,t)$, and all homogeneous symmetries of this form being polynomials in x,t clearly are of negative weight, while $\mathbf{K}$ is of positive weight. This result is valid e.g. for *any* NWD system of the form $\mathbf{u}_t=\Phi(x)\mathbf{u}_n+\Psi(x)\mathbf{u}_{n-1}+\mathbf{f}(x,\mathbf{u},\dots,\mathbf{u}_{n-2})$, where Φ, Ψ are $s\times s$ matrices, provided this system possesses a scaling symmetry $\mathbf{D}=\alpha t\mathbf{F}+x\mathbf{u}_1+\beta\mathbf{u}$ with $\alpha>0$.

Corollary 2. *Let* $\alpha\neq 0$, $\partial\Phi/\partial t=0$ *and* $\partial X_\gamma/\partial t=\partial T_\gamma/\partial t=0$, $\gamma=1,\dots,c$*, and let all entries of* Φ *be* $\mathbf{D}$*-homogeneous for some* $\mathbf{D}=\alpha t\mathbf{F}+x\mathbf{u}_1+\beta\mathbf{u}$ *with* $\alpha\neq 0$*. Suppose that there exist a* $\mathbf{D}$*-homogeneous* $\mathbf{Q}\in\operatorname{Ann}_F(\mathcal{A})$ *and a* $\mathbf{D}$*-homogeneous* $\boldsymbol{\tau}\in\mathcal{A}^s$ *such that* $\partial\boldsymbol{\tau}/\partial t=0$, $\partial[\boldsymbol{\tau},\mathbf{F}]/\partial t=0$, $\mathbf{K}=\boldsymbol{\tau}+t[\boldsymbol{\tau},\mathbf{F}]\in S_F(\mathcal{A})$, $q\equiv\operatorname{ford}\mathbf{Q}>\max(n+n_0-2,0)$, $b\equiv\operatorname{ford}[\boldsymbol{\tau},\mathbf{F}]>\max(\operatorname{ford}\boldsymbol{\tau},n)$*, the formal series* $([\boldsymbol{\tau},\mathbf{F}])'$ *is nondegenerate,* $[[\boldsymbol{\tau},\mathbf{F}],\mathbf{Q}]\in\mathcal{L}$*,* *where* $\mathcal{L}$ *is a subspace of* $\mathcal{A}^s$, $[\boldsymbol{\tau}, \mathbf{Q}] \in \mathcal{A}^s$. *Let also* $\mathrm{wt}(\mathbf{H}) \neq (b+q)\alpha/n$ *for all homogeneous* $\mathbf{H} \in \mathcal{L} \cap S_F^{(n_0-1)}(\mathcal{A}) \cap \mathrm{Ann}_F(\mathcal{A})$.

Then $\mathbf{Q}_1 = [\boldsymbol{\tau}, \mathbf{Q}] \in \mathrm{Ann}_F(\mathcal{A})$, *and* $\mathrm{ford}\, \mathbf{Q}_1 > q$.

Proof. Under the assumptions made, Eq. (4) for $\mathbf{G} = \mathbf{K}$ implies that $[\boldsymbol{\tau}, \mathbf{F}] \in \mathrm{Ann}_F(\mathcal{A})$. Therefore, by Proposition 3 we have $[[\boldsymbol{\tau}, \mathbf{F}], \mathbf{Q}] = 0$, whence, using $[\mathbf{F}, \mathbf{Q}] = 0$ and the Jacobi identity, we find that $[\mathbf{F}, [\boldsymbol{\tau}, \mathbf{Q}]] = 0$, and thus $[\boldsymbol{\tau}, \mathbf{Q}] \in \mathrm{Ann}_F(\mathcal{A})$. By (19) the nondegeneracy of $([\boldsymbol{\tau}, \mathbf{F}])'$ readily implies $\mathrm{ford}[\boldsymbol{\tau}, \mathbf{Q}] = \mathrm{ford}[\mathbf{K}, \mathbf{Q}] = b + q - n > q$. □

Proposition 5. *Suppose that the conditions of Corollary 2 are satisfied,* $\mathrm{ad}^j_{[\boldsymbol{\tau},\mathbf{F}]}(\mathbf{Q}) \in \mathcal{L}_j$, *where* $\mathcal{L}_j$ *are some subspaces of* $\mathcal{A}^s$, $\mathbf{Q}_j \equiv \mathrm{ad}^j_{\boldsymbol{\tau}}(\mathbf{Q}) \in \mathcal{A}^s$, *and* $\mathrm{wt}(\mathbf{H}) \neq ((b-n)j + q + n)\alpha/n$ *for all (homogeneous)* $\mathbf{H} \in \mathcal{L}_j \cap S_F^{(n_0-1)}(\mathcal{A}) \cap \mathrm{Ann}_F(\mathcal{A})$, $j = 2, \ldots, i$.

Then $\mathbf{Q}_j \in \mathrm{Ann}_F(\mathcal{A})$ *and* $\mathrm{ford}\, \mathbf{Q}_j > \mathrm{ford}\, \mathbf{Q}_{j-1}$, $j = 1, \ldots, i$.

This proposition is proved by the repeated application of Corollary 2 for $\mathbf{Q} = \mathbf{Q}_j = \mathrm{ad}^j_{\boldsymbol{\tau}}(\mathbf{Q})$ and repeated use of this corollary for $j = 2, \ldots, i$. Note that we can easily verify that $\mathrm{ad}^j_{\boldsymbol{\tau}}(\mathbf{Q}) \in \mathcal{A}^s$, using Proposition 1.

Using Propositions 1 and 5 and Corollary 2, we can check that $\boldsymbol{\tau}$ indeed is a nontrivial master symmetry, and that it produces a sequence of symmetries of infinitely growing formal orders. Let us stress that in order to apply these results we do not need to assume *a priori* the existence of hereditary recursion operator [5] or e.g. of “negative” master symmetries $\boldsymbol{\tau}_j$, $j < 0$ [19] for the system in question. Thus, the results of the present paper provide a useful complement to the general theory of master symmetries of integrable evolution equations, cf. e.g. [3, 4, 5, 19].

Let us mention that in general the symmetries $\mathbf{Q}_i$ are not obliged to commute. However, using either the results of present paper or other methods, see e.g. [1, 4, 5, 19], one can easily check the commutativity of $\mathbf{Q}_i$ and pick out the commutative subset in the set of $\mathbf{Q}_i$, if necessary.

Note that it is often possible to take $[\boldsymbol{\tau}, \mathbf{F}]$ or $\mathbf{F}$ for $\mathbf{Q}$. In this case the only additional ingredient required for the use of Proposition 5 is a suitable ‘candidate’ $\boldsymbol{\tau}$ for the master symmetry.

As an example, consider integrable Harry Dym equation $u_t = u^3 u_3 \equiv H$, see e.g. [1, 14]. H is homogeneous with respect to $\mathbf{D} = 3tH + xu_1$ and satisfies the conditions of Propositions 1 and 5 for all $i = 2, 3, \ldots$ with $\alpha = 3$, $b = 5$, $\mathcal{A} = \mathcal{A}(\Omega_{\mathrm{UAC},H})$, $\boldsymbol{\tau} = u^3 D^3(u\omega_1) \equiv \boldsymbol{\tau}_0 + u^3 u_3 \omega_1$, $\boldsymbol{\tau}_0 \in \mathcal{A}_{\mathrm{loc}}$, $\mathbf{Q} = [\boldsymbol{\tau}, u^3 u_3] = 3u^5 u_5 + \cdots \in \mathrm{Ann}_H(\mathcal{A})$. The nonlocal variable ω_1 in $\boldsymbol{\tau}$ is defined by means of the relations $\partial\omega_1/\partial t = -uu_2 - u_1^2/2$, $\partial\omega_1/\partial x = u^{-1}$ (informally, $\omega_1 = D^{-1}(u^{-1})$). Thus, by Proposition 5 $\mathbf{Q}_j = \mathrm{ad}^j_{\boldsymbol{\tau}}(\mathbf{Q}) \in \mathrm{Ann}_H(\mathcal{A})$, $j = 1, 2, \ldots$, together with $\mathbf{Q}_{-1} \equiv u^3 u_3 \in \mathrm{Ann}_H(\mathcal{A}_{\mathrm{loc}})$ and $\mathbf{Q}_0 \equiv \mathbf{Q}$ form the infinite hierarchy of time-independent symmetries for the Harry Dym equation. Proposition 3 readily implies the commutativity of $\mathbf{Q}_j$, $j = -1, 0, 1, \ldots$. Moreover, one can show that $\mathbf{Q}_j$, $j = 0, 1, \ldots$, are in fact *local* generalized symmetries of Harry Dym equation and coincide (up to the constant multiples) with the members of hierarchy generated by means of the recursion operator $\mathfrak{R} = u^3 D^3 \circ u \circ D^{-1} \circ u^{-2}$ from the seed symmetry $u^3 u_3$.

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