July 7, 2001

Continuity of the measure of the spectrum for discrete quasiperiodic operators

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Abstract. We study discrete Schrödinger operators $(H_{\alpha,\theta}\psi)(n) = \psi(n-1) + \psi(n+1) + f(\alpha n+\theta)\psi(n)$ on $l^2(Z)$, where $f(x)$ is a real analytic periodic function of period 1. We prove a general theorem relating the measure of the spectrum of $H_{\alpha,\theta}$ to the measures of the spectra of its canonical rational approximants under the condition that the Lyapunov exponents of $H_{\alpha,\theta}$ are positive. For the almost Mathieu operator ($f(x) = 2\lambda\cos 2\pi x$) it follows that the measure of the spectrum is equal to $4|1-|\lambda||$ for all real θ, $\lambda \neq \pm 1$, and all irrational α.

1 Introduction

Consider quasiperiodic operators acting on $l^2(\mathbb{Z})$ and given by:

$$(H_{\alpha,\theta}\psi)(n) = \psi(n-1) + \psi(n+1) + f(\alpha n + \theta)\psi(n), \qquad n = \ldots, -1, 0, 1, \ldots, \tag{1.1}$$

where $f(x)$ is a real analytic periodic function of period 1. Denote by $S(\alpha,\theta)$ the spectrum of $H_{\alpha,\theta}$. For rational $\alpha = p/q$ the spectrum consists of at most q intervals. Let $S(\alpha) = \bigcup_{\theta\in\mathbb{R}} S(\alpha,\theta)$. Note that for irrational α the spectrum of H (as a set) is independent of θ (see, e.g., [6]), and therefore $S(\alpha,\theta) = S(\alpha)$. In this paper we study continuity in α of $S(\alpha)$ and its measure. For sets, we will use $|\cdot|$ to denote the Lebesgue measure.

The fact that various quantities could be easier to analyse and sometimes are even computable for periodic operators, $H_{p/q,\theta}$, makes results on continuity in α particularly important. For example, the Aubry-Andre conjecture on the measure of the spectrum [1] states that for the almost Mathieu operator given by (1.1) with $f(\theta) = 2\lambda\cos 2\pi\theta$, for irrational α and all real λ, θ there is an equality

$$|S_\lambda(\alpha,\theta)| = 4|1-|\lambda||. \tag{1.2}$$

Avron,van Mouche, Simon [3] proved that, for $|\lambda| \neq 1$, $|S_\lambda(p_n/q_n)| \to 4|1-|\lambda||$ as $q_n \to \infty$, and Last [15] established this fact for $|\lambda| = 1$. Given these theorems, the proof of the Aubry-Andre conjecture was reduced to a continuity result.

The continuity in α of $S(\alpha)$ was proven in [2, 7]. Continuity of the measure of the spectrum is a more delicate issue, since, in particular, $|S(\alpha)|$ can be discontinuous at rational α. Establishing continuity at irrational α requires quantitative estimates on the continuity of the spectrum. The first such result, namely the Hölder-$\frac{1}{3}$ continuity was proved in [5], where it was used to establish a zero-measure spectrum (and therefore the Aubry-Andre conjecture) for the almost Mathieu operator with Liouville frequencies α at the critical coupling $\lambda = 1$. That argument was improved to the Hölder-1/2 continuity (for arbitrary $f \in C^1$) in [3] and subsequently used in [14, 15] to establish (1.2) for the almost Mathieu operator for α with unbounded continuous fraction expansion, therefore proving the Aubry-Andre conjecture for a.e. (but not all) α. It was essentially argued in [3] that Hölder continuity of any order larger than 1/2 would imply the desired continuity property of the measure of the spectrum for all α. Such continuity (more precisely, almost Lipschitz continuity, as in Theorem 3) was proved in [11] for the almost Mathieu operator with $|\lambda| > 14.5$ (or dual regime), using exponential localization for Diophantine frequencies and delicate analysis of generalized eigenfunctions (positions of resonances and behavior between them).

In the present paper we prove almost Lipschitz continuity of the spectrum for operators (1.1) with arbitrary analytic f under the assumption of positivity of the Lyapunov exponents. We would like to note that, unlike in [11], we do not rely on the theorem on exponential localization, and the present paper is essentially self-contained. In fact, the localization theorem for general analytic potentials, proved in [4], establishes pure point spectrum for a.e. α without explicit Diophantine control, and therefore would not be applicable for our purposes. On the other hand, we would like to mention that, as the proof below shows, continuity of the spectrum requires simpler arguments than localization.

Let M_n be the transfer-matrix of $H\psi = E\psi$:

$$\begin{pmatrix} \psi(n+1) \\ \psi(n) \end{pmatrix} = M_n(\theta, E) \begin{pmatrix} \psi(n) \\ \psi(n-1) \end{pmatrix}.$$

Then

$$M_n(\theta, E) = \begin{pmatrix} E - f(\alpha n + \theta) & -1 \\ 1 & 0 \end{pmatrix}, \qquad n = \ldots, -1, 0, 1, \ldots.$$

Let $T_n(\theta, E) = M_{n-1}(\theta, E) \cdots M_1(\theta, E) M_0(\theta, E)$ be the n-step transfer-matrix. The Lyapunov exponent $\gamma(E, \alpha)$ of the family (1.1) is defined as follows:

$$\gamma(E, \alpha) \equiv \lim_{n \to \infty} \frac{1}{n} \int_0^1 \ln ||T_n(\theta, E)|| d\theta = \inf_n \frac{1}{n} \int_0^1 \ln ||T_n(\theta, E)|| d\theta. \tag{1.3}$$

It is well defined by (1.3) by Kingman's subadditive ergodic theorem (see, e.g., [6]). Note that $\det M_n(\theta, E) = 1$ whence it follows that $\gamma(E, \alpha) \geq 0$. Our main result is

Theorem 1. *Let $\gamma(E, \alpha) > 0$ for Lebesgue a.e. E. Then for any real θ,*

$$|S(\alpha, \theta)| = \lim_{n \to \infty} \left| S\left(\frac{p_n}{q_n}\right) \right|, \tag{1.4}$$

where p_n/q_n is the sequence of canonical rational approximants to α.

Remarks.

1. Note that a.e. positivity of γ implies that α is irrational.

2. For α whose coefficients of the continued fraction expansion form an unbounded sequence, the condition $\gamma(E, \alpha) > 0$ is redundant. The result of Theorem 1 follows in this case (by a simple generalization of the argument in [14]) from the Hölder-1/2 continuity of the spectrum established in [3]. For α's whose continued fraction coefficients form a bounded sequence (henceforth, we denote the set of such α by Ω) this continuity is not sufficient to imply Theorem 1. For such α we prove the theorem by first establishing the almost-Lipschitz continuity of the spectrum for which, in turn, we use positivity of the Lyapunov exponent.

3. Note that it is not clear apriori that the limit above exists.

4. While our proof can be easily adapted for arbitrary α we, for the reason above, concentrate only on the case $\alpha \in \Omega$, which slightly simplifies certain arguments.

We would like to add that recently a number of remarkable properties of quasiperiodic operators with analytic potentials have been established based on the positivity of the Lyapunov exponents [10, 4, 12, 9]. Theorem 1, therefore, adds to the collection of general corollaries of positive Lyapunov exponents for analytic potentials.

For the almost Mathieu operator Theorem 1 immediately implies

Theorem 2. *For every $\lambda,\ \theta \in \mathbb{R}$, $|\lambda| \neq 1$, and every irrational α, the measure of the spectrum of the almost Mathieu operator $H_{\lambda,\alpha,\theta}$ is equal to $4|1-|\lambda||$.*

This improves some results of [14, 11] and establishes the Aubry-Andre conjecture on the measure of the spectrum for all values of parameters in the noncritical case.

Proof: It was shown in [3] that on a sequence of rational approximants p_n/q_n to any real α we have that $|S_\lambda(p_n/q_n)|$ ($|\lambda| \neq 1$) converges to $4|1-|\lambda||$. Assume $\lambda > 0$ as the case $\lambda < 0$ is easily obtained by the transform $\theta \to \theta + 1/2$ and the case $\lambda = 0$ is trivial. We further assume $\lambda > 1$ as the case $0 < \lambda < 1$ can than be analyzed using duality (see, e.g. [6]). For $\lambda > 1$, it is known (e.g., [6]) that $\gamma(E, \alpha) > 0$ for all real E and irrational α, which concludes the proof in view of Theorem 1. ∎

In Section 2 we establish almost Lipschitz continuity of the spectrum and in Section 3 we complete the proof of Theorem 1.

2 Continuity of the spectrum.

In what follows we shall often omit the indices α, θ, or E from the notation. Denote the basis in which H has the tridiagonal structure (1.1) by $\{e_i\}_{i=-\infty}^{\infty}$. Let $(E-H)_{[x_1,x_2]}$ be the operator $E-H$ restricted to the subspace spanned by $\{e_i\}_{i=x_1}^{x_2}$. Set $k = x_2+1-x_1$. Consider the polynomial in E:

$$\widetilde{P}_k(\alpha,\theta,E) = \det((E-H)_{[0,k-1]}), \qquad \widetilde{P}_0 = 1.$$

Let f_k be the Fourier truncation of $f(\theta)$:

$$f_k(\theta) = \sum_{j=-k^2}^{k^2} \hat{f}_j e^{2\pi i j\theta}. \tag{2.5}$$

By analyticity we have $|f(\theta) - f_k(\theta)| < \exp(-dk^2)$, where $d > 0$ depends on $f(\theta)$, for k sufficiently large. Set $z = \exp(2\pi i\theta)$. Let H_k be the operator H with $f(\theta)$ replaced by $f_k(\theta)$ and consider

$$P_k(\alpha,z,E) = z^{k^3}\det((E-H_k)_{[0,k-1]}). \tag{2.6}$$

Obviously, $P_k(z)$ is a polynomial of degree $2k^3$ in z. Using induction for the three-term recurrence relation (which connects $\widetilde{P}_n$, $\widetilde{P}_{n-1}$, and $\widetilde{P}_{n-2}$) and boundedness of $f(\theta)$, we obtain for all z, $|z| = 1$, and E in a bounded set:

$$|P_k| - e^{-dk^2} < |\widetilde{P}_k| < |P_k| + e^{-dk^2}, \qquad |P_k| < e^{Dk}, \tag{2.7}$$

where $D < \infty$, d is somewhat decreased from the prior value but still positive, and k is larger than some k_1.

We shall now obtain the upper and lower bounds on $|P_k(z, E)|$ which we use later to establish the continuity properties of the spectrum.

Lemma 1. (Estimate of $|P_k(z, E)|$ from below) *For every $\alpha \in \Omega$, $E \in S$, $\varepsilon' > 0$ and at least one integer k out of each three consecutive integers s, $s+1$, $s+2$, $s > k_1$, there exists $C(\varepsilon', \alpha)$ such that in any interval of length Ck^3 there is an integer m such that $|P_k(e^{2\pi i(\theta+m\alpha)}, E)| \geq e^{(\gamma(E)-\varepsilon')k}$. The constant C is independent of E.*

Remark. This statement, appropriately adjusted, holds under any Diophantine condition on α.

Proof: Since, as is easy to verify,

$$T_k(\theta, E) = \begin{pmatrix} \widetilde{P}_k(\theta) & -\widetilde{P}_{k-1}(\theta+\alpha) \\ \widetilde{P}_{k-1}(\theta) & -\widetilde{P}_{k-2}(\theta+\alpha) \end{pmatrix}, \qquad k \geq 2,$$

we can rewrite (1.3) in the form

$$\gamma(E) = \inf_k \frac{1}{k} \int_0^1 \ln \mu_k(\theta) d\theta, \tag{2.8}$$

where $\mu_k(\theta) = \max\{|P_k(\theta)|, |P_{k-1}(\theta)|, |P_{k-1}(\theta+\alpha)|, |P_{k-2}(\theta+\alpha)|\}$. Let $B_k = \{\theta \in [0,1) : \mu_k(\theta) \geq e^{(\gamma(E)-\varepsilon')k}\}$. Then, using (2.7) and (2.8), we obtain

$$k\gamma(E) \leq \int_0^1 \ln \mu_k(\theta) d\theta = \int_{B_k} + \int_{[0,1)\setminus B_k} \leq |B_k| kD + (1-|B_k|)(\gamma(E)-\varepsilon')k.$$

Therefore, $|B_k| \geq \varepsilon'/(D - \gamma(E) + \varepsilon') \geq 3c_1(\varepsilon')$ for all E in any bounded set (in particular, the spectrum of H.) Hence among each three consecutive indices $s, s+1, s+2 > k_1$, there is at least one, denote it k, such that

$$|N_k := \{\theta \in [0,1) : |P_k(e^{2\pi i\theta})| \geq e^{(\gamma(E)-\varepsilon')k}\}| \geq c_1(\varepsilon') > 0$$

Denote by M the set of all such k.

N_k consists of less than $9k^3$ intervals. Let J be the interval in N_k with the maximum length. We have

$$|J| \geq \frac{c_1}{9k^3}. \tag{2.9}$$

It follows from the Diophantine properties of Ω that $\theta + m\alpha (\mathrm{mod}\, 1) \in J$ for at least one $m \in I$, where I is any interval of length Ck^3. For reader's convenience we give a simple argument for the above.

Let $\alpha \in [0,1)$ be an irrational and consider the continued fraction expansion $\alpha = [0; a_1, a_2, \ldots]$, where a_i are positive integers. Recall (e.g.,[13]) that the convergents $p_n/q_n = [0; a_1, a_2, \ldots, a_n]$ have the properties

$$\alpha = \frac{p_n}{q_n} + \frac{\widetilde{\delta}}{q_n q_{n+1}}, \qquad |\widetilde{\delta}| < 1, \qquad q_n = a_n q_{n-1} + q_{n-2}, \qquad n = 3, 4, \ldots. \tag{2.10}$$

and the greatest common divisor $gcd(p_n, q_n) = 1$.

For a fixed n, consider the set $\theta_j = \theta + \alpha j (\mathrm{mod}\, 1)$, $j = 0, 1, \ldots, q_n - 1$. Let θ_j and $\theta_{j'}$ be nearest neighbors. We have, using (2.10),

$$\left|\alpha j - \frac{j p_n}{q_n}\right| = \left|\frac{j\tilde{\delta}}{q_n q_{n+1}}\right| < \frac{j}{q_n q_{n+1}} < \frac{1}{q_n}. \tag{2.11}$$

Since the points $\frac{jp_n}{q_n}(\mathrm{mod}\, 1)$, $j = 0, 1, \ldots, q_n - 1$ are the same as $\frac{j}{q_n}(\mathrm{mod}\, 1)$, $j = 0, 1, \ldots, q_n - 1$, up to the ordering, we have, using (2.11):

$$\max |\theta_j - \theta_{j'}| < \frac{3}{q_n}, \qquad j = 1, 2, \ldots, q_n. \tag{2.12}$$

Therefore, if $c_1/9k^3 \geq 3/q_n$, there exists $0 \leq m \leq q_n - 1$ such that $\theta_m \in J$. Accordingly, choose n so that

$$q_{n-1} < \frac{27k^3}{c_1} \leq q_n. \tag{2.13}$$

For $\alpha \in \Omega$ we have that $a_n \leq B(\alpha)$ for all n. Then

$$q_n = a_n q_{n-1} + q_{n-2} < 2B(\alpha) q_{n-1} < Ck^3. \tag{2.14}$$

where $C = 60B(\alpha)/c_1$. Therefore, at one of any q_n (as defined by (2.13)) of the points θ_m with consecutive indices we have the needed estimate for $P_k(e^{2\pi i \theta})$. Inequality (2.14) implies the statement of the Lemma. ∎

Lemma 2. (Estimate of $|P_k(z,E)|$ from above) *For any $\delta, \varepsilon > 0$, there exists a set $F(\delta,\varepsilon) \subset S$ such that $|F(\delta,\varepsilon)| < \delta$ and for any $\alpha \in \Omega$, $E \in S \setminus F(\delta,\varepsilon)$, and sufficiently large k ($k > k_2(\delta,\alpha,\varepsilon)$) we have $|P_k(z,E)| \leq e^{(\gamma(E)+\varepsilon)k}$. The constant k_2 is independent of z and E.*

Remark. Lemma 2 actually holds uniformly, for all E in a bounded set, as follows from a result of [9]. For a precise statement see [8]. However, this result requires full complicated machinery of [9], while a nonuniform statement above has a simple proof and is sufficient for our purposes.

Proof: Let $\gamma_k(E,\alpha) = \frac{1}{k}\int_0^1 \ln ||T_k(\theta,E)|| d\theta$. It is proved in [4] (Lemma 2.1) that for large k we have $|P_k(z)| < e^{(\gamma_k(E,\alpha)+\varepsilon)k}$ for all $|z| = 1$. Note that Lemma 2.1 from [4] is a relatively simple statement, independent from the rest of the argument in [4].

Let $D_n = \{E \in S : \forall k > n\ \forall |z| = 1,\ |P_k(z,E)| \leq e^{(\gamma(E)+\varepsilon)k}\}$. Then, since $\gamma_k(E) \to \gamma(E)$ for every E, we have $|S \setminus D_n| \to 0$ as $n \to \infty$, which completes the proof of Lemma 2. ∎

We are now ready to formulate the result about continuity of the spectrum.

Theorem 3. *Suppose $\alpha \in \Omega$ and $\gamma(E,\alpha) > 0$ for Lebesgue a.e. E. Then for any $\delta > 0$, there exists a set $A(\delta) \subset S(\alpha)$ such that $|A(\delta)| < \delta$ and for any $E \in S(\alpha) \setminus A(\delta)$ and sufficiently small difference $|\alpha - \alpha'| < h(\alpha, \delta)$ we can find $E' \in S(\alpha')$ such that*

$$|E' - E| < c(\alpha, \delta)|(\alpha - \alpha') \ln^3 |\alpha - \alpha'||. \tag{2.15}$$

The constant c is independent of E, E', α'.

Remarks.

1. The exponent 3 in (2.15) can be replaced by $2 + \epsilon$ with any $\epsilon > 0$, as the reader can easily verify starting with equation (2.5). For the almost Mathieu case, 3 can be replaced by 1.
2. A similar statement (with 3 replaced by a higher power) holds under a standard Diophantine condition on α.
3. This statement can be made uniform in $E \in S$ if we require $\gamma(E)$ to be bounded away from 0 (as in the case of the almost Mathieu operator) and use the uniform version of Lemma 2.

Proof: As in [5, 3, 14, 11], for a given $E \in S(\alpha)$ we construct an approximate eigenvector for H_α. In order to obtain almost-Lipschitz continuity we need the error of approximation to be exponentially small, as in [11].

Fix $\alpha \in \Omega$. Let $G = \{E \in S : \gamma(E) < g\}$ for some $g > 0$. Then $|G| \to 0$ as $g \to 0$. Let g be such that $|G| < \delta/2$. Choose positive ε', ε'' so that $\varepsilon' + \varepsilon'' \le g/16$. Set $A(\delta) = F(\frac{\delta}{2}, \varepsilon'') \cup G$. Fix $E \in S \setminus A(\delta)$. Set $K = \{k \in M, k > 2k_2(\delta, \alpha, \varepsilon'') + 3\}$ with k_2 from Lemma 2, and M defined in the proof of Lemma 1.

For any $E_0 \in S$ let $G(x,y)$ be the matrix elements of $G = ((H - E_0)_{[x_1,x_2]})^{-1}$. Using Cramer's rule and (2.7) at the last step, we obtain:

$$|G(x,x_1)| < \frac{|P_{x_2-x}(e^{2\pi i\theta_{x+1}}, E_0)| + e^{-dk^2}}{|P_k(e^{2\pi i\theta_{x_1}}, E_0)| - e^{-dk^2}}, \qquad |G(x,x_2)| < \frac{|P_{x-x_1}(e^{2\pi i\theta_{x_1}}, E_0)| + e^{-dk^2}}{|P_k(e^{2\pi i\theta_{x_1}}, E_0)| - e^{-dk^2}}, \tag{2.16}$$

where $k = x_2 - x_1 + 1 > k_1$.

We shall now fix x_1, k, and E_0. First take a $k \in K$. By Lemma 2, $|P_{[(k+j)/2]}(z,E)| < \exp{(\gamma(E) + \varepsilon'')(k/2+1)}$ for $j = -3, -2, \ldots, 2$, all z. We can find a neighborhood of E, $r_k(E)$ of diameter smaller than $\exp(-kg/4)$, so that for any $E'' \in r_k(E)$ we have for all z ($|z|$=1): $|P_{[(k+j)/2]}(E'')| < |P_{[(k+j)/2]}(E)| \exp{(\varepsilon''(k/2+1))}$ and $|P_k(E'')| > |P_k(E)| \exp(-\varepsilon' k)$. Now take a generalized eigenvalue $E_0 \in r_k(E)$. Let ψ be a corresponding generalized eigenfunction, that is a formal solution to the equation $H\psi = E_0\psi$ with $|\psi(x)| < \widetilde{C}(|x|+1)$ for all x. By [16] the set of E_0 which admit such solutions supports the spectral measure and is, therefore, dense in the spectrum. There holds:

$$\max_x \frac{|\psi(x)|}{|x|+1} = \frac{|\psi(x_{\max})|}{|x_{\max}|+1} = R < \infty \tag{2.17}$$

for some point $x_{\max}$. We normalize ψ so that $R=1$.

As is easily seen by considering $(H-E_0)_{[x_1,x_2]}\psi$, we have for $x \in [x_1, x_2]$:

$$-\psi(x) = G(x,x_1)\psi(x_1-1) + G(x,x_2)\psi(x_2+1). \tag{2.18}$$

Define the interval $I = [x_{\max} - k - Ck^3, x_{\max} - k]$, where $C = C(\varepsilon', \alpha)$ is the constant from Lemma 1. Then $|I| = Ck^3$, and in view of Lemma 1, we can find $x_1 = m$ in I (which fixes position of the interval $[x_1, x_2]$) such that $|P_k(e^{2\pi i\theta_m}, E)| \geq \exp{(\gamma(E)-\varepsilon')k}$ and hence $|P_k(e^{2\pi i\theta_m}, E_0)| > \exp{(\gamma(E)-2\varepsilon')k}$. We now evaluate $\psi(x)$ at the midpoint of the interval $[x_1,x_2]$: $x_0 = x_1 + [(k-1)/2]$ and at its nearest neighbors. Using Lemma 1 to evaluate the denominator in (2.16), we obtain $|G(x_0,x_1)| < 2\exp{((-\gamma(E)+\varepsilon)k/2 + \gamma(E)+\varepsilon)}$, $\varepsilon = 4(\varepsilon'+\varepsilon'')$ and the same estimate for $G(x_0,x_2)$ for sufficiently large $k \in K$ (depending on d). Applying (2.18), we get

$$|\psi(x_0-1)|, |\psi(x_0)| < 4e^{-(g-\varepsilon)k/2+D+\varepsilon}(|x_{\max}| + (C+2)k^3) \tag{2.19}$$

for sufficiently large $k \in K$.

Now let $I' = [x_{\max}+k, x_{\max}+k+Ck^3]$. Similarly, we choose an interval $[x_1', x_2']$ so that at $x_1' \in I'$ we have a lower bound of Lemma 1. We apply again Lemmas 1 and 2 to get the same estimate (2.19) for $\psi(x_0')$ and $\psi(x_0'+1)$ at the midpoint x_0' of $[x_1', x_2']$ and at $x_0'+1$. By construction, $L = |x_0 - x_0'|$ satisfies $k < L < 2(C+1)k^3$. Set

$$\psi_L(x) = \begin{cases} \psi(x), & x \in [x_0, x_0'] \\ 0, & \text{otherwise} \end{cases}.$$

and $\phi_L(x) = \psi_L(x)/||\psi_L||$. Since $x_{\max} \in [x_0, x_0']$, we have $||\psi_L|| > |x_{\max}|+1$. Hence

$$\begin{aligned} &\frac{|\psi(x_0-1)|}{||\psi_L||}, |\phi_L(x_0)| < 4e^{-(g-\varepsilon)k/2+D+\varepsilon}\frac{|x_{\max}|+(C+2)k^3}{|x_{\max}|+1} < \\ &(C+2)e^{-(g-2\varepsilon)k/2} \end{aligned} \tag{2.20}$$

for $k \in K$, $k > K_0$, where $K_0(\varepsilon, D, d)$ is sufficiently large.

By the variational principle, there exists a point E' in the spectrum of $H_{\alpha',\theta'}$ (here $\theta' = (\alpha-\alpha')x_{\max}+\theta$) such that

$$\begin{aligned} &|E'-E| \leq ||(H_{\alpha',\theta'}-E)\phi_L|| \leq \\ &||(H_{\alpha',\theta'}-H_{\alpha,\theta})\phi_L|| + ||(H_{\alpha,\theta}-E_0)\phi_L|| + |E-E_0| < \\ &C'|\alpha-\alpha'|L + 4(C+2)e^{-(g-2\varepsilon)k/2} + e^{-kg/4} < \\ &C'2(C+1)|\alpha-\alpha'|k^3 + 4(C+3)e^{-gk/4}, \end{aligned} \tag{2.21}$$

where we applied the estimate $|f(\alpha n+\theta)-f(\alpha' n+\theta')|<C'|\alpha-\alpha'|L$ with some $C'>0$ and used that, by our choice of parameters, $2\varepsilon \le g/2$. Let α' be sufficiently close to α, so that $k=[4|\ln|\alpha-\alpha'||/g]$ (or $k\pm 1$) is in K and larger than K_0 and $|\ln|\alpha-\alpha'||$ is sufficiently large depending on values of C, C', and g. Then we obtain from (2.21) the statement of Theorem 3 with $c(\alpha,\delta)=2^7C'(C+1)/g^3+1$. ∎

3 Proof of Theorem 1

As we noted in the introduction, the theorem for $\alpha \notin \Omega$ is easy to prove following [3, 14]. Take $\alpha\in\Omega$ and consider the sequence of its canonical rational approximants p_n/q_n. Because of continuity in θ, the set $S(p_n/q_n)$ consists of at most q_n disjoint intervals, say $S(p_n/q_n)=\cup_{i=1}^{q'_n}[a_i,b_i]$, $q'_n\le q_n$. Given the continuity result of Theorem 3, the proof of Theorem 1 differs from the corresponding analysis for the almost Mathieu operator in [14] only in one detail. By [17, 15] we have $|S(\alpha)|\ge \limsup_{n\to\infty}|S(p_n/q_n)|$. We are going to show that $|S(\alpha)|\le \liminf_{n\to\infty}|S(p_n/q_n)|$ when $\gamma(E)>0$ for a.e. E. For all n sufficiently large (such that $|p_n/q_n-\alpha|<h(\alpha,\delta)$), Theorem 3 says that, except perhaps for the points belonging to $A(\delta)$, all the other points in the spectrum of H_α are necessarily inside the set ($\alpha'=p_n/q_n$)

$$\cup_{i=1}^{q'_n}[a_i-c|(\alpha-\alpha')\ln^3|\alpha-\alpha'||,b_i+c|(\alpha-\alpha')\ln^3|\alpha-\alpha'||].$$

Therefore,

$$|S(\alpha)|<|S(p_n/q_n)|+2cq_n|(\alpha-p_n/q_n)\ln^3|\alpha-p_n/q_n||+\delta \tag{3.22}$$

As $n\to\infty$, the second addend at the right hand side tends to 0 for any irrational α because of (2.10). The constant $\delta>0$ can be taken arbitrary small by Theorem 3. Hence, $|S(\alpha)|=\lim_{n\to\infty}|S(p_n/q_n)|$. ∎

4 Acknowledgements

We would like to acknowledge the support of the NSF grant DMS-9800860. S.J. was also supported under NSF grant DMS-0070755, and I.K. by Sonderforschungsbereich 288. I.K. is grateful to S. Jitomirskaya and A. Klein for their hospitality in the University of California, Irvine, where part of this work was written.

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