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CLEBSCH-GORDAN AND RACAH-WIGNER COEFFICIENTS FOR A CONTINUOUS SERIES OF REPRESENTATIONS OF $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$

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1. INTRODUCTION

Noncompact quantum groups can be expected to lead to very interesting generalizations of the rich and beautiful subject of harmonic analyis on noncompact groups. Important progress has recently been made concerning an abstract (C^*-algebraic) theory of noncompact quantum groups, see [1] for a nice overview and further references. However, an important problem is still the rather limited supply of interesting examples. Results on the harmonic analysis are so far only known for the quantum deformation of the group of motions on the euclidean plane[2, 3], the quantum Lorentz group [5, 6] and $SU_q(1,1)$ [7][8]. Moreover, there sometimes exist subtle analytical obstacles to construct quantum deformations of classical groups such as $SU(1,1)$ on the C^*-algebraic level, cf. [4].

Recently some evidence was presented in [9] that a certain noncompact quantum group with deformation parameter $q = e^{\pi i b^2}$ should describe a crucial internal structure of Liouville theory, a two-dimensional conformal field theory (CFT) that can be seen to be as much a prototype for a CFT with continuous spectrum of Virasoro representations as the harmonic analysis on $SL(2,\mathbb{C})$ is a protoype for noncompact groups. The relation between Liouville theory and that quantum group which was proposed in [9] generalizes the known equivalences between fusion categories of chiral algebras in conformal field theories and braided tensor categories of quantum group representations, cf. e.g. [12, 13]. These equivalences concern the isomorphisms that represent the operation of commuting tensor factors as well as the associativity of tensor products, and can be boiled down to the comparison of certain numerical data, the most non-trivial being some generalization of the Racah-Wigner coefficients (or fusion coefficients in CFT terminology).

The quantum group in question is $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$. A class of "well-behaved" representation of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ on Hilbert-spaces was defined and classified in [10]. We will study a certain subclass of the representations listed there. Some of the representations found in [10] reproduce known represenions of principal or discrete series of $\mathfrak{sl}(2,\mathbb{R})$ in the classical limit $b \to 0$, others do not have a classical limit at all. The representations we will consider are of the latter type. Let us remark that representations that are essentially equivalent to the class of representations dicussed in our paper were recently also discussed in [14]. The main result of the latter paper is a very interesting proposal for a braiding operation on such representations.

In our present paper we will present explicit descriptions for the decomposition of tensor products of these representations into irreducibles, as well as the isomorphism relating two canonical bases for triple tensor products. What appears to be remarkable is the fact that the subseries we have picked out is actually closed under forming tensor products, which one would generally not expect if there exist other unitary representation. The maps describing the decomposition of tensor products lead to the definition and explicit calculation of the generalization of the Racah-Wigner coefficients which represent the central ingredient for the approach of [9] from the mathematics of quantum groups.

From the mathematical point of view one may view our results as providing a technical basis for further studies of a C^* algebraic quantum group that may be generated[1] from $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ and its dual object, which is expected to be a C^* algebraic quantum group generated from $SL_q(2,\mathbb{R})$. In [9] we presented the definition of $SL_q^+(2,\mathbb{R})$ as a quantum space, a C^* algebra $\mathcal{A}^+$ that is generated from $SL_q(2,\mathbb{R})$ and is acted on by analogues of left and right regular representation of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$. An L^2-space was introduced there, and the result describing its decomposition into irreducible representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ (Plancherel decomposition) was announced.

Two aspects of these constructions were unusual: $\mathcal{A}^+$ was introduced such that the elements a, b, c, d generating $SL_q(2,\mathbb{R})$ have *positive* spectrum and the L^2-space was introduced by a measure that has no classical $q \to 1$ limit. It turns out that it is *precisely* the subset of unitary $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ representations studied in the present paper which appears in the Plancherel decomposition of that L^2-space. We view these results as hints towards existence of a rather interesting C^*-algebraic quantum group related to $SL_q(2,\mathbb{R})$ that has no classical counterpart, but other beautiful properties such as a self-duality under $b \to b^{-1}$ which are crucial for the application to Liouville theory [9].

A first hint towards this self-duality can be found in the observation made in [9][14] (see also [15] for closely related earlier observations) that the representations that we consider may alternatively be seen as representations of $\mathcal{U}_{\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$, where $\tilde{q} = e^{\pi i/b^2}$. This led L. Faddeev to the proposal [14] to unify $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ and $\mathcal{U}_{\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ into an object called "modular double", which exhibits the self-duality under $b \to b^{-1}$ in a manifest way. And indeed, it is found in the present paper that the Clebsch-Gordon intertwining maps, as well as the Racah-Wigner coefficients can be constructed in terms of a remarkable special function $S_b(x)$. This special function is closely related to the Barnes Double Gamma function [28], and was more recently independently introduced under the names of "Quantum Dilogarithm" in [16], and as "Quantum Exponential function" in [17]. The function $S_b(x)$ has the property to be self-dual in the sense that it satisfies $S_b(x) = S_{1/b}(x)$. It follows from this self-duality of the function S_b that the Clebsch-Gordan maps constructed in the present paper can be seen as intertwining maps for the "modular double" of L. Faddeev.

We would finally like to point out that our techniques for dealing with finite difference operators that involve shifts by imaginary amounts, in particular the method for determining the spectrum of such an operator, seem to be new and should have generalizations to a variety of other problems where such operators appear. Moreover, the investigation of the class of special functions that we use is fairly recent, so we will need to deduce several previously unknown properties.

[1] In a similar sense as the bounded operators on $L^2(\mathbb{R})$ are generated by the unbounded operators p and q that satisfy $[p, q] = -i$, cf. [11] for more details

The paper is organized as follows: In the following section we will introduce some technical preliminaries. Since we have to deal with finite difference operators that shift the arguments of functions by *imaginary* amounts, a lot of what follows will be based on the theory of functions analytic in certain strips around the real axis, and the description of their Fourier-transforms via results of Paley-Wiener type.

The third section introduces the class of representations that will be studied in the present paper and discusses some of their properties.

This is followed by a section describing the decomposition of tensor products of representations into irreducibles.

We then define and calculate b-Racah Wigner coefficients as the kernel that appears in the integral transformation that establishes the isomorphism between two canonical decompositions of triple tensor products.

Appendix A is in some sense the technical heart of the paper: It contains the spectral analysis of a finite difference operator of second order that is related to the Casimir on tensor products of two representations.

Appendices B and C contain some information on the special functions that are used in the body of the paper.

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2. PRELIMINARIES

We collect some basic conventions, definitions and standard results that will be used throughout the paper.

2.1. Finite difference operators

The quantum group will be realized in terms of finite difference operators that shift the arguments by an *imaginary* amount. On functions $f(x)$, $x \in \mathbb{R}$ that have an analytic continuation to a strip containing $\{x \in \mathbb{C}; \mathrm{Im}(x) \in [-a_-, a_+]\}$, $a_\pm \geq 0$ one may define the finite difference operators T_x^{ia}, $a \in [-a_-, a_+]$ by

$$T_x^{ia} f(x) = f(x + ia). \tag{1}$$

As convenient notation we will use

$$[x]_b \equiv \frac{\sin(\pi b x)}{\sin(\pi b^2)}, \qquad d_x \equiv \frac{1}{2\pi}\partial_x, \qquad [d_x + a]_b \equiv \frac{e^{\pi i b a} T_x^{\frac{ib}{2}} - e^{-\pi i b a} T_x^{-\frac{ib}{2}}}{e^{\pi i b^2} - e^{-\pi i b^2}}. \tag{2}$$

2.2. Fourier-transformation

Our notation and conventions concerning the Fourier-transformations are as follows: Let $\mathcal{S}(\mathbb{R})$ denote the usual Schwartz-space of functions on the real line. The Fourier-transformation of a function

$f \in \mathcal{S}(\mathbb{R})$ will be defined as

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} dx\; e^{-2\pi i\omega x} f(x). \tag{3}$$

The corresponding inversion formula is then

$$f(x) = \int_{-\infty}^{\infty} d\omega\; e^{2\pi i\omega x} \tilde{f}(\omega). \tag{4}$$

The Fourier-transformation maps the finite difference operator T_x^{ia} to the operator of multiplication with $e^{-2\pi a\omega}$. It will therefore be a useful tool for dealing with these operators. Of fundamental importance will be the connection between analyticity of functions in a strip to exponential decay properties of its Fourier-transform and vice versa that is expressed by the classical Paley-Wiener theorem:

THEOREM 1. — *(Paley-Wiener) Let f be in $L^2(\mathbb{R})$. Then $(e^{2\pi x a_+} + e^{-2\pi x a_-})f \in L^2(\mathbb{R})$, $a_\pm > 0$ if and only if $\tilde{f}$ has an analytic continuation to the strip $\{\omega \in \mathbb{C}; Im(\omega) \in (-a_-, a_+)\}$ such that for any $\omega_2 \in (-a_-, a_+)$, $\tilde{f}(. + i\omega_2) \in L^2(\mathbb{R})$ and*

$$\sup_{\omega_2 \leq b} \int_{-\infty}^{\infty} d\omega_1\; |\tilde{f}(\omega_1 + i\omega_2)|^2 \;<\; \infty \quad \textit{for any} \quad b \in (-a_-, a_+). \tag{5}$$

Proof. — Cf. e.g. [19]. □

The following simple variant of this result will often be useful:

LEMMA 1. — *For $f \in \mathcal{S}(\mathbb{R})$, the following two conditions are equivalent:*

(1) *f is the restriction to $\mathbb{R}$ of a function F that is meromorphic in the strip $\{z \in \mathbb{C}; Im(z) \in (-a_-, a_+)\}$, $a_+, a_- > 0$ with finitely many poles in the upper (lower) half plane at $\mathcal{P}_\pm \equiv \{z_j; j \in \mathcal{I}_\pm\}$, $|\mathrm{Im}(z_j)| > 0$, and all functions $F_y(x) \equiv F(x + iy)$, $y \in (-a_-, a_+)$ are of rapid decrease, and*
(2) *one has the following asymptotic behavior of the Fourier-transform $\tilde{f}(\omega)$ for $\omega \to \pm\infty$:*

$$\begin{aligned} \tilde{f}(\omega) \;&=\; -\,2\pi i \sum_{j \in \mathcal{I}_-} e^{-2\pi i z_j \omega} \operatorname*{Res}_{z=z_j} F(z) + \tilde{f}_{a_+}(\omega) \\ \tilde{f}(\omega) \;&=\; +\,2\pi i \sum_{j \in \mathcal{I}_+} e^{-2\pi i z_j \omega} \operatorname*{Res}_{z=z_j} F(z) + \tilde{f}_{a_-}(\omega), \end{aligned}$$

where $\tilde{f}_{a_\pm}(\omega)$ decay as $x \to \pm\infty$ faster than $e^{-2\pi a|\omega|}$ for any $a \in (-a_-, a_+)$.

2.3. **Distributions**

Let $\mathcal{S}'(\mathbb{R})$ be the space of tempered distributions on $\mathcal{S}(\mathbb{R})$. The dual pairing between a distributions $\Phi \in \mathcal{S}'(\mathbb{R})$ and a function $f \in \mathcal{S}(\mathbb{R})$ will be denoted by $\langle \Phi, f \rangle$. The Fourier transformation on $\mathcal{S}'(\mathbb{R})$ is defined by $\langle\tilde{\Phi},\tilde{f}\rangle \equiv \langle\Phi,f\rangle$ for any $f\in\mathcal{S}(\mathbb{R})$. It should be noted that if a distribution $\Phi\in\mathcal{S}'(\mathbb{R})$ actually happens to be represented by a function $\Phi(x)$ via

$$\langle\Phi,f\rangle = \int_{-\infty}^{\infty} dx\ \Phi(x)f(x)$$

then our definition of the Fourier-transform of Φ implies that instead of (4) one has the following inversion formula for $\Phi(x)$:

$$\Phi(x) = \int_{-\infty}^{\infty} d\omega\ e^{-2\pi i\omega x}\tilde{\Phi}(\omega). \tag{6}$$

The distributions that appear below will all be defined in terms of meromorphic functions by means of the so-called $i\epsilon$-prescription: Assume given a familiy of functions Φ_ϵ, $\epsilon>0$ that are meromorphic in some strip containing $\mathbb{R}$, rapidly decreasing at infinity and have finitely many poles with ϵ-independent residues at a distance ϵ from the real axis. The limit $\Phi\equiv\lim_{\epsilon\to0}\Phi_\epsilon$ then defines a distribution $\Phi\in\mathcal{S}'(\mathbb{R})$. We will often use the symbolic notation $\Phi(x)$ for the resulting distribution, keeping in mind that $\Phi(x)$ will not be defined for all $x\in\mathbb{R}$.

There is a simple generalization of Lemma 1 to such distributions in $\mathcal{S}'(\mathbb{R})$: Poles on the real axis correspond to asymptotic behavior of the form $e^{2\pi i\omega x}$ of the Fourier-transform:

LEMMA 2. — *For $\Phi\in\mathcal{S}'(\mathbb{R})$, the following two conditions are equivalent:*

(1) $\Phi=\lim_{\epsilon\to0}\Phi_\epsilon$, *where Φ_ϵ is for $\epsilon>0$ represented as the restriction to $\mathbb{R}$ of a function $\Phi_\epsilon(x)$ that is meromorphic in the strip $\{z\in\mathbb{C}; Im(z)\in(-a_-,a_+)\}$, $a_+,a_->0$ with finitely many poles in the upper (lower) half plane at $\mathcal{P}^\epsilon_\pm\equiv\{z_j\pm i\epsilon; j\in\mathcal{I}_\pm\}$, $\pm\mathrm{Im}(z_j)\geq0$, and all functions $\Phi_{\epsilon,y}(x)\equiv\Phi_\epsilon(x+iy)$, $x,y\in\mathbb{R}$, $y\in(-a_+,a_-)$ are of rapid decrease, and*

(2) *$\tilde{\Phi}$ is represented by a function $\tilde{\Phi}(\omega)\in\mathcal{C}^\infty(\mathbb{R})$ that has the following asymptotic behavior:*

$$\begin{aligned}\tilde{\Phi}(\omega) &= +2\pi i\sum_{j\in\mathcal{I}_+} e^{2\pi i z_j\omega}\operatorname*{Res}_{z=z_j}\Phi(z)+\tilde{\Phi}_{a_+}(\omega)\\ \tilde{\Phi}(\omega) &= -2\pi i\sum_{j\in\mathcal{I}_-} e^{2\pi i z_j\omega}\operatorname*{Res}_{z=z_j}\Phi(z)+\tilde{\Phi}_{a_-}(\omega),\end{aligned}$$

where $\tilde{\Phi}_{a_\pm}(\omega)$ decay faster than than $e^{-2\pi a|\omega|}$ for any $a\in(-a_-,a_+)$.

REMARK 1. — The sign flips between Lemma 1 and Lemma 2 are due to the different inversion formulae for functions and distributions.

2.4. **A useful Lemma from complex analysis**

The following Lemma is useful for determining the analytic properties of convolutions of meromorphic functions:

LEMMA 3. — *Let $f(z_0;z_1,z_2)$ be meromorphic in its variables in some open strip $\mathcal{S}$ around the real axis, with singular behavior near $z_0=z_1=z_2$ of the form $R_{12}(z_1)(z_0-z_1)^{-1}(z_0-z_2)^{-1}$.*

The function $I(z_1, z_2)$, defined by the integral

$$I(z_1,z_2) \;\equiv\; \int_{-\infty}^{\infty} dz_0 \;\; f(z_0; z_1, z_2), \tag{7}$$

will then be a function that has a meromorphic continuation w.r.t. z_i, $i = 1,2$ to the whole strip $\mathcal{S}$. If z_1 and z_2 were initially seperated by the real axis one will find a pole with residue $R_{12}(z_1)$ at $z_1 = z_2$. If not, $I(z_1, z_2)$ will be nonsingular at $z_1 = z_2$ as well.

Proof. — To define the meromorphic continuation of $I(z_1, z_2)$ in cases where the poles z_i, $i = 1,2$ cross the contour of integration of the integral (7) one just needs to deform the contour accordingly. This will obviously always be possible as long as z_i, $i = 1,2$ were initially not separated by the real axis. We will therefore turn to the case that they were initially seperated, and consider w.l.o.g. the case that z_1 was initially in the upper, z_2 in the lower half plane. In this case one may deform the contour into a contour that passes *above* z_1 plus a small circle around z_1. The residue contribution from the integral over that small circle is

$$2\pi i \frac{R_{12}(z_1)}{z_1 - z_2} + (\text{contributions regular as } z_1 - z_2 \to 0) \tag{8}$$

The Lemma is proven. $\square$

3. A CLASS OF REPRESENTATIONS OF $U_q(sl(2,\mathbb{R}))$

3.1. Definintion

$U_q(sl(2,\mathbb{R})$ is a Hopf-algebra with

$$\begin{aligned}
&\text{generators:} \quad E, \quad F, \quad K, \quad K^{-1};\\
&\text{relations:} \quad KE = qEK, \qquad KF = q^{-1}FK, \qquad [E,F] = -\frac{K^2 - K^{-2}}{q - q^{-1}};\\
&\text{star-structure:} \quad K^* = K, \qquad E^* = E, \qquad F^* = F;\\
&\text{co-product:} \quad \Delta(K) = K \otimes K, \qquad \begin{aligned}\Delta(E) =& E \otimes K + K^{-1} \otimes E,\\ \Delta(F) =& F \otimes K + K^{-1} \otimes F.\end{aligned}
\end{aligned} \tag{9}$$

The center of $U_q(sl(2,\mathbb{R})$ is generated by the q-Casimir

$$C = FE - \frac{qK^2 + q^{-1}K^{-2} - 2}{(q - q^{-1})^2}. \tag{10}$$

We will consider the case that $q = e^{\pi i b^2}$, $b \in (0,1) \cap (\mathbb{R} \setminus \mathbb{Q})$.

Unitary representations of $U_q(sl(2,\mathbb{R}))$ by operators on a Hilbert-space have been studied in [10]. Since there are no unitary representations in terms of bounded operators some care is needed in order to single out an interesting class of "well-behaved" representations. A natural notion of "well-behaved" was introduced in [10], where the corresponding unitary representations of $U_q(sl(2,\mathbb{R}))$ were classified.

In the present paper we will study a one-parameter subclass $\mathcal{P}_\alpha$, $\alpha \in Q/2 + i\mathbb{R}$, $Q = b + b^{-1}$ of the representations listed in [10] which are constructed as follows: The representation will be realized on the space $\mathcal{P}_\alpha$ of entire analytic functions $f(x)$ that have a Fourier-transform $f(\omega)$ which is meromorphic in $\mathbb{C}$ with possible poles at

$$\begin{aligned} \omega &= i(\alpha - Q - nb - mb^{-1}) \\ \omega &= i(Q - \alpha + nb + mb^{-1}) \end{aligned} \qquad n, m \in \mathbb{Z}^{\geq 0}. \tag{11}$$

REMARK 2. — It can be shown that $\mathcal{P}_\alpha$ is a Frechet-space.

One may then introduce the following finite difference operators

$$\begin{aligned} \pi_\alpha(E) &\equiv e^{+2\pi bx}[d_x + Q - \alpha]_b \\ \pi_\alpha(F) &\equiv e^{-2\pi bx}[d_x + \alpha - Q]_b \end{aligned} \qquad \pi_\alpha(K) \equiv T_x^{\frac{ib}{2}}. \tag{12}$$

As shorthand notation we will also use $u_\alpha \equiv \pi_\alpha(u)$.

LEMMA 4. — (i) *The operators $\pi_\alpha(u)$, $u = E, F, K$ map $\mathcal{P}_\alpha$ into itself.*
(ii) *$\pi_\alpha(u)$, $u = E, F, K$ generate a representation of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ on $\mathcal{P}_\alpha$.*

Proof. — To verify (i), note that Fourier-transformation maps E_α, F_α, K_α into the following operators:

$$\begin{aligned} \tilde{E}_\alpha &= [-i\omega + \alpha]_b T_\omega^{ib} \\ \tilde{F}_\alpha &= [-i\omega - \alpha]_b T_\omega^{-ib} \end{aligned} \qquad K_\alpha = e^{-\pi b\omega}. \tag{13}$$

The claim follows from the fact that $[x]_b = 0$ for $x = nb^{-1}$, $n \in \mathbb{Z}$.
(ii) is checked by straightforward calculation. □

PROPOSITION 1. — *The operators (12) generate an* integrable *operator representation of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ in the sense of* [10], *i.e.*

1. *E_α, F_α, K_α have self-adjoint extensions in $L^2(\mathbb{R})$,*
2. *the corresponding unitary operators E_α^{it}, F_α^{it}, K_α^{it} satisfy*

$$K_\alpha^{is} E_\alpha^{it} = q^{-ts} E_\alpha^{it} K_\alpha^{is}, \qquad K_\alpha^{is} F_\alpha^{it} = q^{ts} F_\alpha^{it} K_\alpha^{is}, \quad \textit{and}$$

3. *the q-Casimir strongly commutes with E_α, F_α and K_α.*

Proof. — It suffices to show that the representation $\mathcal{P}_\alpha$ is unitarily equivalent to one of the representations listed in [10]. Consider the operator J_α defined as $(J_\alpha \tilde{f})(\omega) = S_b(\alpha - i\omega)\tilde{f}(\omega)$ in terms of the special function $S_b(x)$ (cf. Appendix B). J_α is unitary since $|S_b^{-1}(\alpha - i\omega)|^2 = 1$ which follows from eqn. (134) in Appendix B. Moreover, it follows from the analytic and asymptotic properties of $S_b(x)$ given in the Appendix that J_α maps $\mathcal{P}_\alpha$ to the space $\mathcal{R}_\alpha$ of entire analytic functions which have a Fourier-transform that is meromorphic in $\mathbb{C}$ with possible poles at

$$\begin{aligned} \omega &= i(\alpha - Q - nb - mb^{-1}) \\ \omega &= i(-\alpha - nb - mb^{-1}) \end{aligned} \qquad n, m \in \mathbb{Z}^{\geq 0}. \tag{14}$$

One finally finds from the functional relations of the S_b-functions, eqn. (133) that

$$\begin{aligned} J_\alpha^{-1}\tilde{E}_\alpha J_\alpha &= T_\omega^{ib} \\ J_\alpha^{-1}\tilde{F}_\alpha J_\alpha &= [\alpha + i\omega]_b T_\omega^{-ib}[\alpha - i\omega]_b \end{aligned} \qquad J_\alpha^{-1} K_\alpha J_\alpha = e^{-\pi b\omega}. \tag{15}$$

Our representation is thereby easily recognized as the representation denoted by $(I)_{1,-1,c}$ in Corollary 5 of [10], where $c = [\alpha - \frac{Q}{2}]_b^2 + 2(q - q^{-1})^{-2}$. Note that our notation Q is different from that in [10] and $c \leq 2(q - q^{-1})^{-2}$. □

REMARK 3. — The representations considered here form a subset of the representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ that appear in the classification of [10]. This subset has the following remarkable property: If one introduces generators $\tilde{E}$, $\tilde{F}$, $\tilde{K}$ by replacing $b \to b^{-1}$ in the expressions for E, F, K given above, one obtains a representation of $\mathcal{U}_{\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ $\tilde{q} = \exp(\pi i b^{-2})$ on the same space $\mathcal{P}_\alpha$. The generators $\tilde{E}$, $\tilde{F}$, $\tilde{K}^2$ commute with E, F, K^2 on the space $\mathcal{P}_\alpha$. This does not mean, however, that these operators commute as self-adjoint operators on $L^2(\mathbb{R})$. This self-duality property of our representations $\mathcal{P}_\alpha$ is related to the fact that the representations $(\mathcal{P}_\alpha, \pi_\alpha)$ do *not* have a classical ($b \to 0$) limit.

3.2. **Intertwining operators**

The representations with labels α and $Q - \alpha$ are equivalent. The unitary operator establishing this equivalence can be most easily found by considering the Fourier-transform of the representation (12), as already done in the proof of Proposition 1, eqns. (13): Define the operator $\tilde{\mathcal{I}}_\alpha : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ as

$$(\tilde{\mathcal{I}}_\alpha \tilde{f})(\omega) \;=\; \tilde{B}_\alpha(\omega) f(\omega), \quad \tilde{B}_\alpha(\omega) \;\equiv\; \frac{S_b(\alpha - i\omega)}{S_b(Q - \alpha - i\omega)}. \tag{16}$$

The operator $\tilde{\mathcal{I}}_\alpha$ is unitary since $|\tilde{B}_\alpha(\omega)| = 1$. It maps $\mathcal{P}_\alpha$ to $\mathcal{P}_{Q-\alpha}$ as follows from the analytic and asymptotic properties of the S_b-function summarized in Appendix B. The fact that

$$\pi_{Q-\alpha}(u)\tilde{\mathcal{I}}_\alpha = \tilde{\mathcal{I}}_\alpha \pi_\alpha(u), \quad u \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R})) \tag{17}$$

is a simple consequence of the functional relations (133), Appendix B of the S_b-functions.

By inverse Fourier-transformation one finds the representation of the intertwining operator on functions $f(x)$. It takes the form

$$(\mathcal{I}_\alpha f)(x) \;=\; \int_{\mathbb{R}} dx' \; B_\alpha(x - x') f(x), \tag{18}$$

where the inverse Fourier-transform defining the kernel $B_\alpha(x - x')$ may be found by means of eqn. (136), Appendix B to be given by

$$B_\alpha(x - x') \;=\; S_b(2\alpha) \frac{S_b\big(\frac{Q}{2} + i(x - x') - \alpha\big)}{S_b\big(\frac{Q}{2} + i(x - x') + \alpha\big)}. \tag{19}$$

4. **THE CLEBSCH-GORDAN DECOMPOSITION OF TENSOR PRODUCTS**

The co-product allows us to define the tensor product of representations: For any $u \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ let $\pi_{21}(u) \equiv (\pi_{\alpha_2} \otimes \pi_{\alpha_1})\Delta(u)$. The operators $\pi_{21}(u)$ generate a representation of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ on $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$. Our aim is to determine the decomposition of this representation into irreducible representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$.

LEMMA 5. — $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$ *is dense in* $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$.

Proof. — Any two-variable Hermite-function is contained in $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$. $\square$

DEFINITION 1. — *Define a distributional kernel* $\left[\begin{smallmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{smallmatrix}\right]$ *(the "Clebsch-Gordan coefficients") by an expression of the form*

$$\left[\begin{smallmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{smallmatrix}\right] \equiv \lim_{\epsilon \downarrow 0} \left[\begin{smallmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{smallmatrix}\right]_\epsilon, \tag{20}$$

where the meromorphic function $\left[\begin{smallmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{smallmatrix}\right]_\epsilon$ *is defined as*

$$\begin{aligned} \left[\begin{smallmatrix} Q-\alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{smallmatrix}\right]_\epsilon = {} & e^{-\frac{\pi i}{2}(\Delta_{\alpha_3} - \Delta_{\alpha_2} - \Delta_{\alpha_1})} \\ & \times D_b(\beta_{32}; y_{32} + \epsilon) D_b(\beta_{31}; y_{31} + \epsilon) D_b(\beta_{21}; y_{21} + \epsilon), \end{aligned} \tag{21}$$

$\Delta_\alpha = \alpha(Q - \alpha)$, *the distribution* $D_b(\alpha; y)$ *is defined in terms of the Double Sine function* $S_b(y)$ *(cf. Appendix) as*

$$D_b(\alpha; y) = \frac{S_b(y)}{S_b(y + \alpha)}, \tag{22}$$

and the coefficients y_{ji}, β_{ji}, $j > i \in \{1, 2, 3\}$ *are given by*

$$\begin{aligned} y_{32} &= i(x_3 - x_2) - \tfrac{1}{2}(\alpha_3 + \alpha_2 - Q) \qquad & \beta_{32} &= \alpha_2 + \alpha_3 - \alpha_1 \\ y_{31} &= i(x_1 - x_3) - \tfrac{1}{2}(\alpha_3 + \alpha_1 - Q) \qquad & \beta_{31} &= \alpha_3 + \alpha_1 - \alpha_2 \\ y_{21} &= i(x_1 - x_2) - \tfrac{1}{2}(\alpha_2 + \alpha_1 - 2\alpha_3) \qquad & \beta_{21} &= \alpha_2 + \alpha_1 - \alpha_3. \end{aligned} \tag{23}$$

The aim of this section will be to prove

THEOREM 2. — *The* $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$*-representation* π_{21} *defined on* $\pi_{\alpha_2} \otimes \pi_{\alpha_1}$ *decomposes as follows into irreducible representations* $\mathcal{P}_\alpha$*:*

$$\pi_{\alpha_2} \otimes \pi_{\alpha_1} \simeq \int_{\mathbb{S}}^{\oplus} d\alpha\ \pi_\alpha, \qquad \mathbb{S} \equiv \frac{Q}{2} + i\mathbb{R}^+. \tag{24}$$

The isomorphism can be described explicitly in terms of a unitary map $\mathcal{C}_{21}$ *of the form*

$$\mathcal{C}_{21} \quad : \quad \begin{aligned} L^2(\mathbb{R} \times \mathbb{R}) \quad &\to \quad L^2(\mathbb{S} \times \mathbb{R}, d\mu(\alpha_3) dx_3), \qquad d\mu(\alpha) \equiv |S_b(2\alpha)|^2 \\ f(x_2, x_1) \quad &\to \quad F_f(\alpha_3, x_3) \equiv \int_{\mathbb{R}} dx_2 dx_1 \left[\begin{smallmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{smallmatrix}\right] f(x_2, x_1) \end{aligned} \tag{25}$$

such that the corresponding projections $\Pi_{21}(\alpha_3)$, $\big(\Pi_{21}(\alpha_3) f\big)(x_3) = F_f(\alpha_3, x_3)$, *map* $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$ *into* $\mathcal{P}_{\alpha_3}$ *and intertwine the respective* $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ *actions according to*

$$\Pi_{21}(\alpha_3) \pi_{21}(u) = \pi_{\alpha_3}(u) \Pi_{21}(\alpha_3) \qquad u \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R})). \tag{26}$$

REMARK 4. — It follows from Theorem 2 that the representation π_{21} is in fact integrable, which was not clear apriori.

REMARK 5. — It is remarkable and nontrivial that the subset of "self-dual" integrable representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ is actually closed under tensor products.

REMARK 6. — The appearance of the measure $d\mu(\alpha)$ is natural since $d\mu(\alpha)$ is the Plancherel measure for the dual space of functions $L^2(SL_q^+(2,\mathbb{R}))$, cf. [18].

COROLLARY 1. — *The Clebsch-Gordan coefficients* $\left[\begin{smallmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{smallmatrix}\right]$ *satisfy the following orthogonality and completeness relations:*

$$
\begin{aligned}
&\lim_{\epsilon\downarrow 0}\int_{\mathbb{R}} dx_1 dx_2 \, \left[\begin{smallmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{smallmatrix}\right]_\epsilon^* \left[\begin{smallmatrix} \beta_3 & \alpha_2 & \alpha_1 \\ y_3 & x_2 & x_1 \end{smallmatrix}\right]_\epsilon = |S_b(2\alpha_3)|^{-2}\delta(\alpha_3-\beta_3)\delta(x_3-y_3) \\
&\lim_{\epsilon\downarrow 0}\int_{\mathbb{S}} d\alpha_3 \, |S_b(2\alpha_3)|^2 \int_{\mathbb{R}} dx_3 \, \left[\begin{smallmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{smallmatrix}\right]_\epsilon^* \left[\begin{smallmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & y_2 & y_1 \end{smallmatrix}\right]_\epsilon = \delta(x_2-y_2)\delta(x_1-y_1).
\end{aligned}
\tag{27}
$$

The main step in the proof of Theorem 2 will be the construction of a common spectral decomposition for the operators $Q_{21}\equiv(\pi_{\alpha_2}\otimes\pi_{\alpha_1})\Delta(Q)$ and K_{21}. The decomposition of $L^2(\mathbb{R}\times\mathbb{R})$ into eigenspaces of K_{21} is simply obtained by Fourier-transformation:

$$
\mathcal{F} \quad : \quad
\begin{aligned}
L^2(\mathbb{R}\times\mathbb{R}) \quad &\rightarrow \quad L^2(\mathbb{R}\times\mathbb{R}) \\
f(x_2,x_1) \quad &\rightarrow \quad F(\kappa_3,x_-)\equiv \int_{\mathbb{R}} dx_+ \; e^{-\pi i\kappa_3 x_+} f\left(\tfrac{x_++x_-}{2},\tfrac{x_+-x_-}{2}\right)
\end{aligned}
\tag{28}
$$

The q-Casimir Q_{21} is mapped under this Fourier-transformation $\mathcal{F}$ into a second order finite difference operator $C_{21}(\kappa_3)$ that contains shifts w.r.t. the variable x_- only and therefore leaves the eigenspaces of K_{21} invariant:

$$
\begin{aligned}
C_{21}(\kappa_3) - \big[\alpha_3-\tfrac{Q}{2}\big]_b^2 = \\
= \;\; & \big[-ix-\tfrac{1}{2}(\alpha_1+\alpha_2-Q)+(\alpha_3-\tfrac{Q}{2})\big]_b\big[-ix-\tfrac{1}{2}(\alpha_1+\alpha_2-Q)-(\alpha_3-\tfrac{Q}{2})\big]_b \\
& -\big[-ix+\tfrac{1}{2}(\alpha_1+\alpha_2)-Q\big]_b\Big(e^{i\pi b(-ix-\frac{1}{2}(\alpha_1+\alpha_2))}\{\alpha_1-\alpha_2+i\kappa_3\}_b \\
& \qquad\qquad - e^{-i\pi b(-ix-\frac{1}{2}(\alpha_1+\alpha_2))}\{\alpha_1-\alpha_2-i\kappa_3\}_b\Big)T_{x_-}^{-ib} \\
& +\big[-ix+\tfrac{1}{2}(\alpha_1+\alpha_2)-Q\big]_b\big[-ix+\tfrac{1}{2}(\alpha_1+\alpha_2)-2Q\big]_b T_{x_-}^{-2ib},
\end{aligned}
\tag{29}
$$

where the following notation has been used:

$$
[x]_b\equiv\frac{\sin(\pi bx)}{\sin(\pi b^2)}, \qquad \{x\}_b\equiv\frac{\cos(\pi bx)}{i\sin(\pi b^2)}.
\tag{30}
$$

The spectral analysis of the operator C_{21} is performed in Appendix A. The result may be summarized as follows: Eigenfunctions $\Phi_{\alpha_3}(\alpha_2,\alpha_1|\kappa_3|x)$ of C_{21} are given by an expression of the form

$$
\Phi_{Q-\alpha_3}(\alpha_2,\alpha_1|\kappa_3|x) \;=\; M_{\alpha_2,\alpha_1}^{\alpha_3;\kappa_3}\, e^{\pi x(2\alpha_3-2\alpha_2+i\kappa_3)}\;\Theta_b(T,y_-)\;\Psi_b(U,V,W;y_+).
\tag{31}
$$

The special functions $\Theta_b(T;y)$ and $\Psi_b(U,V,W;y)$ are defined in Appendix B, $y_\pm$ are introduced as $y_\pm=-ix-\frac{1}{2}(\alpha_2+\alpha_1-Q)\mp(\alpha_3-\frac{Q}{2})$ and the coefficients T,U,V,W are given as

$$
\begin{aligned}
T &= \alpha_2+\alpha_1-\alpha_3 \qquad & V &= -i\kappa_3+\alpha_3 \\
U &= \alpha_3+\alpha_1-\alpha_2 \qquad & W &= -i\kappa_3+\alpha_1-\alpha_2+Q.
\end{aligned}
\tag{32}
$$

THEOREM 3. — *A complete set of generalized eigenfunctions for the operator* $C_{21}(\kappa_3)$ *is given by* $\{(\Phi_{\alpha_3})^*;\alpha_3\in\mathbb{S}\}$.

By combining Theorem 3 with the usual Plancherel formula for the Fourier-transformation $\mathcal{F}$ one concludes that each function $f(x_2,x_1)\in L^2(\mathbb{R}\times\mathbb{R})$ can be decomposed as $(x_\pm\equiv x_2\pm x_1)$

$$f(x_2,x_1)=\int_{\mathbb{R}} d\kappa_3\;\, e^{\pi i\kappa_3 x_+}\int_{\mathbb{S}} d\mu(\alpha_3)\;\,\big(\Phi_{\alpha_3}(\alpha_2,\alpha_1|\kappa_3|x_-)\big)^* F_f(\alpha_3,\kappa_3), \tag{33}$$

where the generalized Fourier-transformation F_f of f is defined as

$$F_f(\alpha_3,\kappa_3)=\int_{\mathbb{R}} dx_2dx_1\;\, e^{-\pi i\kappa_3 x_+}\;\Phi_{\alpha_3}(\alpha_2,\alpha_1|\kappa_3|x_-)f(x_2,x_1). \tag{34}$$

The measure $d\mu(\alpha_3)$ will be determined later. One may next observe that

LEMMA 6. — *One has*

$$\left[\begin{smallmatrix}\alpha_3 & \alpha_2 & \alpha_1\\ \kappa_3 & x_2 & x_1\end{smallmatrix}\right]\;\equiv\;\int_{\mathbb{R}} dx_3\;\, e^{2\pi i\kappa_3 x_3}\left[\begin{smallmatrix}\alpha_3 & \alpha_2 & \alpha_1\\ x_3 & x_2 & x_1\end{smallmatrix}\right]\;=\;e^{-\pi i\kappa_3 x_+}\;\Phi_{\alpha_3}(\alpha_2,\alpha_1|\kappa_3|x_-), \tag{35}$$

if the normalization factor M in (31) is chosen as

$$M_{\alpha_2,\alpha_1}^{\alpha_3;\kappa_3}\;\equiv\;e^{\pi i\alpha_2(\alpha_2-\alpha_3)}e^{-\pi i(\alpha_3-i\kappa_3)(\alpha_3+\alpha_2-Q)} \tag{36}$$

Proof. — The kernel $\left[\begin{smallmatrix}Q-\alpha_3 & \alpha_2 & \alpha_1\\ x_3 & x_2 & x_1\end{smallmatrix}\right]$ may be rewritten in terms of the function $\Theta_b(\beta;y)$ as follows:

$$\begin{aligned}\left[\begin{smallmatrix}Q-\alpha_3 & \alpha_2 & \alpha_1\\ x_3 & x_2 & x_1\end{smallmatrix}\right]=&\;e^{\pi i\alpha_1\alpha_2}e^{2\pi(x_3(\alpha_2-\alpha_1)+\alpha_1x_1-\alpha_2x_2)}\\ &\times\Theta_b(\beta_{32};y_{32})\Theta_b(\beta_{31};y_{31})\Theta_b(\beta_{21};y_{21}).\end{aligned} \tag{37}$$

The substitution $s=-i(x_3-x_2)+\frac{1}{2}(\alpha_3+\alpha_2-Q)$ then leads to the Euler-type integral (146) for the b-hypergeometric function. The rest is straightforward. □

If follows that the generalized Fourier-transformation defined in Theorem 3 represents a decomposition into eigenspaces of the q-Casimir Q_{21}. Two things remain to be done in order to finish the proof of Theorem 2: On the one hand it remains to calculate the spectral measure $d\mu(\alpha_3)$, and on the other hand one needs to verify the intertwining property (26).

4.1. **Spectral measure**

We will show in this subsection that $d\mu(\alpha_3)=|S_b(2\alpha_3)|^2$. This follows from the combination of the following two results. We first of all determine the asymptotics of the distributional Fourier-transform of Φ_{α_3}:

LEMMA 7. — *The function $\tilde{\Phi}_{\alpha_3}(\omega)$ (defined as in (6)) decays exponentially for $\omega\to\infty$ and has the following asymptotic behavior for $\omega\to-\infty$:*

$$\tilde{\Phi}_{\alpha_3}(\omega)\;=\;N_+(\alpha_3)e^{2\pi i\omega x_+}+N_-(\alpha_3)e^{2\pi i\omega x_-}+R_-(\omega), \tag{38}$$

where $R_-(\omega)$ decays exponentially for $\omega\to-\infty$, x_+ and x_- are defined by

$$x_\pm\equiv+\tfrac{i}{2}\big(\alpha_1+\alpha_2-Q\big)\pm i\big(\alpha_3-\tfrac{Q}{2}\big)$$

and $|N_\pm(\alpha_3)|^2=|S_b(2\alpha_3)|^{-2}$.

Proof. — According to Lemma 2 one just needs to calculate the residues of Φ_{α_3} for the poles at $x = x_\pm$. We will only need the absolute values of these quantities.

The pole at $x = x_-$ comes from the G_b/G_b factor in the expression for Φ. To calculate its residue one needs the following special value of the Ψ-function:

$$\Psi_b(U,V;W;W-U-V) = \frac{G_b(V)G_b(W-U-V)}{G_b(W-U)}, \tag{39}$$

which follows easily from the fact that the representation (146) simplifies to the b-beta integral (136) for $x = W - U - W$. We furthermore note that $|G_b(\frac{Q}{2} + ix)|^2 = 1$ from the reflection property of $S_b(x)$ stated in the Appendix B. It thereby follows that

$$|N_-(\alpha_3)|^2 = |M^{\alpha_3;\kappa_3}_{\alpha_2\alpha_1} G_b(Q - 2\alpha_3)|^2. \tag{40}$$

One has $|M^{\alpha_3;\kappa_3}_{\alpha_2\alpha_1}|^2 = e^{\pi i Q(Q-2\alpha_3)}$, and $|G_b(Q-2\alpha_3)|^2 = e^{-\pi i Q(Q-2\alpha_3)}|S_b(2\alpha_3)|^{-2}$ from the connection between S_b and G_b, as well as the reflection property of S_b (see Appendix B). Therefore $|N_-(\alpha_3)|^2 = |S_b(2\alpha_3)|^{-2}$.

The pole at $x = x_+$ corresponds to the pole at $y = 0$ of $\Psi_b(U,V;W;y)$. One may determine the singular term for $y \to 0$ by applying Lemma 3 to the Euler integral representation (146) for the function Ψ_b:

$$2\pi e^{-2\pi i y\beta}\frac{G_b(-y+\gamma-\beta)}{G_b(\alpha)G_b(-y+Q)} = \frac{1}{y}\frac{G_b(\gamma-\beta)}{G_b(\alpha)} + (\text{contributions regular as } y \to 0). \tag{41}$$

The rest of the calculation proceeds as in the case of $N_-(\alpha_3)$ and yields $|N_+(\alpha_3)|^2 = |S_b(2\alpha_3)|^{-2}$. $\square$

PROPOSITION 2. — *Assume that the generalized eigenfunctions $\tilde{\Phi}_{\alpha_3}$ decay exponentially for $\omega \to \infty$ and have asymptotic behavior of the form (38) with $|N_+(\alpha_3)|^2 = |N_-(\alpha_3)|^2$ for $\omega \to -\infty$. In that case one may define the "inner product" $(\Phi_{\alpha_3}, \Phi_{\alpha_3'})$ as a bi-distribution which is explicitly given by*

$$(\Phi_{\alpha_3}, \Phi_{\alpha_3'}) \;=\; |N_+(\alpha_3)|^2\delta(\alpha_3 - \alpha_3'). \tag{42}$$

Proof. — Consider

$$\begin{aligned}&(C_{21}(\kappa_3)\Phi_{\alpha_3}, \Phi_{\alpha_3'}) - (\Phi_{\alpha_3}, C_{21}(\kappa_3)\Phi_{\alpha_3'}) = \\ &= \lim_{W\to\infty}\sum_{s=\pm}\int_{-W}^{W} d\omega\,\Big(\big(\tilde{\delta}_s(\omega)\tilde{\Phi}_{\alpha_3}(\omega+sib)\big)^*\tilde{\Phi}_{\alpha_3'}(\omega) - \big(\tilde{\Phi}_{\alpha_3}(\omega)\big)^*\tilde{\delta}_s(\omega)\tilde{\Phi}_{\alpha_3'}(\omega+sib)\Big),\end{aligned} \tag{43}$$

where the Fourier-transform of the explicit expression (105) for $C_{21}(\kappa_3)$ has been used. The contour of integration for the second term in (43) can be deformed into $\mathbb{R} - isb$ plus contours from $-W$ to $-W - isb$ and $W - isb$ to W. The integral over $\mathbb{R} - isb$ cancels the first term on the right hand side of (43). Only the contour from $-W$ to $-W - isb$ will give nonvanishing contributions in the limit $W \to \infty$ due to the exponential decay of $\tilde{\Phi}_{\alpha_3}(\omega)$ for $\omega \to \infty$. In the remaining term one gets in the limit $W \to \infty$ contributions only from the leading terms in the asymptotics of $\tilde{\Phi}_{\alpha_3}(\omega)$ for $\omega \to -\infty$ as quoted in Lemma 38. Taking into account that

$$\tilde{\delta}_s(\omega) = \frac{1}{(q-q^{-1})^2}e^{s\pi i b(Q-\alpha_1-\alpha_2)} + \mathcal{O}(e^{2\pi b\omega}) \tag{44}$$

for $\omega \to -\infty$, it follows that ($\alpha_3 = \frac{Q}{2} + ip_3$, $\alpha_3' = \frac{Q}{2} + ip_3'$)

$$
\begin{aligned}
&(C_{21}(\kappa_3)\Phi_{\alpha_3}, \Phi_{\alpha_3'}) - (\Phi_{\alpha_3}, C_{21}(\kappa_3)\Phi_{\alpha_3'}) = \\
&= \frac{1}{(q-q^{-1})^2} \lim_{W\to\infty} \sum_{s=\pm} \sum_{\epsilon_1,\epsilon_2=\pm} \frac{\big(N_{\epsilon_1}(\alpha_3)\big)^* N_{\epsilon_2}(\alpha_3')}{2\pi i(\epsilon_1 p_3 - \epsilon_2 p_3')} e^{2\pi i W(\epsilon_1 p_3 - \epsilon_2 p_3')} \cdot \\
&\qquad \cdot e^{2\pi s\epsilon_2 b p_3'} \big(1 - e^{2\pi s b(\epsilon_1 p_3 - \epsilon_2 p_3')}\big).
\end{aligned}
\tag{45}
$$

The expression on the right hand side of (45) vanishes by the Riemann-Lebesque Lemma for $p_3 \neq p_3'$ as well as $\epsilon_1 \neq \epsilon_2$. The remainder is found to be

$$
\begin{aligned}
&(C_{21}(\kappa_3)\Phi_{\alpha_3}, \Phi_{\alpha_3'}) - (\Phi_{\alpha_3}, C_{21}(\kappa_3)\Phi_{\alpha_3'}) = \\
&= \big([ip_3']_b^2 - [ip_3]_b^2\big)\ |N_+(\alpha_3)|^2 \lim_{W\to\infty} \frac{e^{2\pi i W(p_3-p_3')} - e^{-2\pi i W(p_3-p_3')}}{2\pi i(p_3-p_3')}.
\end{aligned}
\tag{46}
$$

It follows that

$$
\begin{aligned}
(\Phi_{\alpha_3}, \Phi_{\alpha_3'}) &= |N_+(\alpha_3)|^2 \lim_{W\to\infty} \frac{e^{2\pi i W(p_3-p_3')} - e^{-2\pi i W(p_3-p_3')}}{2\pi i(p_3-p_3')} \\
&= |N_+(\alpha_3)|^2\ \delta(\alpha_3 - \alpha_3')
\end{aligned}
\tag{47}
$$

by the corresponding well-known property of the kernel $\sin(Rx)/x$, cf. e.g. [21, Chapter IX, Exercise 14]. □

4.2. **Intertwining property**

PROPOSITION 3. — *The projections $\Pi_{21}(\alpha_3)$, $\alpha_3 \in \mathbb{S}$ map $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$ into $\mathcal{P}_{\alpha_3}$ and satisfy the intertwining property (26).*

Proof. — $F_f(\alpha_3, x_3)$ will be entire analytic w.r.t. x_3 by straightforward application of Lemma 3, using that f is entire analytic in x_2, x_1 and the analytic properties of the Clebsch-Gordan coefficients summarized in Lemma 1, Appendix C. One similarly finds by using Lemma 2, Appendix C that the Fourier-transform $F_f(\alpha_3, \kappa_3)$ will be meromorphic in κ_3 with poles at $\kappa = \pm(Q - \alpha + nb + mb^{-1})$, $n, m \in \mathbb{Z}^{\geq 0}$ for any $\in \mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$. This establishes the first claim in Proposition 3.

Note that the analytic continuation of the integral (25) that defines $F_f(\alpha_3, x_3)$ can be represented by integrating over a deformed contour $C^{(2)} \subset \mathbb{C}^2$. For later use we will present suitable contours for the cases of analytic continuation to $\{x_3 \in \mathbb{C}; \mathrm{Im}(x_3) \in [0, \frac{b}{2}]\}$ and $\{x_3 \in \mathbb{C}; \mathrm{Im}(x_3) \in [-\frac{b}{2}, 0]\}$ respectively: In the first case one may integrate x_1 over the real axis and instead of integrating over x_2 one may integrate $x_{32} \equiv -iy_{32}$, cf. (23), over a contour consisting of the union of the half axes $(-\infty, -\delta]$ and $[\delta, +\infty)$, $b > \delta > b/2$ with a half-circle in the upper half plane around $x_{32} = 0$ of radius δ. In the second case one may integrate x_2 over $\mathbb{R}$, and $x_{31} \equiv -iy_{31}$ over the contour C_1 consisting of the union of the half axes $(-\infty, -\delta]$ and $[\delta, +\infty)$ with a half-circle of radius δ in the *lower* half plane around $x_{31} = 0$.

Now consider the right hand side of (26). The expressions for $\pi_{21}(u)$, $u = E, F, K$ contain the shift operators

$$
T_{x_1}^{+\frac{ib}{2}} T_{x_2}^{+\frac{ib}{2}}, \quad T_{x_1}^{-\frac{ib}{2}} T_{x_2}^{-\frac{ib}{2}} \quad \text{and} \quad T_{x_1}^{-\frac{ib}{2}} T_{x_2}^{+\frac{ib}{2}}.
\tag{48}
$$

The shift operator $T_{x_i}^{\pm\frac{ib}{2}}$ is “partially integrated” by (i) shifting the contour of integration over x_i to the axis $\mathbb{R} \mp \frac{ib}{2}$, where one will pick up a residue contribution from the pole of the Clebsch-Gordan coefficients that lies between these two contours, and (ii) introducing the new variables of integration $x_i' \equiv x_i \pm \frac{ib}{2}$. In this way one rewrites the expression for $\mathcal{C}_{21}\pi_{21}(u)f$ in the form

$$\int\limits_{C_1} dx_2 \int\limits_{C_2} dx_1 \,\left(\pi_{21}^t(u)\left[\begin{smallmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{smallmatrix}\right]\right) f(x_2,x_1), \tag{49}$$

where the π_{21}^t denotes the transpose of π_{21}, and the contours C_i, $i = 1,2$ are just the contours introduced above to represent the analytic continuation w.r.t. x_3. It is important to notice that due to the fact that only the shift operators (48) appear in the expressions for $\pi_{21}(u)$, $u = E, F, K$ one does not need to introduce further deformations of the contours in order to treat the poles from the factor in the Clebsch-Gordan coefficients that depends on $x_2 - x_1$ only.

It is verified by a straightforward calculation using (133) that the Clebsch-Gordan coefficients satisfy the finite difference equations

$$\pi_{21}^t(u)\left[\begin{smallmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{smallmatrix}\right] = \pi_{\alpha_3}(u)\left[\begin{smallmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{smallmatrix}\right], \qquad u = E, F, K. \tag{50}$$

Inserting these relations into (49) yields an expression that is easily identified as $\pi_{\alpha_3}(u)\mathcal{C}_{21}f$. $\square$

5. RACAH-WIGNER COEFFICIENTS FOR $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$

5.1. Canonical decompositions for triple tensor products

Triple tensor products $\mathcal{P}_{\alpha_3} \otimes \mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$ carry a representation π_{321} of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ given by

$$\begin{aligned} \pi_{321} \;&\equiv (\pi_{\alpha_3} \otimes \pi_{\alpha_2} \otimes \pi_{\alpha_1}) \circ \Delta^{(3)}, \\ \Delta^{(3)} \;&\equiv (\Delta \otimes \mathrm{id}) \circ \Delta \;\equiv\; (\mathrm{id} \otimes \Delta) \circ \Delta. \end{aligned} \tag{51}$$

The decomposition of this representation into irreducibles can be constructed by iterating Clebsch-Gordan maps: There are two canonical ways to do so, which will be referred to as “s-channel” and “t-channel” respectively. The first of these corresponds to first decomposing the factor $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$ into a direct sum of irreducible representations $\mathcal{P}_{\alpha_s}$ then performing the Clebsch-Gordan decomposition of $\mathcal{P}_{\alpha_3} \otimes \mathcal{P}_{\alpha_s}$. This extends to a unitary map

$$\mathcal{C}_{3(21)} \quad : \quad \begin{array}{lcl} L^2(\mathbb{R}\times\mathbb{R}\times\mathbb{R}) & \to & L^2(\mathbb{S}^2 \times \mathbb{R}, d\mu(\alpha_4)d\mu(\alpha_s)dx_4) \\ f(x_3,x_2,x_1) & \to & F_f^s(\alpha_4,\alpha_s,x_4), \end{array} \tag{52}$$

The generalized Fourier-transform F_f^s of f is defined as

$$\begin{aligned} F_f^s(\alpha_4,\alpha_s;x_4) \;\equiv\; \lim_{\epsilon_2\downarrow 0}\lim_{\epsilon_1\downarrow 0} \int\limits_{\mathbb{R}^2} dx_3 dx_s \;\left[\begin{smallmatrix} \alpha_4 & \alpha_3 & \alpha_s \\ x_4 & x_3 & x_s \end{smallmatrix}\right]_{\epsilon_2} \times \\ \times \int\limits_{\mathbb{R}^2} dx_2 dx_1 \;\left[\begin{smallmatrix} \alpha_s & \alpha_2 & \alpha_1 \\ x_s & x_2 & x_1 \end{smallmatrix}\right]_{\epsilon_1} f(x_3,x_2,x_1), \end{aligned} \tag{53}$$

which in the notation $\mathfrak{x} \equiv (x_3, x_2, x_1)$, $d\mathfrak{x} \equiv dx_3 dx_2 dx_1$ can be rewritten as

$$
\begin{aligned}
F_f^s(\alpha_4, \alpha_s; x_4) \;\equiv\; & \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}^3} d\mathfrak{x}\; \Phi^s_{\alpha_s}\left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\right]_\epsilon (x_4; \mathfrak{x})\; f(\mathfrak{x}), \\
\text{where} \quad \Phi^s_{\alpha_s}\left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\right]_\epsilon (x_4; \mathfrak{x}) = & \int_{\mathbb{R}} dx_s \; \left[\begin{smallmatrix} \alpha_4 & \alpha_3 & \alpha_s \\ x_4 & x_3 & x_s \end{smallmatrix}\right]_\epsilon \left[\begin{smallmatrix} \alpha_s & \alpha_2 & \alpha_1 \\ x_s & x_2 & x_1 \end{smallmatrix}\right]_\epsilon \quad \alpha_4, \alpha_s \in \mathbb{S}, \; x_4 \in \mathbb{R}.
\end{aligned} \tag{54}
$$

Some useful properties of the functions $\Phi^s_{\alpha_s}\left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\right]_\epsilon (x_4; \mathfrak{x})$ are collected in Appendix C.

The generalized Fourier-transformation $\mathcal{C}_{3(21)}$ is such that the two-parameter family of projections $\Pi^s(\alpha_4, \alpha_s) : \mathcal{P}_{\alpha_3} \otimes \mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1} \to \mathcal{P}_{\alpha_4}(\mathbb{R})$ defined by $f \to F_f^s(\alpha_4, \alpha_s; .)$ intertwine the representation π_{321} with the irreducible representation π_{α_4}. It therefore realizes the following isomorphism of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ representations

$$
\mathcal{P}_{\alpha_3} \otimes \mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1} \;\simeq\; \int_{\mathbb{S}}^{\oplus} d\mu(\alpha_4)\; \mathcal{P}_{\alpha_4} \,\otimes\, \mathcal{S}_\mu, \tag{55}
$$

where the multiplicity space $\mathcal{S}_\mu \simeq L^2(\mathbb{S}, d\mu)$ is considered to be equipped with the trivial action of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$.

A second canonical decomposition of $\mathcal{P}_{\alpha_3} \otimes \mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$ is obtained by first decomposing the factor $\mathcal{P}_{\alpha_3} \otimes \mathcal{P}_{\alpha_2}$ into a direct sum of irreducible representations $\mathcal{P}_{\alpha_t}$ and then performing the Clebsch-Gordan decomposition of $\mathcal{P}_{\alpha_t} \otimes \mathcal{P}_{\alpha_1}$. One obtains a map

$$
\mathcal{C}_{(32)1} \quad : \quad \begin{aligned} L^2(\mathbb{R} \times \mathbb{R} \times \mathbb{R}) \quad &\to \quad L^2(\mathbb{S}^2 \times \mathbb{R}, d\mu(\alpha_4) d\mu(\alpha_t) dx_4) \\ f(x_3, x_2, x_1) \qquad &\to \quad F_f^t(\alpha_4, \alpha_t, x_4), \end{aligned} \tag{56}
$$

where F_f^t is defined by a generalized Fourier-transform of the same form as (53) but with Φ^s_{21} replaced by

$$
\Phi^t_{\alpha_t}\left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\right]_\epsilon (x_4; \mathfrak{x}) = \int_{\mathbb{R}} dx_t \; \left[\begin{smallmatrix} \alpha_4 & \alpha_t & \alpha_1 \\ x_4 & x_t & x_1 \end{smallmatrix}\right]_\epsilon \left[\begin{smallmatrix} \alpha_t & \alpha_3 & \alpha_2 \\ x_t & x_3 & x_2 \end{smallmatrix}\right]_\epsilon . \quad \alpha_4, \alpha_t \in \mathbb{S}, \; x_4 \in \mathbb{R}. \tag{57}
$$

As in the case of the s-channel, one has a corresponding two-parameter family of projections $\Pi^s(\alpha_4, \alpha_s) : \mathcal{P}_{\alpha_3} \otimes \mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1} \to \mathcal{P}_{\alpha_4}$ that intertwine the representation π_{321} with the irreducible representation π_{α_4}.

REMARK 7. — The unitarity of the maps $\mathcal{C}_{3(21)}$ and $\mathcal{C}_{(32)1}$ ensures *existence* of self-adjoint extensions for the operators $\pi_{3(21)}(u)$, $\pi_{(32)1}(u)$, $u = E, F, K, Q$: Simply take the image of the self-adjoint extensions on $L^2(\mathbb{S}^2 \times \mathbb{R})$ under $\mathcal{C}^{-1}_{3(21)}$ or $\mathcal{C}^{-1}_{(32)1}$.

However, it is not a priori clear that such self-adjoint extensions are *unique*. In particular, it could be that the self-adjoint extensions that are defined in terms of the maps $\mathcal{C}_{3(21)}$ and $\mathcal{C}_{(32)1}$ are inequivalent. This disturbing possibility will be excluded shortly.

5.2. **Relation between $\mathcal{C}_{3(21)}$ and $\mathcal{C}_{(32)1}$**

It will be convenient to also consider the Fourier-transforms $\Phi^\flat_{\alpha_s}\left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\right]_\epsilon (k_4; \mathfrak{x})$, $\flat = s, t$ that are defined as

$$
\Phi^\flat_{\alpha_s}\left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\right]_\epsilon (k_4; \mathfrak{x}) \;=\; \int_{\mathbb{R}} dx_4 \; e^{2\pi i k_4 x_4}\; \Phi^\flat_{\alpha_s}\left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\right]_\epsilon (x_4; \mathfrak{x}). \tag{58}
$$

Unitarity of the maps $\mathcal{C}_{3(21)}$ and $\mathcal{C}_{(32)1}$ allows us to relate the transforms F_f^s and F_f^t by a transformation of the form

$$F_f^s(\alpha_4,\alpha_s,k_4) \;=\; \int\limits_{\mathbb{S}^2} d\alpha_4' d\alpha_t \int\limits_{\mathbb{R}} dk_4 \;\; \mathcal{K}\big[\begin{smallmatrix} \alpha_4 & \alpha_s & k_4 \\ \alpha_4' & \alpha_t & k_4' \end{smallmatrix}\big]\; F_f^t(\alpha_4',\alpha_t,k_4'). \tag{59}$$

The distribution $\mathcal{K}$ appearing in (59) can be represented as

$$\begin{aligned} &\mathcal{K}\big[\begin{smallmatrix} \alpha_4 & \alpha_s & k_4 \\ \alpha_4' & \alpha_t & k_4' \end{smallmatrix}\big] = \\ &\quad = \lim_{\rho\to\infty}\lim_{\epsilon\downarrow 0} \int\limits_{-\infty}^{\infty} dx_2 \int\limits_{-\rho}^{\rho} dx_3 dx_1 \;\; \big(\Phi^t_{\alpha_t}\big[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4' & \alpha_1 \end{smallmatrix}\big]_\epsilon(k_4';\mathfrak{x})\big)^* \Phi^s_{\alpha_s}\big[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\big]_\epsilon(k_4;\mathfrak{x}). \end{aligned} \tag{60}$$

We will first prove

PROPOSITION 4. — *The distribution $\mathcal{K}$ is of the form*

$$\mathcal{K}\big[\begin{smallmatrix} \alpha_4 & \alpha_s & k_4 \\ \alpha_4' & \alpha_t & k_4' \end{smallmatrix}\big] \;=\; \delta(\alpha_4-\alpha_4')\delta(k_4-k_4')\; K\big[\begin{smallmatrix} \alpha_4 & \alpha_s \\ k_4 & \alpha_t \end{smallmatrix}\big]. \tag{61}$$

Proof. — This will be a consequence of the following result: $\mathcal{K}$ satisfies

$$\begin{aligned} \Big(\big[\alpha_4-\tfrac{Q}{2}\big]_b^2 - \big[\alpha_4'-\tfrac{Q}{2}\big]_b^2\Big)\; \mathcal{K}\big[\begin{smallmatrix} \alpha_4 & \alpha_s & k_4 \\ \alpha_4' & \alpha_t & k_4' \end{smallmatrix}\big] \;&=\; 0 \\ (k_4-k_4')\; \mathcal{K}\big[\begin{smallmatrix} \alpha_4 & \alpha_s & k_4 \\ \alpha_4' & \alpha_t & k_4' \end{smallmatrix}\big] \;&=\; 0. \end{aligned} \tag{62}$$

To see that (62) implies the claim, consider the simplified case of a distribution $T \in \mathcal{S}'(\mathbb{R})$ that satisfies $Tf = 0$, where f is a function that vanishes only at x_0 and such that $fg \in \mathcal{S}(\mathbb{R})$ if $g \in \mathcal{S}(\mathbb{R})$. This distribution has support only at x_0. By Theorem V.11 of [20] one has $T = \sum_{n=0}^N a_n(x_0)\partial_x^n\delta(x-x_0)$. It is then easy to see that $Tf = 0$ implies $a_n = 0$ for $n \neq 0$. The generalization to the case at hand is clear.

To verify (62) one may note that the functions $\Phi^\flat_{\alpha_t}\big[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\big]_\epsilon(k_4;\mathfrak{x})$, $\flat = s,t$ satisfy eigenvalue equations for the operators $Q_{321} \equiv \pi_{321}(Q)$ and $K_{321} \equiv \pi_{321}(K)$ up to an error of order $\mathcal{O}(\epsilon)$. It follows that

$$\begin{aligned} &\Big(\big[\alpha_4-\tfrac{Q}{2}\big]_b^2 - \big[\alpha_4'-\tfrac{Q}{2}\big]_b^2\Big)\mathcal{K}\big[\begin{smallmatrix} \alpha_4 & \alpha_s & k_4 \\ \alpha_4' & \alpha_t & k_4' \end{smallmatrix}\big] = \\ &\quad = \lim_{\epsilon_1,\epsilon_2\downarrow 0}\;\lim_{\rho\to\infty} \int\limits_{\mathbb{R}} dx_2 \int\limits_{-\rho}^{\rho} dx_3 dx_1 \Big(\big(\Phi^t_{\alpha_t}\big[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4' & \alpha_1 \end{smallmatrix}\big]_{\epsilon_1}(k_4';\mathfrak{x})\big)^* \;\; Q_{321}\Phi^s_{\alpha_s}\big[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\big]_{\epsilon_2}(k_4;\mathfrak{x}) \\ &\qquad\qquad - \big(Q_{321}\Phi^t_{\alpha_t}\big[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4' & \alpha_1 \end{smallmatrix}\big]_{\epsilon_1}(k_4';\mathfrak{x})\big)^* \;\; \Phi^s_{\alpha_s}\big[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\big]_{\epsilon_2}(k_4;\mathfrak{x})\Big). \end{aligned} \tag{63}$$

The right hand side of (63) will vanish if Q_{321} can be "partially integrated". To show that this is the case, one needs some information on the form that Q_{321} takes when acting on functions $f(\mathfrak{x})$. By straightforward evaluation of its definition one obtains an expression in terms of shift operators

$$T_1^{is_1 b}T_2^{is_2 b}T_3^{is_3 b}, \quad \text{where } \; T_i = T_{x_i}, \;\; s_i \in \{+,-\}, \;\; i = 1,2,3.$$

It is convenient to introduce an alternative set of shift operators

$$T_+^3 = T_1T_2T_3, \qquad T_{21}^2 = T_2T_1^{-1} \qquad T_{32}^2 = T_3T_2^{-1}.$$

The crucial point now is that the expression for Q_{321} when rewritten in terms of T_+, T_{21}, T_{32} takes the following form

$$Q_{321} = \sum_{n_+=-3}^{3}\sum_{n_{21}=0}^{3}\sum_{n_{32}=0}^{3} P_{n_+n_{21}n_{32}}(\mathfrak{x})\, T_+^{in+b}\, T_{21}^{\frac{2}{3}ibn_{21}}\, T_{32}^{\frac{2}{3}ibn_{32}}, \tag{64}$$

so it contains shifts of x_{21}, x_{32}, x_{31} by *positive* imaginary amounts up to $2ib$ only. Furthermore note that in (63) one may replace T_+ by $e^{-2\pi i k_4}$. The analytic properties of the integrand in (63) as following from Lemma 20 in Appendix C now allow to partially integrate Q_{321} by appropriate shifts of the contours of integration over x_3, x_2, x_1 (cf. proof of Proposition 3).

The verification of the second equation in (62) is similar. □

REMARK 8. — This result implies that the self-adjoint extensions of $\pi_{321}(u)$, $u = K, Q$ that are defined by the maps $\mathcal{C}_{3(21)}$ and $\mathcal{C}_{(32)1}$ indeed coincide. A similar argument as in the proof of the previous proposition will also cover the two other cases $u = E, F$.

5.3. **Calculation of the Racah-Wigner coefficients I**

It will be useful to also introduce

$$\begin{aligned}&\mathcal{X}\left[\begin{smallmatrix}\alpha_4 & \alpha_s & x_4\\ \alpha_4' & \alpha_t & x_4'\end{smallmatrix}\right] = \\ &= \lim_{\epsilon\to 0+}\int_{-\infty}^{\infty} dx_3dx_2dx_1\ \left(\Phi^t_{\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]_\epsilon(x_4';\mathfrak{x})\right)^*\ \Phi^s_{\alpha_s}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]_\epsilon(x_4;\mathfrak{x}).\end{aligned} \tag{65}$$

Proposition 4 has an obvious counterpart for $\mathcal{X}$:

PROPOSITION 5. — *The distribution* $\mathcal{X}$ *is of the form*

$$\mathcal{X}\left[\begin{smallmatrix}\alpha_4 & \alpha_s & x_4\\ \alpha_4' & \alpha_t & x_4'\end{smallmatrix}\right] = \delta(\alpha_4-\alpha_4')\delta(x_4-x_4')\left\{\begin{smallmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{smallmatrix}\middle|\begin{smallmatrix}\alpha_s\\ \alpha_t\end{smallmatrix}\right\}_b. \tag{66}$$

Proof. — Introduce

$$\mathcal{K}_{\epsilon,\rho}\left[\begin{smallmatrix}\alpha_4 & \alpha_s & k_4\\ \alpha_4' & \alpha_t & k_4'\end{smallmatrix}\right] = \int_{-\infty}^{\infty} dx_2\int_{-\rho}^{\rho} dx_3dx_1\ \left(\Phi^s_{\alpha_s}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]_\epsilon(k_4;\mathfrak{x})\right)^*\Phi^t_{\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]_\epsilon(k_4;\mathfrak{x}). \tag{67}$$

The coefficient of $\delta(k_4-k_4')$ in the expression for $\mathcal{K}$ coinicides with the sum of the coefficients with which $e^{-2\pi i(k_4-k_4')x_1}$ and $e^{-2\pi i(k_4-k_4')x_3}$ appear in the asymptotic expansion of the integrand in (67), cf. Lemma 20. Lemma 2 identifies the origin of these terms in the asymptotic expansion of $\Phi^\flat$, $\flat = s,t$, with the poles in the dependence of $\Phi^\flat[\ldots]_\epsilon(x_4;\mathfrak{x})$, $\flat = s,t$ on their variable x_4. It follows that the coefficient of $\delta(k_4-k_4')$ in the expression for $\mathcal{K}$ is independent of k_4. The result now follows from standard properties of the Fourier transformation. □

PROPOSITION 6. — *We have*

$$\begin{aligned}\left\{\begin{smallmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{smallmatrix}\middle|\begin{smallmatrix}\alpha_s\\ \alpha_t\end{smallmatrix}\right\}_b &= N\ \frac{S_b(\alpha_2+\alpha_s-\alpha_1)S_b(\alpha_t+\alpha_1-\alpha_4)}{S_b(\alpha_2+\alpha_t-\alpha_3)S_b(\alpha_s+\alpha_3-\alpha_4)}\cdot\\ &\cdot|S_b(2\alpha_t)|^2\int_{-i\infty}^{i\infty} ds\ \frac{S_b(U_1+s)S_b(U_2+s)S_b(U_3+s)S_b(U_4+s)}{S_b(V_1+s)S_b(V_2+s)S_b(V_3+s)S_b(V_4+s)},\end{aligned} \tag{68}$$

where the coefficients U_i and V_i, $i = 1, \ldots, 4$ are given by

$$
\begin{aligned}
U_1 &= \alpha_s + \alpha_1 - \alpha_2 & V_1 &= 2Q + \alpha_s - \alpha_t - \alpha_2 - \alpha_4 \\
U_2 &= Q + \alpha_s - \alpha_2 - \alpha_1 & V_2 &= Q + \alpha_s + \alpha_t - \alpha_4 - \alpha_2 \\
U_3 &= \alpha_s + \alpha_3 - \alpha_4 & V_3 &= 2\alpha_s \\
U_4 &= Q + \alpha_s - \alpha_3 - \alpha_4 & V_4 &= Q,
\end{aligned} \tag{69}
$$

and N is a constant.

Proof. — Let

$$
\mathcal{K}_\epsilon\big[\begin{smallmatrix} \alpha_4 & \alpha_s & x_4 \\ \alpha_4' & \alpha_t & x_4' \end{smallmatrix}\big] = \int_{-\infty}^{\infty} dx_3 dx_2 dx_1 \;\big(\Phi^t_{\alpha_t}\big[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4' & \alpha_1 \end{smallmatrix}\big]_\epsilon(x_4'; \mathfrak{x})\big)^* \, \Phi^s_{\alpha_s}\big[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\big]_\epsilon(x_4; \mathfrak{x}). \tag{70}
$$

The analytic and asymptotic properties of the integrand follow from Lemma 19 in Appendix C. Let us observe that for $\epsilon > 0$ one is dealing with absolutely convergent integrals, the integrand being meromorphic both w.r.t. the integration variables and the parameters. The integral (70) therefore does not depend on the order in which the integrations are performed, so we will assume that it is first integrated over x_2.

Singular behavior will emerge in the limit $\epsilon \to 0$. We will call a pole relevant if it has distance of $\mathcal{O}(\epsilon)$ from the real axis, irrelevant otherwise[2]. It then easily follows from Lemma 3 that the integration over x_2 does not introduce any new relevant poles since all the relevant poles in the x_2 dependence that have distance of $\mathcal{O}(\epsilon)$ are lying on the same side of the contour.

Next one may integrate over x_1. We find from Lemma 19 in Appendix C that

$$
\begin{aligned}
\Phi^s_{\alpha_s}\big[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\big]_\epsilon(x_4, \mathfrak{x}) &= \frac{\mathcal{R}^s_{13}}{x_1 - x_3 + \alpha_{13} - 2i\epsilon} + \frac{\mathcal{R}^s_{14}}{x_1 - x_4 + \alpha_{14} - 2i\epsilon} + (\mathrm{Reg}_s), \\
\big(\Phi^t_{\alpha_t}\big[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4' & \alpha_1 \end{smallmatrix}\big]_\epsilon(x_4', \mathfrak{x})\big)^* &= \frac{\mathcal{R}^t_{13}}{x_1 - x_3 + \alpha_{13}' + 2i\epsilon} + \frac{\mathcal{R}^t_{14}}{x_1 - x_4' + \alpha_{14}' + i\epsilon} + (\mathrm{Reg}_t),
\end{aligned} \tag{71}
$$

where $(\mathrm{Reg}_\flat)$, $\flat = s, t$ are terms that do not lead to relevant poles in the variable x_1 after having integrated over x_2. The following abbreviations have been used:

$$
\begin{aligned}
\alpha_{13} &= \tfrac{i}{2}(\alpha_1 + \alpha_3 - 2(Q - \alpha_4)), & \alpha_{13}' &= \tfrac{i}{2}(\alpha_1 + \alpha_3 - 2(Q - \alpha_4')), \\
\alpha_{14} &= \tfrac{i}{2}(\alpha_1 - \alpha_4), & \alpha_{14}' &= \tfrac{i}{2}(\alpha_1 - \alpha_4').
\end{aligned} \tag{72}
$$

It is then easily found by using Lemma 3 that the result of the integration over x_1 will have poles at the following locations:

$$
\begin{aligned}
& i(\alpha_4 - \alpha_4') - 4i\epsilon = 0, && x_3 - x_4 - \tfrac{i}{2}(\alpha_3 + \alpha_4 - 2(Q - \alpha_4')) - 4i\epsilon = 0, \\
& x_4' - x_4 + \tfrac{i}{2}(\alpha_4' - \alpha_4) - 3i\epsilon = 0, && x_4' - x_3 + \tfrac{i}{2}(\alpha_3 + \alpha_4' - 2(Q - \alpha_4)) - 3i\epsilon = 0.
\end{aligned} \tag{73}
$$

The relevant residues can easily be assembled from the expressions given in Appendix C. Moreover, it is straightforward to work out their poles. By again using Lemma 3 one then finds that all four poles listed in (73) will, after doing the x_3 integration, produce terms that are singular for $x_4 = x_4'$, $\alpha_4 = \alpha_4'$ and $\epsilon \to 0$. The terms that lead to $\delta(x_4 - x_4')\delta(\alpha_4 - \alpha_4')$ are easily identified by means of

$$
\lim_{\epsilon \to 0+} \Big(\frac{1}{x - i\epsilon} - \frac{1}{x + i\epsilon}\Big) = 2\pi i \delta(x). \tag{74}
$$

[2]We of course assume that ϵ has been chosen to be much smaller than b

All these terms have as residue an expression proportional to

$$\underset{y_{31}=0}{\mathrm{Res}}\ \underset{y_{21}=0}{\mathrm{Res}}\big[\begin{smallmatrix}\alpha_4 & \alpha_3 & \alpha_s\\ * & * & *\end{smallmatrix}\big]\ \underset{y_{31}=0}{\mathrm{Res}}\ \underset{y_{21}=0}{\mathrm{Res}}\big[\begin{smallmatrix}\alpha_4 & \alpha_t & \alpha_1\\ * & * & *\end{smallmatrix}\big] \tag{75}$$
$$\int_{\mathbb{R}} dx_2\ \underset{y_{31}=0}{\mathrm{Res}}\big[\begin{smallmatrix}\alpha_s & \alpha_2 & \alpha_1\\ * & x_2 & x_1\end{smallmatrix}\big]_{x_1=x_3-\alpha_{13}}\ \underset{y_{32}=0}{\mathrm{Res}}\big[\begin{smallmatrix}\alpha_t & \alpha_3 & \alpha_2\\ x_t & * & x_2\end{smallmatrix}\big]_{x_t=x_3-\frac{i}{2}(\alpha_3-\alpha_t)}.$$

One just needs to assemble the ingredients to check that the expression (75) coincides with what one finds on the right hand side of (68) □

REMARK 9. — With more patience, one could of course also fix the constant N by the method used in the previous proof. We refrain from doing so since we will present a less tedious and more illuminating way of calculating it in the next subsection. What will be needed there, however, is the information on analyticity of the coefficients $\{\ldots\}$ w.r.t. α_t that follows from Proposition 6.

5.4. **Relation between the distributions Φ^s and Φ^t**

PROPOSITION 7. — *Φ^s and Φ^t are related by a linear transformation of the form*

$$\Phi^s_{\alpha_s}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big](x_4;\mathfrak{x}) \;=\; \int_{\mathbb{S}} d\alpha_t\ \big\{\begin{smallmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{smallmatrix}\big|\begin{smallmatrix}\alpha_s\\ \alpha_t\end{smallmatrix}\big\}_b\ \Phi^t_{\alpha_t}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big](x_4;\mathfrak{x}). \tag{76}$$

The relation (76) can be read either as (i) relation between function analytic in

$$\mathcal{A}^{(4)} \equiv \{\underline{x}=(x_4,x_3,x_2,x_1)\in\mathbb{C}^4;\ Im(x_1)<Im(x_2)<Im(x_3),$$
$$Im(x_1)<Im(x_4)<Im(x_3),\ Im(x_3-x_1)<Q\},$$

or (ii) as relation between functions meromorphic w.r.t. $\underline{x}\in\mathbb{C}^4$, or (iii) as relation between distributions defined as boundary values of $\Phi^\flat$, $\flat=s,t$ for $(x_4,\mathfrak{x})\in\mathbb{R}^4$.

Proof. — We will start from equation (59). By using Fourier-transformation w.r.t. the variable k_4 and equation (66) one may rewrite (59) as follows:

$$F^s_f(\alpha_4,\alpha_s,x_4) \;=\; \int_{\mathbb{S}} d\alpha_t\ \big\{\begin{smallmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{smallmatrix}\big|\begin{smallmatrix}\alpha_s\\ \alpha_t\end{smallmatrix}\big\}_b\ F^t_f(\alpha_4',\alpha_t,x_4). \tag{77}$$

Let us introduce sequences of test-functions that tend towards delta-distributions:

$$t_n(\mathfrak{y};\mathfrak{x}) \;=\; \Big(\frac{n}{2\pi}\Big)^{\frac{3}{2}} e^{-\frac{n}{2}\|\mathfrak{x}-\mathfrak{y}\|^2}, \qquad \mathfrak{y}=(y_3,y_2,y_1). \tag{78}$$

LEMMA 8. — *Let $\underline{y}\equiv(x_4,\mathfrak{y})\in\mathcal{A}^{(4)}$ with $Im(y_1)<0$. In this case one has*

$$\lim_{n\to\infty} F^\flat_{t_n(\mathfrak{y};.)}(\alpha_4,\alpha_\flat,x_4) = \Phi^\flat_{\alpha_\flat}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big](x_4;\mathfrak{y}). \tag{79}$$

Proof. — By writing out the definition of $F^\flat_{t_n}$ and shifting the contours of integration over x_i to $\mathbb{R}+i\mathrm{Im}(y_i)$, $i=1,2,3$, one reduces the claim to the standard result that

$$\lim_{n\to\infty} t_n(\mathfrak{y};\mathfrak{x}) = \delta^3(\mathfrak{x}-\mathfrak{y})$$

for $\mathrm{Im}(y_i)=0$, $i=1,2,3$ (Note that $\Phi^\flat$ is regular for these values of its arguments as follows from Lemma 19, Appendix C). □

We will now consider the sequence with elements

$$\int_{\mathbb{S}} d\alpha_t \,\left\{ \begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{smallmatrix} \middle| \begin{smallmatrix} \alpha_s \\ \alpha_t \end{smallmatrix} \right\}_b F^t_{t_n(\mathfrak{y},.)}(\alpha_4, \alpha_t, x_4). \tag{80}$$

It converges for $n \to \infty$ due to Lemma 8 and equation (77). We would like to show that one may exchange the limit $n \to \infty$ with the integration over α_t so that the limit of (80) is given by the integral

$$\int_{\mathbb{S}} d\alpha_t \,\left\{ \begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{smallmatrix} \middle| \begin{smallmatrix} \alpha_s \\ \alpha_t \end{smallmatrix} \right\}_b \Phi^t_{\alpha_t}\left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix} \right](x_4; \mathfrak{y}). \tag{81}$$

To this aim it is useful to note that

LEMMA 9. — *Under the conditions on the variable $\mathfrak{y}$ introduced in Lemma 8 one finds that the integrand in (81) decays exponentially for $p_t \equiv -i(\alpha_t - \frac{Q}{2}) \to \pm\infty$. The integrand in (80) decays at least as fast as the integrand in (81).*

Proof. — By a straightforward calculation using the method in the proof of Lemma 17, Appendix B and eqn. (135) one finds that

$$\begin{aligned} &\Phi^t_{\alpha_t}\left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix} \right](x_4; \mathfrak{y}) \quad \text{decays stronger than } e^{\mp \pi Q p_t} \text{ and} \\ &\left\{ \begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{smallmatrix} \middle| \begin{smallmatrix} \alpha_s \\ \alpha_t \end{smallmatrix} \right\}_b \quad \text{grows as } e^{\pm \pi Q p_t} \end{aligned} \tag{82}$$

for $p_t \to \infty$. The first statement in Lemma 9 follows.

The second statement follows from the first by shifting the contour of integration over x_1 in the definition of $F^t_{t_n(\mathfrak{y},.)}$ to $\mathbb{R} + i\mathrm{Im}(y_1)$. □

The integrals (80)(81) can therefore be transformed into integrals over a compact set, e.g. the interval $[0, 1]$. In order to justify the exchange of limit and integration it therefore suffices to prove the following

LEMMA 10. — *The convergence of $F^t_{t_n(\mathfrak{y},.)}(\alpha_4, \alpha_t, x_4)$ is uniform in α_t.*

Proof. — To shorten the exposition, let us consider a slightly simplified situation. Assume that $f_p(x)$ is analytic w.r.t. both p and x in open strips that contain the real axis and decays exponentially for either $|p|$ or $|x|$ going to infinity. Let $t_n(x) = \sqrt{\frac{n}{2\pi}} e^{-nx^2/2}$ and study the convergence of $f_{p,n} \equiv \int_{\mathbb{R}} dx f_p(x) t_n(x)$ for $n \to \infty$. Upon writing $f_p(x) = f_p(0) + x g_p(x)$, the task reduces to the study of

$$\int_{\mathbb{R}} dx \;\, g_p(x) \;\, x t_n(x) \;=\; \frac{1}{\sqrt{2\pi n}} \int_{\mathbb{R}} dx \;\, e^{-\frac{n}{2}x^2} \;\partial_x g_p(x). \tag{83}$$

Convergence for $n \to \infty$ will be uniform in p provided that $\partial_x g_p(x)$ is bounded as function of both p and x. But this is a consequence of our assumptions: The exponential decay allows us to transform $f_p(x)$ (resp. $\partial_x g_p(x)$) to a function that is analytic on a compact rectangle in $\mathbb{C}^2$, and therefore bounded.

The regularity properties of Φ^t necessary to extend the argument to the present situation follow from Lemma 19, Appendix C. □

We have proved (76) provided $(x_4, \mathfrak{x})$ satisfies the same conditions as $(x_4, \mathfrak{y})$ in Lemma 8. Proposition 7 follows by analytic continuation. $\square$

5.5. **Calculation of Racah-Wigner coefficients II**

We have shown that the meromorphic functions Φ^s and Φ^t are related by an integral transformation of the form (76). If one fixes the values of three of the four variables $x_4, \dots, x_1$ in (76) one obtains an integral transformation for a function of a single variable. In fact, the analytic properties of $\Phi^s_{\alpha_s}$ and $\Phi^t_{\alpha_t}$ even allow one to choose complex values. It will be convenient to consider

$$\Psi^s_{\alpha_s}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right](x) = \lim_{x_4\to\infty} e^{2\pi\alpha_4 x_4} \lim_{x_2\to-\infty} \prod_{j=1}^{3} e^{-2\pi\alpha_j x_j}\, \Phi^s_{\alpha_s}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right](\mathfrak{x})\Big|^{x_1=x}_{x_3=\frac{i}{2}(Q+\alpha_2-\alpha_4)}, \tag{84}$$

where $\bar{\alpha} = Q - \alpha$, and the same for $\Psi^t_{\alpha_t}$. The integral that defines $\Phi^s_{\alpha_s}$ and $\Phi^t_{\alpha_t}$, (54)(57) can be done explicitly in this limit by using (146). One finds expressions of the form

$$\begin{aligned}
&\Psi^s_{\alpha_s}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right](x) = N^s_{\alpha_s}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right]\Theta^s_{\alpha_s}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right](x)\\
&\qquad\Theta^s_{\alpha_s}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right](x) = e^{+2\pi x(\alpha_s-\alpha_2-\alpha_1)} F_b(\alpha_s+\alpha_1-\alpha_2, \alpha_s+\alpha_3-\alpha_4; 2\alpha_s; -ix)\\
&\Psi^t_{\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right](x) = N^t_{\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right]\Theta^t_{\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right](x)\\
&\qquad\Theta^t_{\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right](x) = e^{-2\pi x(\alpha_t+\alpha_1-\alpha_4)} F_b(\alpha_t+\alpha_3-\alpha_2, \alpha_t+\alpha_1-\alpha_4; 2\alpha_t; +ix),
\end{aligned}\tag{85}$$

where F_b is the b-hypergeometric function defined in the Appendix, and $N^s_{\alpha_s}$, $N^t_{\alpha_t}$ are certain normalization factors.

The linear transformation following from (76) can now be calculated as follows: One observes that $\Psi^s_{\alpha_s}$ (resp. $\Psi^t_{\alpha_t}$) are eigenfunctions of the finite difference operators $\mathcal{Q}_s$ and $\mathcal{Q}_t$ defined respectively by

$$\begin{aligned}
\mathcal{Q}_s &= \big[d_x+\alpha_1+\alpha_2-\tfrac{Q}{2}\big]^2 - e^{+2\pi bx}\big[d_x+\alpha_1+\alpha_2+\alpha_3-\alpha_4\big]\big[d_x+2\alpha_1\big]\\
\mathcal{Q}_t &= \big[d_x+\alpha_1-\alpha_4+\tfrac{Q}{2}\big]^2 - e^{-2\pi bx}\big[d_x+\alpha_1+\alpha_2-\alpha_3-\alpha_4\big]\big[d_x\big].
\end{aligned}\tag{86}$$

It can be shown that

THEOREM 4. — *The operators $\mathcal{Q}_s$ and $\mathcal{Q}_t$ have unique self-adjoint extensions in $L^2(\mathbb{R}, dx e^{2\pi Qx})$. Bases of $L^2(\mathbb{R}, dx e^{2\pi Qx})$ in the sense of generalized eigenfunctions are given by the sets of functions $\{\Theta^s_{\alpha_s}; \alpha_s \in \mathbb{S}\}$ and $\{\Theta^t_{\alpha_t}; \alpha_t \in \mathbb{S}\}$, where the normalization is given by*

$$\int_{\mathbb{R}} dx\; e^{2\pi Qx}\, \big(\Theta^\flat_{\alpha'_\flat}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right](x)\big)^* \Theta^\flat_{\alpha_\flat}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right](x) = \delta(\alpha_\flat - \alpha'_\flat), \quad \flat = s,t. \tag{87}$$

The proof is omitted as it is very similar to the proof of Theorem 3. It follows that the Racah-Wigner coefficients can be evaluated in terms of the overlap between these two bases:

$$\left\{\begin{smallmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \bar{\alpha}_4\end{smallmatrix}\middle|\begin{smallmatrix}\alpha_s\\ \alpha_t\end{smallmatrix}\right\}_b = \frac{N^s_{\alpha_s}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right]}{N^t_{\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right]} \int_{\mathbb{R}} dx\; e^{2\pi Qx}\, \big(\Theta^t_{\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right](x)\big)^* \Theta^s_{\alpha_s}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right](x). \tag{88}$$

The integral can be done by using the representation (143) for the b-hypergeometric function. The result is just equation (68) with $N = 1$.

5.6. **Properties the Racah-Wigner coefficients**

First of all let us note that orthogonality and completeness of the bases $\{\Phi^s_{\alpha_s}; \alpha_s \in \mathbb{S}\}$ and $\{\Phi^t_{\alpha_t}; \alpha_t \in \mathbb{S}\}$ imply the following orthogonality relations for the b-Racah-Wigner symbols

$$\int_{\mathbb{S}} d\alpha_s \, |S_b(2\alpha_s)|^2 \left\{ \begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{smallmatrix} \middle| \begin{smallmatrix} \alpha_s \\ \alpha_t \end{smallmatrix} \right\}_b \left(\left\{ \begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{smallmatrix} \middle| \begin{smallmatrix} \alpha_s \\ \beta_t \end{smallmatrix} \right\}_b \right)^* = |S_b(2\alpha_t)|^2 \, \delta(\alpha_t - \beta_t). \tag{89}$$

This may be verified e.g. by rewriting

$$\begin{aligned} \left(\Phi^t_{\alpha_t} \left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix} \right]_\epsilon (x_4; .) \, , \, \Phi^t_{\alpha'_t} \left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha'_4 & \alpha_1 \end{smallmatrix} \right]_\epsilon (x'_4; .) \right) & = \\ = |S_b(2\alpha_t)|^{-2} \delta(\alpha_t - \alpha'_t) \delta(\alpha_4 - \alpha'_4) \delta(x_4 - x'_4) & \end{aligned} \tag{90}$$

with the help of the inversion formula to (76)

$$\Phi^t_{\alpha_s} \left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix} \right] (x_4; \mathfrak{x}) = \int_{\mathbb{S}} d\alpha_s \left| \frac{S_b(2\alpha_s)}{S_b(2\alpha_t)} \right|^2 \left(\left\{ \begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{smallmatrix} \middle| \begin{smallmatrix} \alpha_s \\ \alpha_t \end{smallmatrix} \right\}_b \right)^* \Phi^s_{\alpha_t} \left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix} \right] (x_4; \mathfrak{x}), \tag{91}$$

and finally using (90) with subscripts t replaced by s.

Second, by considering quadruple products of representations one finds the so-called pentagon equation in the usual way:

$$\int_{\mathbb{S}} d\delta_1 \left\{ \begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \beta_2 \end{smallmatrix} \middle| \begin{smallmatrix} \beta_1 \\ \delta_1 \end{smallmatrix} \right\}_b \left\{ \begin{smallmatrix} \alpha_1 & \delta_1 \\ \alpha_4 & \alpha_4 \end{smallmatrix} \middle| \begin{smallmatrix} \beta_2 \\ \gamma_2 \end{smallmatrix} \right\}_b \left\{ \begin{smallmatrix} \alpha_2 & \alpha_3 \\ \alpha_4 & \gamma_2 \end{smallmatrix} \middle| \begin{smallmatrix} \delta_1 \\ \gamma_1 \end{smallmatrix} \right\}_b = \left\{ \begin{smallmatrix} \beta_1 & \alpha_3 \\ \alpha_4 & \alpha_5 \end{smallmatrix} \middle| \begin{smallmatrix} \beta_2 \\ \gamma_1 \end{smallmatrix} \right\}_b \left\{ \begin{smallmatrix} \alpha_1 & \alpha_2 \\ \gamma_1 & \alpha_5 \end{smallmatrix} \middle| \begin{smallmatrix} \beta_1 \\ \gamma_2 \end{smallmatrix} \right\}_b. \tag{92}$$

5.7. **From intertwiners to coinvariants**

Let us consider coinvariants on tensor products of representations. These will be maps $\mathcal{B} : \mathcal{P}_{\alpha_n} \otimes \ldots \otimes \mathcal{P}_{\alpha_1.} \to \mathbb{C}$ that satisfy the coinvariance property

$$\mathcal{B} \circ \big((\pi_{\alpha_n} \otimes \ldots \otimes \pi_{\alpha_1}) \Delta^{(n)}(u) \big) = 0, \quad u \in \mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R})), \tag{93}$$

where $\Delta^{(n)}$ is defined recursively by $\Delta^{(n)} = (\mathrm{id} \otimes \Delta)(\Delta^{(n-1)}) = (\Delta \otimes \mathrm{id})(\Delta^{(n-1)})$, $\Delta^{(2)} \equiv \Delta$.

The basic case to consider is $n = 2$. Let $\mathcal{B}_\alpha : \mathcal{P}_{Q-\alpha} \otimes \mathcal{P}_\alpha \to \mathbb{C}$ be defined by

$$\mathcal{B}_\alpha(f \otimes g) \equiv \langle f \, , \mathcal{T} g \rangle, \qquad \mathcal{T} \equiv T_x^{-i\frac{Q}{2}} \tag{94}$$

PROPOSITION 8. — *$\mathcal{B}_\alpha$ satisfies the coinvariance property (93).*

Proof. — Let us note that

$$\langle T_x^{i\alpha} f \, , \, g \rangle = \langle f \, , T_x^{-i\alpha} g \rangle \tag{95}$$

if $f \in \mathcal{P}_{Q-\alpha}$ and $g \in \mathcal{P}_\alpha$. A straightforward calculation then shows that

$$\langle \pi_{Q-\alpha}(u) f \, , \, g \rangle = \langle f \, , \pi_\alpha(u) g \rangle, \quad u \in \mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R})). \tag{96}$$

It is useful to also note the commutation relations

$$\mathcal{T} E_\alpha = e^{-i\pi bQ} E_\alpha \mathcal{T}, \qquad \mathcal{T} F_\alpha = e^{+i\pi bQ} F_\alpha \mathcal{T}, \qquad \mathcal{T} K_\alpha = K_\alpha \mathcal{T}. \tag{97}$$

We may then calculate in the case $u = E$

$$\begin{aligned} \mathcal{B}_\alpha\big(((\pi_{Q-\alpha}\otimes\pi_\alpha)\circ\Delta(E))f\otimes g\big) \;=\;& \\ =\;& \langle\, E_{Q-\alpha}f\,,\,\mathcal{T}K_\alpha g\,\rangle + \langle\, K_{Q-\alpha}f\,,\,\mathcal{T}E_\alpha g\,\rangle \\ =\;& \langle\, E_{Q-\alpha}f\,,\,K_\alpha\mathcal{T}g\,\rangle + e^{-i\pi bQ}\langle\, K_{Q-\alpha}f\,,\,E_\alpha\mathcal{T}g\,\rangle \\ =\;& \langle\, f\,,\,E_\alpha K_\alpha\mathcal{T}g\,\rangle - q^{-1}\langle\,\mathcal{T}f\,,\,K_\alpha E_\alpha\mathcal{T}g\,\rangle \\ =\;& 0. \end{aligned} \tag{98}$$

The calculation for the case $u = F$ is identical and the case $u = K$ is trivial. □

A coinvariant $\mathcal{B}'_\alpha : \mathcal{P}_\alpha\otimes\mathcal{P}_\alpha$ is then obtained by combining $\mathcal{B}_\alpha$ with the intertwining operator $\mathcal{I}_\alpha$:

$$\mathcal{B}'_\alpha \;\equiv\; \mathcal{B}_\alpha\circ(\mathcal{I}_\alpha\otimes\mathrm{id}). \tag{99}$$

In order to construct coinvariants $\mathcal{B}^{(n)}$ for $n > 2$ one may use intertwining maps

$$\mathcal{C}\in\mathrm{Hom}_{\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))}(\mathcal{P}_{\alpha_{n-1}}\otimes\ldots\otimes\mathcal{P}_{\alpha_1},\mathcal{P}_{\alpha_n}).$$

Such maps can be constructed by iterating Clebsch-Gordan maps, as has been discussed explicitly in the case $n = 4$ at the beginning of the present Section. One may associate a coinvariant $\mathcal{B}_\mathcal{C}$ to any $\mathcal{C}\in\mathrm{Hom}_{\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))}(\mathcal{P}_{\alpha_{n-1}}\otimes\ldots\otimes\mathcal{P}_{\alpha_1},\mathcal{P}_{\alpha_n})$ via

$$\mathcal{B}_\mathcal{C} \;\equiv\; \mathcal{B}\circ(\mathrm{id}\otimes\mathcal{C}). \tag{100}$$

The maps $\mathcal{C}$ can be represented explicitly with the help of meromorphic integral kernels $\Phi_\mathcal{C}(x_n;\mathfrak{x})$, $\mathfrak{x}\equiv(x_{n-1},\ldots,x_1)$ that generalize $\Phi^\flat_{\alpha_\flat}$ and the Clebsch-Gordan coefficients. It follows that the corresponding coinvariant $\mathcal{B}_\mathcal{C}$ can be represented as

$$\mathcal{B}_\mathcal{C}(f_n\otimes\ldots\otimes f_1) \;=\; \int\limits_{\mathbb{R}} dx_n\; T_{x_n}^{i\frac{Q}{2}} f_n(x_n) \int\limits_{\mathbb{R}^{n-1}} d\mathfrak{x}\;\Phi_\mathcal{C}(x_n;\mathfrak{x})\; f_{n-1}(x_{n-1})\ldots f_1(x_1). \tag{101}$$

It is possible to rewrite (101) as a convolution of $f_n(x_n)\ldots f_1(x_1)$ against a kernel $\Psi_\mathcal{C}(\underline{x})$, $\underline{x}\equiv(x_n,\ldots,x_1)$: To this aim it is necessary to “partially” integrate the finite difference operator in (101) to let it act on $\Phi_\mathcal{C}$. One should note that the analytic continuation of the integral over $\mathfrak{x}$ to complex values of x_n may in general be represented by integrating the variable $\mathfrak{x}$ over deformed contours, cf. e.g. the proof of Proposition 3. One arrives at a representation of the form

$$\mathcal{B}_\mathcal{C}(f_n\otimes\ldots\otimes f_1) \;=\; \int\limits_{C^n} dx_n\ldots dx_1\;\Psi_\mathcal{C}(x_n,\ldots,x_1)f_n(x_n)\ldots f_1(x_1), \tag{102}$$

where

$$\Psi_\mathcal{C}(x_n,\ldots,x_1) \;=\; T_{x_n}^{-i\frac{Q}{2}}\;\Phi_\mathcal{C}(x_n;x_{n-1},\ldots,x_1). \tag{103}$$

REMARK 10. — The kernels that represent the coinvariants are in some respects analogous to functional realizations of the conformal blocks in conformal field theory. We strongly suspect that we are touching upon the tip of an iceberg at this point: Quantization of Teichmüller space, as developed in [22][23] conjecturally leads to a construction of spaces of conformal blocks in Liouville theory. One may expect this to be equivalent to a quantization of certain moduli spaces of flat $SL(2,\mathbb{R})$ connections on Riemann surfaces with marked points. In analogy to results of [24] one would expect spaces of conformal blocks in the case of the punctured Riemann sphere to be represented by spaces of coinvariants in tensor products of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ representations. A class of these has been constructed in the present subsection. It would certainly be rather interesting and far-reaching if one could establish a direct relation between these spaces and the Hilbert spaces constructed via quantization of Teichmüller space.

In this regard we find the following observation quite intriguing: Consider the case of $n = 4$. There is a canonical way to define a Hilbert space $\mathcal{H}^{(0,4)}$ of coinvariants by taking the sets $\{\Phi^{\flat}_{\alpha}; \alpha \in \mathbb{S}\}$ for either $\flat = s$ or $\flat = t$ as basis in the sense of generalized functions with the normalization given by

$$(\Phi^{\flat}_{\alpha}\,,\, \Phi^{\flat}_{\alpha'}\,) \;=\; |S_b(2\alpha)|^{-2}\delta(\alpha-\alpha'). \tag{104}$$

The observation made in subsection 5.6. now implies that $\mathcal{H}^{(0,4)}$ is in a canonical way isomorphic to $L^2(\mathbb{R})$ such that multiplication with $[\alpha_s - \frac{Q}{2}]_b^2$ (resp. $[\alpha_s - \frac{Q}{2}]_b^2$) gets mapped into the self-adjoint finite difference operator $\mathcal{Q}_s$ (resp. $\mathcal{Q}_t$). Maybe there is a rather direct connection of these operators to the geodesic length operators appearing in the quantization of Teichmüller space. This would establish a direct relation between the latter and our quantum group results.

6. APPENDIX A: SPECTRAL ANALYSIS OF $C_{21}(\kappa_3)$

This appendix is devoted to the proof of Theorem 3.

6.1. Preliminaries

The difference operator to be considered is of the form

$$C_{21}(\kappa_3) - [\alpha_3 - \tfrac{Q}{2}]_b^2 \;=\; \delta_+ e^{\pi i b Q} e^{2\pi b x} - \delta_0 + \delta_- e^{-\pi i b Q} e^{-2\pi b x}, \tag{105}$$

where δ_s, $s = -, 0, +$ are x-independent finite difference operators given by
(106)

$$\begin{aligned}
\delta_+ &= T_x^{-ib}[d_x - \alpha_2 - ik_3]_b[d_x - \alpha_1 + ik_3]_b \\
2\delta_0 &= \{0\}_b\Big(\{Q\}_b T_x^{-2ib} - \big(e^{-2\pi b k_3}\{2\alpha_2 - Q\}_b + e^{2\pi b k_3}\{2\alpha_1 - Q\}_b\big)T_x^{-ib} + \{2\alpha_3 - Q\}_b\Big) \\
\delta_- &= T_x^{-ib}[d_x + \alpha_2 - ik_3]_b[d_x + \alpha_1 + ik_3]_b,
\end{aligned}$$

and $\kappa_3 = -2k_3$. It will initially be defined on the domain $\mathfrak{D} \subset L^2(\mathbb{R})$ consisting of functions with the following property: There exists a function $F(z)$ that is

(1) holomorphic in the strip $\{z \in \mathbb{C} | \mathrm{Im}(z) \in [-2b, 0]\}$ and
(2) the functions $F_y(x) \equiv F(x + iy)$ are in $L^2(\mathbb{R}, dx \cosh(2\pi b x))$ for any $y \in [-2b, 0]$.

PROPOSITION 9. — *The operator $(C_{21}(k_3), \mathfrak{D})$ is a symmetric, densely defined operator in $L^2(\mathbb{R})$. The domain $\mathfrak{D}^\dagger$ of its adjoint is dense as well.*

Proof. — First of all note that one has

$$(f, T_x^{-ib} g) = (T_x^{-ib} f, g) \tag{107}$$

for any $f, g \in \mathfrak{D}$. This follows by shifting the contour of the integration that represents $(f, T_- g)$ to the line $\mathbb{R} + ib$. The fact that $C_{21}(\kappa_3)$ is symmetric is then seen by a simple calculation remembering that $\alpha_i^* = Q - \alpha_i$, $i = 1, 2$.

The fact that $\mathfrak{D}$ and $\mathfrak{D}^\dagger$ are dense in $L^2(\mathbb{R})$ is easily seen by noting that any Hermite-function is contained in these sets. □

The Paley-Wiener theorem provides a characterization of the Fourier-transform $\tilde{\mathfrak{D}}$ of the domain $\mathfrak{D}$ of $C_{21}(\kappa_3)$. The action of $C_{21}(\kappa_3)$ on functions in $\mathfrak{D}$ then corresponds to acting on $\tilde{\mathfrak{D}}$ with the following operator:

$$\begin{aligned}
C_{21}(\kappa_3) - \big[\alpha_3 - \tfrac{Q}{2}\big]_b^2 \;&\equiv\; \Delta_0 - e^{2\pi b\omega}\Delta_1 + e^{4\pi b\omega}\Delta_2 \\
\Delta_0 &= [d_\omega + \alpha_3 - Q - \tfrac{1}{2}(\alpha_1 + \alpha_2)]_b[d_\omega - \alpha_3 - \tfrac{1}{2}(\alpha_1 + \alpha_2)]_b \\
\Delta_1 &= [d_\omega + \tfrac{1}{2}(\alpha_1 + \alpha_2)]_b\Big(e^{i\pi b(d_\omega - \frac{1}{2}(\alpha_1+\alpha_2) + Q)}\{\alpha_1 - \alpha_2 - 2ik\}_b \\
&\qquad - e^{-i\pi b(d_\omega - \frac{1}{2}(\alpha_1+\alpha_2)+Q)}\{\alpha_1 - \alpha_2 + 2ik\}_b\Big) \\
\Delta_2 &= [d_\omega + \tfrac{1}{2}(\alpha_1 + \alpha_2)]_b[d_\omega + \tfrac{1}{2}(\alpha_1 + \alpha_2) + Q]_b.
\end{aligned} \tag{108}$$

6.2. **Strategy**

The key to the proof of Theorem 3 is the following result characterizing regularity and asymptotic properties of distributional solutions to the eigenvalue equation of the operator $C_{21}(\kappa_3)$:

THEOREM 5. — *Let $\Phi \in \mathcal{S}'(\mathbb{R})$ be a distributional solution of $(C_{21}(\kappa_3) - [\alpha_3 - \frac{Q}{2}]^2)^t\, \Phi = 0$.*

(1) *$\tilde{\Phi}$ is represented by a function $\tilde{\Phi}(\omega)$ that can be continued to a meromorphic function on $\mathbb{C}$, with simple poles within $\mathfrak{S}_{Q/2}$ only at*

$$\begin{aligned} \omega &= -\,k_3 + i(\alpha_1 + nb + mb^{-1}), \quad \omega = -\,k_3 - i(\alpha_1 + nb + mb^{-1}), \\ \omega &= +\,k_3 + i(\alpha_2 + nb + mb^{-1}), \quad \omega = +\,k_3 - i(\alpha_2 + nb + mb^{-1}), \end{aligned} \qquad n, m \in \mathbb{Z}^{\geq 0}.$$

(2) *Φ can be represented as $\Phi = \lim_{\epsilon\to 0} \Phi_\epsilon$ where Φ_ϵ is for $\epsilon > 0$ represented as the restriction to $\mathbb{R}$ of a function $\Phi_\epsilon(x)$ that is meromorphic on $\mathbb{C}$ with poles only at*

$$\begin{aligned} x &= +\,\tfrac{i}{2}\big(\alpha_1 + \alpha_2 - Q\big) \pm i\big(\alpha_3 - \tfrac{Q}{2}\big) - i(\epsilon + nb + mb^{-1}), \\ x &= -\,\tfrac{i}{2}\big(\alpha_1 + \alpha_2 - Q\big) + i\big(\tfrac{Q}{2} + nb + mb^{-1}\big), \end{aligned} \qquad n, m \in \mathbb{Z}^{\geq 0}.$$

In fact, given these properties it is not very difficult to show that for any given eigenvalue $[\alpha_3 - \frac{Q}{2}]^2$ there is at most one tempered distributional solution to the eigenvalue equation (Proposition 13). Moreover, no such solution exists for $\mathrm{Re}(2\alpha_3 - Q) \neq 0$. It follows [25] that the deficiency indices vanish and $C_{21}(\kappa_3)$ has a unique self-adjoint extension. The spectral decomposition can be written as expansion into generalized eigenfunctions [26]. It can be shown on rather general grounds that only tempered distributions can appear in the spectral decomposition, as nicely discussed in [27]. The combination of Theorem 5 and Proposition 13 therefore also yields a characterization of the support of the Plancherel measure.

These remarks reduce the proof of Theorem 3 to that of Theorem 5 and Proposition 13.

6.3. **Preparations**

In view of the explicit expressions for $C_{21}(\kappa_3)$ (cf. (105)) resp. its Fourier-transform (108) one may anticipate that the analysis of the asymptotic behavior of Φ and $\tilde{\Phi}$ will require some information about properties of the operators δ_+, δ_- resp. Δ_0, Δ_2. The information that will be needed is contained in the following Lemmas:

LEMMA 11. — *$\delta_\pm$ is invertible on $\mathcal{C}_c^\infty(\mathbb{R})$. The image $f(x)$ of a function $g \in \mathcal{C}_c^\infty(\mathbb{R})$ under $\delta_\pm^{-1}$ has the following properties:*

(1) *$f(x)$ is analytic in the strip $\{x \in \mathbb{C}; Im(x) \in (-2b, 0)\}$ and $f(x) \in \mathcal{C}^\infty(\mathbb{R})$, $f(x - 2ib) \in \mathcal{C}^\infty(\mathbb{R})$.*

(2) *$\tilde{f}(\omega)$ is meromorphic in $\mathbb{C}$ with simple poles at*

$$\omega = -k_3 + i(\mp\alpha_1 + nb^{-1}) \qquad \omega = +k_3 + i(\mp\alpha_2 + nb^{-1}) \qquad n \in \mathbb{Z}.$$

Proof. — The action of $\delta_\pm^{-1}$ is represented on the Fourier transform $\tilde{f}$ as multiplication with

$$(\tilde{\delta}_\pm)^{-1}(\omega) \equiv e^{-2\pi b\omega}[i\omega \mp \alpha_2 - ik_3]_b^{-1}[i\omega \mp \alpha_1 + ik_3]_b^{-1}.$$

The statement on the analyticity properties of $\tilde{f}$ is then clear after recalling that the function $\tilde{g}(\omega)$ is entire analytic and of rapid decay being the Fourier transform of a $\mathcal{C}_c^\infty$ function [21, Theorem IX.11].

The statement that $(\delta_+^{-1}g)(x)$ is analytic in the strip $\{x \in \mathbb{C}; \mathrm{Im}(x) \in (-2b, 0)\}$ follows from the asymptotic decay properties of $(\tilde{\delta}_\pm^{-1})(\omega)$ by means of the Paley-Wiener Theorem. In fact, the rapid decay of $\tilde{g}(\omega)$ ensures convergence of the inverse Fourier transformation for any x-derivative of $(\delta_+^{-1}g)(x)$ even in the extremal cases $\mathrm{Im}(x) = 0$ and $\mathrm{Im}(x) = -2b$. □

We will furthermore need similar statements about the inverses of Δ_0 and Δ_2.

LEMMA 12. — *Δ_2 is invertible on $\mathcal{C}_c^\infty(\mathbb{R})$. The image $f(\omega)$ of a function $g \in \mathcal{C}_c^\infty(\mathbb{R})$ under Δ_2^{-1} has the following properties:*

(1) *$\tilde{f}(x)$ is meromorphic in $\mathbb{C}$ with simple poles at*

$$x = -\tfrac{i}{2}(\alpha_1 + \alpha_2) - i(Q + nb^{-1}) \qquad x = -\tfrac{i}{2}(\alpha_1 + \alpha_2) + inb^{-1} \qquad n \in \mathbb{Z}.$$

(2) *$f(\omega)$ is analytic in the strip $\{\omega \in \mathbb{C}; Im(x) \in (-b, b)\}$ and $f(\omega \pm ib) \in \mathcal{C}^\infty(\mathbb{R})$.*

LEMMA 13. — *Δ_0 is invertible on the space of functions*

$$\mathcal{D}(\Delta_0) \equiv \big(d_\omega + \alpha_3 - Q - \tfrac{1}{2}(\alpha_1 + \alpha_2)\big)\big(d_\omega - \alpha_3 - \tfrac{1}{2}(\alpha_1 + \alpha_2\big)h, \quad h \in \mathcal{C}_c^\infty(\mathbb{R}).$$

The image $f(\omega)$ of a function $g \in \mathcal{D}(\Delta_0)$ under Δ_0^{-1} has the following properties:

(1) *$\tilde{f}(x)$ is meromorphic in $\mathbb{C}$ with simple poles at*

$$x = +\tfrac{i}{2}(\alpha_1 + \alpha_2 - Q) \pm i(\alpha_3 - \tfrac{Q}{2}) - inb^{-1} \qquad n \in \mathbb{Z} \setminus \{0\}.$$

(2) *$f(\omega)$ is analytic in the strip $\{\omega \in \mathbb{C}; Im(x) \in (-b, b)\}$ and $f(\omega \pm ib) \in \mathcal{C}^\infty(\mathbb{R})$.*

6.4. **Asymptotic estimates**

We now want to show that the Fourier-transform $\tilde{\Phi}$ of Φ may actually be represented by integration against a function $\tilde{\Phi}(\omega)$. For technical reasons it will be necessary to start by considering the distribution $\Phi_R \in \mathcal{S}'(\mathbb{R})$ defined by

$$\tilde{\Phi}_R \equiv \tilde{\delta}_{\mathrm{tr,R}}(\omega)\tilde{\Phi} \equiv \prod_{\substack{\omega' \in \mathcal{I}_+ \cup \mathcal{I}_- \\ |\mathrm{Im}(\omega')| < R}} (\omega - \omega')\ \tilde{\Phi},$$

where $\mathcal{I}_+$ (resp. $\mathcal{I}_-$) are the sets of values for ω where either $\tilde{\delta}_+(\omega)$ or $\tilde{\delta}_-(\omega)$ have a pole in the upper (resp. lower) half plane. The following result characterizes the asymptotic behavior of Φ_R.

PROPOSITION 10. — *Let $\tau_n \in \mathcal{C}_c^\infty(\mathbb{R})$ have support only in $[n-1, n+1]$. For sufficiently large value of R there exists some $N > 0$ such that*

$$\cosh(2\pi bn)\langle \Phi_R, \tau_n \rangle < N \quad \textit{for all } n \in \mathbb{Z}. \tag{109}$$

Proof. — We will rewrite $\langle \Phi_R, \tau_n \rangle$ in a form that allows us to estimate its asymptotics for large n. One may write

$$\begin{aligned} \langle \Phi_R, \tau_n \rangle = & \langle \Phi, \delta_{\mathrm{tr,R}} \tau_n \rangle, \\ = & \langle \Phi, \delta_+ e^{2\pi bx} \sigma_{n,R} \rangle, \qquad \text{where} \quad \sigma_{n,R} \equiv e^{-2\pi bx}(\delta_+)^{-1}\delta_{\mathrm{tr,R}}\tau_n; \\ = & \langle \Phi, \delta_+^c \sigma_{n,R} \rangle, \qquad \text{where} \quad \delta_+^c \equiv (\delta_0 - \delta_- e^{-2\pi bx}). \end{aligned} \tag{110}$$

In the last step we have used that Φ weakly solves the eigenvalue equation, for which one needs to check that $\sigma_{n,R} \in \mathfrak{D}$: One point of having introduced $\delta_{\rm tr,R}$ is that it improves the asymptotic behavior of $(\delta_+)^{-1}\delta_{\rm tr,R}\tau_n$ for $x \to -\infty$ by cancelling the poles of its Fourier transform in $\{\omega \in \mathbb{C}; \mathrm{Im}(\omega) < R\}$.

The regularity theorem for tempered distributions [20, Theorem V.10] allows us to furthermore write

$$(111) \qquad \langle \Phi_R, \tau_n \rangle = \int_{-\infty}^{\infty} dx \; \Theta(x) \; \rho_{n,R}(x) \qquad \text{where} \quad \rho_{n,R} \equiv \partial_x^k \; \delta_+^c e^{-2\pi bx} (\delta_+)^{-1} \delta_{\rm tr,R} \tau_n .$$

for some positive integer k and a polynomially bounded continuous function $\Theta(x)$. The functions $\rho_{n,R}(x)$ may be represented by expressions of the form

$$(112) \qquad \rho_{n,R}(x) \; = \; \sum_{k=1,2} C_k e^{-2\pi bx} \int_{-\infty}^{\infty} d\omega \; e^{2\pi i \omega x} \frac{P_{k,R}(\omega)\tilde{\tau}_n(\omega)}{(1-e^{2\pi b(\omega - k + i\alpha_1)})(1-e^{2\pi b(\omega + i\alpha_2)})},$$

where $P_{k,R}(\omega)$ $k = 1,2$ are some polynomials in ω. The functions $\rho_{n,R}(x)$ have main support around $x = n$, and by choosing R large enough one can achieve decay stronger than $e^{-2\pi\lambda|x-n|}$ for any $\lambda > 0$. It is then convenient to split the integral in (111) into an integral J_n obtained by integrating over $[\frac{n}{2}, \frac{3n}{2}]$ and the remainder J_n^c.

In order to estimate J_n^c one may use the polynomial boundedness of $\Theta(x)$ to estimate its absolute value by some constant times $\cosh(\epsilon x)$, where ϵ can be as small as one likes. The absolute value of $\rho_{n,R}(x)$ can in $\mathbb{R} \setminus [\frac{n}{2}, \frac{3n}{2}]$ be estimated by some inverse power of $\cosh(x)$, which is bounded by the chosen value of R. It follows that the exist D_1, N_1 such that

$$(113) \qquad |J_n^c| \; \leq \; D_1 e^{-2\pi\mu n} \quad \text{for any } n > N_1,$$

where μ can be made arbitrarily large by choosing R large enough.

In the case of J_n one may estimate $|\rho_{n,R}(x)|$ by some constant times $e^{-2\pi bn}e^{-2\pi b|x-n|}$ and $\Theta(x)$ simply by a constant, which easily gives existence of D_2, N_2 such that

$$(114) \qquad |J_n| \; \leq \; D_2 e^{-2\pi bn} \quad \text{for any } n > N_1.$$

This proves the claim about the asymptotics for $n \to \infty$. In the case of $n \to -\infty$ one uses the operator δ_- in a completely analogous fashion $\square$

6.5. **Representation of $\tilde{\Phi}$**

Assume that the set $\{\tau_n; n \in \mathbb{Z}\}$ represents a $\mathcal{C}_c^\infty(\mathbb{R})$-partition of unity. It will be convenient to choose the τ_n as translates of τ_0: $\tau_n(x) = \tau_0(x-n)$. This can always be achieved: Let

$$\tau_0(x) = \begin{cases} 0 & \text{if} \quad |x| > \frac{3}{4} \\ 1 & \text{if} \quad |x| < \frac{1}{4} \\ \chi(x+\frac{1}{2}) & \text{if} \quad x \in [-\frac{3}{4}, -\frac{1}{4}] \\ 1-\chi(x-\frac{1}{2}) & \text{if} \quad x \in [+\frac{1}{4}, +\frac{3}{4}], \end{cases} \tag{115}$$

$$\chi(x) \;=\; N^{-1} \int\limits_{-\frac{1}{4}}^{x} dt \; \exp\Bigg(\frac{1}{(x-\frac{1}{4})(x+\frac{1}{4})}\Bigg) \qquad N = \int\limits_{-\frac{1}{4}}^{\frac{1}{4}} dt \; \exp\Bigg(\frac{1}{(x-\frac{1}{4})(x+\frac{1}{4})}\Bigg)$$

The result of Proposition 10 implies convergence of the following sum

$$\tilde{\Phi}_R(\omega) \;\equiv\; \sum_{n\in\mathbb{Z}} \langle \Phi_R, \tau_n e^{-2\pi i\omega x} \rangle \tag{116}$$

which defines $\tilde{\Phi}_R(\omega)$ as a function that is analytic in the strip $\{\omega \in \mathbb{C}; \mathrm{Im}(\omega) \in (-b, b)\}$.

PROPOSITION 11. — *The function $\tilde{\Phi}_R(\omega)$ represents the distribution Φ_R in the sense that*

$$\langle \Phi_R, f \rangle \;=\; \int\limits_{-\infty}^{\infty} d\omega \; \tilde{\Phi}_R(\omega) \tilde{f}(\omega). \tag{117}$$

Proof. — To begin with, note that $\Phi_{R,n}(\omega) \equiv \langle \Phi_R, \tau_n e^{-2\pi i\omega x} \rangle$ represents the Fourier-transform of the distribution $\tau_n \Phi_R \in \mathcal{S}'(\mathbb{R})$ of compact support [21, Theorem IX.12]. It follows that $\langle \Phi_R, \tau_n e^{-2\pi i\omega x} \rangle$ is polynomially bounded. Since the convergence in (116) is absolute, one concludes that $\tilde{\Phi}_R(\omega)$ is polynomially bounded as well. In the evaluation of $\tilde{\Phi}_R(\omega)$ against a test-function $f \in \mathcal{S}(\mathbb{R})$ one may therefore insert definition (117) and exchange the orders of integration and summation to get

$$\begin{aligned} \int\limits_{-\infty}^{\infty} d\omega \; \tilde{\Phi}_R(\omega) \tilde{f}(\omega) \;&=\; \sum_{n\in\mathbb{Z}} \int\limits_{-\infty}^{\infty} d\omega \; \tilde{\Phi}_{R,n}(\omega) \tilde{f}(\omega) \\ &=\; \sum_{n\in\mathbb{Z}} \langle \Phi_R, \tau_n f \rangle \;=\; \langle \Phi_R, f \rangle, \end{aligned} \tag{118}$$

where we used that fact that the set $\{\tau_n; n \in \mathbb{Z}\}$ represents a partition of unity in the last step. $\square$

In order to recover the sought-for distribution Φ from Φ_R one only has to divide $\tilde{\Phi}_R(\omega)$ by $\tilde{\delta}_{\mathrm{tr,R}}(\omega)$. The resulting function is *meromorphic* in the strip $\{\omega \in \mathbb{C}; \mathrm{Im}(\omega) \in (-b, b)\}$, with poles at distance $\frac{1}{2}(b^{-1} - b)$ from the real axis.

6.6. **Representation of** Φ

In order to get a similar result on the representation of Φ in x-space we will analogously consider the asymptotics of $\tilde{\Phi}$ in ω-space. Here it will be convenient to start by considering

$$\Phi'_R \equiv \tilde{\delta}'_{\mathrm{tr,R}}(x)\Phi \equiv \prod_{s\in\{+,-\}}(x-x_s)\prod_{\substack{y\in\mathcal{I}_+\cup\mathcal{I}_-\\ |\mathrm{Im}(z)|<R}}(x-y)\;\Phi,$$

where $\mathcal{I}_+$ (resp. $\mathcal{I}_-$) denotes the union of the sets of zeros of $\tilde{\Delta}_2(z)$ and $\tilde{\Delta}_0(z)$ which lie in the upper (resp. lower) half plane, and $x_\pm$ are the zeros of $\tilde{\Delta}_0(z)$ that lie *on* the real axis, given by

$$x_\pm \equiv +\tfrac{i}{2}\big(\alpha_1+\alpha_2-Q\big) \pm i\big(\alpha_3-\tfrac{Q}{2}\big).$$

For the asymptotics of $\tilde{\Phi}'_R$ one has a result completely analogous to Proposition 10:

PROPOSITION 12. — *Let $\{\tau_n; n\in\mathbb{Z}\}$ be a sequence of functions in $\mathcal{C}_c^\infty(\mathbb{R})$ that have support only in $[n-1,n+1]$. For sufficiently large R there exists some $N>0$ such that*

$$\cosh(2\pi bn)\langle\tilde{\Phi}'_R,\tau_n\rangle < N \quad \textit{for all } n\in\mathbb{Z}. \tag{119}$$

Proof. — The proof is to a large extend analgous to that of Proposition 10, so we will only sketch some necessary modifications.

In order to get an estimate of $\langle\tilde{\Phi}'_R,\tau_n\rangle$ for $n\to-\infty$ one may use the eigenvalue equation to rewrite it as

$$\begin{aligned}\langle\tilde{\Phi}'_R,\tau_n\rangle =&\langle\tilde{\Phi},\Delta_0\Delta_0^{-1}\delta'_{\mathrm{tr,R}}\tau_n\rangle\\ =&\langle\tilde{\Phi},\Delta_0^c\Delta_0^{-1}\delta'_{\mathrm{tr,R}}\tau_n\rangle \qquad \text{where } \Delta_0^c=e^{2\pi b\omega}\Delta_1-e^{4\pi b\omega}\Delta_2.\end{aligned} \tag{120}$$

It follows as in the proof of Proposition 10 that $\langle\tilde{\Phi}'_R,\tau_n\rangle\sim e^{+2\pi bn}$ for $n\to-\infty$.

In the case of $n\to\infty$ one may use instead

$$\begin{aligned}\langle\tilde{\Phi}'_R,\tau_n\rangle =&\langle\tilde{\Phi},e^{4\pi b\omega}\Delta_2\Delta_2^{-1}e^{-4\pi b\omega}\delta'_{\mathrm{tr,R}}\tau_n\rangle\\ =&\langle\tilde{\Phi},\Delta_2^c\Delta_2^{-1}e^{-4\pi b\omega}\delta'_{\mathrm{tr,R}}\tau_n\rangle \qquad \text{where } \Delta_0^c=e^{2\pi b\omega}\Delta_1-\Delta_0,\end{aligned} \tag{121}$$

which gives $\langle\tilde{\Phi}'_R,\tau_n\rangle\sim e^{-2\pi bn}$ for $n\to\infty$. □

It follows as in the previous section that Φ'_R is represented by convolution against a function $\Phi'_R(x)$ which is holomorphic in $\{x\in\mathbb{C};\mathrm{Im}(x)\in(-b,b)\}$. In this case, however, recovering Φ from Φ'_R is more subtle since $\tilde{\delta}'_{\mathrm{tr,R}}(x)$ has two simple zeros on the real axis. The resulting ambiguity in the definition of Φ in terms of $\Phi'_R(x)$ is well-known (cf. e.g. [20, Chapter V, Example 9]) and may be parametrized as follows:

$$\Phi=\prod_{s\in\{+,-\}}\left(\frac{C_s}{x-x_s+i0}+\frac{1-C_s}{x-x_s-i0}\right)\prod_{\substack{y\in\mathcal{I}_+\cup\mathcal{I}_-\\ |\mathrm{Im}(z)|<R}}\frac{1}{x-y}\;\Phi'_R(x). \tag{122}$$

Lemma 2 then describes the corresponding asymptotic behavior of $\tilde{\Phi}(\omega)$. In general one would find terms with exponential decay weaker than $e^{-2\pi b|\omega|}$ for $\omega\to\infty$ that come either from zeros of $\tilde{\delta}'_{\mathrm{tr,R}}(x)$ strictly above the real axis, or from $x_\pm$ in the case of $C_s\neq 0$. The occurrence of such terms can be excluded by means of the following argument:

LEMMA 14. — *Let $\Phi \in \mathcal{S}'(\mathbb{R})$ be a distributional solution of $(C_{21}(\kappa_3) - [\alpha_3 - \frac{Q}{2}]^2)^t\,\Phi = 0$ that is represented by a function $\tilde{\Phi}(\omega)$ which has asymptotic behavior for $\omega \to \infty$ of the form*

$$\tilde{\Phi}(\omega) \;=\; +2\pi i \sum_{j\in\mathcal{I}_-} e^{-2\pi i z_j \omega} R_j + \tilde{\Phi}_{a_-}(\omega),$$

where $\tilde{\Phi}_b(\omega)$ decays at least as fast as $e^{-2\pi b\omega}$ for $\omega \to \infty$. Then $R_j = 0$ if Im$(z_j) < b$.

Proof. — Consider $\langle\tilde{\Phi}, \tau_n\rangle$, where now τ_n is chosen proportional to $e^{-\kappa(x-n)^2}$. One has

$$\big[\alpha_3 - \tfrac{Q}{2}\big]_b^2 \,\langle\tilde{\Phi}, \tau_n\rangle \;=\; \Big\langle\tilde{\Phi}, \Big(\Delta_0 - e^{2\pi b\omega}\Delta_1 + e^{4\pi b\omega}\Delta_2 + \big[\alpha_3 - \tfrac{Q}{2}\big]_b^2\Big)\tau_n\Big\rangle. \tag{123}$$

Now if there were terms with exponential decay weaker than $e^{-2\pi b\omega}$ in the asymptotic expansion of $\tilde{\Phi}(\omega)$ for $\omega \to \infty$ one would find terms terms that grow exponentially with $n \to \infty$ on the right hand side of (123). But polynomial boundedness of $\tilde{\Phi}$ excludes the occurrence of such terms on the left hand side of (123). □

6.7. **Completing the proof of Theorem 5**

Concerning the distribution Φ, we previously found that away from its singular support at $x = x_\pm$ it is represented by a function $\Phi(x)$. The asymptotic behavior of $\Phi(x)$ is via Lemma 2 given by the analytic properties of $\tilde{\Phi}$ that were stated after the proof of Proposition 11. The possible poles of $\tilde{\Phi}$ at distance $\frac{1}{2}(b^{-1} - b)$ from the real axis would lead to terms which decay more slowly as $e^{-2\pi b|x|}$ for $|x| \to \infty$. The appearance of such terms can now easily be excluded by an argument analogous to the proof of Lemma 14 in the x-representation.

Furthermore, knowing that the function $\Phi(x)$ that represents Φ away from its singular support decays exponentially for $|x| \to \infty$ allows us to use an argument very similar to the proof of Proposition 10 to further improve upon the estimate of the rate of decay as given in Proposition 10: In estimating J_n one may for large enough n replace $\Theta(x)$ by $\Phi(x)$. The exponential decay of the latter may then be used to improve (114) to

$$|J_n| \;\leq\; D_2 e^{-2\pi\nu n} \quad \text{for any } n > N_1. \tag{124}$$

for some $\nu > b$, implying that $\Phi(x)$ decays faster than $e^{-2\pi b|x|}$ for $|x| \to \infty$.

But this means via Lemma 2 that the Fourier-transformation $\tilde{\Phi}(\omega)$ is analytic in an open strip containing $\{\omega \in \mathbb{C}; |\text{Im}(\omega)| < b\}$, and that $\tilde{\Phi}(\omega)$ solves $(\tilde{C}_{21}(k_3) - [\alpha_3 - \frac{Q}{2}]_b^2)^t\tilde{\Phi}(\omega) = 0$ in the ordinary sense. The meromorphic extension to all of $\mathbb{C}$ is then easily obtained by using the eigenvalue equation to define the values of $\tilde{\Phi}(\omega)$ outside $\{\omega \in \mathbb{C}; |\text{Im}(\omega)| < b\}$ in terms of those inside. This finishes the proof of the first half of Theorem 5. The completion of the proof of the second half proceeds along very similar lines.

6.8. **Uniqueness of generalized eigenfunctions**

Theorem 3 also implies that the meromorphic function $\Phi(x)$ that represents the distribution Φ must solve the transpose of the eigenvalue equation in the usual sense.

PROPOSITION 13. — *There is at most one solution to $(C_{21}(\kappa_3) - [\alpha_3 - \frac{Q}{2}]^2)^t\,\Phi(x) = 0$ that has the analytic and asymptotic properties that follow from Theorem 5.*

Proof. — If one introduces $\Xi(x)$ via (recall $\kappa_3 = -2k_3$)

$$\Phi(x) = e^{\pi x(\alpha_3+\alpha_1-\alpha_2-i\kappa_3)} \frac{S_b(-ix - \frac{1}{2}(\alpha_1+\alpha_2)+\alpha_3)}{S_b(-ix+\frac{1}{2}(\alpha_1+\alpha_2))} \times \Xi\big(x - \tfrac{i}{2}(\alpha_1+\alpha_2-2(Q-\alpha_3))\big), \tag{125}$$

one may verify by direct calculation using the functional equation of the function $S_b(x)$ that the equation $(C_{21}(\kappa_3) - [\alpha_3 - \frac{Q}{2}]^2)^t\,\Phi(x) = 0$ is equivalent to the following equation for $\Xi(x)$:

$$\Big((1-e^{2\pi ib(\alpha_3+\alpha_1-\alpha_2)}T_x^{ib})(1-e^{2\pi ib(\alpha_3-i\kappa_3)}T_x^{ib}) - e^{-2\pi bx}(1-T_x^{ib})(1-e^{2\pi ib(\alpha_1-\alpha_2-i\kappa_3)}T_x^{ib})\Big)\Xi(x) = 0. \tag{126}$$

By using Lemma 2 and the properties of $S_b(x)$ that are summarized in Appendix B one may deduce the following properties of the Fourier transform $\tilde{\Xi}(\omega)$ of $\Xi(x)$ from Theorem 5:

(1) $\Xi(x)$ has a Fourier transform $\tilde{\Xi}(\omega)$ that is analytic in $\{\omega \in \mathbb{C}; \mathrm{Im}(\omega) \in (-Q/2, 0)\}$, and
(2) $\tilde{\Xi}(\omega)$ has the following asymptotic behavior for $\omega \to \pm\infty$:

$$\tilde{\Xi}(\omega) = R_+(\omega), \qquad \tilde{\Xi}(\omega) = K_- + R_-(\omega),$$

where K_- is a constant, $R_-(\omega)$ has exponential decay for $\omega \to -\infty$ and $R_+(\omega)$ has exponential decay stronger than $e^{-4\pi b\omega}$ for $\omega \to \infty$.

Equation (126) is equivalent to the following *first order* difference equation for $\tilde{\Xi}(\omega)$:

$$\Big((1-e^{2\pi ib(\alpha_3+\alpha_1-\alpha_2-i\omega)})(1-e^{2\pi ib(\alpha_3-i\kappa_3-i\omega)}) - (1-e^{2\pi ib(Q-i\omega)})(1-e^{2\pi ib(Q+\alpha_1-\alpha_2-i\kappa_3-i\omega)})T_\omega^{ib}\Big)\tilde{\Xi}(\omega) = 0. \tag{127}$$

Now there exists a solution to (127), namely

$$\tilde{\Xi}(\omega) = \frac{G_b(\alpha_3+\alpha_1-\alpha_2-i\omega)G_b(\alpha_3-i\kappa_3-i\omega)}{G_b(Q-i\omega)G_b(Q+\alpha_1-\alpha_2-i\kappa_3-i\omega)}, \tag{128}$$

that has all the required analytic and asymptotic properties. If there was a second solution $\tilde{\Xi}'(\omega)$ of these conditions one could consider the ratio $Q(\omega) \equiv \tilde{\Xi}'(\omega)/\tilde{\Xi}(\omega)$. This ratio must be a solution to $(T_\omega^{ib} - 1)Q(\omega) = 0$. Since $\tilde{\Xi}(\omega)$ has no zeros in the open strip $\{\omega \in \mathbb{C}; \mathrm{Im}(\omega) \in (-Q/2, 0)\}$ one concludes that $Q(\omega)$ is *holomorphic* in any such strip. The function $Q(\omega)$ must furthermore be asymptotic to the constant function for $\omega \to \pm\infty$. But this implies that $Q = \mathrm{const.}$: The function $P(z) \equiv Q(\frac{b}{2\pi}\ln(z))$ is holomorphic and regular on the whole Riemann sphere, therefore constant. □

7. APPENDIX B: SPECIAL FUNCTIONS

The basic building block for the class of special functions to be considered is the Double Gamma function introduced by Barnes [28], see also [29]. The Double Gamma function is defined as

$$\log \Gamma_2(s|\omega_1, \omega_2) = \left(\frac{\partial}{\partial t} \sum_{n_1,n_2=0}^{\infty} (s + n_1\omega_1 + n_2\omega_2)^{-t} \right)_{t=0} . \tag{129}$$

Let $\Gamma_b(x) = \Gamma_2(x|b, b^{-1})$, and define the Double Sine function $S_b(x)$ and the Upsilon function $\Upsilon_b(x)$ respectively by

$$S_b(x) = \frac{\Gamma_b(x)}{\Gamma_b(Q-x)} \qquad \Upsilon_b(x) = \frac{1}{\Gamma_b(x)\Gamma_b(Q-x)}. \tag{130}$$

It will also be useful to introduce

$$G_b(x) = e^{\frac{\pi i}{2}x(x-Q)} S_b(x). \tag{131}$$

7.1. Useful properties of S_b, G_b

7.1.1. *Self-duality.*

$$S_b(x) = S_{b^{-1}}(x) \qquad G_b(x) = G_{b^{-1}}(x). \tag{132}$$

7.1.2. *Functional equations.*

$$S_b(x+b) = 2\sin(\pi bx) S_b(x) \qquad G_b(x+b) = (1 - e^{2\pi ibx}) G_b(x). \tag{133}$$

7.1.3. *Reflection property.*

$$S_b(x) S_b(Q-x) = 1 \qquad G_b(x) G_b(Q-x) = e^{\pi i(x^2 - xQ)}. \tag{134}$$

7.1.4. *Analyticity.* $S_b(x)$ and $G_b(x)$ are meromorphic functions with poles at $x = -nb - mb^{-1}$ and zeros at $x = Q + nb + mb^{-1}$, $n, m \in \mathbb{Z}^{\geq 0}$.

7.1.5. *Asymptotic behavior.*

$$S_b(x) \sim \begin{cases} e^{-\frac{\pi i}{2}(x^2 - xQ)} & \text{for } \mathrm{Im}(x) \to +\infty \\ e^{+\frac{\pi i}{2}(x^2 - xQ)} & \text{for } \mathrm{Im}(x) \to -\infty \end{cases} \qquad G_b(x) \sim \begin{cases} 1 & \text{for } \mathrm{Im}(x) \to +\infty \\ e^{+\pi i(x^2 - xQ)} & \text{for } \mathrm{Im}(x) \to -\infty \end{cases} \tag{135}$$

7.2. b-beta integral

LEMMA 15. — *We have*

$$B_b(\alpha, \beta) \equiv \frac{1}{i} \int_{-i\infty}^{i\infty} d\tau \; e^{2\pi i\tau\beta} \frac{G_b(\tau + \alpha)}{G_b(\tau + Q)} = \frac{G_b(\alpha) G_b(\beta)}{G_b(\alpha + \beta)} \tag{136}$$

Proof. — From the relation (recall $T_\tau f(\tau) \equiv f(\tau+b)$)

$$0 \;=\; \int\limits_{-i\infty}^{i\infty} d\tau (1-T_\tau^b)\; e^{2\pi i\tau\beta} \frac{G_b(\tau+\alpha)}{G_b(\tau+Q)}, \tag{137}$$

which easily follows from the analyticity and asymptotic properties of the G_b-function by means of Cauchy's theorem one finds the following functional equation for $B_b(\alpha,\beta)$:

$$\frac{B_b(\alpha,\beta+b)}{B_b(\alpha+b,\beta)} = \frac{1-e^{2\pi ib\beta}}{1-e^{2\pi ib\beta}}. \tag{138}$$

By the $b \to b^{-1}$ self-duality of B_b one also has the same equation with $b \to b^{-1}$. For irrational values of b it follows that (138) and its $b \to b^{-1}$ counterpart determine B_b uniquely up to a function of $\alpha + \beta$. The expression on the left hand side of course satisfies (138). To fix the remaining ambiguity one may note that the integral defining B_b can be evaluated in the special case of $\alpha = b^{-1}$ by means of [31, Chapt. 1.5., eqn. (28)]:

$$B_b(b^{-1},\beta) \;=\; \frac{b^{-1}}{1-e^{2\pi ib^{-1}\beta}}. \tag{139}$$

The equation (136) follows. □

Let us also introduce the combination

$$\Theta_b(y;\alpha) \;\equiv\; \frac{G_b(y)}{G_b(y+\alpha)}. \tag{140}$$

The b-beta-integral (136) can be read as a formula for the Fourier-transform of $\Theta_b(y;\alpha)$:

$$\Theta_b(y;\alpha) \;=\; \frac{1}{G_b(y)}\frac{1}{i}\int\limits_{-i\infty}^{i\infty} d\tau\; e^{2\pi i\alpha\tau}\Theta_b(\tau+y;Q+y). \tag{141}$$

An expansion describing the asymptotic behavior of $\Theta_b(y;\alpha)$ for $|\mathrm{Im}(y)| \to \infty$ can therefore easily be obtained from Lemma (2): One finds

$$\begin{aligned} \Theta_b(y;\alpha) &\underset{\mathrm{Im}(y)\to+\infty}{\simeq} \sum_{n,m\geq 0} \Theta_{b,+}^{(n,m)}(\alpha)e^{2\pi i(nb+mb^{-1})y} \\ \Theta_b(y;\alpha) &\underset{\mathrm{Im}(y)\to-\infty}{\simeq} \sum_{n,m\geq 0} \Theta_{b,-}^{(n,m)}(\alpha)e^{-2\pi i(\alpha+nb+mb^{-1})y}, \end{aligned} \tag{142}$$

where $\Theta_{b,+}^{(0,0)}(\alpha) = 1$, $\Theta_{b,-}^{(0,0)}(\alpha) = e^{-\pi i\alpha(\alpha-Q)}$.

7.3. **b-hypergeometric function**

The b-hypergeometric function will be defined by an integral representation that resembles the Barnes integral for the ordinary hypergeometric function:

$$F_b(\alpha,\beta;\gamma;y) = \frac{1}{i}\frac{S_b(\gamma)}{S_b(\alpha)S_b(\beta)}\int\limits_{-i\infty}^{i\infty} ds\;\, e^{2\pi isy}\frac{S_b(\alpha+s)S_b(\beta+s)}{S_b(\gamma+s)S_b(Q+s)}, \tag{143}$$

where the contour is to the right of the poles at $s = -\alpha - nb - mb^{-1}$ and $s = -\beta - nb - mb^{-1}$ and to the left of the poles at $s = nb + mb^{-1}$ and $s = Q - \gamma + nb + mb^{-1}$, $n, m = 0, 1, 2, \ldots$. The function $F_b(\alpha, \beta; \gamma; -ix)$ is a solution of the q-hypergeometric difference equation

$$\big([\delta_x + \alpha][\delta_x + \beta] - e^{-2\pi bx}[\delta_x][\delta_x + \gamma - Q]\big) F_b(\alpha, \beta; \gamma; -ix) = 0, \qquad \delta_x = \tfrac{1}{2\pi}\partial_x \tag{144}$$

This definition of a q-hypergeometric function is closely related to the one first given in [30].

LEMMA 16. — *Consider the case that* $Re(\alpha) = Re(\beta) = Q/2$, $Re(\gamma) = Q$. $F_b(\alpha, \beta; \gamma; y)$ *is analytic in* y *in the strip* $\{y \in \mathbb{C}; Re(y) \in (-Q/2, Q/2)\}$. *The leading asymptotic behavior for* $|Im(y)| \to \infty$ *is given by*

$$\begin{aligned} F_b(\alpha, \beta; \gamma; y) =& 1 + \mathcal{O}(e^{2\pi iby}) + \\ & + e^{2\pi i(Q-\gamma)y} \frac{S_b(\gamma)}{S_b(2Q-\gamma)} \frac{S_b(Q+\beta-\gamma)S_b(Q+\alpha-\gamma)}{S_b(\alpha)S_b(\beta)}(1 + \mathcal{O}(e^{2\pi iby})) \\ F_b(\alpha, \beta; \gamma; y) =& e^{-2\pi i\alpha y} \frac{S_b(\gamma)S_b(\alpha-\beta)}{S_b(\beta)S_b(\gamma-\alpha)}(1 + \mathcal{O}(e^{-2\pi iby})) \\ & + e^{-2\pi i\beta y} \frac{S_b(\gamma)S_b(\beta-\alpha)}{S_b(\alpha)S_b(\gamma-\beta)}(1 + \mathcal{O}(-e^{2\pi iby})). \end{aligned} \tag{145}$$

There is also a kind of deformed Euler-integral for the hypergeometric function [30]:

$$\Psi_b(\alpha, \beta; \gamma; y) = \frac{1}{i} \int_{-i\infty}^{i\infty} ds \;\, e^{2\pi is\beta} \frac{G_b(s+y)G_b(s+\gamma-\beta)}{G_b(s+y+\alpha)G_b(s+Q)}. \tag{146}$$

For the case of main interest, $\mathrm{Re}(\alpha) = \mathrm{Re}(\beta) = Q/2$, $\mathrm{Re}(\gamma) = Q$ and $\mathrm{Re}(x) = 0$ one needs to deform the contour such that it passes the pole at $s = 0$ in the right half plane, the pole at $s = -y$ in the left half plane respectively. It then defines a function that is analytic in the right y half plane and develops a pole on the imaginary axis at $x = 0$ (Lemma 3).

LEMMA 17. — $\Psi_b(\alpha, \beta; \gamma; y)$ *has the following asymptotic behavior for* $|Im(y)| \to \infty$*:*

$$\begin{aligned} \Psi_b(\alpha, \beta; \gamma; y) =& \frac{G_b(\gamma-\beta)G_b(\beta)}{G_b(\gamma)}(1 + \mathcal{O}(e^{2\pi iby})) \\ & + e^{\pi i(\gamma-\beta)(\gamma-\beta-Q)} e^{2\pi i(Q-\gamma)y} \frac{G_b(Q+\alpha-\gamma)}{G_b(2Q-\gamma)G_b(\alpha)}(1 + \mathcal{O}(e^{2\pi iby})) \\ \Psi_b(\alpha, \beta; \gamma; y) =& e^{-2\pi i\alpha y} e^{-\pi i\alpha(\alpha-Q)} \frac{G_b(\beta-\alpha)G_b(\gamma-\beta)}{G_b(\gamma-\alpha)}(1 + \mathcal{O}(e^{-2\pi iby})) \\ & + e^{-2\pi i\beta y} e^{-\pi i\beta(\beta-Q)} \frac{G_b(\alpha-\beta)G_b(\beta)}{G_b(\alpha)}(1 + \mathcal{O}(e^{-2\pi iby})). \end{aligned} \tag{147}$$

Proof. — In order to study the limit $\mathrm{Im}(y) \to \infty$ it is convenient to split the integral into two integrals I_+ and I_- over the intervals $(-y/2, \infty)$ and $(-\infty, -y/2)$ respectively. In the case of I_+ one may use the asymptotics of the Θ_b functions containing y for imaginary part of their argument going to $+\infty$, eqn. (142), to get

$$\lim_{\mathrm{Im}(y)\to\infty} I_+ = \lim_{\mathrm{Im}(y)\to\infty} \frac{1}{i} \int_{-\frac{y}{2}}^{i\infty} ds \;\, e^{2\pi is\beta} \frac{G_b(s+\gamma-\beta)}{G_b(s+Q)} = \frac{G_b(\beta)G_b(\gamma-\beta)}{G_b(\gamma)}, \tag{148}$$

where (136) was used in the second step.

To study the behavior of I_- for $\mathrm{Im}(y) \to \infty$ it is convenient to change the integration variable in the second integral to $t = s + y$. One gets

$$(149) \qquad I_- = \frac{1}{i} \int_{-i\infty}^{\frac{y}{2}} dt \;\; e^{2\pi i(t-y)\beta} \frac{G_b(t)G_b(t-y+\gamma-\beta)}{G_b(t+\alpha)G_b(t-y+Q)}.$$

In this expression one may now use the asymptotics of the Θ_b functions containing y for imaginary part of their argument going to $-\infty$, eqn. (142), which yields as previously

$$(150) \qquad \lim_{\mathrm{Im}(y)\to\infty} e^{-2\pi i y(Q-\gamma)} I_- \;=\; e^{\pi i(\gamma-\beta)(\gamma-\beta-Q)} e^{2\pi i(Q-\gamma)y} \frac{G_b(Q+\alpha-\gamma)}{G_b(2Q-\gamma)G_b(\alpha)}.$$

The behavior for $\mathrm{Im}(y) \to -\infty$ is studied similarly. □

LEMMA 18. — $\Psi_b(\alpha,\beta;\gamma;y)$ *is a solution of the finite difference equation* $\mathcal{L}_b\Psi_b = 0$*, where*

$$(151) \qquad \mathcal{L}_b \equiv e^{-2\pi i b y}(1-T_y^b)(1-e^{2\pi i b(\gamma-Q)}T_y^b) - (1-e^{2\pi i b\alpha}T_y^b)(1-e^{2\pi i b\beta}T_y^b).$$

Proof. — Abbreviate the integrand in (146) by I. A direct calculation shows that it satisfies the equation

$$(152) \qquad \mathcal{L}_b I \;=\; -(1-e^{2\pi i b\alpha})(1-T_s^b)e^{2\pi i s\beta} \frac{G_b(s+x)G_b(s+\gamma-\beta)}{G_b(s+x+\alpha+b)G_b(s+b^{-1})}.$$

The Lemma follows from Cauchy's theorem. □

The finite difference equation allows us to define the meromorphic continuation of Ψ_b into the right y half plane. The precise relation between Ψ_b and F_b is

$$(153) \qquad \Psi_b(\alpha,\beta;\gamma;y) = \frac{G_b(\beta)G_b(\gamma-\beta)}{G_b(\gamma)} F_b(\alpha,\beta;\gamma;y'), \qquad y' = y - \tfrac{1}{2}(\gamma-\alpha-\beta+Q).$$

This follows as in the proof of Proposition (13) from the facts that (i) the finite difference equations satisfied by left and right hand sides of (153) are equivalent, and (ii) analytic and asymptotic properties of the functions of y appearing on both sides of (153) coincinde.

8. **APPENDIX C**

This appendix collects some results on the analytic and asymptotic properties of Clebsch-Gordan coefficients, the kernels $\Phi^\flat$, $\flat = s, t$ and the Racah-Wigner coefficients.

8.1. **Clebsch-Gordan coefficients**

LEMMA 1. — The analytic and asymptotic properties of the Clebsch-Gordan coefficients $\left[\begin{smallmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{smallmatrix}\right]$ may be summarized as follows:

(1) $\left[\begin{smallmatrix} Q-\alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{smallmatrix}\right]$ decays exponentially as $e^{-2\pi\alpha_i|x_i|}$ if any one of $|x_i| \to \infty$, $i = 1, 2, 3$.

(2) the Clebsch-Gordan coefficients are meromorphic w.r.t. each variable x_i, $i = 1, 2, 3$ with poles w.r.t. x_1 at

$$\begin{aligned}
&\text{Upper half plane:} && x_1 = x_2 - \tfrac{i}{2}(\alpha_1 + \alpha_2 - 2\alpha_3) + i(\epsilon + nb + mb^{-1}) \\
& && x_1 = x_3 - \tfrac{i}{2}(\alpha_3 + \alpha_1 - Q) + i(\epsilon + nb + mb^{-1}) \\
&\text{Lower half plane:} && x_1 = x_2 - \tfrac{i}{2}(Q - \alpha_1 - \alpha_2) - i(Q + nb + mb^{-1}) \\
& && x_1 = x_3 - \tfrac{i}{2}(2\alpha_2 - \alpha_3 - \alpha_1) - i(Q + nb + mb^{-1}),
\end{aligned}$$

where $n, m \in \mathbb{Z}^{\geq 0}$, and w.r.t. x_2 at

$$\begin{aligned}
&\text{Upper half plane:} && x_2 = x_1 + \tfrac{i}{2}(Q - \alpha_1 - \alpha_2) + i(Q + nb + mb^{-1}) \\
& && x_2 = x_3 + \tfrac{i}{2}(2\alpha_1 - \alpha_3 - \alpha_2) + i(Q + nb + mb^{-1}) \\
&\text{Lower half plane:} && x_2 = x_1 - \tfrac{i}{2}(2\alpha_3 - \alpha_1 - \alpha_2) - i(\epsilon + nb + mb^{-1}) \\
& && x_2 = x_3 - \tfrac{i}{2}(Q - \alpha_3 - \alpha_2) - i(\epsilon + nb + mb^{-1}).
\end{aligned}$$

Proof. — Direct consequence of analytic and asymptotic properties of the S_b-function given in Appendix B. $\square$

LEMMA 2. — The dependence of $\left[\begin{smallmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ \kappa_3 & \kappa_2 & \kappa_1 \end{smallmatrix}\right]$ w.r.t. variables $\kappa_3, \kappa_2, \kappa_1$ is of the following form:

$$\left[\begin{smallmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ \kappa_3 & \kappa_2 & \kappa_1 \end{smallmatrix}\right] \;=\; \delta(\kappa_3 - \kappa_2 - \kappa_1)\; Z\left(\begin{smallmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ \kappa_3 & \kappa_2 & \kappa_1 \end{smallmatrix}\right), \tag{154}$$

where $Z\left(\begin{smallmatrix} Q-\alpha_3 & \alpha_2 & \alpha_1 \\ \kappa_3 & \kappa_2 & \kappa_1 \end{smallmatrix}\right)$ is defined on the hypersurface $\kappa_3 - \kappa_2 - \kappa_1 = 0$ only and is meromorphic w.r.t. κ_i, $i = 1, 2, 3$ with poles only at

$$\kappa_i = \pm i(\alpha_i + nb + mb^{-1}), \qquad i = 1, 2, 3, \qquad n, m \in \mathbb{Z}^{\geq 0}. \tag{155}$$

Proof. — One needs to calculate

$$\left[\begin{smallmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ \kappa_3 & \kappa_2 & \kappa_1 \end{smallmatrix}\right] \;=\; \int_{\mathbb{R}} dx_2 dx_1\; e^{2\pi i k_1 x_1} e^{2\pi i k_2 x_2} \left[\begin{smallmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ \kappa_3 & x_2 & x_1 \end{smallmatrix}\right]. \tag{156}$$

By inserting (35) and changing variables $(x_1,x_2)\to(x_+,x_-)$, $x_\pm\equiv x_2\pm x_1$ one finds that the integration over x_+ produces $\delta(\kappa_3-\kappa_2-\kappa_1)$. $Z\big(\begin{smallmatrix}\alpha_3 & \alpha_2 & \alpha_1\\ \kappa_3 & \kappa_2 & \kappa_1\end{smallmatrix}\big)$ is therefore given by the integral

$$(157)\qquad Z\big(\begin{smallmatrix}\alpha_3 & \alpha_2 & \alpha_1\\ \kappa_3 & \kappa_2 & \kappa_1\end{smallmatrix}\big) \;=\; \int_{\mathbb{R}} dx_-\; e^{\pi i x_-(k_2-k_1)}\;\Phi_{\alpha_3}(\alpha_2,\alpha_1|\kappa_3|x_-).$$

It is then useful to employ the Barnes integral representation (143) for the b-hypergeometric function that appears in the definition (31) of the function Φ_{α_3}. The order of integrals in the resulting double integral may be exchanged, and the x_- integration carried out by means of (136). Up to prefactors that are entire analytic in k_i, $i=1,2,3$ one is left with the following integral:

$$(158)\qquad \frac{1}{i}\int_{-i\infty}^{i\infty} ds\; e^{2\pi i s Q}\,\frac{G_b(s+A_1)G_b(s+A_2)G_b(s+A_3)}{G_b(s+B_1)G_b(s+B_2)G_b(s+B_3)},$$

where the coefficients are given by

$$(159)\qquad \begin{aligned} A_1 &= Q-\alpha_3+\alpha_1-\alpha_2 & \qquad B_1 &= Q+\alpha_1-\alpha_2-i\kappa_3\\ A_2 &= Q-\alpha_3-i\kappa_3 & B_2 &= 2Q-\alpha_3-\alpha_2+i\kappa_1\\ A_3 &= \alpha_1+i\kappa_1 & B_3 &= Q. \end{aligned}$$

The claim now follows by straightforward application of Lemma 3. □

8.2. Kernels $\Phi^\flat_{\alpha_\flat}$, $\flat=s,t$

LEMMA 19. — *Analytic and asymptotic properties of* $\Phi^\flat_{\alpha_s}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big]_\epsilon(x_4;\mathfrak{x})$ *can be summarized as follows:*

(1) $\Phi^s_{\alpha_s}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big]_\epsilon(x_4;\mathfrak{x})$ *is meromorphic w.r.t.*

$$\begin{aligned} &x_1 \quad in \quad \{x_1\in\mathbb{C};Im(x_1)\in(-Q,b)\} && x_3 \quad in \quad \{x_3\in\mathbb{C};Im(x_1)\in(-b,Q)\}\\ &x_2 \quad in \quad \{x_2\in\mathbb{C};Im(x_1)\in(-b,Q)\} && x_4 \quad in \quad \{x_4\in\mathbb{C};Im(x_1)\in(-b,b)\}. \end{aligned}$$

The poles are located at (notation: $x_{ij}\equiv x_i-x_j$)

$$\begin{aligned} &x_{12}+\tfrac{i}{2}(\alpha_2+\alpha_1-2\alpha_s)-2i\epsilon=0, && \\ &x_{12}+\tfrac{i}{2}(\alpha_2+\alpha_1-2(Q-\alpha_s))-i\epsilon=0, && x_{14}+\tfrac{i}{2}(\alpha_1-\alpha_4)-2i\epsilon=0,\\ &x_{13}+\tfrac{i}{2}(\alpha_3+\alpha_1-2(Q-\alpha_4))-2i\epsilon=0, && x_{34}+\tfrac{i}{2}(\alpha_4-\alpha_3)+i\epsilon=0. \end{aligned}$$

It decays exponentially for $|x_i|\to\infty$ as $e^{-\pi Q|x_i|}$.

(2) $\Phi^t_{\alpha_s}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big]_\epsilon(x_4;\mathfrak{x})$ *is analytic w.r.t.*

$$\begin{aligned} &x_1 \quad in \quad \{x_1\in\mathbb{C};Im(x_1)\in(-Q,b)\} && x_3 \quad in \quad \{x_3\in\mathbb{C};Im(x_1)\in(-b,Q)\}\\ &x_2 \quad in \quad \{x_2\in\mathbb{C};Im(x_1)\in(-Q,b)\} && x_4 \quad in \quad \{x_4\in\mathbb{C};Im(x_1)\in(-b,b)\}. \end{aligned}$$

The poles are located at

$$x_{32}-\tfrac{i}{2}(\alpha_3+\alpha_2-2\alpha_t)+2i\epsilon=0,\qquad x_{14}+\tfrac{i}{2}(\alpha_1-\alpha_4)-i\epsilon=0,$$
$$x_{32}-\tfrac{i}{2}(\alpha_3+\alpha_2-2(Q-\alpha_t))+i\epsilon=0,\qquad x_{34}+\tfrac{i}{2}(\alpha_4-\alpha_3)+2i\epsilon=0.$$
$$x_{13}+\tfrac{i}{2}(\alpha_3+\alpha_1-2(Q-\alpha_4))-2i\epsilon=0,$$

It decays exponentially for $|x_i|\to\infty$ *as* $e^{-\pi Q|x_i|}$.

The residues of these poles that are needed in Section 5 can be represented as follows:

$$
\begin{aligned}
\mathcal{R}^s_{13} &\propto \operatorname*{Res}_{y_{21}=0}\left[\begin{smallmatrix}\alpha_4 & \alpha_3 & \alpha_s\\ x_4 & x_3 & *\end{smallmatrix}\right]\operatorname*{Res}_{y_{31}=0}\left[\begin{smallmatrix}\alpha_s & \alpha_2 & \alpha_1\\ x_s & x_2 & *\end{smallmatrix}\right]_{x_s=x_3-\frac{i}{2}(\alpha_s+\alpha_3-2(Q-\alpha_4))+i\epsilon}\\
\mathcal{R}^s_{14} &\propto \operatorname*{Res}_{y_{31}=0}\left[\begin{smallmatrix}\alpha_4 & \alpha_3 & \alpha_s\\ x_4 & x_3 & *\end{smallmatrix}\right]\operatorname*{Res}_{y_{31}=0}\left[\begin{smallmatrix}\alpha_s & \alpha_2 & \alpha_1\\ x_s & x_2 & *\end{smallmatrix}\right]_{x_s=x_4-\frac{i}{2}(\alpha_s-\alpha_4)+i\epsilon}\\
\mathcal{R}^t_{13} &\propto \operatorname*{Res}_{y_{32}=0}\left[\begin{smallmatrix}\alpha_t & \alpha_3 & \alpha_2\\ * & x_3 & x_2\end{smallmatrix}\right]\operatorname*{Res}_{y_{21}=0}\left[\begin{smallmatrix}\alpha_4 & \alpha_t & \alpha_1\\ x_4' & x_t & *\end{smallmatrix}\right]_{x_s=x_3-\frac{i}{2}(\alpha_3-\alpha_s)+i\epsilon}\\
\mathcal{R}^t_{14} &\propto \int_{\mathbb{R}} dx_t \operatorname*{Res}_{y_{31}=0}\left[\begin{smallmatrix}\alpha_4 & \alpha_t & \alpha_1\\ x_4' & x_t & *\end{smallmatrix}\right]\left[\begin{smallmatrix}\alpha_t & \alpha_3 & \alpha_2\\ x_t & x_3 & x_2\end{smallmatrix}\right],
\end{aligned}
\tag{160}
$$

where the undetermined prefactor does not depend on any of the variables and the $*$ appearing in the arguments indicates the variable of the b-Clebsch-Gordan coefficients that is to be expressed in terms of the others. The necessary residues are

$$
\begin{aligned}
\operatorname*{Res}_{y_{21}=0}\left[\begin{smallmatrix}\alpha_3 & \alpha_2 & \alpha_1\\ x_3 & x_2 & *\end{smallmatrix}\right] &= \frac{1}{2\pi S_b(\alpha_3+\alpha_2+\alpha_1-Q)}\frac{S_b\big(i(x_3-x_2)-\frac{1}{2}(\alpha_2-\alpha_3)\big)}{S_b\big(i(x_3-x_2)-\frac{1}{2}(\alpha_2-\alpha_3)+\beta_{32}\big)}\\
&\qquad\frac{S_b\big(i(x_2-x_3)+\frac{1}{2}(\alpha_2+\alpha_3-2(Q-\alpha_3))\big)}{S_b\big(i(x_2-x_3)+\frac{1}{2}(\alpha_2+\alpha_3-2(Q-\alpha_3))+\beta_{31}\big)}\\
\operatorname*{Res}_{y_{31}=0}\left[\begin{smallmatrix}\alpha_3 & \alpha_2 & \alpha_1\\ * & x_2 & x_1\end{smallmatrix}\right] &= \frac{S_b(\alpha_3+\alpha_2-\alpha_1)}{2\pi}\frac{S_b\big(i(x_1-x_2)-\frac{1}{2}(\alpha_1+\alpha_2-2\alpha_3)\big)}{S_b\big(i(x_1-x_2)-\frac{1}{2}(\alpha_1+\alpha_2-2\alpha_3)+\beta_{31}\big)}\\
&\qquad\frac{S_b\big(i(x_1-x_2)-\frac{1}{2}(\alpha_1+\alpha_2-2(Q-\alpha_3))\big)}{S_b\big(i(x_1-x_2)-\frac{1}{2}(\alpha_1+\alpha_2-2(Q-\alpha_3))+\beta_{32}\big)}\\
\operatorname*{Res}_{y_{32}=0}\left[\begin{smallmatrix}\alpha_3 & \alpha_2 & \alpha_1\\ x_3 & x_2 & x_1\end{smallmatrix}\right] &= \frac{S_b(\alpha_3+\alpha_1-\alpha_2)}{2\pi}\frac{S_b\big(i(x_1-x_2)-\frac{1}{2}(\alpha_1+\alpha_2-2\alpha_3)\big)}{S_b\big(i(x_1-x_2)-\frac{1}{2}(\alpha_1+\alpha_2-2\alpha_3)+\beta_{31}\big)}\\
&\qquad\frac{S_b\big(i(x_1-x_2)-\frac{1}{2}(\alpha_1+\alpha_2-2(Q-\alpha_3))\big)}{S_b\big(i(x_1-x_2)-\frac{1}{2}(\alpha_1+\alpha_2-2(Q-\alpha_3))+\beta_{21}\big)}\\
\operatorname*{Res}_{y_{32}=0}\operatorname*{Res}_{y_{21}=0}\left[\begin{smallmatrix}\alpha_3 & \alpha_2 & \alpha_1\\ * & * & *\end{smallmatrix}\right] &= \operatorname*{Res}_{y_{31}=0}\operatorname*{Res}_{y_{21}=0}\left[\begin{smallmatrix}\alpha_3 & \alpha_2 & \alpha_1\\ * & * & *\end{smallmatrix}\right] = \frac{S_b(2\alpha_3-Q)}{(2\pi)^2 S_b(\alpha_1+\alpha_2+\alpha_3-Q)}.
\end{aligned}
\tag{161}
$$

LEMMA 20. — *Analytic and asymptotic properties of* $\Phi^\flat_{\alpha_s}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]_\epsilon(k_4;\mathfrak{x})$, $\flat=s,t$ *can be summarized as follows:*

(1) $\Phi^s_{\alpha_s}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]_\epsilon(k_4;\mathfrak{x})$ *is meromorphic w.r.t.*

$$x_1 \;\; in \;\; \{x_1\in\mathbb{C}; Im(x_1)\in(-Q,b)\},\qquad x_3 \;\; in \;\; \{x_3\in\mathbb{C}; Im(x_1)\in(-b,Q)\},$$
$$x_2 \;\; in \;\; \{x_2\in\mathbb{C}; Im(x_1)\in(-b,Q)\},\qquad k_4 \;\; in \;\; \{k_4\in\mathbb{C}; Im(x_1)\in(-\tfrac{Q}{2},\tfrac{Q}{2})\}.$$

(2) $\Phi^t_{\alpha_s}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]_\epsilon(k_4;\mathfrak{x})$ *is meromorphic w.r.t.*

$$\begin{array}{llllll} x_1 & \textit{in} & \{x_1\in\mathbb{C}; \textit{Im}(x_1)\in(-Q,b)\} & x_3 & \textit{in} & \{x_3\in\mathbb{C}; \textit{Im}(x_1)\in(-b,Q)\}\\ x_2 & \textit{in} & \{x_2\in\mathbb{C}; \textit{Im}(x_1)\in(-Q,b)\} & k_4 & \textit{in} & \{k_4\in\mathbb{C}; \textit{Im}(x_1)\in(-\frac{Q}{2},\frac{Q}{2})\}.\end{array}$$

The poles in their dependence on x_1,x_2,x_3 are those poles of $\Phi^\flat_{\alpha_s}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]_\epsilon(x_4;\mathfrak{x})$, $\flat=s,t$, which are at positions independent of x_4. Both behave asymptotically

$$\begin{array}{ll} \textit{for } |x_1|\to\infty \textit{ as } e^{-2\pi i k_4 x_1}, & \textit{for } |x_3|\to\infty \textit{ as } e^{-2\pi i k_4 x_3},\\ \textit{for } |x_2|\to\infty \textit{ as } e^{-2\pi \alpha_2 |x_2|}, & \textit{for } |k_4|\to\infty \textit{ as } e^{-2\pi \epsilon k_4}.\end{array}$$

8.3. **Racah-Wigner coefficients**

LEMMA 21. — $\left\{\begin{smallmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{smallmatrix}\middle|\begin{smallmatrix}\alpha_s\\ \alpha_t\end{smallmatrix}\right\}_b$ *is meromorphic w.r.t. all six variables and has poles at $\beta=-nb-mb^{-1}$ where $n,m\in\mathbb{Z}^{\geq 0}$ and β may be any of the following:*

$$\begin{array}{llll} \alpha_2+\alpha_1-\alpha_s & Q-\alpha_s-\alpha_2+\alpha_1 & Q-\alpha_s-\alpha_4+\alpha_3 & 2Q-\alpha_3-\alpha_4-\alpha_s\\ \alpha_s+\alpha_1-\alpha_2 & 2Q-\alpha_1-\alpha_2-\alpha_s & Q-\alpha_s-\alpha_3+\alpha_4 & Q-\alpha_3-\alpha_4+\alpha_s\\ \alpha_3+\alpha_2+\alpha_t-Q & Q-\alpha_3-\alpha_t-\alpha_2 & \alpha_1+\alpha_4+\alpha_t-Q & \alpha_t+\alpha_4-\alpha_1\\ \alpha_3+\alpha_2-\alpha_t & Q-\alpha_2-\alpha_t-\alpha_3 & \alpha_1+\alpha_4-\alpha_t & Q-\alpha_1+\alpha_4-\alpha_t\end{array}$$

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