Hyper-Hermitian quaternionic Kähler manifolds

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Abstract

We call a quaternionic Kähler manifold with non-zero scalar curvature, whose quaternionic structure is trivialized by a hypercomplex structure, a hyper-Hermitian quaternionic Kähler manifold. We prove that every locally symmetric hyper-Hermitian quaternionic Kähler manifold is locally isometric to the quaternionic projective space or to the quaternionic hyperbolic space. We describe locally the hyper-Hermitian quaternionic Kähler manifolds with closed Lee form and show that the only complete simply connected such manifold is the quaternionic hyperbolic space.

Keywords: quaternionic Kähler manifold, hyper-Hermitian structure, Lee form
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1 Introduction

A $4n$-dimensional ($n > 1$) Riemannian manifold is quaternionic Kähler if its holonomy group is contained in $Sp(n)Sp(1)$. On every such manifold the bundle of endomorphisms of the tangent bundle has a parallel 3-dimensional subbundle, denoted by S^2H (see Sections 2,3), which is locally trivialized by a triple of orthogonal almost complex structures satisfying the quaternionic identities. Every quaternionic Kähler manifold is Einstein and Alekseevsky [1] has proved that its curvature tensor has a form which resembles that of a 4-dimensional oriented self-dual Einstein Riemannian manifold. This similarity allows the extension of many constructions and results about self-dual Einstein manifolds to quaternionic Kähler manifolds. Because of this, a 4-dimensional quaternionic Kähler manifold is defined to be an oriented self-dual Einstein Riemannian manifold.

There is a series of possible additional structures on a quaternionic Kähler manifold: Salamon [20] has shown that S^2H always has local sections which are complex structures; Alekseevsky, Marchiafava and Pontecorvo [17, 6] have studied quaternionic Kähler manifolds with a global section of S^2H which is almost complex or complex structure; they have proved [4, 6] that if on a compact quaternionic Kähler manifold S^2H is trivialized globally by an almost hypercomplex structure, then the scalar curvature is zero, that is, the manifold is locally hyper-Kähler.

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In the present paper we study the quaternionic Kähler manifolds on which S^2H is trivialized by a hypercomplex structure. These manifolds are simultaneously quaternionic Kähler and hyper-Hermitian, so we call them *hyper-Hermitian quaternionic Kähler (hHqK)* manifolds. To avoid the situation of locally hyper-Kähler manifolds, we require in addition that the scalar curvature is not zero.

The simplest examples of hHqK manifolds are the the quaternionic hyperbolic space $\mathbb{H}H^n$ and the domain of non-homogeneous quaternionic coordinates on the quaternionic projective space $\mathbb{H}P^n$. The whole $\mathbb{H}P^n$ cannot be hHqK because it does not admit any almost complex structure [15]. In fact, there are no complete hHqK manifolds with positive scalar curvature. This follows from the above mentioned result of Alekseevsky and Marchiafava [4] (since every complete quaternionic Kähler manifold with positive scalar curvature is compact) or, alternatively, from Theorem 6.3 in [19]. On the other hand, it is conjectured in [6] that the only complete simply connected hHqK manifold with negative scalar curvature is $\mathbb{H}H^n$.

Further examples, generalizing the above two, are the Swann bundles [21]. These are principal $\mathbb{H}^*/\mathbb{Z}_2$-bundles over a quaternionic Kähler base and have quaternionic Kähler metrics and a pseudo-hyper-Kähler metric with hyper-Kähler potential, which share the same quaternionic structure.

It is well-known that all underlying complex structures of a hyper-Hermitian structure have the same Lee form. The condition that a quaternionic Kähler manifold is hHqK can be expressed as a differential equation for this form (see [6] or Proposition 3.4 below).

The above mentioned hHqK structures on $\mathbb{H}P^n$, $\mathbb{H}H^n$ and the Swann bundles all have exact Lee forms. On the other hand, Apostolov and Gauduchon [8] have classified locally the 4-dimensional hHqK manifolds with non-closed Lee form, which in addition have an orthogonal complex structure compatible with the given orientation (and therefore not a section of S^2H): Every such manifold is locally isometric to $\mathbb{R}_+ \times S^3$ with one of the Pedersen-LeBrun metrics [16, 14]. More generally, Calderbank [9] has shown that every 4-dimensional hHqK manifold M is locally of the form $\mathbb{R} \times B$, where B is a 3-dimensional hyper-CR Einstein-Weil space and the metric on M is explicitly given by the geometry of B (in particular, the projection $\pi : M \longrightarrow B$ is a conformal submersion).

The two main goals of the present paper are to describe locally the hHqK manifolds, which

A. are locally symmetric.

B. have closed Lee form.

With respect to problem **A**, we prove in Theorem 4.1 that every locally symmetric hHqK manifold is locally homothetic to $\mathbb{H}P^n$ or $\mathbb{H}H^n$, thus giving a positive answer to a question of Alekseevsky and Marchiafava [3]. The idea of the proof is to find, in addition to the above mentioned equation for the Lee form, a differential equation for its exterior differential and then differentiate it until enough algebraic equations are obtained, so that the curvature tensor can be determined. In dimension 4 Theorem 4.1 is a direct consequence of a result of Eastwood and Tod [10] about Einstein-Weyl structures on locally-symmetric manifolds (see also [8]).

It is well-known that every 4-dimensional hHqK manifold with closed Lee form is locally homothetic to $S^4 \cong \mathbb{H}P^1$ or $\mathbb{R}H^4 \cong \mathbb{H}H^1$. Thus, it is enough to consider problem **B** in dimension $4n$ with $n > 1$.

The above mentioned Swann bundles have hHqK structure with close Lee form. In Theorem 5.3 we show that, conversely, every hHqK manifold, whose Lee form is closed and has non-constant length, is locally isometric to a Swann bundle. The proof relies on the close relation between this type of hHqK structures and pseudo-hyper-Kähler metrics with hyper-Kähler potential, given in Theorem 5.1.

In the remaining case of hHqK manifolds with closed Lee form of constant length the scalar curvature is necessarily negative. Such manifolds are constructed in Theorem 5.6 in a way which resembles the construction of the Swann bundles: They are $\mathbb{R}_+ \times \mathbb{R}^3$-bundles over a hyper-Kähler base. The converse is also true (Theorem 5.7): Every hHqK manifold with closed Lee form of constant length is locally isometric to such a bundle.

In the last section we show that a complete simply connected hHqK manifold with closed Lee form is homothetic to $\mathbb{H}H^n$, thus giving support to the above mentioned conjecture of Alekseevsky, Marchiafava and Pontecorvo [6].

2 Algebraic preliminaries

Let E and H be the following complex representations of $Sp(n)$ and $Sp(1)$: $E = \mathbb{H}^n$ with $A \cdot x = Ax$ for $A \in Sp(n)$, $x \in \mathbb{H}^n$ and $H = \mathbb{H}$ with $q \cdot y = y\bar{q}$ for $q \in Sp(1)$, $y \in \mathbb{H}$. The tensor products $E^{\otimes r} \otimes H^{\otimes s}$ are representations of $Sp(n)Sp(1) \cong Sp(n) \times_{\mathbb{Z}_2} Sp(1)$ if $r+s$ is even, as $(-1,-1) \in Sp(n) \times Sp(1)$ acts trivially in this case. Since E and H are quaternionic, these even tensor products are complexifications of real representations of $Sp(n)Sp(1)$. For example, $E \otimes H = T^{\mathbb{C}}$, where $T = \mathbb{H}^n$ with $[A,q] \in Sp(n)Sp(1)$ acting on $\xi \in \mathbb{H}^n \cong \mathbb{R}^{4n}$ by $[A,q]\cdot\xi = A\xi\bar{q}$. This exhibits $Sp(n)Sp(1)$ as a subgroup of $SO(4n)$.

From now on, although expressing the representations of $Sp(n)Sp(1)$ in terms of E and H, we shall think of them as the corresponding underlying real representations. Identifying T and T^* by the scalar product g, the space of bilinear forms over T is

$$T^* \otimes T^* = S^2H \otimes S^2E \oplus \mathbb{R}g \oplus \Lambda_0^2 E \oplus S^2E \oplus S^2H \oplus S^2H \otimes \Lambda_0^2 E. \tag{2.1}$$

The first three summands form S^2T^* and the last three form Λ^2T^*. The space S^2H is isomorphic to the Lie algebra of $Sp(1)$. Considered as a subspace of $End(T)$, $S^2H = span\{I,J,K\}$, where

$$I\xi = -\xi i, \quad J\xi = -\xi j, \quad K\xi = -\xi k, \qquad \xi \in \mathbb{H}^n. \tag{2.2}$$

The endomorphisms I, J, K satisfy the quaternionic identities

$$I^2 = J^2 = K^2 = -\mathbf{1} = IJK, \tag{2.3}$$

where $\mathbf{1}$ is the identity operator.

Let $\boldsymbol{L} = I\otimes I + J \otimes J + K \otimes K$, considered as an operator on $T^* \otimes T^*$. $\boldsymbol{L}$ is $Sp(n)Sp(1)$-invariant and $\boldsymbol{L}^2 = 2\boldsymbol{L} + \mathbf{3}$. The eigenspace for the eigenvalue 3 is $\mathbb{R}g \oplus \Lambda_0^2E \oplus S^2E$ (the remaining summands in (2.1) form the eigenspace for the eigenvalue -1). We call the bilinear forms belonging to this eigenspace $\mathbb{H}$-*Hermitian* since they are characterized by the property of being Hermitian with respect to each of I, J, K. The space of skew-symmetric $\mathbb{H}$-Hermitian forms is S^2E; it is isomorphic to the Lie algebra of $Sp(n)$. The space of symmetric $\mathbb{H}$-Hermitian forms is $\Lambda^2E = \mathbb{R}g \oplus \Lambda_0^2E$, with Λ_0^2E being the space of symmetric trace-free $\mathbb{H}$-Hermitian bilinear forms (alternatively, as a complex space $\Lambda^2E = \mathbb{C}\sigma_E \oplus \Lambda_0^2E$, where σ_E is the $Sp(n)$-invariant symplectic form on E and Λ_0^2E is the space of 2-forms, whose contraction with σ_E is zero). The projector on the space of $\mathbb{H}$-Hermitian bilinear forms is obviously $\frac{1}{4}(\mathbf{1}+\boldsymbol{L})$.

An algebraic curvature tensor is called *hyper-Kähler* if it has the algebraic properties of a curvature tensor of a hyper-Kähler manifold, that is, an algebraic curvature tensor which is $\mathbb{H}$-Hermitian with respect to the first pair of arguments (and therefore also with respect to the second pair). The space of hyper-Kähler curvature tensors is [20] $S^4E \subset S^2(S^2E)$.

Let π and π_h be the projections on the spaces of algebraic curvature tensors and hyper-Kähler algebraic curvature tensors respectively. We need the explicit forms of π and π_h only in some special cases, which we list below.

Let $R \in T^{*\otimes 4}$, $\Phi \in T^* \otimes T^*$. We define $\Pi R, \tau R, c(\Phi, R) \in T^{*\otimes 4}$ by

$$\Pi R(X,Y,Z,W) = R(X,Y,Z,W) + R(Y,X,Z,W) - R(W,Z,X,Y) - R(Z,W,Y,X),$$

$$\tau R(X,Y,Z,W) = R(Y,Z,X,W),$$

$$c(\Phi,R)(X,Y,Z,W) = R(X,Y,Z,FW) = \Phi(W,R(X,Y)Z),$$

where FX and $R(X,Y)Z$ are defined by

$$g(FX,Y) = \Phi(X,Y), \quad g(R(X,Y)Z,W) = R(X,Y,Z,W). \tag{2.4}$$

If R is skew-symmetric with respect to the first two arguments and satisfies the Bianchi identity with respect to the first three arguments (that is, $(\mathbf{1} + \tau + \tau^2)R = 0$), then $\pi R = \frac{1}{4}\Pi R$. In particular, if R satisfies these conditions, then $c(\Phi, R)$ satisfies them as well, and therefore $\pi c(\Phi,R) = \frac{1}{4}\Pi c(\Phi,R)$. If, furthermore, R is $\mathbb{H}$-Hermitian with respect to both first pair and second pair of arguments, then

$$\pi_h c(\Phi,R) = \frac{1}{4}\Pi c(\Phi,R). \tag{2.5}$$

For $\Phi, \Psi \in \Lambda^2 T^*$

$$\pi\tau\,\Phi\otimes\Psi = \frac{1}{12}(-2\Phi\otimes\Psi - 2\Psi\otimes\Phi + \Pi\tau\,\Phi\otimes\Psi) \tag{2.6}$$

and for $\Phi, \Psi \in S^2T^*$

$$\pi\tau\,\Phi\otimes\Psi = \frac{1}{4}\Pi\tau\,\Phi\otimes\Psi. \tag{2.7}$$

Let ι_X denote the contraction by the vector X: for $R \in T^{*\otimes k}$ the tensor $\iota_X R \in T^{*\otimes(k-1)}$ is defined by

$$\iota_X R\,(X_1,\dots,X_{k-1}) = R(X,X_1,\dots,X_{k-1}).$$

It follows from (2.6) and (2.7) that for $\Phi, \Psi \in S^2E$

$$\begin{aligned}\pi_h\tau\,\Phi\otimes\Psi\,(X,Y,Z,W) &= \frac{1}{12}(-2(\Phi\otimes\Psi + \Psi\otimes\Phi)(X,Y,Z,W) \\ &\quad +(\mathbf{1}+\boldsymbol{L})\iota_Y\iota_X\Pi\tau\,\Phi\otimes\Psi\,(Z,W)).\end{aligned} \tag{2.8}$$

Finally, recall that the curvature tensor R_0 of the quaternionic projective space $\mathbb{H}P^n$ is

$$R_0(X,Y,Z,W) = \frac{1}{2}(\mathbf{1}+\boldsymbol{L})\iota_Y\iota_X\Pi\tau\, g\otimes g\,(Z,W) - 2\boldsymbol{L}\,X^\flat\otimes Z^\flat\,(Y,W), \tag{2.9}$$

where for a vector X (resp. 1-form φ) $X^\flat$ (resp. $\varphi^\#$) denotes the dual 1-form (resp. vector). Notice that R_0 is a quaternionic Kähler curvature tensor, but not a hyper-Kähler curvature tensor. Its scalar curvature is $s_0 = 16n(n+2)$.

3 HHqK manifolds: definition and general considerations

We begin this section with some well-known definitions and facts in order to fix the notations.

An *almost Hermitian structure* (g, I) on a manifold M consists of a Riemannian metric g and an orthogonal almost complex structure I, that is, $I^2 = -\mathbf{1}$ and $g(I\cdot, I\cdot) = g$. The *Kähler form* Ω_I and the *Lee form* φ of (g, I) (or of I with respect to g) are defined by

$$\Omega_I(X,Y) = g(IX,Y), \quad \varphi(X) = \frac{1}{2} trace\{I(\nabla_\bullet I)X\}.$$

In other words, $\varphi = \frac{1}{2} I d^* \Omega_I$, where d^* is the adjoint of the exterior differentiation and the action of I on an arbitrary 1-form ψ is defined by the identification of TM and T^*M by g, that is, $I\psi = -\psi \circ I$. Notice that different normalization factors are more often used in the definition of the Lee form.

The almost complex structure I is complex (or integrable) if and only if

$$(\nabla_{IX} I)IY = (\nabla_X I)Y. \tag{3.10}$$

In this case (g, I) is called a *Hermitian structure.* The structure (g, I) is *Kähler* if I is parallel.

An *almost hypercomplex structure* on a manifold is defined by a triple (I, J, K) of almost complex structures, satisfying (2.3). If g is a Riemannian metric and I, J, K are orthogonal with respect to g, then (g, I, J, K) is an *almost hyper-Hermitian structure.* If (I, J, K) is a *hypercomplex structure*, that is, if I, J, K are integrable, the structure (g, I, J, K) is called *hyper-Hermitian.* It is well-known that in this case I, J, K share the same Lee form φ, which is also the Lee form of each of the complex structures in the S^2-family, determined by them.

An almost hyper-Hermitian structure is *hyper-Kähler* if each of I, J, K is parallel. Every hyper-Kähler manifold is Ricci flat.

An *almost quaternionic Hermitian structure* on a manifold M consists of a Riemannian metric and a 3-dimensional subbundle of the bundle of endomorphisms of TM, which is locally trivialized by a triple of orthogonal almost complex structures, satisfying (2.3). If $\dim M = 4n$ with $n > 1$, the structure is called *quaternionic Kähler* when the subbundle is parallel. Equivalently, a $4n$-dimensional $(n > 1)$ Riemannian manifold is quaternionic Kähler if its holonomy group is contained in $Sp(n)Sp(1)$.

We use the same notation for a $Sp(n)Sp(1)$-representation and the corresponding bundle, associated to the principal $Sp(n)Sp(1)$-bundle given by the holonomy reduction. For example, the defining 3-dimensional subbundle is S^2H.

The condition that S^2H is parallel can be expressed in terms of a local trivializing almost hypercomplex structure (I, J, K) by the equation

$$\big(\nabla_X I, \nabla_X J, \nabla_X K\big) = \big(I, J, K\big) D(a,b,c)(X), \tag{3.11}$$

where $D(a,b,c)$ is a $\mathfrak{so}(3)$-valued 1-form, given by

$$D(a,b,c) = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}, \qquad a, b, c \in \Gamma(T^*M).$$

Lemma 3.1 *i) A $4n$-dimensional ($n > 1$) almost quaternionic Hermitian manifold is quaternionic Kähler if and only if the Kähler forms of a local trivializing almost hypercomplex structure (I, J, K) satisfy*

$$(3.12) \qquad \big(d\Omega_I, d\Omega_J, d\Omega_K\big) = \big(\Omega_I, \Omega_J, \Omega_K\big) \wedge D(a, b, c).$$

The forms a, b, c coincide with those in (3.11).

ii) An almost hyper-Hermitian manifold is hyper-Kähler if and only if $d\Omega_I = d\Omega_J = d\Omega_K = 0$.

Proof: i) It follows from (3.11) that (3.12) is satisfied on a quaternionic Kähler manifold. Conversely, if (3.12) is satisfied, then the algebraic ideal of the exterior algebra ΛT^*M, generated by S^2H, is a differential ideal and the fundamental form $\Omega_I \wedge \Omega_I + \Omega_J \wedge \Omega_J + \Omega_K \wedge \Omega_K$ is parallel. Thus, by Theorem 2.2 in [21], the manifold is quaternionic Kähler.

That the forms a, b, c coincide with those in (3.11) follows from the injectivity of the map

$$T^*M \oplus T^*M \oplus T^*M \ni (\alpha, \beta, \gamma) \mapsto \alpha \wedge \Omega_I + \beta \wedge \Omega_J + \gamma \wedge \Omega_K \in \Lambda^3 T^*M$$

under the given dimension assumption.

Part ii) is proved by Hitchin [13]. □

Every quaternionic Kähler manifold is Einstein. Following [3, 4], we denote by ν the *reduced scalar curvature*, $\nu = \frac{4s}{s_0} = \frac{s}{4n(n+2)}$, where s is the (constant) scalar curvature. The curvature tensor of a quaternionic Kähler manifold has the form [1, 19]

$$(3.13) \qquad R = \frac{1}{4}\nu R_0 + R',$$

where R_0 is the (parallel) curvature tensor of $\mathbb{H}P^n$, given by (2.9), and $R' \in \Gamma(S^4E)$ (that is, R' is a hyper-Kähler curvature tensor).

When the dimension is 4, the definition of a quaternionic Kähler manifold gives nothing more than an oriented Riemannian manifold. But its curvature tensor has the form (3.13) only if it is self-dual and Einstein. Because of this, we define, as is usually done, a 4-dimensional quaternionic Kähler manifold to be an oriented self-dual Einstein manifold.

The following fact is well-known.

Lemma 3.2 *On a quaternionic Kähler manifold the Kähler forms of a local trivializing almost hypercomplex structure (I, J, K) satisfy*

$$\big(da + b \wedge c, db + c \wedge a, dc + a \wedge b\big) = -\nu\big(\Omega_I, \Omega_J, \Omega_K\big),$$

where a, b, c are the 1-forms in (3.11).

A quaternionic Kähler manifold with vanishing scalar curvature is locally hyper-Kähler. Since we would like to avoid this situation, we assume in the sequel that the quaternionic Kähler manifolds satisfy the additional requirement of having non-zero scalar curvature.

Let (I, J, K) be a local almost hypercomplex structure on a quaternionic Kähler manifold, trivializing S^2H. Then it follows from (3.11) that the Lee forms of I, J, K are $-\frac{1}{2}(Jb + Kc)$, $-\frac{1}{2}(Kc + Ia)$, $-\frac{1}{2}(Ia + Jb)$ respectively. Using also (3.10), we see that I, J, K are integrable if and only if $Ia = Jb = Kc$. Thus we obtain (see also [6, 7])

Proposition 3.3 *On a quaternionic Kähler manifold a local trivializing almost hypercomplex structure* (I, J, K) *is hypercomplex if and only if there exists a 1-form* φ *such that in (3.11)* $a = I\varphi$, $b = J\varphi$, $c = K\varphi$. *In this case* φ *is the common Lee form of the hypercomplex structure.*

The next proposition gives the necessary and sufficient condition under which S^2H is locally trivialized by a hypercomplex structure (see also [6]).

Proposition 3.4 *The bundle* S^2H *on a quaternionic Kähler manifold is locally trivialized by a hypercomplex structure with Lee form* φ *if and only if there exists a 2-form* $\Phi \in \Gamma(S^2E)$ *such that*

$$\nabla\varphi = \frac{1}{2}\big((-\mathbf{1} + \boldsymbol{L})\,\varphi \otimes \varphi - \nu g\big) + \Phi. \tag{3.14}$$

In this case necessarily $\Phi = \frac{1}{2}d\varphi$.

Notice that the operator $\boldsymbol{L}$ on a quaternionic Kähler manifold is independent of the choice of a local trivializing almost hypercomplex structure and is parallel by (3.11).

Proof: Let (I, J, K) be a local trivializing almost hypercomplex structure. We want to find a local trivializing hypercomplex structure (I', J', K'). It follows from (2.3) that $\big(I', J', K'\big) = \big(I, J, K\big)S$ for some $S \in SO(3)$. By (3.11) and Proposition 3.3, we get

$$dS + DS = SD', \tag{3.15}$$

where $D = D(a, b, c)$, $D' = D(I'\varphi, J'\varphi, K'\varphi)$. Let $\widetilde{D} = D(I\varphi, J\varphi, K\varphi)$. Then $SD' = \widetilde{D}S$ and (3.15) becomes

$$dS = (\widetilde{D} - D)S.$$

This equation has a solution locally if and only if

$$d(\widetilde{D} - D) - (\widetilde{D} - D) \wedge (\widetilde{D} - D) = 0.$$

Using Lemma 3.2, this is equivalent to

$$\nabla_X\varphi(Y) + \nabla_{IY}\varphi(IX) = J\varphi(X)J\varphi(Y) + K\varphi(X)K\varphi(Y) - \nu g(X, Y)$$

and the two similar equations obtained by cyclic permutations of I, J, K. Then it is easy to see that these equations are equivalent to the requirement that the symmetric part of $\nabla\varphi$ is

$$\frac{1}{2}\big((-\mathbf{1} + \boldsymbol{L})\,\varphi \otimes \varphi - \nu g\big)$$

and its skew-symmetric part is $\mathbb{H}$-Hermitian. □

Definition: A *hyper-Hermitian quaternionic Kähler (hHqK)* manifold is a quaternionic Kähler manifold such that S^2H is trivialized by a hypercomplex structure. The Lee form of the hypercomplex structure is called the *Lee form* of the hHqK manifold.

Proposition 3.5 *Let* M *be a hHqK manifold with Lee form* φ, $\xi = \varphi^{\#}$ *and* $\Phi = \frac{1}{2}d\varphi$. *Then*

$$\nabla_Z\Phi = \iota_\xi\iota_Z R' - 3\varphi(Z)\Phi + \frac{1}{2}(\mathbf{1} + \boldsymbol{L})\,\iota_Z\,\varphi \wedge \Phi; \tag{3.16}$$

$$\nabla_\xi R' = \nu R' + A_1 + A_2 + A_3, \tag{3.17}$$

where

$$A_1 = -\frac{1}{2}\Pi c\big((\mathbf{1}+\boldsymbol{L})\,\varphi\otimes\varphi, R'\big),$$
$$A_2 = -\Pi c(\Phi, R'),$$
$$A_3 = 6\pi_h\tau\,\Phi\otimes\Phi$$

(the right-hand side is given by (2.8));

$$\nabla^2_{\xi,U} R' = \frac{3}{2}\nu\nabla_U R' + B_{1U} + B_{2U} + B_{3U} + B_{4U} + B_{5U}, \tag{3.18}$$

where

$$\begin{aligned} B_{1U}(X,Y,Z,W) &= \frac{1}{2}(-\mathbf{1}+\boldsymbol{L})\big(\varphi\otimes\nabla_{\bullet}R'(X,Y,Z,W)\big)(U,\xi) \\ &\quad -\frac{1}{2}\Pi c\big((\mathbf{1}+\boldsymbol{L})\,\varphi\otimes\varphi, \nabla_U R'\big)(X,Y,Z,W), \end{aligned}$$
$$B_{2U} = -\nabla_{FU}R' - \Pi c(\Phi, \nabla_U R')$$

(F is the endomorphism corresponding to Φ by (2.4)),

$$B_{3U} = \nu\varphi(U)R' + \frac{1}{2}\nu\Pi c\big((\mathbf{1}+\boldsymbol{L})\,\varphi\otimes U^\flat, R'\big),$$
$$B_{4U} = -\Pi c\big((\mathbf{1}+\boldsymbol{L})\,\iota_U\Phi\otimes\varphi, R'\big) + 12\pi_h\tau\,\iota_\xi\iota_U R'\otimes\Phi,$$
$$B_{5U} = -\frac{1}{2}\Pi c\big((\mathbf{1}+\boldsymbol{L})\,\varphi\otimes U^\flat, A_3\big).$$

Proof: The first equality is proved by using (3.14) to calculate $\nabla_X\nabla_Y\varphi(Z)$, then antisymmetrizing with respect to X and Y to get $R(X,Y,Z,\xi)$ and using (3.13), (2.9) and $d\Phi = 0$.

The second equality is proved in a similar fashion by using (3.16) and (3.14) to calculate $\nabla_Z\nabla_W\Phi$ and then antisymmetrizing to get $R(Z,W)\Phi$.

The equality (3.18) is obtained by differentiating (3.17) with respect to U and using (3.14) and (3.16) to substitute $\nabla_U\varphi$ and $\nabla_U\Phi$ and (3.17) to express A_2 through $\nabla_\xi R'$, A_1 and A_3. □

Remarks:

1) By (3.14), we can determine all components of $\nabla\varphi$ with respect to the decomposition (2.1). For example,

$$d^*\varphi = 2n\nu - |\varphi|^2. \tag{3.19}$$

2) It is clear from the proof of Proposition 3.5 that it is derived only from (3.14). This means that it remains true on the whole set where the solution φ of (3.14) is defined, although the hyper-Hermitian structure, corresponding to φ, may exist only on a smaller set.

3) It follows from (3.16) that Φ is co-closed and therefore harmonic, a result obtained in [6].

4) By (2.5) and (2.8), A_1, A_2, A_3, B_{1U},…,B_{5U} are hyper-Kähler curvature tensors. It is also easily seen that B_1,…,B_5 satisfy the second Bianchi identity and therefore they have the algebraic properties of a covariant derivative of a hyper-Kähler curvature tensor.

4 Locally symmetric hHqK manifolds

In this section we give a positive answer to a question of Alekseevsky and Marchiafava [3] concerning the symmetric quaternionic Kähler manifolds, which are locally hypercomplex. In dimension 4 our theorem is a direct consequence of a result of Eastwood and Tod [10].

Theorem 4.1 *A locally symmetric hHqK manifold is locally homothetic to $\mathbb{H}P^n$ or $\mathbb{H}H^n$ and its Lee form is closed.*

Proof: For a vector X we denote $span\,\{X, IX, JX, KX\}$ by $span_{\mathbb{H}}\{X\}$ and the orthogonal complement of $span_{\mathbb{H}}\{X\}$ by $span_{\mathbb{H}}\{X\}^{\perp}$.

The vanishing of $\nabla R'$ means that (3.17) and (3.18) are reduced to

$$0 = \nu R' + A_1 + A_2 + A_3, \tag{4.20}$$

$$0 = B_3 + B_4 + B_5. \tag{4.21}$$

The Lee form φ cannot be zero on an open set (otherwise Proposition 3.3 and Lemma 3.2 imply $\nu = 0$). Thus, it is enough to prove that R' and Φ vanish at the points where $\varphi \neq 0$. We do this in five consecutive steps. At every step we put in (4.21) arguments U, X, Y, Z, W of certain type and prove that certain components of R' and Φ vanish.

Step 1. $U, X, Y, Z, W \in span_{\mathbb{H}}\{\xi\}$.
In this case it follows that Φ and R' vanish if all their arguments are in $span_{\mathbb{H}}\{\xi\}$. This completes the proof if the dimension is 4.

Step 2. $U \in span_{\mathbb{H}}\{\xi\}^{\perp}$, $X, Y, Z, W \in span_{\mathbb{H}}\{\xi\}$.
Then we see that R' vanishes if three of its arguments lie in $span_{\mathbb{H}}\{\xi\}$.

Step 3. $U, X, Z \in span_{\mathbb{H}}\{\xi\}$, $Y, W \in span_{\mathbb{H}}\{\xi\}^{\perp}$.
First we take $Y = W = F\xi$ ($F\xi$ is orthogonal to $span_{\mathbb{H}}\{\xi\}$ since F is a antisymmetric endomorphism commuting with I, J, K). Then we obtain

$$|F\xi|^4 = \frac{1}{3}\nu R'(\xi, F\xi, \xi, F\xi).$$

It follows from (3.14), (3.16) and (4.20) that

$$\xi(|F\xi|^4) = -2(4|\varphi|^2 + \nu)|F\xi|^4,$$

$$\xi\big(R'(\xi, F\xi, \xi, F\xi)\big) = -6|\varphi|^2 R'(\xi, F\xi, \xi, F\xi).$$

The last three equalities yield

$$(|\varphi|^2 + \nu)|F\xi|^4 = 0.$$

Thus $F\xi = 0$ or $|\varphi|^2$ is constant, $|\varphi|^2 = -\nu$. In the latter case, using (3.14), we obtain

$$0 = (F\xi)(|\varphi|^2) = -2|F\xi|^2.$$

Hence $F\xi = 0$, which means that Φ vanishes if one of its arguments belongs to $span_{\mathbb{H}}\{\xi\}$.

Now we take $U = X = Z = \xi$ and arbitrary $Y, W \in span_{\mathbb{H}}\{\xi\}^{\perp}$. Then we get $R'(\xi, Y, \xi, W) = 0$. Hence, R' vanishes if two of its arguments are in $span_{\mathbb{H}}\{\xi\}$.

Step 4. $U, X, Z \in span_{\mathbb{H}}\{\xi\}^{\perp}$, $Y = W = \xi$.
In this case it follows that R' vanishes if one of its arguments lies in $span_{\mathbb{H}}\{\xi\}$.

Step 5. $U = \xi$, $X, Y, Z, W \in span_{\mathbb{H}}\{\xi\}^{\perp}$.
Then we obtain

$$A_3(X,Y,Z,W) = \nu R'(X,Y,Z,W). \tag{4.22}$$

This is also true for arbitrary X, Y, Z, W, since if any of them belongs to $span_{\mathbb{H}}\{\xi\}$, then both A_3 and R' vanish. Hence,

$$\nabla_\xi A_3 = \nu \nabla_\xi R' = 0.$$

But from (3.16) we have $\nabla_\xi \Phi = -|\varphi|^2 \Phi$, and therefore $\nabla_\xi A_3 = -2|\varphi|^2 A_3$. Thus $A_3 = 0$ and by (4.22), $R' = 0$. The vanishing of A_3 implies also $\Phi = 0$. $\square$

5 HHqK manifolds with closed Lee form

In this section M is a hHqK manifold with closed Lee form φ and the hypercomplex structure is (I, J, K).

Because of $\Phi = \frac{1}{2} d\varphi = 0$, (3.14) and (3.16) become

$$\nabla \varphi = \frac{1}{2}\big((-\mathbf{1} + \boldsymbol{L})\,\varphi \otimes \varphi - \nu g\big), \tag{5.23}$$

$$R'(X,Y,Z,\xi) = 0. \tag{5.24}$$

If the dimension is 4, (5.24) implies $R' = 0$. Thus a 4-dimensional hHqK manifold with closed Lee form is locally homothetic to $\mathbb{H}P^1 \cong S^4$ or $\mathbb{H}H^1 \cong \mathbb{R}H^4$, a fact, which is well-known. Hence, for the rest of this section we can assume that $\dim M = 4n$ with $n > 1$.

By (5.23),

$$\nabla_\xi \xi = -\frac{1}{2}(|\varphi|^2 + \nu)\xi. \tag{5.25}$$

Thus, after a change of the parameter, the integral curves of ξ are geodesics.

From (5.23) and Proposition 3.3, we get

$$\nabla(I\varphi) = \frac{1}{2}\big(-\varphi \otimes I\varphi - I\varphi \otimes \varphi - J\varphi \wedge K\varphi - \nu\Omega_I\big) \tag{5.26}$$

and similarly by cyclic permutations of I, J, K. Equation (5.23) also implies

$$d(|\varphi|^2 + \nu) = -(|\varphi|^2 + \nu)\varphi. \tag{5.27}$$

Since our considerations will be local, we can assume that $\varphi = df$ for some function f. Thus from (5.27) it follows that

$$(|\varphi|^2 + \nu)e^f = C, \tag{5.28}$$

where C is a constant.

Let $\psi = de^f$ and $\eta = \psi^\# = e^f \xi$. By (5.23) and (5.26), we obtain

$$\nabla \psi = \frac{1}{2} e^f \big((\mathbf{1} + \boldsymbol{L})\,\varphi \otimes \varphi - \nu g\big), \tag{5.29}$$

$$\nabla(I\psi) = \frac{1}{2}e^f\left(\varphi\wedge I\varphi - J\varphi\wedge K\varphi - \nu\Omega_I\right) \tag{5.30}$$

and similarly by cyclic permutations of I, J, K.

Equation (5.29) shows that $\nabla\psi$ is $\mathbb{H}$-Hermitian and therefore η is an infinitesimal quaternionic automorphism. Even more, using also Proposition 3.3, we see that η is an infinitesimal automorphism of I, J, K, that is, an infinitesimal hypercomplex automorphism. It follows again from (5.29) that, similarly to ξ, after a change of the parameter the integral curves of η are geodesics.

By (5.30) $I\eta$, $J\eta$, $K\eta$ are Killing vector fields, which are also infinitesimal quaternionic automorphisms (in fact, every Killing vector field on quaternionic Kähler manifold is an infinitesimal quaternionic automorphism, see [20]). They are infinitesimal hypercomplex automorphisms only if $|\varphi|^2+\nu=0$.

It follows from (5.29), (5.30), (3.13), (2.9) and (5.24) that $span_{\mathbb{H}}\{\eta\}$ is a totally geodesic distribution with integral manifolds of constant curvature ν (larger totally geodesic quaternionic distributions, containing $span_{\mathbb{H}}\{\eta\}$, exist on hHqK manifolds with closed Lee form, see [5, 6]). The commutators of η, $I\eta$, $J\eta$, $K\eta$ are given by

$$[\eta, I\eta]=0, \quad [I\eta, J\eta] = CK\eta \tag{5.31}$$

and the same with cyclic permutations of I, J, K.

Hence, if $C\neq 0$ (that is, if $|\varphi|^2+\nu\neq 0$), $I\eta$, $J\eta$, $K\eta$ induce an infinitesimal isometric action of $Sp(1)$ and together with η they give rise to an infinitesimal quaternionic action of $\mathbb{H}^*$ on M. This situation very much resembles the one in the case of hyper-Kähler manifold with a hyper-Kähler potential, described by Swann [21]. Below we show that these two situations are closely related.

We recall the definition of a hyper-Kähler potential in the pseudo-Riemannian settings.

A function μ on a pseudo-Kähler manifold (M, g_0, I) is called a *Kähler potential* if

$$\frac{1}{2}dId\mu = \Omega_I^0. \tag{5.32}$$

A function μ on a pseudo-hyper-Kähler manifold (M, g_0, I, J, K) is called a *hyper-Kähler potential* if it is a Kähler potential for each of the underlying pseudo-Kähler structures. As shown by Swann [21], this is equivalent to

$$\nabla d\mu = g_0. \tag{5.33}$$

Hence, $d(g_0(d\mu, d\mu)) = 2d\mu$, that is,

$$\mu - \frac{1}{2}g_0(d\mu, d\mu) = D, \tag{5.34}$$

where D is a constant.

Theorem 5.1 *i) Let (M,g) be a hHqK manifold with closed Lee form φ and reduced scalar curvature ν, such that $g(\varphi,\varphi)+\nu\neq 0$. Then with respect to the same hypercomplex structure*

$$g_0 = \frac{1}{\nu}(g(\varphi,\varphi)+\nu)\left(\frac{1}{\nu}(\mathbf{1}+\boldsymbol{L})\,\varphi\otimes\varphi + g\right) \tag{5.35}$$

is a pseudo-hyper-Kähler metric with hyper-Kähler potential

$$\mu = \frac{2}{\nu^2}(g(\varphi,\varphi)+\nu). \tag{5.36}$$

The signature of g_0 is Riemannian when $\nu(g(\varphi,\varphi)+\nu)>0$ (and therefore always when $\nu>0$), and $(4,4(n-1))$ with positive sign on $span_{\mathbb{H}}\{\varphi\}$ when $\nu(g(\varphi,\varphi)+\nu)<0$.

ii) Let (M,g_0) be a pseudo-hyper-Kähler manifold with hyper-Kähler potential μ. Then for each $p\neq 0$

$$g_p = -\frac{p}{(pg_0(d\mu,d\mu)+1)^2}(\mathbf{1}+\boldsymbol{L})\, d\mu\otimes d\mu + \frac{1}{pg_0(d\mu,d\mu)+1}g_0 \tag{5.37}$$

(where defined) forms together with the given hypercomplex structure a pseudo-hHqK structure with Lee form $\varphi_p = -d\ln|pg_0(d\mu,d\mu)+1|$ and reduced scalar curvature $4p$. The metric g_p is positive definite when $pg_0(d\mu,d\mu)+1>0$ and g_0 is positive definite, and when $pg_0(d\mu,d\mu)+1<0$ and g_0 has signature $(4,4(n-1))$ with positive sign on $span_{\mathbb{H}}\{d\mu\}$.

iii) If g_0 is constructed from g as in i) and g_p from g_0 as in ii), then $g=g_{\frac{1}{4}\nu}$. Similarly, if we start with a pseudo-hyper-Kähler metric and construct g_p, then the pseudo-hyper-Kähler metric constructed from g_p is just the initial one.

Proof: i) To simplify the computations, we rewrite (5.35) and (5.36), using (5.28), in the form

$$g_0 = \frac{Ce^{-f}}{\nu}\left(\frac{1}{\nu}(\mathbf{1}+\boldsymbol{L})\,\varphi\otimes\varphi + g\right), \qquad \mu = \frac{2Ce^{-f}}{\nu^2}.$$

Let (I,J,K) be the hypercomplex trivialization of S^2H on (M,g). Obviously, I, J, K are orthogonal with respect to g_0. The Kähler form of I with respect to g_0 is

$$\Omega_I^0 = \frac{Ce^{-f}}{\nu}\left(\frac{1}{\nu}(\varphi\wedge I\varphi + J\varphi\wedge K\varphi)+\Omega_I\right).$$

Then, using (5.23), (5.26), Lemma 3.1 i) and Proposition 3.3, it is easily verified that $d\Omega_I^0=0$ and $\frac{1}{2}dId\mu=\Omega_I^0$, and similarly for J and K. Hence, if g_0 is non-degenerate, then, by Lemma 3.1 ii), it is pseudo-hyper-Kähler with hyper-Kähler potential μ. On the orthogonal complement of $span_{\mathbb{H}}\{\xi\}$ we have $g_0=\frac{1}{\nu}(g(\varphi,\varphi)+\nu)g$ and $g_0(\xi,\xi)=\frac{1}{\nu^2}(g(\varphi,\varphi)+\nu)^2g(\varphi,\varphi)$, which is positive if $\varphi\neq 0$. This proves the non-degeneracy of g_0 and the assertion about its signature.

ii) Using (5.34), we see that (5.37) is equivalent to

$$g_p = -\frac{p}{(2p(\mu-D)+1)^2}(\mathbf{1}+\boldsymbol{L})\, d\mu\otimes d\mu + \frac{1}{2p(\mu-D)+1}g_0.$$

Obviously, the given hypercomplex structure and g_p form an almost quaternionic Hermitian structure. The Kähler form of I with respect to g_p is

$$\Omega_I^p = -\frac{p}{(2p(\mu-D)+1)^2}(d\mu\wedge Id\mu + Jd\mu\wedge Kd\mu) + \frac{1}{2p(\mu-D)+1}\Omega_I^0$$

and similarly for J and K. Now, using (5.32), it is easy to see that Ω_I^p, Ω_J^p, Ω_K^p satisfy (3.12) for $a=I\varphi_p$, $b=J\varphi_p$, $c=K\varphi_p$. Thus, by Lemma 3.1 i) and Proposition 3.3,

(M, g_p) is a hHqK manifold with Lee form φ_p. The statement about the reduced scalar curvature follows from Lemma 2.2 by a straightforward computation.

Let ζ be the vector field dual to $d\mu$ with respect to g_0. Then $g_p(\zeta,\zeta) = \frac{g_0(d\mu,d\mu)}{(pg_0(d\mu,d\mu)+1)^2}$ and hence if g_0 is positive definite on $span_{\mathbb{H}}\{d\mu\}$, then so is g_p. On the orthogonal complement of $span_{\mathbb{H}}\{\zeta\}$ we have $g_p = \frac{1}{pg_0(d\mu,d\mu)+1} g_0$. This completes the proof of the non-degeneracy of g_p and the statement about its positive definiteness.

Part iii) is straightforward, after noticing that $g(\varphi,\varphi) = \frac{1}{4}\nu^2 g_0(d\mu, d\mu)$. □

Now we summarize some results of Swann [21].

Let M' be a quaternionic Kähler manifold and P' be the principal $SO(3)$-bundle over M', whose points are the frames (I', J', K') trivializing S^2H and satisfying (2.3). The *Swann bundle* over M' is the principal $\mathbb{R}_+ \times SO(3)$-bundle $\mathcal{U}(M') = \mathbb{R}_+ \times P'$. The Levi-Civita connection defines a horizontal distribution on P' and hence also on $\mathcal{U}(M')$. A hypercomplex structure (I, J, K) is defined on $\mathcal{U}(M')$ in the following way. The projection $\pi : \mathcal{U}(M') \longrightarrow M'$ induces an isomorphism of the horizontal space on $\mathcal{U}(M')$ at the point (r, I', J', K') and the tangent space of M' at the corresponding point. On the horizontal space I, J, K are defined to correspond respectively to I', J', K' under this isomorphism. On the fibres (I, J, K) is the standard hypercomplex structure, that is, I, J, K are given by (2.2) after identifying the tangent spaces of $\mathbb{R}_+ \times SO(3) = \mathbb{H}^*/\mathbb{Z}_2$ with its Lie algebra $\mathbb{H}$.

There exist also quaternionic Kähler metrics compatible with this hypercomplex structure.

Theorem 5.2 *[21] Let (M', g') be a $4(n-1)$-dimensional quaternionic Kähler manifold with reduced scalar curvature ν' and r be the radial coordinate on $\mathcal{U}(M')$. Then for $p \neq 0$ the above hypercomplex structure and*

$$g_p = \frac{1}{4r^2(pr^2+1)^2}(\mathbf{1} + \boldsymbol{L})\, dr^2 \otimes dr^2 + \frac{\nu' r^2}{4(pr^2+1)}\pi^* g' \tag{5.38}$$

form on the submanifold $\nu'(pr^2+1) > 0$ a (positive definite) hHqK structure with Lee form $\varphi_p = -d\ln|pr^2+1|$ and reduced scalar curvature $\nu_p = 4p$. The metric g_0 is pseudo-hyper-Kähler with respect to the same hypercomplex structure and has hyper-Kähler potential $\mu = \frac{r^2}{2}$. It has Riemannian signature if $\nu' > 0$ and signature $(4, 4(n-1))$ if $\nu' < 0$.

Notice that $(\mathcal{U}(M'), g_p)$ and $(\mathcal{U}(M'), g_q)$ are homothetic if p and q have the same sign.

It is easily seen that $g_p(\varphi_p, \varphi_p) + \nu_p = 4p(pr^2+1)$. Thus, Theorem 5.2 gives examples of hHqK manifolds with exact Lee form φ such that φ is everywhere non-zero and $|\varphi|^2 + \nu \neq 0$. It turns out, as suggested by Theorem 5.1, that these examples exhaust locally all such manifolds.

Theorem 5.3 *Let (M, g) be a hHqK manifold with closed Lee form φ and reduced scalar curvature ν such that $\varphi \neq 0$ and $|\varphi|^2 + \nu \neq 0$. Then (M, g) is locally isometric to $(\mathcal{U}(M'), g_{\frac{1}{4}\nu})$ for some quaternionic Kähler manifold M'.*

Proof: From the results of Swann [21] it follows that every pseudo-hyper-Kähler manifold with hyper-Kähler potential μ, such that $d\mu$ does not vanish, is locally homothetic to $(\mathcal{U}(M'), g_0)$ for some pseudo-quaternionic Kähler manifold M'. From (5.38) we get

$$g_p = -\frac{p}{4(pr^2+1)^2}(\mathbf{1} + \boldsymbol{L})\, dr^2 \otimes dr^2 + \frac{1}{pr^2+1} g_0.$$

Since the hyper-Kähler potential of g_o is $\mu = \frac{r^2}{2}$, we obtain the result by applying Theorem 5.1. $\square$

Next, we consider the case when φ vanishes at some point.

The condition $R' \equiv 0$ ensures that the Cauchy problem for (5.23) locally has a solution for any initial data. Therefore, on $\mathbb{H}P^n$ and $\mathbb{H}H^n$ the bundle S^2H can be locally trivialized by a hyper-complex structure, whose Lee form is closed and vanishes at some point. In fact, these are the only such manifolds:

Theorem 5.4 *i) A hHqK manifold, whose Lee form is closed and vanishes at some point, is locally homothetic to $\mathbb{H}P^n$ or $\mathbb{H}H^n$.*

ii) A hyper-Kähler manifold with hyper-Kähler potential μ, such that $d\mu$ vanishes at some point, is flat.

Proof: i) Differentiating (5.24) and using also (5.23), we see that

$$\nabla_X R'(\xi, Y, Z, W) = \frac{1}{2}\nu R'(X, Y, Z, W). \tag{5.39}$$

Thus, at the point p, where φ vanishes, we have $R' = 0$.

We shall prove that all the covariant derivatives of the curvature tensor also vanish at p. Then, because the metric is Einstein and hence analytic, it will follow that the curvature is $R = \frac{1}{4}\nu R_0$, that is, the manifold is locally homothetic to $\mathbb{H}P^n$ or $\mathbb{H}H^n$.

Using (5.23), (5.24) and (5.39), it is easily proved by induction that

$$\begin{aligned}\nabla^{k+1}_{X_1,\ldots,X_{k+1}} R'(\xi, Y, Z, W) &= \frac{1}{2}\nu \sum_{s=1}^{k+1} \nabla^k_{X_1,\ldots,\widehat{X}_s,\ldots,X_{k+1}} R'(X_s, Y, Z, W) \\ &\quad + P_k(\varphi, R', \ldots, \nabla^{k-1}R'),\end{aligned} \tag{5.40}$$

where P_k is a polynomial without term of order zero. The notation $\widehat{X}_s$ is used to indicate that the argument X_s is omitted. Now, supposing that $\nabla^l R' = 0$ at p for $l < k$, we see by (5.40) that

$$\sum_{s=1}^{k+1} \nabla^k_{X_1,\ldots,\widehat{X}_s,\ldots,X_{k+1}} R'(X_s, Y, Z, W)(p) = 0. \tag{5.41}$$

Since the antisymmetrization of $\nabla^k_{X_1,\ldots,X_k} R'(X_{k+1}, Y, Z, W)$ with respect to X_1 and X_2 is $(R(X_1, X_2)\nabla^{k-2}R')(X_3, \ldots, X_{k+1}, Y, Z, W)$, it follows that at p it is symmetric with respect to X_1 and X_2. Similarly, it is symmetric at p with respect to X_s and X_{s+1} for every $s < k$, since its antisymmetrization with respect to these two arguments is expressed by the covariant derivatives of R' of order less than $k - 1$.

Hence, $\nabla^k R'(p)$ is symmetric with respect to the first k arguments and the proof of i) is completed by the following algebraic lemma:

Lemma 5.5 *Let $T \in S^kT^* \otimes \Lambda^2T^*$ satisfy the Bianchi identity with respect to the last three arguments and*

$$\sum_{s=1}^{k+1} T(X_1, \ldots, \widehat{X}_s, \ldots, X_{k+1}, X_s, X_{k+2}) = 0. \tag{5.42}$$

Then $T = 0$.

Proof: Antisymmetrizing (5.42) with respect to X_{k+1} and X_{k+2} and using the Bianchi identity with respect to the last three arguments, we obtain

$$2T(X_1,\ldots,X_{k+2})+\sum_{s=1}^{k}T(X_1,\ldots,\widehat{X}_s,\ldots,X_k,X_s,X_{k+1},X_{k+2})=0.$$

The symmetry with respect to the first k arguments now implies $T=0$. □

ii) It follows from (5.33) that $R(X,Y,Z,\zeta)=0$, where $\zeta=(d\mu)^{\#}$. Now ii) can be proved in the same way as i), the polynomials P_k being identically zero. □

Remarks:

5) Lemma 5.5 in fact verifies that the condition (5.41) forces the vanishing of the component of $\nabla^k R'$ in the subspace of highest dominant weight in the space of tensors with the symmetries of the kth covariant derivative of the curvature tensor of an Einstein manifold. Thus Theorem 5.4 follows from Theorem 10.2 in [20] or from the similar result for the special case of a quaternionic Kähler manifold, given in the proof of Theorem 2.6 in [18].

6) Part ii) of Theorem 5.4 also follows from the results in [2, 22]. By (5.33), we see that ζ is an infinitesimal conformal transformation with non-vanishing divergency. If $d\mu$ vanishes somewhere, then ζ is an essential infinitesimal conformal transformation, that is, it is not an infinitesimal isometry for any conformal metric. Then it follows from the "obvious" parts of Proposition 2 in [2] or the Theorem in [22] that the manifold is locally conformally flat (notice that in this obvious part it is not necessary to have a global conformal transformation). But it is also Ricci-flat, and hence flat.

7) Part i) of Theorem 5.4 can also be proved using part ii) and Theorem 5.1. This is straightforward if $\nu>0$, but if $\nu<0$, we need part ii) for pseudo-hyper-Kähler manifolds of signature $(4,4(n-1))$. The same proof will work in this case if the metric is analytic. By Theorem 5.1, to a pseudo-hyper-Kähler metric of signature $(4,4(n-1))$ corresponds a (positive definite) hHqK structure with closed Lee form φ. But a quaternionic Kähler metric is analytic in geodesic normal coordinates and since by (3.19) and (5.28)

$$\Delta f=d^*\varphi=(2n+1)\nu-Ce^{-f},$$

f and $\varphi=df$ are also analytic in these coordinates. Thus, by (5.35) the pseudo-hyper-Kähler metric is analytic in the same coordinates.

Finally, we focus our attention on the case of hHqK manifold with closed Lee form φ, satisfying $|\varphi|^2+\nu=0$. Clearly, this is possible only if $\nu<0$. First we construct examples of such manifolds.

Let (M',g') be a $4(n-1)$-dimensional hyper-Kähler manifold, the hypercomplex structure being (I',J',K'). Then the Kähler forms $\Omega_{I'}$, $\Omega_{J'}$, $\Omega_{K'}$ are closed. Thus, restricting our considerations on a small open set, we can fix 1-forms $\alpha_{I'}$, $\alpha_{J'}$, $\alpha_{K'}$ such that

$$\Omega_{I'}=d\alpha_{I'},\quad \Omega_{J'}=d\alpha_{J'},\quad \Omega_{K'}=d\alpha_{K'}.$$

Let (t,u,v,w) be the standard coordinates on $\mathbb{H}\cong\mathbb{R}^4$, that is, $\mathbb{H}\ni q=t+ui+vj+wk$. Let $M=\{q\in\mathbb{H}\,:\,t>0\}\times M'$ and $\pi:M\longrightarrow M'$ be the projection. We fix a negative constant ν and define on M an almost hypercomplex structure (I,J,K) by

$$Idt=-du+\nu\pi^*\alpha_{I'},\quad Jdt=-dv+\nu\pi^*\alpha_{J'},\quad Kdt=-dw+\nu\pi^*\alpha_{K'}, \tag{5.43}$$

$$I\pi^*\beta = \pi^* I'\beta, \quad J\pi^*\beta = \pi^* J'\beta, \quad K\pi^*\beta = \pi^* K'\beta, \qquad \beta \in T^*M' \tag{5.44}$$

and a Riemannian metric g by

$$g = -\frac{1}{\nu t^2}(\mathbf{1} + \boldsymbol{L})\, dt \otimes dt + \frac{1}{t}\pi^* g'. \tag{5.45}$$

Theorem 5.6 *The manifold (M, g, I, J, K) is hHqK with reduced scalar curvature ν and Lee form $\varphi = -d\ln t$, which satisfies $|\varphi|^2 + \nu = 0$.*

Proof: It is clear from the definitions that I, J, K are orthogonal with respect to g. The Kähler form of I is

$$\Omega_I = -\frac{1}{\nu t^2}\big(dt \wedge (-du + \nu\pi^*\alpha_{I'}) + (-dv + \nu\pi^*\alpha_{J'}) \wedge (-dw + \nu\pi^*\alpha_{K'})\big) + \frac{1}{t}\pi^*\Omega_{I'}$$

and similarly for Ω_J and Ω_K. Now a straightforward computation shows that (3.12) is satisfied with $a = -Id\ln t$, $b = -Jd\ln t$, $c = -Kd\ln t$. Thus, by Lemma 3.1 and Proposition 3.3, (g, I, J, K) is a hHqK structure on M with Lee form $\varphi = -d\ln t$. That the reduced scalar curvature is ν is verified using Lemma 3.2. The equality $|\varphi|^2 + \nu = 0$ is an obvious consequence of the definition of g. □

Remark 8 It is easy to see that $I\frac{\partial}{\partial t} = -\frac{\partial}{\partial u}$, $J\frac{\partial}{\partial t} = -\frac{\partial}{\partial v}$, $K\frac{\partial}{\partial t} = -\frac{\partial}{\partial w}$, that is, on the fibres of π the hypercomplex structure is the standard hypercomplex structure on $\mathbb{H}$, given by (2.2). Also, $\pi : (M, tg) \longrightarrow (M', g')$ is a Riemannian submersion.

Now we show that the converse of Theorem 5.6 is also true.

Theorem 5.7 *A hHqK manifold M with closed Lee form φ and negative reduced scalar curvature ν, such that $|\varphi|^2 + \nu = 0$, is locally isometric to one of the manifolds in Theorem 5.6.*

Proof: Since $\nabla\psi$ is symmetric, the distribution $\eta^\perp = \{X : g(\eta, X) = 0\}$ is integrable. Further, the integral curves of η are geodesics (up to a change of the parameter) and therefore $X \in \eta^\perp$ implies $[\eta, X] \in \eta^\perp$. Hence, in a neighbourhood of a fixed point $p_0 \in M$ we can choose coordinates $(t, u, v, w, x^1, \dots, x^{4(n-1)})$ such that

$$\eta = -\frac{\partial}{\partial t}, \quad \eta^\perp = span\left\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial w}, \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{4(n-1)}}\right\}. \tag{5.46}$$

By (5.31), we have that η, $I\eta$, $J\eta$, $K\eta$ commute ($C = 0$ by (5.28)) and therfore we can take the above coordinates in such a way that

$$I\eta = \frac{\partial}{\partial u}, \quad J\eta = \frac{\partial}{\partial v}, \quad K\eta = \frac{\partial}{\partial w}. \tag{5.47}$$

From now on we restrict our considerations to this coordinate neighbourhood.

Since $|\varphi|^2 + \nu = 0$, we have $|\psi|^2 = -\nu e^{2f}$, and it follows from (5.46) that $\psi = \nu e^{2f} dt$. On the other hand, $\psi = de^f$ and therefore $e^f = \frac{1}{-\nu t + D}$, where D is a constant. Changing the coordinate t by a translation, we can assume that $e^f = -\frac{1}{\nu t}$ (and hence $t > 0$).

Let $M' = \{p \in M : t(p) = t(p_0), u(p) = u(p_0), v(p) = v(p_0), w(p) = w(p_0)\}$ and $\pi : M \longrightarrow M'$ be the projection. We call $\mathcal{V} = span_{\mathbb{H}}\{\eta\}$ *the vertical distribution* and its orthogonal complement $\mathcal{H}$ *the horizontal distribution.* $\mathcal{V}$ and $\mathcal{H}$ are invariant under the action of the complex structures I, J, K. As seen before, since $|\varphi|^2+\nu=0$, each of η, $I\eta$, $J\eta$, $K\eta$ is an infinitesimal hypercomplex automorphism. Therefore I, J, K project down to almost complex structures I', J', K' on M', that is, (5.44) is satisfied.

Let $h: TM \longrightarrow \mathcal{H}$ be the orthogonal projection. We define a Riemannian metric g' on M' by

$$g'(X,Y) = tg(hX,hY), \qquad X,Y \in T_pM' \subset T_pM.$$

It is not difficult to see that the tensor $tg(h\cdot,h\cdot)$ on M projects down to a tensor on M', that is, $\pi:(M,tg)\longrightarrow(M',g')$ is a Riemannian submersion. Thus, g is given by (5.45).

It is obvious that I', J', K' are orthogonal with respect to g'. So,

$$\Omega_I = -\frac{1}{\nu t^2}\big(dt\wedge Idt + Jdt\wedge Kdt\big) + \frac{1}{t}\pi^*\Omega_{I'}. \tag{5.48}$$

Now, it follows from (5.47) that

$$Idt = -du + \sum_{s=1}^{4(n-1)} f_s dx^s. \tag{5.49}$$

Hence,

$$dIdt = \sum_{s=1}^{4(n-1)} df_s\wedge dx^s. \tag{5.50}$$

But $\varphi = df$ and therefore $dt = t\varphi$. Thus, using Lemma 3.2, Proposition 3.3 and (5.48), we obtain

$$dIdt = \nu\pi^*\Omega_{I'}. \tag{5.51}$$

It follows by (5.50) and (5.51) that the coefficients f_s do not depend on t, u, v, w and therefore

$$\sum_{s=1}^{4(n-1)} f_s dx^s = \nu\pi^*\alpha_{I'}. \tag{5.52}$$

for some form $\alpha_{I'}$ on M'. Now, by (5.49), (5.51) and (5.52), we obtain $\pi^*d\alpha_{I'} = \pi^*\Omega_{I'}$, that is, $d\alpha_{I'} = \Omega_{I'}$. Repeating the same argument for J and K, we see by Lemma 3.1 ii) that M' is hyper-Kähler and (5.43) is satisfied. □

Remark 9 The above proof can be easily modified when $|\varphi|^2+\nu\neq 0$ to get a proof of Theorem 5.2. In this case we can again take $\eta = -\frac{\partial}{\partial t}$. The vector fields $I\eta$, $J\eta$, $K\eta$ do not commute, but they form an integrable distribution and the coordinates can be taken so that $span\{I\eta,J\eta,K\eta\} = span\{\frac{\partial}{\partial u},\frac{\partial}{\partial v},\frac{\partial}{\partial w}\}$. Again we have a Riemannian submersion $\pi:(M,\frac{1}{e^f|\varphi|^2}g)\longrightarrow(M',g')$, where

$$g' = \frac{1}{e^f|\varphi|^2}g(h\cdot,h\cdot)_{|M'} = \frac{1}{C-\nu|\varphi|^2}g(h\cdot,h\cdot)_{|M'}.$$

The complex structures I, J, K do not project down to M', but their span does and together with g' forms a quaternionic Kähler structure with reduced scalar curvature $C\nu$. In the chosen coordinates

$$e^f = \frac{C}{e^{Ct}+\nu}.$$

Now, after changing the coordinate t by $e^{Ct}=\nu^2r^2$, it is easily seen that g is locally isometric to the metric$g_{\frac{1}{4}\nu}$ on $\mathcal{U}(M')$ in Theorem 5.2.

6 Complete hHqK manifolds

It is proved in [4, 6] that on a compact quaternionic Kähler manifold the bundle S^2H cannot be globally trivialized by an almost hypercomplex structure. In particular, this is true for the complete quaternionic Kähler manifolds with positive scalar curvature. It is conjectured by Alekseevsky, Marchiafava and Pontecorvo [6] that the only complete simply connected hHqK manifold is $\mathbb{H}H^n$. In support of this they prove that if in addition the Lee form is closed, then there exists a (possibly singular) integrable quaternionic distribution, whose regular orbits are locally homothetic to $\mathbb{H}H^k$. Below we show that under this additional assumption the manifold is indeed $\mathbb{H}H^n$.

Proposition 6.1 *A complete simply connected hHqK manifold with closed Lee form is homothetic to* $\mathbb{H}H^n$.

Proof: If φ vanishes somewhere, then by Theorem 5.4 the manifold is homothetic to $\mathbb{H}H^n$ (as remarked above, $\nu < 0$). Thus, it is enough to consider the case when $\varphi \neq 0$ everywhere.

Let x_t be an integral curve of ξ, parametrized with respect to its length t. Then x_t is geodesic and because of the completeness, it is defined for all $t \in \mathbb{R}$. Thus, $\xi_{x_t} = h(t)\dot{x}_t$ with $h(t) = |\xi_{x_t}| \neq 0$ and (5.25) becomes

$$\frac{dh}{dt} = -\frac{1}{2}(h^2 + \nu).$$

This equation has no solutions with $h^2 + \nu > 0$, defined on the whole $\mathbb{R}$, while every solution with $h^2 + \nu < 0$ vanishes somewhere.

Hence, it remains to consider the case $|\varphi|^2 + \nu = 0$. As seen before, the distribution $span_{\mathbb{H}}\{\eta\}$ is totally geodesic and its integral manifolds are of constant (negative) curvature ν. Every quaternionic Kähler manifold is analytic and hence, by Proposition 7 in [5], these totally geodesic submanifolds can be extended to complete (immersed) totally geodesic submanifolds. By (5.31), we have that $I\eta$, $J\eta$, $K\eta$ are three commuting Killing vector fields on them. This is a contradiction, since the algebra $\mathfrak{so}(1,4)$ of Killing vector fields of $\mathbb{R}H^4$ has rank 2. □

Remark 10 Using (5.33), it can be easily proved in the same way as above that on a complete simply connected hyper-Kähler manifold with hyper-Kähler potential μ there exists a point at which $d\mu$ vanishes. Hence, by Theorem 5.4 ii), the only such manifold is $\mathbb{H}^n$ with the flat metric.

The proof of Proposition 6.1 shows that there are no complete hHqK manifolds whose Lee form satisfies $|\varphi|^2 + \nu = 0$. On the other hand, there exists a global solution of (5.23) on $\mathbb{H}H^n$, satisfying this condition. This follows from the local existence on $\mathbb{H}H^n$ of a solution of the Cauchy problem for (5.23) with any initial data. If $|\varphi|^2 + \nu = 0$ at one point, then this is true everywhere, where the solution is defined, and since the isometry group of $\mathbb{H}H^n$ acts transitively on the unit tangent bundle, this local solution can be extended on the whole $\mathbb{H}H^n$. The corresponding vector field η is an infinitesimal quaternionic automorphism which is not a Killing vector field. It can not be a complete vector field since the group of quaternionic automorphisms of $\mathbb{H}H^n$ coincides with the group of its isometries [5].

It follows from the above discussion that on $\mathbb{H}H^n$ the bundle S^2H can be locally trivialized by hypercomplex structures which have the same (globally defined) Lee form φ. Such a situation cannot occur on a compact quaternionic Kähler manifold:

Proposition 6.2 *On a compact quaternionic Kähler manifold the equation (3.14) has no global solutions.*

Proof: By Remark 3, the exact form Φ is harmonic and therefore $\Phi = 0$. Now (5.27) shows that at every critical point of $|\varphi|^2$ we have $(|\varphi|^2 + \nu)\varphi = 0$. Integrating (3.19), we see that $\nu > 0$. Hence, at a point of maximum of $|\varphi|^2$ the form φ must vanish. Thus, $\varphi \equiv 0$ and therefore $\nu = 0$, which is a contradiction. $\square$

The quaternionic projective space is the only complete locally hHqK manifold with positive scalar curvature in dimensions 4 and 8. This follows from Theorem 4.1 and the results of Hitchin [12], Friedrich and Kurke [11] and Poon and Salamon [18], according to which every complete quaternionic Kähler manifold with positive scalar curvature in these dimensions is symmetric. In fact, there are no known examples of non-symmetric complete quaternionic Kähler manifold with positive scalar curvature. Thus, it seems reasonable to expect that $\mathbb{H}P^n$ could be characterized by the above property in all dimensions.

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