Hexagonal circle patterns and integrable systems. Patterns with constant angles

Alexander I. Bobenko[1] and Tim Hoffmann[2]

Fakultät II, Institut für Mathematik, Technische Universität Berlin,
Strasse des 17 Juni 136, 10623 Berlin, Germany

1 Introduction

The theory of circle packings and, more generally, of circle patterns enjoys in recent years a fast development and a growing interest of specialists in complex analysis and discrete mathematics. This interest was initiated by Thurston's rediscovery of the Koebe-Andreev theorem [K] about circle packing realizations of cell complexes of a prescribed combinatorics and by his idea about approximating the Riemann mapping by circle packings (see [T1, RS]). Since then many other remarkable facts about circle patterns were established, such as the discrete maximum principle and Schwarz's lemma [R] and the discrete uniformization theorem [BS]. These and other results demonstrate surprisingly close analogy to the classical theory and allow one to talk about an emerging of the "discrete analytic function theory" [DS], containing the classical theory of analytic functions as a small circles limit.

Approximation problems naturally lead to infinite circle patterns for an analytic description of which it is advantageous to stick with fixed regular combinatorics. The most popular are hexagonal packings where each circle touches exactly six neighbors. The C^∞ convergence of these packings to the Riemann mapping was established in [HS]. Another interesting and elaborated class with similar approximation properties to be mentioned here are circle patterns with the combinatorics of the square grid introduced by Schramm [S]. The square grid combinatorics of Schramm's patterns results in an analytic description which is closer to the Cauchy-Riemann equations of complex analysis then the one of the packings with hexagonal combinatorics. Various other regular combinatorics also have similar properties [H].

Although computer experiments give convincing evidence for the existence of circle packing analogs of many standard holomorphic functions [DS], the only circle packings that have been described explicitly are Doyle spirals [BDS] (which are analogs of the exponential function) and conformally symmetric packings [BH] (which are analogs of a quotient of Airy functions). Schramm's patterns are richer with explicit examples: discrete analogs of the functions $\exp(z), \mathrm{erf}(z)$, Airy [S] and $z^c, \log(z)$ [AB] are known. Moreover $\mathrm{erf}(z)$ is also an entire circle pattern.[3]

A natural question is: what property is responsible for this comparative richness of Schramm's patterns? Is it due to the packing - pattern or (hexagonal - square) combinatorics difference? Or maybe it is the integrability of Schramm's patterns which is crucial. Indeed, Schramm's square grid circle patterns in conformal setting are known to be described by an integrable system [BP2] whereas for the packings it is still unknown[4].

[1]E–mail: `bobenko @ math.tu-berlin.de`

[2]E–mail: `timh @ sfb288.math.tu-berlin.de`

[3]Doyle conjectured that the Doyle spirals are the only entire circle packings. This conjecture remains open.

[4]It should be said that, generally, the subject of discrete integrable systems on lattices different from $\mathbb{Z}^n$ is underdeveloped at present. The list of relevant publications is almost exhausted by [Ad, KN, ND].

In the present paper we introduce and study *hexagonal circle patterns with constant angles*, which merge features of the two circle patterns discussed above. Our circle patterns have the combinatorics of the regular hexagonal lattice (i.e. of the packings) and intersection properties of Schramm's patterns. Moreover the latter are included as a special case into our class. An example of a circle patterns studied in this paper is shown in Figure 1. Each elementary hexagon of the honeycomb lattice corresponds to a circle, and each common vertex of two hexagons corresponds to an intersection point of the corresponding circles. In particular, each circle carries six intersection points with six neighboring circles and at each point there meet three circles. To each of the three types of edges of the regular hexagonal lattice (distinguished by their directions) we associate an angle $0 \leq \alpha_n < \pi,\ n = 1, 2, 3$ and set that the corresponding circles of the hexagonal circle pattern intersect at this angle. It is easy to see that α_n's are subject to the constraint $\alpha_1 + \alpha_2 + \alpha_3 = \pi$.

We show that despite of the different combinatorics the properties and the description of the hexagonal circle patterns with constant angles are quite parallel to those of Schramm's circle patterns. In particular, the intersection points of the circles are described by a discrete equation of Toda type known to be integrable [Ad]. In Section 4 we present a conformal (i.e. invariant with respect to Möbius transformations) description of the hexagonal circle patterns with constant angles and show that one can vary the angles α_n arbitrarily preserving the cross-ratios of the intersection points on circles, thus each circle pattern generates a two-parameter deformation family. Analytic reformulation of this fact provides us with a new integrable system possessing a Lax representation in 2 by 2 matrices on the regular hexagonal lattice.

Conformally symmetric hexagonal circle patterns are introduced in Section 5. Those are defined as patterns with conformally symmetric flowers, i.e. each circle with its six neighbors is invariant under a Möbius involution (Möbius 180^0 rotation). A similar class of circle packings was investigated in [BH]. Let us mention also that a different subclass of hexagonal circle patterns - with the multi-ratio property instead of the angle condition - was introduced and discussed in detail in [BHS]. In particular it was shown that this class is also described by an integrable system. Conformally symmetric circle patterns comprise the intersection set of the two known integrable classes of hexagonal circle patterns: "with constant angles" and "with multi-ratio property". The corresponding equations are linearizable and can be easily solved.

Further in Section 6 we establish a rather remarkable fact. It turns out that Doyle circle packings and analogue hexagonal circle patterns with constant angles are build out of the same circles. Moreover, given such a pattern one can arbitrarily vary the intersection angles α_n preserving the circle radii.

Extending the intersection points of the circles by their centers one embeds hexagonal circle patterns with constant angles into an integrable system on the dual Kagome lattice (see Section 7). Having included hexagonal circle patterns with constant angles into the framework of the theory of integrable systems, we get an opportunity of applying the immense machinery of the latter to study the properties of the former. This is illustrated in Section 8, where we introduce and study some isomonodromic solutions of our integrable systems on the dual Kagome lattice. The corresponding circle patterns are natural discrete versions of the analytic functions z^c and $\log z$. The results of Section 8 constitute an extension to the present, somewhat more intricate, situation of the similar construction for Schramm's circle patterns with the combinatorics of the square grid [BP2, AB].

Acknowledgements The authors thank S.I. Agafonov and Yu.B. Suris for collaboration and helpful discussions.

2 Hexagonal circle patterns with constant angles

The present paper deals with *hexagonal circle patterns*, i.e. circle patterns with the combinatorics of the regular hexagonal lattice (the honeycomb lattice). An example is shown in Figure 1. Each elementary hexagon of the honeycomb lattice corresponds to a circle, and each common vertex of two hexagons corresponds to an intersection point of the corresponding circles. In particular, each circle carries six intersection points with six neighboring circles and at each intersection point exactly three circles meet.

For analytic description of hexagonal circle patterns we introduce some convenient lattices and

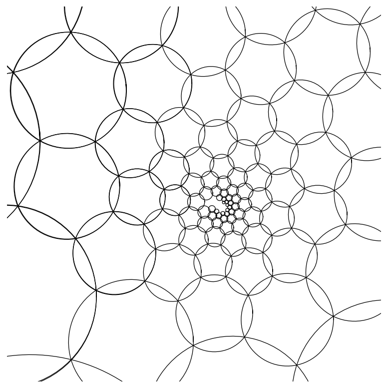

Figure 1: Doyle hexagonal isotropic circle pattern.

variables. First of all we define the *regular triangular lattice* $\mathcal{TL}$ as the cell complex whose vertices are

$$V(\mathcal{TL}) = \Big\{\mathfrak{z} = k + \ell\omega + m\omega^2 : \ k, \ell, m \in \mathbb{Z}\Big\}, \quad \text{where} \quad \omega = \exp(2\pi i/3), \tag{1}$$

whose edges are all non–ordered pairs

$$E(\mathcal{TL}) = \Big\{[\mathfrak{z}_1, \mathfrak{z}_2] : \ \mathfrak{z}_1, \mathfrak{z}_2 \in V(\mathcal{TL}), \ |\mathfrak{z}_1 - \mathfrak{z}_2| = 1\Big\}, \tag{2}$$

and whose 2-cells are all regular triangles with the vertices in $V(\mathcal{TL})$ and the edges in $E(\mathcal{TL})$. We shall use triples $(k, \ell, m) \in \mathbb{Z}^3$ as coordinates of the vertices. On the regular triangular lattice two such triples are equivalent and should be identified if they differ by the vector (n, n, n) with $n \in \mathbb{Z}$.

The vertices of the regular triangular lattice correspond to centers and intersection points of hexagonal circle patterns. Associating one of the centers with the point $k = \ell = m = 0$ we obtain the *regular hexagonal sublattice* $\mathcal{HL}$ with all 2-cells being the regular hexagons with the vertices in

$$V(\mathcal{HL}) = \Big\{\mathfrak{z} = k + \ell\omega + m\omega^2 : \ k, \ell, m \in \mathbb{Z}, \ k + \ell + m \not\equiv 0 \pmod 3\Big\}, \tag{3}$$

and the edges in

$$E(\mathcal{HL}) = \Big\{[\mathfrak{z}_1, \mathfrak{z}_2] : \ \mathfrak{z}_1, \mathfrak{z}_2 \in V(\mathcal{HL}), \ |\mathfrak{z}_1 - \mathfrak{z}_2| = 1\Big\}. \tag{4}$$

The cells and the vertices correspond to circles and to the intersection points of the hexagonal circle patterns respectively. Natural labelling of the faces

$$F(\mathcal{HL}) = \Big\{\mathfrak{z} = k + \ell\omega + m\omega^2 : \ k, \ell, m \in \mathbb{Z}, \ k + \ell + m \equiv 0 \pmod 3\Big\}$$

yields

$$V(\mathcal{HL}) \bigcup F(\mathcal{HL}) = V(\mathcal{TL}).$$

Definition 2.1 *We say that a map* $w : V(\mathcal{HL}) \mapsto \hat{\mathbb{C}}$ *defines a* hexagonal circle pattern*, if the following condition is satisfied:*

- *Let*
$$\mathfrak{z}_k = \mathfrak{z}' + \varepsilon^k \in V(\mathcal{HL}), \quad k = 1, 2, \ldots, 6, \quad \textit{where} \quad \varepsilon = \exp(\pi i/3),$$
be the vertices of any elementary hexagon in $\mathcal{HL}$ *with the center* $\mathfrak{z}' \in F(\mathcal{HL})$*. Then the points* $w(\mathfrak{z}_1), w(\mathfrak{z}_2), \ldots, w(\mathfrak{z}_6) \in \hat{\mathbb{C}}$ *lie on a circle, and their circular order is just the listed one. We denote the circle through the points* $w(\mathfrak{z}_1), w(\mathfrak{z}_2), \ldots, w(\mathfrak{z}_6)$ *by* $C(\mathfrak{z}')$*, thus putting it into a correspondence with the center* $\mathfrak{z}'$ *of the elementary hexagon above.*

As a consequence of this condition, we see that if two elementary hexagons of $\mathcal{HL}$ with the centers in $\mathfrak{z}', \mathfrak{z}'' \in F(\mathcal{HL})$ have a common edge $[\mathfrak{z}_1, \mathfrak{z}_2] \in E(\mathcal{HL})$, then the circles $C(\mathfrak{z}')$ and $C(\mathfrak{z}'')$ intersect in the points $w(\mathfrak{z}_1)$, $w(\mathfrak{z}_2)$. Similarly, if three elementary hexagons of $\mathcal{HL}$ with the centers in $\mathfrak{z}', \mathfrak{z}'', \mathfrak{z}''' \in F(\mathcal{HL})$ meet in one point $\mathfrak{z}_0 \in V(\mathcal{HL})$, then the circles $C(\mathfrak{z}')$, $C(\mathfrak{z}'')$ and $C(\mathfrak{z}''')$ also have a common intersection point $w(\mathfrak{z}_0)$.

Remark. We will consider also circle patterns defined not on the whole of $\mathcal{HL}$, but rather on some connected subgraph of the regular hexagonal lattice.

To each pair of intersecting circles we associate the corresponding edge $\mathfrak{e} \in E(\mathcal{HL})$ and denote by $\phi(\mathfrak{e})$ the intersection angle of the circles, $0 \le \phi < 2\pi$. The edges of $E(\mathcal{HL})$ can be decomposed into three classes

$$\begin{aligned} E_1^H &= \Big\{\mathfrak{e} = [\mathfrak{z}', \mathfrak{z}''] \in E(\mathcal{HL}) : \mathfrak{z}' - \mathfrak{z}'' = \pm 1\Big\}, \\ E_2^H &= \Big\{\mathfrak{e} = [\mathfrak{z}', \mathfrak{z}''] \in E(\mathcal{HL}) : \mathfrak{z}' - \mathfrak{z}'' = \pm \omega\Big\}, \\ E_3^H &= \Big\{\mathfrak{e} = [\mathfrak{z}', \mathfrak{z}''] \in E(\mathcal{HL}) : \mathfrak{z}' - \mathfrak{z}'' = \pm \omega^2\Big\}. \end{aligned} \tag{5}$$

These three sets correspond to three possible directions of the edges of the regular hexagonal lattice.

We shall study in this paper a subclass of hexagonal circle patterns intersecting at given angles defined globally on the whole lattice.

Definition 2.2 *We say that a map* $w : V(\mathcal{HL}) \mapsto \hat{\mathbb{C}}$ *defines a* hexagonal circle pattern with constant angles *if in addition to the condition of Definition 2.1 the intersection angles of circles are constant within their class* E_n^H*, i.e.*

$$\phi(\mathfrak{e}) = \alpha_n \qquad \forall \mathfrak{e} \in E_n^H,\ n = 1, 2, 3.$$

This angle condition implies in particular

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi. \tag{6}$$

We call the circle pattern *isotropic* if all its intersection angles are equal $\alpha_1 = \alpha_2 = \alpha_3 = \pi/3$.

The existence of hexagonal circle patterns with constant angles can be easily demonstrated via solving a suitable Cauchy problem. For example, one can start with the initial circles $C(n(1 - \omega)), C(n(\omega - \omega^2)), C(n(\omega^2 - 1))$, $n \in \mathbb{N}$.

Remark. In the special case $\alpha_1 = \alpha_2 = \pi/2, \alpha_3 = 0$ at each vertex one obtains two pairs of touching circles intersecting orthogonally. The hexagonal circle pattern becomes in this case a circle patterns of Schramm [S] with the combinatorics of the square grid. The generalization $\alpha_1 + \alpha_2 = \pi, \alpha_3 = 0$ was introduced in [BP2].

Figure 2 shows a nearly Schramm pattern.

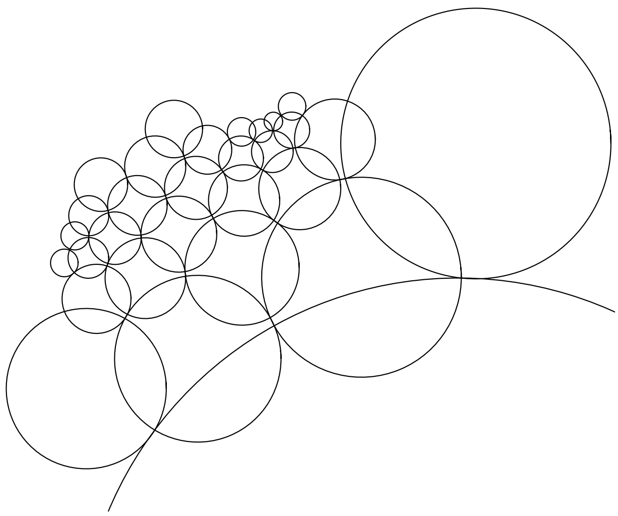

Figure 2: A nearly Schramm pattern.

3 Point and radii descriptions

In this paper three different analytic descriptions are used to investigate hexagonal circle patterns with constant angles. Obviously, these circle patterns can be characterized through the *radii* of the circles. On the other hand, they can be described through the coordinates of some natural *points*, such as the intersection points or centers of circles. Finally, note that the class of circle patterns with constant angles is invariant with respect to arbitrary fractional-linear transformations of the Riemann sphere $\hat{\mathbb{C}}$ (Möbius transformations). Factorizing with respect to this group we naturally come to a *conformal* description of the hexagonal circle patterns with constant angles in Section 4.

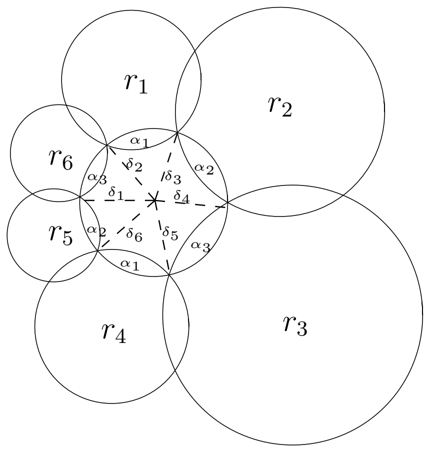

Figure 3: Circle flower.

The basic unit of a hexagonal circle pattern is the *flower*, illustrated in Figure 3 and consisting of a center circle surrounded by six petals. The radius r of the central circle and the radii r_n, $n =$

$1, \ldots, 6$ of the petals satisfy

$$\arg \prod_{n=1}^{6} (r + e^{i\alpha_n} r_n) = \pi, \tag{7}$$

where α_n is the angle between the circles with radii r and r_n. Specifying this for the hexagonal circle patterns with constant angles and using (6) one obtains the following theorem.

Theorem 3.1 *The mapping $r : F(\mathcal{HL}) \to \mathbb{R}_+$ is the radius function of a hexagonal circle pattern with constant angles $\alpha_1, \alpha_2, \alpha_3$ iff it satisfies*

$$\arg \prod_{n=1}^{3} (1 + e^{i\alpha_n} R_n)(e^{-i\alpha_n} + R_{n+3}) = 0, \qquad R_n = \frac{r_n}{r} \tag{8}$$

for every flower.

Conjugating (8) and dividing it by the product $R_1 \ldots R_6$ one observes that for every hexagonal circle pattern with constant angles there exists a *dual* one.

Definition 3.2 *Let $r : F(\mathcal{HL}) \to \mathbb{R}_+$ be the radius function of a hexagonal circle pattern CP with constant angles. The hexagonal circle pattern CP^* with the same constant angles and the radii function $r^* : F(\mathcal{HL}) \to \mathbb{R}_+$ given by*

$$r^* = \frac{1}{r} \tag{9}$$

is called dual *to CP.*

For deriving the point description of the hexagonal circle patterns with constant angles it is convenient to extend the intersection points of the circles by their centers. Fix some point $P_\infty \in \hat{\mathbb{C}}$. The reflections of this point in the circles of the pattern will be called *conformal centers* of the circles. In the particular case $P_\infty = \infty$ the conformal centers become the centers of the corresponding circles. We call an extension of a circle pattern by conformal centers a *center extension.*

Möbius transformations play crucial role for the considerations in this paper. Recall that the *cross-ratio* of four points

$$q(z_1, z_2, z_3, z_4) := \frac{(z_2 - z_1)(z_4 - z_3)}{(z_3 - z_2)(z_1 - z_4)} \tag{10}$$

is invariant with respect to these transformations. We start with a simple

Lemma 3.3 *Let z_2, z_4 be the intersection points and z_1, z_3 the conformal centers of two circles intersecting with the angle α as in Figure 4. The cross-ratio of these points is*

$$q(z_1, z_2, z_3, z_4) = e^{-2i\alpha}. \tag{11}$$

The claim is obvious for the Euclidean centers. These can be mapped to conformal centers by an appropriate Möbius transformation which preserves the cross-ratio.

Thus a circle pattern provides a solution $z : V(\mathcal{TL}) \to \hat{\mathbb{C}}$ to (11) with the corresponding angles α. This solution is defined at the vertices of the lattice $\mathcal{TL}$ with quadrilateral sites comprised by the pairs of points as in Lemma 3.3. For a hexagonal circle pattern with constant angles one can derive from (11) equations for the intersection points $z : V(\mathcal{HL}) \to \hat{\mathbb{C}}$ and conformal centers $z : V(\mathcal{TL} \setminus \mathcal{HL}) \to \hat{\mathbb{C}}$.

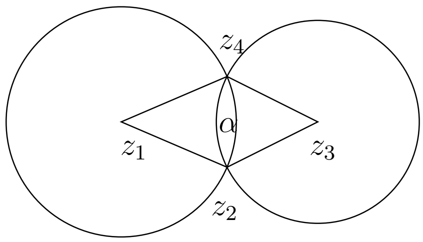

Figure 4: Cross-ratio of an elementary quadrilateral.

Theorem 3.4 *Let $z, z_1, z_2, z_3, z_4, z_5, z_6$ be conformal centers of a flower of a hexagonal circle pattern with constant angles $\alpha_1, \alpha_2, \alpha_3$, where $\alpha_n, n = 1, 2, 3$ are the angles of pairs of circles corresponding to z, z_n and z, z_{n+3}. Define $\delta_1, \delta_2, \delta_3$ through*

$$2\alpha_n = \delta_{n+2} - \delta_{n+1}, \qquad \pmod{2\pi},\ n \in \{1, 2, 3\} \pmod 3. \tag{12}$$

Then z_n satisfy a discrete equation of Toda type on the hexagonal lattice

$$\sum_{n=1}^{3} A_n \left(\frac{1}{z - z_n} + \frac{1}{z - z_{n+3}} \right) = 0, \tag{13}$$

where

$$A_n = e^{i\delta_{n+2}} - e^{i\delta_{n+1}} \qquad n \in \{1, 2, 3\} \pmod 3.$$

Let w_1, w_2, w_3 be the neighboring intersection points to the point w of a hexagonal circle pattern and let α_n be the angle between the circles intersecting at w, w_n. Then the following identity holds

$$\sum_{n=1}^{3} A_n \left(\frac{1}{w - w_n} \right) = 0, \tag{14}$$

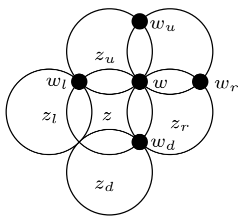

Figure 5: Intersection points and conformal centers of a Schramm circle pattern.

In the special case of Schramm's patterns $\alpha_3 = 0, \alpha_1 + \alpha_2 = \pi$ flowers contain only four petals and we arrive at the following

Theorem 3.5 *The intersection points $w, w_r, w_u, w_l, w_d \in \mathbb{C}$ and the conformal centers $z, z_r, z_u, z_l, z_d \in \mathbb{C}$ of the neighboring circles of a Schramm pattern (labelled as in Figure 5) satisfy the discrete equation of Toda type on $\mathbb{Z}^2$ lattice*

$$\frac{1}{w - w_r} + \frac{1}{w - w_l} = \frac{1}{w - w_u} + \frac{1}{w - w_d}, \tag{15}$$

$$\frac{1}{z - z_r} + \frac{1}{z - z_\ell} = \frac{1}{z - z_u} + \frac{1}{z - z_d}. \tag{16}$$

The proofs of these two theorems are presented in Appendix B (in the case of complex $\alpha \in \mathbb{C}$).

It should be noticed that both equations (16) and (13) equations appeared in the theory of integrable equations in a totally different context [Su, Ad]. The geometric interpretation in the present paper is new.

The sublattices of the centers and of the intersection points are dependent and one can be essentially uniquely reconstructed from the other. The corresponding formulas, which are natural to generalize for complex cross-ratios (10), hold for both discrete equations of Toda type discussed above. We present these relations, which are of independent interest in the theory of discrete integrable systems, in Appendix B.

In Section 7 we will show that the hexagonal circle patterns with constant angles are described by an integrable system on a regular lattice closely related to the lattices introduced in Section 2.

4 Conformal description

Let us now turn our attention to a conformal description of the hexagonal circle patterns. This description will be used in the construction of conformally symmetric circle patterns in the next section. We derive equations for the cross-ratios of the points of the hexagonal lattice that allow to reconstruct the lattice up to Möbius transformations.

First we will investigate the relations of cross-ratios inside one hexagon of the hexagonal lattice as shown in Fig. 6.

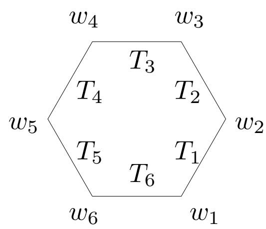

Figure 6: Cross-ratios in a hexagon.

Lemma 4.1 *Given a map $w : V(\mathcal{HL}) \to \hat{\mathbb{C}}$ let $\mathfrak{z}_1, \dots, \mathfrak{z}_6$ be six points of a hexagon cyclically ordered (see Fig. 6). To each edge $[\mathfrak{z}_i, \mathfrak{z}_{i+1}]$ of the hexagon, let us assign a cross-ratio T_i of successive points $w_i = w(\mathfrak{z}_i)$:*

$$T_i := q(w_i, w_{i+1}, w_{i+2}, w_{i-1}), \qquad (i \mod 6). \tag{17}$$

Then the equations

$$\frac{T_1}{T_4} = \frac{T_3}{T_6} = \frac{T_5}{T_2} = \frac{T_1 + T_3 - 1 - T_1 T_2 T_3}{1 - T_2} \tag{18}$$

hold.

Proof. Let m_i be the Möbius transformation that maps w_{i-1}, w_i, and w_{i+1} to 0, 1, and ∞. Then $M_i := m_{i+1}^{-1} m_i$ maps w_{i-1}, w_i, and w_{i+1} to w_i, w_{i+1}, and w_{i+2} respectively and has the form

$$M_i = \begin{pmatrix} 1 & -1 \\ 1 & -T_i \end{pmatrix}$$

Now $(M_6 M_5 M_4)^{-1} = \rho M_3 M_2 M_1$ for some ρ gives the desired identities. $\square$

Since every edge in $E(\mathcal{HL})$ belongs to two hexagons it carries two cross-ratios in general. We will now investigate the relation between them and show that in case of a hexagonal circle pattern with constant angles the two cross-ratios coincide. This way Lemma 4.1 will furnish a map $T : E(\mathcal{HL}) \to \mathbb{C}$.

Besides the cross-ratios of successive points in each hexagon (the T_i) we will need the cross-ratios of a point and its three neighbors:

Definition 4.2 *Let $\mathfrak{z}_1, \mathfrak{z}_2$, and $\mathfrak{z}_3$ be the neighbors of $\mathfrak{z} \in V(\mathcal{HL})$ counterclockwise ordered and $[\mathfrak{z}_i, \mathfrak{z}] \in E_i^H, i = 1, 2, 3$. Any map $w : V(\mathcal{HL}) \to \hat{\mathbb{C}}$ furnishes three cross-ratios in each point $\mathfrak{z} \in V(\mathcal{HL})$:*

$$S_{\mathfrak{z}}^{(i)} := q(w_i, w_{i+1}, w, w_{i+2}), \quad i = 1, 2, 3 \quad (\bmod 3). \tag{19}$$

They are linked by the modular transformation

$$S_{\mathfrak{z}}^{(i+1)} = 1 - \frac{1}{S_{\mathfrak{z}}^{(i)}}, \qquad (i \bmod 3) \tag{20}$$

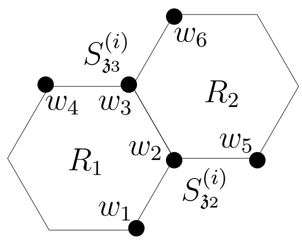

Figure 7: Six points around one edge.

Of course the two types of cross-ratios are not independent: When we look at an edge from $E(\mathcal{HL})$ with its four neighboring edges the six points form four cross-ratios (two T and two S). Three points fix the Möbius transformation and the other three can be calculated from only three cross-ratios. Therefore one expects one equation:

Lemma 4.3 *Given $w : V(\mathcal{HL}) \to \hat{\mathbb{C}}$ and $\mathfrak{z}_1, \ldots, \mathfrak{z}_6$ as shown in Fig. 7 and the cross-ratios*

$$R_1 := q(w_1, w_2, w_4, w_3), \quad R_2 := q(w_5, w_2, w_6, w_3) \tag{21}$$

$$S_{\mathfrak{z}_3}^{(i)} := q(w_3, w_4, w_2, w_6), \quad S_{\mathfrak{z}_2}^{(i)} := q(w_2, w_5, w_3, w_1) \tag{22}$$

with $[\mathfrak{z}_2, \mathfrak{z}_3] \in E_i^H$. Then the following identity holds:

$$\frac{1 - T_2}{1 - T_1} = \frac{R_1}{R_2} = \frac{S_{\mathfrak{z}_3}^{(i)}}{S_{\mathfrak{z}_2}^{(i)}}. \tag{23}$$

Proof. Use that T_i and R_i are linked by the relation

$$R_i := \frac{1}{1 - T_i} = q(w_i, w_{i+1}, w_{i-1}, w_{i+2}) \tag{24}$$

and insert the definition of the cross-ratio. $\square$

Everything written so far holds for any hexagonal lattice, but in case of the hexagonal circle patterns with constant angles we can calculate all S in terms of the α_i only:

$$S_{\mathfrak{z}}^{(i)} := q(w_i, w_{i+1}, w, w_{i+2}) = e^{-i\alpha_i}\frac{\sin\alpha_{i+1}}{\sin\alpha_{i+2}} \tag{25}$$

if $[\mathfrak{z}_i, \mathfrak{z}] \in E_i$. This can be verified easily by applying a Möbius transformation sending $\mathfrak{z}$ to infinity. The three circles become straight lines forming a triangle with angles $\pi - \alpha_i$. The $S^{(i)}$ are now the quotients of two of its edges.

In particular we get $S_{\mathfrak{z}_1}^{(i)} = S_{\mathfrak{z}_2}^{(i)}$ along all edges $[\mathfrak{z}_1, \mathfrak{z}_2] \in E(\mathcal{HL})$. Thus for our hexagonal circle patterns the above formula (25) implies that there is only one T per edge since we get from equation (23)

$$T_1 = T_2. \tag{26}$$

So Lemma 4.1 defines in fact a map $T : E(\mathcal{HL}) \to \hat{\mathbb{C}}$.

Note that we can get back the α_i from say $S^{(1)}$ in equation (25) by the following formulas:

$$e^{2i\alpha_1} = \frac{\overline{S}^{(1)}}{S^{(1)}}, \quad e^{2i\alpha_2} = \frac{1+S^{(1)}}{1+\overline{S}^{(1)}}, \quad e^{2i\alpha_3} = \frac{S^{(1)}}{\overline{S}^{(1)}}\,\frac{1+\overline{S}^{(1)}}{1+S^{(1)}} \tag{27}$$

Now we can formulate the main theorem that states that the equations which link the cross-ratios in a hexagon plus the constantness of $S^{(1)}$ on the hexagonal lattice describe for real-negative T a hexagonal circle pattern with constant angles up to Möbius transformations:

Theorem 4.4

1. *Given $S \in \mathbb{C}$ and a map $w : V(\mathcal{HL}) \to \hat{\mathbb{C}}$ for which $S_{\mathfrak{z}}^{(1)} = S$ for all $\mathfrak{z} \in V(\mathcal{HL})$. Then the cross-ratios T_i of Lemma 4.1 define a map $T : E(\mathcal{HL}) \to \hat{\mathbb{C}}$ and obey equations (18).*

2. *Conversely given a solution $T : E(\mathcal{HL}) \to \hat{\mathbb{C}}$ of (18) then for each $S \in \mathbb{C}$ there is up to Möbius transformations a unique map $w : V(\mathcal{HL}) \to \hat{\mathbb{C}}$ having the map T as cross-ratios in the sense of Lemma 4.1 and having $S_{\mathfrak{z}}^{(1)} = S$ for all $\mathfrak{z} \in V(\mathcal{HL})$.*

   If the solution T is real-negative the resulting w define a hexagonal circle pattern with constant angles.

Proof. 1. is proven above.
2. Starting with the points $w_{2,0,0}, w_{1,1,0}$, and $w_{1,0,1}$ (which fixes the Möbius transformation) we can calculate $w_{1,0,0}$ using $S_{1,0,0}^{(1)}$. Now we have three points for each hexagon touching in $\mathfrak{z}_{1,0,0}$. Using the T's we can determine all other points of them. Every new point has at least two determined neighbors so we can use the $S^{(1)}$ to compute the third. Now we are able to compute all points of the new neighboring hexagons and so on.

Since real-negative cross-ratios imply that the four points lie cyclically ordered on a circle, we have a circle pattern in the case of real-negative map T. But with (25) the constantness of $S^{(1)}$ implies that the intersection angles are constant too. □

We have shown, that hexagonal circle patterns with constant angles come in one complex (or two real) parameter families: one can choose $S \in \mathbb{C}$ arbitrarily preserving T's. In Appendix A it is shown how this observation implies a Lax representation on the hexagonal lattice for the system (18).

Figure 2 shows a nearly Schramm pattern. One can obtain his description by taking combinations of T_i's and $S^{(i)}$'s that stay finite in the limit $\alpha_3 \to 0$.

5 Conformally symmetric circle patterns

The basic notion of conformal symmetry introduced in [BH] for circle packings can be easily generalized to circle patterns: Every elementary flower shall be invariant under the Möbius equivalent of a 180° rotation.

Definition 5.1 *1. An elementary flower of a hexagonal circle pattern with petals C_i is called* conformally symmetric *if there exists a Möbius involution sending C_i to C_{i+3} ($i \bmod 3$).*

2. A hexagonal circle pattern is called conformally symmetric *if all of its elementary flowers are.*

For investigation of conformally symmetric patterns we will need the notion of the multi-ratio of six points:

$$m(z_1, z_2, z_3, z_4, z_5, z_6) := \frac{z_1 - z_2}{z_2 - z_3} \frac{z_3 - z_4}{z_4 - z_5} \frac{z_5 - z_6}{z_6 - z_1}. \tag{28}$$

Hexagonal circle pattern with multi-ratio -1 are discussed in [BHS] (see [KS] for further geometric interpretation of this quantity). It turns out that in the case of conformally symmetric hexagonal circle patterns the two known integrable classes "with constant angles" and "with multi-ratio -1" coincide.

Proposition 5.2 *1. An elementary flower of a hexagonal circle pattern is conformally symmetric if and only if the opposite intersection angles are equal and the six intersection points with the central circle have multi-ratio -1.*

2. A hexagonal circle pattern is conformally symmetric iff and only if it has constant intersection angles and for all circles the six intersection points have multi-ratio -1.

Proof. First let us show that six points z_i have multi-ratio -1 if and only if there is a Möbuis involution sending z_i to z_{i+3}.

If there is such Möbius transformation it is clear, that $q(z_1, z_2, z_3, z_4) = q(z_4, z_5, z_6, z_1)$ and

$$m(z_1, z_2, z_3, z_4, z_5, z_6) = -q(z_1, z_2, z_3, z_4)/q(z_4, z_5, z_6, z_1). \tag{29}$$

implies

$$m(z_1, z_2, z_3, z_4, z_5, z_6) = -1. \tag{30}$$

Conversely let M be the Möbius transformation sending z_1, z_2 and z_3 to z_4, z_5 and z_6. Then for $z_* := M(z_4)$ equation (30) implies $q(z_4, z_5, z_6, z_1) = q(z_1, z_2, z_3, z_4) = q(M(z_1), M(z_2), M(z_3), M(z_4)) = q(z_4, z_5, z_6, z_*)$ and thus $\mathfrak{z}_* = \mathfrak{z}_1$. The same computation yields $\mathfrak{z}_2 = M(\mathfrak{z}_5)$ and $\mathfrak{z}_3 = M(\mathfrak{z}_6)$.

Now the first statement of the theorem is proven, since the intersection points and angles determine the petals completely. For the prof of the second statement the only thing left to show is that all flowers being conformally symmetric implies that the three intersection angles per flower sum up to π. So let us look at a flower around the circle C with petals C_i. Let the angle between C and C_i be α_i (and we know that $\alpha_i = \alpha_{i+3}$). Then the angle $\beta_1 = \pi - \alpha_1 - \alpha_2$ is the angle between C_1 and C_2 and $\beta_3 = \pi - \alpha_3 - \alpha_1$ the one between C_3 and C_4. Since the flowers around C_2 and C_3 are conformally symmetric too we now have two ways to compute the angle β_2 between C_2 and C_3. Namely

$$\pi - \alpha_2 - \alpha_3 = \beta_2 = \pi - (\pi - \alpha_3 - \alpha_1) - (\pi - \alpha_1 - \alpha_2)$$

which implies $\alpha_1 + \alpha_2 + \alpha_3 = \pi$. $\square$

Using (29) we see, that in the case of multi-ratio -1 the opposite cross-ratios T defined in section 4 must be equal: $T_i = T_{(i \bmod 3)}$. Thus on $E(\mathcal{HL})$ the T's must be constant in the direction perpendicular to the edge they are associated with. For the three cross-ratios in a hexagon we get from (18)

$$T_1 + T_2 + T_3 - T_1T_2T_3 = 2. \tag{31}$$

Let us rewrite this equation by using the labelling shown in Fig. 8. We denote by a_k, b_ℓ and c_m

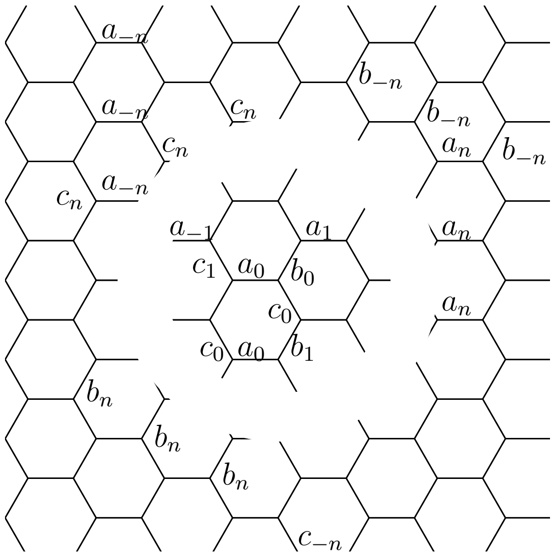

Figure 8: Cross-ratios for conformally symmetric circle patterns

the cross-ratios (24) associated with the edges of the families E_1^H, E_2^H and E_3^H respectively. Note that for the labels in Fig. 8

$$k + \ell + m = 1$$

holds. Written in terms of a_k, b_ℓ, c_m equation (31) is linear

$$a_k + b_\ell + c_m = 1 \tag{32}$$

and can be solved explicitly:

Lemma 5.3 *The general solution to (32) on $E(\mathcal{HL})$ is given by*

$$\begin{aligned} a_k &= a_0 + k\Delta \\ b_\ell &= b_0 + \ell\Delta \\ c_m &= c_0 + m\Delta \end{aligned} \tag{33}$$

with $a_0, b_0, c_0 \in \mathbb{C}$ and $\Delta = 1 - a_0 - b_0 - c_0$.

Proof. Obviously (33) solves (32). On the other hand it is easy to show, that given the cross-ratios on three neighboring edges determine all other cross-ratios recursively. Therefore (33) is the only solution to (32) □

Theorem 5.4 *Conformally symetric circle hexagonal circle patterns are described as follows: Given a_n, b_n, c_n by (33), choose $S^{(1)} \in \mathbb{C}$ (or angles α_1 and α_2) then there is a conformally symmetric circle pattern with intersection angles α_1, α_2, and α_3 given by formula (27).*

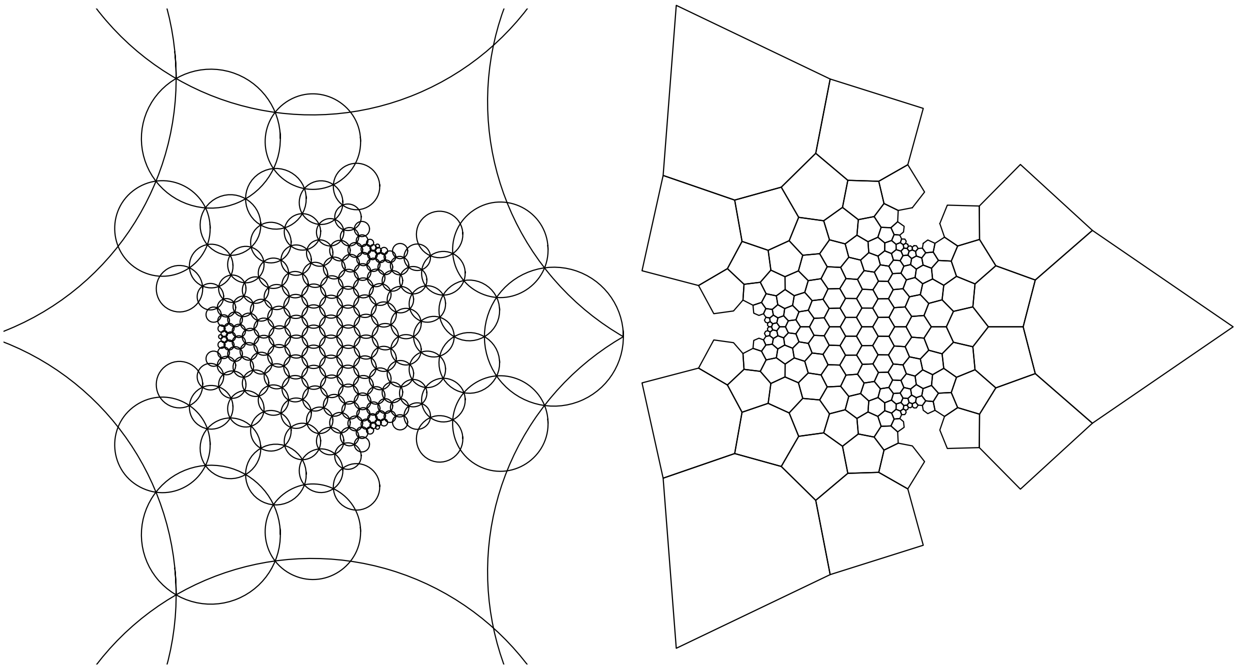

Figure 9: A conformally symmetric circle pattern and $V(\mathcal{HL})$ under the quotient of two Airy functions $f(z) = \frac{\mathrm{Bi}(z)+\sqrt{3}\mathrm{Ai}(z)}{\mathrm{Bi}(z)-\sqrt{3}\mathrm{Ai}(z)}$.

The cross-ratio of four points can be viewed as a discretization of the Schwarzian derivative. In this sense conformally symmetric patterns correspond to maps with a linear Schwarzian. The latter are quotients of two Airy functions [BH]. Figure 9 shows that a symmetric solution of (32) is a good approximation of its smooth counterpart. Figure 2 also shows a conformally symmetric pattern with $S^{(1)} = 10i$ and $a_0 = b_0 = 1/3 - 0.29$, and $c_0 = 1 - 2a_0$.

6 Doyle circle patterns

Doyle circle packings are described through their radii function. The elementary flower of a hexagonal circle packing is a central circle with six touching petals (which in turn touch each other cyclically). Let the radius of the central circle be R and $R_1, \ldots, R_6$ the radii of the petals. Doyle spirals are described through the constraint

$$R_k R_{k+3} = R^2, \quad R_k R_{k+2} R_{k+4} = R^3. \tag{34}$$

There are two degrees of freedom for the whole packing—e. g. R_1/R and R_2/R which are constant for all flowers. The next lemma claims that the same circles that form a Doyle packing build up a circle pattern with constant angles.

Lemma 6.1 *The radii function of a Doyle packing (i. e. a solution to (34)) solves the pattern radii equation (8) for any choice of the angles $\alpha_1, \alpha_2, \alpha_3 = \pi - \alpha_1 - \alpha_2$.*

Proof. Now insert identities (34) into (8). □

Definition 6.2 *A hexagonal circle pattern with constant angles who's radii function obeys the constraint (34) is called a Doyle pattern.*

Figure 1 shows a Doyle pattern. The following lemma and theorem shows how the Doyle patterns fit into our conformal description:

Lemma 6.3 *Doyle patterns are conformally symmetric.*

Proof. We have to show that for Doyle pattern the multi-ratio of each hexagon is -1. One can assume that the circumfencing circle has radius 1 and center 0 and that $w_1 = 1$. Then the other points are given by

$$w_j = w_{j-1}\frac{1 + R_j e^{i\alpha_{(j mod 3)}}}{1 + R_j e^{-i\alpha_{(j mod 3)}}}$$

where R_j are the radii of the petals. Inserting this into the definition of the multi-ratio implies the claim. $\square$

Theorem 6.4 *Doyle patterns and their Möbius transforms can be characterized in the following way: The corresponding solution to (32) is constant*[5]*, i. e.* $a_0 + b_0 + c_0 = 1$ *and* $a_0, b_0, c_0 > 0$.

Proof. Any Doyle pattern gives rise to a constant solution of (32) since all elementary flowers of a Doyle pattern are similar.

On the other hand one sees easily, that all $0 < R_i < 1$ can be realized. $\square$

7 Lax representation and Dual patterns

We start with a general construction of "integrable systems" on graphs which does not hang on the specific features of the lattice. This notion includes the following ingredients:

- An *oriented graph* $\mathcal{G}$ with the vertices $V(\mathcal{G})$ and the edges $E(\mathcal{G})$.
- A loop group $G[\lambda]$, whose elements are functions from $\mathbb{C}$ into some group G. The complex argument λ of these functions is known in the theory of integrable systems as the "spectral parameter".
- A "wave function" $\Psi : V(\mathcal{G}) \mapsto G[\lambda]$, defined on the vertices of $\mathcal{G}$.
- A collection of "transition matrices" $L : E(\mathcal{G}) \mapsto G[\lambda]$ defined on the edges of $\mathcal{G}$.

It is supposed that for any oriented edge $\mathfrak{e} = (\mathfrak{z}_{out}, \mathfrak{z}_{in}) \in E(\mathcal{G})$ the values of the wave functions in its ends are connected via

$$\Psi(\mathfrak{z}_{in}, \lambda) = L(\mathfrak{e}, \lambda)\Psi(\mathfrak{z}_{out}, \lambda). \tag{35}$$

Therefore the following *zero curvature condition* has to be satisfied. Consider any closed contour consisting of a finite number of edges of $\mathcal{G}$:

$$\mathfrak{e}_1 = (\mathfrak{z}_1, \mathfrak{z}_2), \quad \mathfrak{e}_2 = (\mathfrak{z}_2, \mathfrak{z}_3), \quad \ldots, \quad \mathfrak{e}_p = (\mathfrak{z}_p, \mathfrak{z}_1).$$

Then

$$L(\mathfrak{e}_p, \lambda)\cdots L(\mathfrak{e}_2, \lambda)L(\mathfrak{e}_1, \lambda) = I. \tag{36}$$

In particular, for any edge $\mathfrak{e} = (\mathfrak{z}_1, \mathfrak{z}_2)$ one has $\mathfrak{e}^{-1} = (\mathfrak{z}_2, \mathfrak{z}_1)$ and

$$L(\mathfrak{e}^{-1}, \lambda) = \Big(L(\mathfrak{e}, \lambda)\Big)^{-1}. \tag{37}$$

[5]Constant means here $a_k = a_0, b_k = b_0, c_k = c_0$ for all $k \in \mathbb{Z}$.

Actually, in applications the matrices $L(\mathfrak{e}, \lambda)$ depend also on a point of some set X (the "phase space" of an integrable system), so that some elements $x(\mathfrak{e}) \in X$ are attached to the edges $\mathfrak{e}$ of $\mathcal{G}$. In this case the discrete zero curvature condition (36) becomes equivalent to the collection of equations relating the fields $x(\mathfrak{e}_1)$, …, $x(\mathfrak{e}_p)$ attached to the edges of each closed contour. We say that this collection of equations admits a *zero curvature representation*. Such representation may be used to apply analytic methods for finding concrete solutions, transformations or conserved quantities.

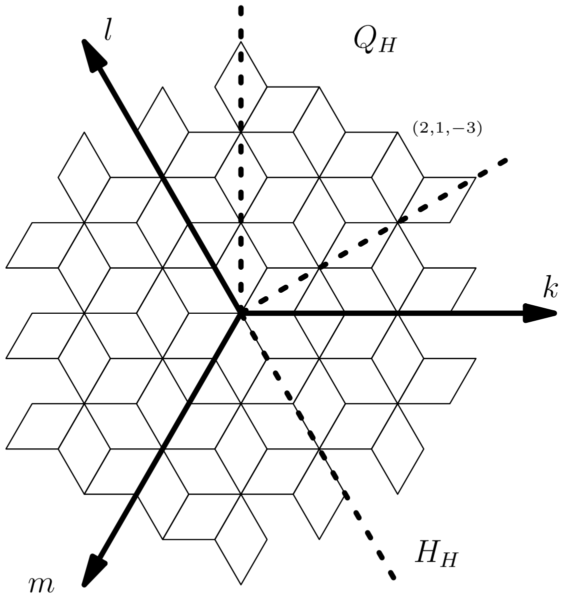

Figure 10: Quadrilateral lattice $\mathcal{QL}$.

In this paper we will deal with zero curvature representations on the hexagonal lattice $\mathcal{HL}$ and especially on a closely related to it a *special quadrilateral lattice* $\mathcal{QL}$ which is obtained from $\mathcal{HL}$ by deleting from $E(\mathcal{TL})$ the edges of the hexagonal lattice $E(\mathcal{HL})$ (i.e. corresponding to the intersection points). So defined lattice $\mathcal{QL}$ has quadrilateral cells, vertices $V(\mathcal{QL}) = V(\mathcal{TL})$ and the edges

$$E(\mathcal{QL}) = E(\mathcal{TL}) \setminus E(\mathcal{HL})$$

as shown in Figure 10. The lattice dual to $\mathcal{QL}$ is known as Kagome lattice [B]. Similar to (5) there are tree types of edges in $E(\mathcal{QL})$ distinguished by their directions:

$$\begin{aligned} E_1^Q &= \Big\{\mathfrak{e} = [\mathfrak{z}', \mathfrak{z}''] \in E(\mathcal{QL}) : \mathfrak{z}' - \mathfrak{z}'' = \pm 1\Big\}, \\ E_2^Q &= \Big\{\mathfrak{e} = [\mathfrak{z}', \mathfrak{z}''] \in E(\mathcal{QL}) : \mathfrak{z}' - \mathfrak{z}'' = \pm \omega\Big\}, \\ E_3^Q &= \Big\{\mathfrak{e} = [\mathfrak{z}', \mathfrak{z}''] \in E(\mathcal{QL}) : \mathfrak{z}' - \mathfrak{z}'' = \pm \omega^2\Big\}. \end{aligned} \tag{38}$$

There is a natural labelling $(k,\ell,m) \in \mathbb{Z}^3$ of the vertices $V(\mathcal{QL})$ which respects the lattice structure of $\mathcal{QL}$. Let us decompose $V(\mathcal{QL}) = V_0 \cup V_1 \cup V_{-1}$ into three sublattices

$$\begin{aligned} V_0 &= \Big\{\mathfrak{z} = k + \ell\omega + m\omega^2 : \ k,\ell,m \in \mathbb{Z}, \ k+\ell+m = 0\Big\}, \\ V_1 &= \Big\{\mathfrak{z} = k + \ell\omega + m\omega^2 : \ k,\ell,m \in \mathbb{Z}, \ k+\ell+m = 1\Big\}, \\ V_{-1} &= \Big\{\mathfrak{z} = k + \ell\omega + m\omega^2 : \ k,\ell,m \in \mathbb{Z}, \ k+\ell+m = -1\Big\}. \end{aligned} \tag{39}$$

Note that this definition associates a unique triple (k,ℓ,m) to each vertex. Neighboring vertices of $\mathcal{QL}$, i.e. those connected by edges, are characterized by the property that their (k,ℓ,m)-labels differ only in one component. Each vertex of V_0 has six edges whereas the vertices of $V_{\pm 1}$ have only three.

The (k,ℓ,m) labelling of the vertices of $\mathcal{QL}$ suggests to consider this lattice as the intersection of $\mathbb{Z}^3$ with the strip $\mid k+\ell+m \mid \le 1$. This description turns out to be useful especially for construction of discrete analogues of z^c and $\log z$ in Section 8. We denote $p = (k,\ell,m) \in \mathbb{Z}^3$ and three different types of edges of $\mathbb{Z}^3$ by

$$E_1 = (p, p + (1,0,0)), \ E_2 = (p, p + (0,1,0)), \ E_3 = (p, p + (0,0,1)), \ p \in \mathbb{Z}^3.$$

The group $G[\lambda]$ we use in our construction is the *twisted loop group* over $\mathrm{SL}(2,\mathbb{C})$:

$$G[\lambda] = \Big\{L : \mathbb{C} \mapsto \mathrm{SL}(2,\mathbb{C}) \Big| \ L(-\lambda) = \sigma_3 L(\lambda) \sigma_3\Big\}, \qquad \sigma_3 = \mathrm{diag}(1,-1). \tag{40}$$

To each type E_n we associate a constant $\Delta_n \in \mathbb{C}$. Let $z_{k,\ell,m}, z^*_{k,\ell,m}$ be fields defined at vertices $z, z^* : \mathbb{Z}^3 \to \mathbb{C}$ and satisfying

$$(z_{in} - z_{out})(z^*_{in} - z^*_{out}) = \Delta_n \tag{41}$$

on all the edges $\mathfrak{e} = (p_{out}, p_{in}) \in E_n, n = 1,2,3$. Here we denote $z_{out,in} = z(p_{out,in}), z^*_{out,in} = z^*(p_{out,in})$. We call the mapping z^* *dual* to z. To each oriented edge $\mathfrak{e} = (p_{out}, p_{in}) \in E_n$ we attach the following element of the group $G[\lambda]$:

$$L^{(n)}(\lambda) = (1 - \lambda^2 \Delta_n)^{-1/2} \begin{pmatrix} 1 & \lambda(z_{in} - z_{out}) \\ \lambda(z^*_{in} - z^*_{out}) & 1 \end{pmatrix}. \tag{42}$$

Note that this form as well as the condition (41) are independent of the orientation of the edge. Substituting (41) into (42) one obtains $L(\lambda)$ in terms of the field z only.

Dual fields z, z^* can be characterized in their own terms. The condition that for any quadrilateral of the dual lattice the oriented edges (z^*_{out}, z^*_{in}) defined by (41) sum up to zero imply the following statements.

Theorem 7.1 *(i) Let $z, z^* : \mathbb{Z}^3 \to \mathbb{C}$ be dual fields, i.e. satisfying the duality condition (41). Then three types of elementary quadrilaterals of $\mathbb{Z}^3$ have the following cross-ratios:*

$$\begin{aligned} q(z_{k,\ell,m}, z_{k+1,\ell,m}, z_{k+1,\ell,m-1}, z_{k,\ell,m-1}) &= \frac{\Delta_1}{\Delta_3}, \\ q(z_{k,\ell,m}, z_{k,\ell,m+1}, z_{k,\ell-1,m+1}, z_{k,\ell-1,m}) &= \frac{\Delta_3}{\Delta_2}, \\ q(z_{k,\ell,m}, z_{k,\ell+1,m}, z_{k-1,\ell+1,m}, z_{k-1,\ell,m}) &= \frac{\Delta_2}{\Delta_1}, \end{aligned} \tag{43}$$

Same identities hold with z replaced by z^*.

(ii) Given a solution $z : \mathbb{Z}^3 \to \mathbb{C}$ to (43) formulas (41) determine (uniquely up to translation) a solution $z^* : \mathbb{Z}^3 \to \mathbb{C}$ of the same system (43).

It is obvious that the zero curvature condition (36) is fulfilled for every closed contour in $\mathbb{Z}^3$ if and only if it holds for all elementary quadrilaterals. It is easy to check that the transition matrices $L^{(n)}$ defined above satisfy the zero curvature condition.

Theorem 7.2 *Let $z, z^* : \mathbb{Z}^3 \to \mathbb{C}$ be a solution to (41) and $\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3, \mathfrak{e}_4$ be consecutive positively oriented edges of an elementary quadrilateral of $\mathcal{QL}$. Then the zero curvature condition*

$$L(\mathfrak{e}_4, \lambda)L(\mathfrak{e}_3, \lambda)L(\mathfrak{e}_2, \lambda)L(\mathfrak{e}_1, \lambda) = I$$

holds with $L(\mathfrak{e}, \lambda)$ defined by (42). Moreover let the elements

$$L^{(n)}(\mathfrak{e}, \lambda) = (1 - \lambda^2\Delta_n)^{-1/2} \begin{pmatrix} 1 & \lambda f \\ \lambda g & 1 \end{pmatrix}, \qquad fg = \Delta_n, n = 1, 2, 3$$

of $G[\lambda]$ be attached to oriented edges $\mathfrak{e} = (p_{out}, p_{in}) \in E_n$. Then the zero curvature condition on $\mathbb{Z}^3$ is equivalent to existence of $z, z^* : \mathbb{Z}^3 \to \mathbb{C}$ such that the factorization

$$f(\mathfrak{e}) = z(p_{in}) - z(p_{out}), \qquad g(\mathfrak{e}) = z^*(p_{in}) - z^*(p_{out})$$

holds. So defined z, z^* satisfy (41, 43).

The zero curvature condition in Theorem 7.2 is a generalization of the Lax pair found in [NC] for the discrete conformal mappings [BP1]. The zero curvature condition implies the existence of the wave function $\Psi : \mathbb{Z}^3 \to G[\lambda]$. The last one can be used to restore the fields z and z^*. There holds the following result having many analogs in the differential geometry described by integrable systems ("Sym formula", see, e.q., [BP2]).

Theorem 7.3 *Let $\Psi(p, \lambda)$ be the solution of (35) with the initial condition $\Psi(p = 0, \lambda) = I$. Then the fields z, z^* may be found as*

$$\left.\frac{d\Psi_{k,\ell,m}}{d\lambda}\right|_{\lambda=0} = \begin{pmatrix} 0 & z_{k,\ell,m} - z_{0,0,0} \\ z^*_{k,\ell,m} - z^*_{0,0,0} & 0 \end{pmatrix}. \tag{44}$$

This simple observation turns out to be useful for analytic constructions of solutions in particular in Section 8.

Interpreting the lattice $\mathcal{QL}$ as $\{(k, \ell, m) \in \mathbb{Z}^3 :\mid k + \ell + m \mid \leq 1\}$ one arrives at the following

Corollary 7.4 *Theorems 7.1, 7.2, 7.3 hold for the lattice $\mathcal{QL}$, i.e. if in the statements one replaces $\mathbb{Z}^3$ by $\mathcal{QL}$, p by $\mathfrak{z}$, E_n by E_n^Q and assumes $k + \ell + m = 0$ in (43).*

Let us return to hexagonal circle patterns and explain their relation to the discrete integrable systems of this Section. To obtain the lattice $\mathcal{QL}$ we extend the intersection points of a hexagonal circle pattern by the conformal centers of the circles. The image of so defined mapping $z : V(\mathcal{QL}) \to \hat{\mathbb{C}}$ consists of the intersection points $z(V_1 \cup V_{-1})$ and the conformal centers $z(V_0)$. The edges connecting the points on the circles with their centers correspond to the edges of the quadrilateral lattice $\mathcal{QL}$. Whereas the angles α_n are associated to three types $E_n^H, n = 1, 2, 3$ of the edges of the hexagonal lattice, constants δ_n defined in (12) are associated to three types $E_n^Q, n = 1, 2, 3$ of the edges of the quadrilateral lattice. Identifying them with

$$\Delta_n = e^{-i\delta_n} \tag{45}$$

in the matrices $L^{(n)}(\lambda)$ above one obtains a zero curvature representation for hexagonal circle patterns with constant angles.

Theorem 7.5 *The intersection points and the conformal centers of the circles $z : \mathcal{QL} \to \hat{\mathbb{C}}$ of a hexagonal circle pattern with constant angles $\alpha_n, n = 1, 2, 3$ satisfy the cross-ratio system (43) on the lattice $\mathcal{QL}$ with $\Delta_n, n = 1, 2, 3$ determined by (12, 45).*

The theorem follows from the identification of (43) with (11) using (12) and (45).

The duality transformation introduced above for arbitrary mappings satisfying the cross-ratio equations preserves the class of such mappings coming from circle patterns with constant angles if one extends the intersection points by the Euclidean centers of the circles. The last one is nothing but the dual circle pattern of Definition 3.2.

Theorem 7.6 *Let $z : \mathcal{QL} \to \hat{\mathbb{C}}$ be a hexagonal circle pattern CP with constant angles extended by the Euclidean centers of the circles. Then the dual circle pattern CP^* together with the Euclidean centers of the circles is given by the dual mapping $z^* : \mathcal{QL} \to \hat{\mathbb{C}}$.*

Since $\Delta_n, n = 1, 2, 3$ are unitary (45), relation (9) follows directly from (41). The patterns have the same intersection angles since the cross-ratio equation (43) for z and z^* coincide.

8 z^c and $\log z$ patterns

To construct hexagonal circle patterns analogs of holomorphic functions z^c and $\log z$ we use the analytic point description presented in Section 7. Recall that we interpret the lattice $\mathcal{QL}$ of the intersection points and the centers of the circles as a subset $\mid k + \ell + m \mid \leq 1$ of $\mathbb{Z}^3$ and thus study the zero-curvature representation and Ψ-function on $\mathbb{Z}^3$.

A fundamental role in the presentation of this Section is played by a *non-autonomous constraint* for the solutions of the cross-ratio system (43)

$$\begin{aligned} bz_{k,\ell,m}^2 + cz_{k,\ell,m} + d = \; & 2(k - a_1)\frac{(z_{k+1,\ell,m} - z_{k,\ell,m})(z_{k,\ell,m} - z_{k-1,\ell,m})}{z_{k+1,\ell,m} - z_{k-1,\ell,m}} + \\ & 2(\ell - a_2)\frac{(z_{k,\ell+1,m} - z_{k,\ell,m})(z_{k,\ell,m} - z_{k,\ell-1,m})}{z_{k,\ell+1,m} - z_{k,\ell-1,m}} + \\ & 2(m - a_3)\frac{(z_{k,\ell,m+1} - z_{k,\ell,m})(z_{k,\ell,m} - z_{k,\ell,m-1})}{z_{k,\ell,m+1} - z_{k,\ell,m-1}}. \end{aligned} \tag{46}$$

where $b, c, d, a_1, a_2, a_3 \in \mathbb{C}$ are arbitrary. Note that the form of the constraint is invariant with respect to Möbius transformations.

Our presentation in this section consists of three parts. First, we explain the origin of the constraint (46) deriving it in the context of isomonodromic solutions of integrable systems. Then we show that it is compatible with the cross-ratio system (43), i.e. there exist non-trivial solutions of (43,46). And finally we specify parameters of these solutions to obtain circle pattern analogs of holomorphic mappings z^c and $\log z$.

We chose a different gauge of the transition matrices to simplify formulas. Let us orient the edges of $\mathbb{Z}^3$ in the directions of increasing $k + \ell + m$. We conjugate $L^{(n)}(\lambda)$ of positively oriented edges with the matrix $\mathrm{diag}(1, \lambda)$, and then multiply by $(1 - \Delta_n\lambda^2)^{1/2}$ in order to get rid of the normalization of the determinant. Writing then μ for λ^2, we end up with the matrices

$$\mathcal{L}^{(n)}(\mathfrak{e}, \mu) = \begin{pmatrix} 1 & z_{in} - z_{out} \\ \mu\dfrac{\Delta_n}{z_{in} - z_{out}} & 1 \end{pmatrix}. \tag{47}$$

associated to the edge $\mathfrak{e} = (p_{out}, p_{in}) \in E_n$ oriented in the direction of increasing of $k + \ell + m$. Each elementary quadrilateral of $\mathbb{Z}^3$ has two consecutive positively oriented pairs of edges $\mathfrak{e}_1, \mathfrak{e}_2$ and $\mathfrak{e}_3, \mathfrak{e}_4$. The zero-curvature condition turns into

$$\mathcal{L}^{(n_1)}(\mathfrak{e}_2)\mathcal{L}^{(n_2)}(\mathfrak{e}_1) = \mathcal{L}^{(n_2)}(\mathfrak{e}_4)\mathcal{L}^{(n_1)}(\mathfrak{e}_3).$$

Then the values of the wave function Φ in neighboring vertices are related by the formulas

$$\begin{cases} \Phi_{k+1,\ell,m}(\mu) = \mathcal{L}^{(1)}(\mathfrak{e}, \mu)\Phi_{k,\ell,m}(\mu), & \mathfrak{e} \in E_1, \\ \Phi_{k,\ell+1,m}(\mu) = \mathcal{L}^{(2)}(\mathfrak{e}, \mu)\Phi_{k,\ell,m}(\mu), & \mathfrak{e} \in E_2, \\ \Phi_{k,\ell,m+1}(\mu) = \mathcal{L}^{(3)}(\mathfrak{e}, \mu)\Phi_{k,\ell,m}(\mu), & \mathfrak{e} \in E_3. \end{cases} \tag{48}$$

We call a solution $z : \mathbb{Z}^3 \mapsto \mathbb{C}$ of the equations (43) *isomonodromic* (cf. [I]), if there exists a wave function $\Phi : \mathbb{Z}^3 \mapsto \mathrm{GL}(2, \mathbb{C})[\mu]$ satisfying (48) and some linear differential equation in μ:

$$\frac{d}{d\mu}\Phi_{k,\ell,m}(\mu) = \mathcal{A}_{k,\ell,m}(\mu)\Phi_{k,\ell,m}(\mu), \tag{49}$$

where $\mathcal{A}_{k,\ell,m}(\mu)$ are 2×2 matrices, meromorphic in μ, with the poles whose position and order do not depend on k, ℓ, m.

It turns out that the simplest non-trivial isomonodromic solutions satisfy the constraint (46). Indeed, since $\det \mathcal{L}^{(n)}(\mu)$ vanishes at $\mu = 1/\Delta_n$ the logarithmic derivative of $\Phi(\mu)$ must be singular in these points. We assume that these singularities are as simple as possible, i.e. simple poles.

Theorem 8.1 *Let $z : \mathbb{Z}^3 \to \mathbb{C}$ be an isomonodromic solution to (43) with the matrix $\mathcal{A}_{k,\ell,m}$ in (49) of the form*

$$\mathcal{A}_{k,\ell,m}(\mu) = \frac{C_{k,\ell,m}}{\mu} + \sum_{n=1}^{3} \frac{B^{(n)}_{k,\ell,m}}{\mu - \dfrac{1}{\Delta_n}} \tag{50}$$

with μ-independent matrices $C_{k,\ell,m}, B^{(n)}_{k,\ell,m}$ and normalized[6] *trace* $\mathrm{tr}\mathcal{A}_{0,0,0}(\mu) = 0$. *Then these matrices have the following form:*

$$\begin{aligned}
C_{k,\ell,m} &= \frac{1}{2}\begin{pmatrix} -bz_{k,\ell,m} - c/2 & bz^2_{k,\ell,m} + cz_{k,\ell,m} + d \\ b & bz_{k,\ell,m} + c/2 \end{pmatrix} \\
B^{(1)}_{k,\ell,m} &= \frac{k - a_1}{z_{k+1,\ell,m} - z_{k-1,\ell,m}}\begin{pmatrix} z_{k+1,\ell,m} - z_{k,\ell,m} & (z_{k+1,\ell,m} - z_{k,\ell,m})(z_{k,\ell,m} - z_{k-1,\ell,m}) \\ 1 & z_{k,\ell,m} - z_{k-1,\ell,m} \end{pmatrix} + \frac{a_1}{2}I \\
B^{(2)}_{k,\ell,m} &= \frac{\ell - a_2}{z_{k,\ell+1,m} - z_{k,\ell-1,m}}\begin{pmatrix} z_{k,\ell+1,m} - z_{k,\ell,m} & (z_{k,\ell+1,m} - z_{k,\ell,m})(z_{k,\ell,m} - z_{k,\ell-1,m}) \\ 1 & z_{k,\ell,m} - z_{k,\ell-1,m} \end{pmatrix} + \frac{a_2}{2}I \\
B^{(3)}_{k,\ell,m} &= \frac{m - a_3}{z_{k,\ell,m+1} - z_{k,\ell,m-1}}\begin{pmatrix} z_{k,\ell,m+1} - z_{k,\ell,m} & (z_{k,\ell,m+1} - z_{k,\ell,m})(z_{k,\ell,m} - z_{k,\ell,m-1}) \\ 1 & z_{k,\ell,m} - z_{k,\ell,m-1} \end{pmatrix} + \frac{a_3}{2}I
\end{aligned}$$

and $z_{k,\ell,m}$ satisfies (46).

Conversely, any solution $z : \mathbb{Z}^3 \to \mathbb{C}$ to the system (43,46) is isomonodromic with $\mathcal{A}_{k,\ell,m}(\mu)$ given by the formulas above.

The proofs of this theorem as well as of the next one are presented in Appendix C. We prove Theorem 8.1 by computing the compatibility conditions of (48) and (49) and Theorem 8.2 by showing the solvability of a reasonably posed Cauchy problem.

[6] As explained in the proof of the theorem this normalization can be achieved without loss of generality.

Theorem 8.2 *For arbitrary $b, c, d, a_1, a_2, a_3 \in \mathbb{C}$ the constraint (46) is compatible with the cross-ratio equations (43), i.e. there are non-trivial solutions of (43,46).*

Further we deal with the special case of (46) where

$$b = a_1 = a_2 = a_3 = 0.$$

If $c \neq 0$ one can always assume $d = 0$ in (46) by shifting $z \to z - \frac{d}{c}$:

$$\begin{aligned} cz_{k,\ell,m} = \quad & 2k\frac{(z_{k+1,\ell,m} - z_{k,\ell,m})(z_{k,\ell,m} - z_{k-1,\ell,m})}{z_{k+1,\ell,m} - z_{k-1,\ell,m}} + \\ & 2\ell\frac{(z_{k,\ell+1,m} - z_{k,\ell,m})(z_{k,\ell,m} - z_{k,\ell-1,m})}{z_{k,\ell+1,m} - z_{k,\ell-1,m}} + \\ & 2m\frac{(z_{k,\ell,m+1} - z_{k,\ell,m})(z_{k,\ell,m} - z_{k,\ell,m-1})}{z_{k,\ell,m+1} - z_{k,\ell,m-1}}. \end{aligned} \tag{51}$$

To define a circle pattern analogue of z^c it is natural to restrict the mapping to the following two subsets of $\mathbb{Z}^3$:

$$Q = \{(k,\ell,m) \in \mathbb{Z}^3 \mid k \geq 0, \ell \geq 0, m \leq 0\}, \ H = \{(k,\ell,m) \in \mathbb{Z}^3 \mid m \leq 0\} \subset \mathbb{Z}^3.$$

Corollary 8.3 *The solution $z : Q \to \mathbb{C}$ of the system (43) satisfying the constraint (51) is uniquely determined by its values*

$$z_{1,0,0}, \ z_{0,1,0}, \ z_{0,0,-1}. \tag{52}$$

Proof. Using the constraint one determines the values along the coordinate lines $z_{n,0,0}, z_{0,n,0}, z_{0,0,-n}$ $\forall n \in \mathbb{N}$. Then all other $z_{k,\ell,m}, \ k, \ell, -m \in \mathbb{N}$ in consecutive order are determined through the cross-ratios (43). Computations using different cross-ratios give the same result due to the following lemma about the eighth point.

Lemma 8.4 *Let $H(p_1 + 1/2, p_2 + 1/2, p_3 - 1/2), p = (p_1, p_2, p_3) \in \mathbb{Z}^3$ be the elementary hexahedron (lying in $\mathbb{C}$) with the vertices $z_{p+(k,\ell,-m)}, \ k, \ell, m \in \{0, 1\}$ and let the cross-ratios of the opposite cites of $H(p_1 + 1/2, p_2 + 1/2, p_3 - 1/2)$ be equal*

$$\begin{aligned} q(z_p, z_{p+(0,0,-1)}, z_{p+(0,1,-1)}, z_{p+(0,1,0)}) = q(z_{p+(1,0,0)}, z_{p+(1,0,-1)}, z_{p+(1,1,-1)}, z_{p+(1,1,0)}) &=: q_1 \\ q(z_p, z_{p+(1,0,0)}, z_{p+(1,0,-1)}, z_{p+(0,0,-1)}) = q(z_{p+(0,1,0)}, z_{p+(1,1,0)}, z_{p+(1,1,-1)}, z_{p+(0,1,-1)}) &=: q_2 \\ q(z_p, z_{p+(0,1,0)}, z_{p+(1,1,0)}, z_{p+(1,0,0)}) = q(z_{p+(0,0,-1)}, z_{p+(0,1,-1)}, z_{p+(1,1,-1)}, z_{p+(1,0,-1)}) &=: q_3 \end{aligned}$$

with $q_1 q_2 q_3 = 1$. Then all the vertices of the hexahedron are uniquely determined through four given z_p, $z_{p+(1,0,0)}$, $z_{p+(0,1,0)}$, $z_{p+(0,0,-1)}$.

The relation of solutions of (43,51) to circle patterns is established in the following

Theorem 8.5 *The solution $z : Q \to \mathbb{C}$ of the system (43,51) with the initial data*

$$z_{1,0,0} = 1, \ z_{0,1,0} = e^{i\beta}, \ z_{0,0,-1} = e^{i\gamma} \tag{53}$$

and unitary cross-ratios $q_n = e^{-2i\alpha_n}$ determines a circle pattern. For all $(k,\ell,m) \in Q$ with even $k + \ell + m$ the points $z_{k\pm1,\ell,m}, z_{k,\ell\pm1,m}, z_{k,\ell,m\pm1}$ lie on a circle with the center $z_{k,\ell,m}$.

Proof. We say that a quadrilateral is of the *kite form* if it has two pairs of equal adjacent edges, (and that a hexahedron is of the kite form if all its cites are of the kite form). These quadrilaterals have unitary cross-ratios with the argument equal to the angle between the edges (see Lemma 3.3). Our proof of the theorem is based on two simple observations:

(i) If a quadrilateral has a pair of adjacent edges of equal length and unitary cross-ratio then it is of the kite form,

(ii) If the cross-ratio of a quadrilateral is equal to $q(z_1, z_2, z_3, z_4) = e^{-2i\alpha}$ where α is the angle between the edges (z_1, z_2) and (z_2, z_3) (as in Figure 4) then the quadrilateral is of the kite form.

Applying these observation to the elementary hexahedron in Lemma 8.4 one obtains

Lemma 8.6 *Let three adjacent edges of the elementary hexahedron in Lemma 8.4 have equal length and the cross-ratios of the cites be unitary $q_n = e^{-2i\alpha_n}$. Then the hexahedron is of the kite form.*

The constraint (51) with $\ell = m = 0$ implies by induction

$$| z_{2n+1,0,0} - z_{2n,0,0} |=| z_{2n,0,0} - z_{2n-1,0,0} |,$$

and all the points $z_{n,0,0}$ lie on the real axis. The same equidistance and straight line properties hold true for ℓ and m axes. Also by induction one shows that all elementary hexahedra of the mapping $z : Q \to \mathbb{C}$ are of the kite form. Indeed, due to Lemma 8.6 the initial conditions (53) imply that the hexahedron $H(1/2, 1/2, -1/2)$ is of the kite form. Since the axis vertices lie on the straight lines, we get, for example, that the angle between the edges $(z_{1,1,0}, z_{1,0,0})$ and $(z_{1,0,0}, z_{2,0,0})$ is α_3. Now applying the observation (ii) to the cites of $H(3/2, 1/2, -1/2)$ we see that

$$| z_{2,0,0} - z_{1,0,0} |=| z_{2,0,0} - z_{2,1,0} |=| z_{2,0,0} - z_{2,0,-1} | .$$

Lemma 8.6 implies $H(3/2, 1/2, -1/2)$ is of the kite form. Proceeding further this way and controlling the lengths of the edges meeting at $z_{k,\ell,m}$ with even $k + \ell + m$ and the angles at $z_{k,\ell,m}$ with odd $k + \ell + m$ one proves the statement for all hexahedra.

Denote the vertices of the hexagonal grid by (see Figure 10)

$$\begin{aligned} Q_H &= \{(k, \ell, m) \in Q : | k + \ell + m |\le 1\}, \\ H_H &= \{(k, \ell, m) \in H : | k + \ell + m |\le 1\}. \end{aligned}$$

The half-plane H_H consists of the sector Q_H and its two images under the rotations with the angles $\pm\pi/3$.

Corollary 8.7 *The mapping $z : H_H \to \mathbb{C}$ given by (43,51) with unitary cross-ratios $q_n = e^{-2i\alpha_n}$ and unitary initial data*

$$| z_{\pm 1,0,0} |=| z_{0,\pm 1,0} |=| z_{0,0,0} |= 1$$

is a hexagonal circle pattern with constant angles α_n extended by the Euclidean centers of the circles. The centers of the corresponding circles are the images of the points with $k + \ell + m = 0$.

The circle patterns determined by most of the initial data $\beta, \gamma \in \mathbb{R}$ are quite irregular. But for a special choice of these parameters one obtains a regular circle pattern which we call the hexagonal circle pattern z^c motivated by the asymptotic of the constraint (51) as $k, \ell, m \to \infty$.

Definition 8.8 *The hexagonal circle pattern z^c, $0 < c < 2$ (extended by the Euclidean centers of the circles) is the solution $z : H_H \to \mathbb{C}$ of (43,51) with*

$$\frac{\Delta_{n+1}}{\Delta_{n+2}} = e^{2i\alpha_n}, \qquad n \pmod 3,$$

and the initial conditions

$$z_{1,0,0} = 1,\ z_{0,1,0} = e^{ic(\alpha_1+\alpha_2)},\ z_{0,0,-1} = e^{ic\alpha_2},\ z_{-1,0,0} = e^{ic\pi},\ z_{0,-1,0} = e^{-ic\alpha_3}. \tag{54}$$

Motivated by our computer experiments (see in particular Figure 11) and the corresponding result for Schramm's circle patterns with the combinatorics of the square grid [AB, A] we conjecture that the hexagonal z^c is embedded, i.e. the interiors of all elementary quadrilaterals $(z(\mathfrak{z}_1), z(\mathfrak{z}_2), z(\mathfrak{z}_3), z(\mathfrak{z}_4)), |\, \mathfrak{z}_{i+1} - \mathfrak{z}_i \,| = 1, i \pmod 4$ are disjoint.

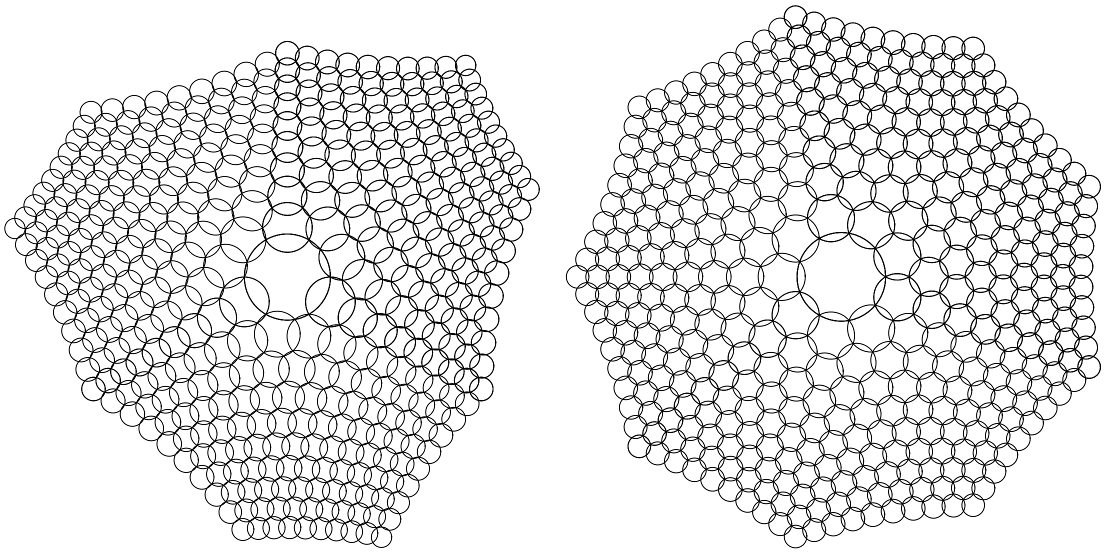

Figure 11: Non-isotropic and isotropic circle patterns $z^{2/3}$.

In the isotropic case when all the intersection angles are the same

$$\alpha_1 = \alpha_2 = \alpha_3 = \frac{\pi}{3}.$$

one has

$$\Delta_1 = \omega\Delta_2 = \omega^2\Delta_3, \tag{55}$$

and due to the symmetry it is enough to restrict the mapping to Q_H. The initial conditions (54) become

$$z_{1,0,0} = 1,\ z_{0,1,0} = e^{2\pi ic/3},\ z_{0,0,-1} = e^{\pi ic/3}. \tag{56}$$

Like for the smooth z^c the images of the coordinate axes $\arg \mathfrak{z} = \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ are the axes with the arguments $\frac{\pi c}{6}, \frac{\pi c}{3}$ and $\frac{\pi c}{2}$ respectively:

$$\arg z_{n,0,-n} = \frac{\pi c}{6},\ \arg z_{0,n,-n} = \frac{\pi c}{2},\ \arg z_{n,n,m} = \frac{\pi c}{3},\ m = -2n; -2n \pm 1, \qquad \forall n \in \mathbb{N}.$$

For $c = \frac{6}{q}$, $q \in \mathbb{N}$ the circle pattern $z^c(Q_H)$ together with its additional $q-1$ copies rotated by the angles $2\pi n/q$, $n = 1, \dots, q-1$ comprise a circle pattern covering the whole complex plane. This pattern is hexagonal at all circles beside the central one which has q neighboring circles. The circle pattern $z^{6/5}$ is shown in Figure 12.

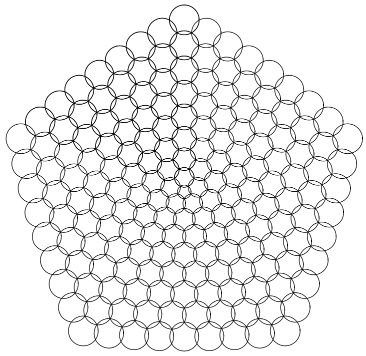

Figure 12: Circle pattern $z^{6/5}$.

Definition 8.8 is given for $0 < c < 2$. For $c = 2$ the radii of the circles intersecting the central circle of the pattern become infinite. To get finite radii the central circle should degenerate to a point

$$z_{0,0,0} = z_{1,0,0} = z_{0,0,-1} = z_{0,1,0} = 0. \tag{57}$$

The mapping $z : Q \to \mathbb{C}$ is then uniquely determined by $z_{2,0,0}, z_{0,2,0}, z_{0,0,-2}, z_{1,1,0}, z_{1,0,-1}, z_{0,1,-1}$, the values of which can be derived taking the limit $c \to 2-0$. Indeed, consider the quader $m = 0$ with the cross-ratios of all elementary quadrilaterals equal to $q = e^{-2i\alpha}$. Choosing $z_{1,0,0} = \epsilon$, $z_{0,1,0} = \epsilon e^{ic\alpha}$ as above we get $z_{2,0,0} = \frac{2\epsilon}{2-c}$. Normalization

$$c = 2 - 2\epsilon$$

yields

$$z_{2,0,0} = 1, \quad z_{0,2,0} = e^{ic\alpha}. \tag{58}$$

Since the angles of the triangle with the vertices $z_{0,0,0}, z_{1,0,0}, z_{1,1,0}$ are $\frac{c\alpha}{2}, \pi - \alpha, \alpha\epsilon$ respectively, one obtains

$$z_{1,1,0} = \epsilon + R_\epsilon e^{i\alpha}, \qquad R_\epsilon = \epsilon \frac{\sin(c\alpha/2)}{\sin(\alpha\epsilon)}.$$

In the limit $\epsilon \to 0$ we have

$$z_{1,1,0} = \frac{\sin\alpha}{\alpha} e^{i\alpha}. \tag{59}$$

Observe that in our previous notations $\alpha = \alpha_1 + \alpha_2 = \pi - \alpha_3$. Finally, the same computations for the sectors $\{\ell = 0, k \geq 0, m \leq 0\}$ and $\{k = 0, \ell \geq 0, m \leq 0\}$ provide us with the following data:

$$\begin{gathered} z_{2,0,0} = 1,\ z_{0,0,-2} = e^{2i\alpha_2},\ z_{0,2,0} = e^{2i(\alpha_1+\alpha_2)}, \\ z_{1,0,-1} = \frac{\sin\alpha_2}{\alpha_2} e^{i\alpha_2},\ z_{0,1,-1} = \frac{\sin\alpha_1}{\alpha_1} e^{i(\alpha_1+2\alpha_2)},\ z_{1,1,0} = \frac{\sin(\alpha_1+\alpha_2)}{\alpha_1+\alpha_2} e^{i(\alpha_1+\alpha_2)}. \end{gathered} \tag{60}$$

In the same way the initial data for other two sectors of H_H are specified. The hexagonal circle pattern with these initial data and $c = 2$ is an analog of the holomorphic mapping z^2 .

The duality transformations preserves the class of circle patterns we defined.

Theorem 8.9
$$(z^c)^* = z^{c^*}, \qquad c^* = 2 - c,$$
where $(z^c)^*$ *is the hexagonal circle pattern dual to the circle pattern* z^c *and normalized to vanish at the origin* $(z^c)^*(\mathfrak{z} = 0) = 0$.

Proof. Let us consider the mapping on the whole Q. It is easy to see that on the axes the duality transformation (41) preserves the form of the constraint with c being replaced by $c^* = 2 - c$. Then the constraint with c^* holds for all points of Q due to the compatibility in Theorem 8.2. Restriction to $\mid k + \ell + m \mid \leq 1$ implies the claim.

The smooth limit of the duality transformation of the hexagonal patterns is the following transformation $f \mapsto f^*$ of holomorphic functions:

$$(f^*(z))' = \frac{1}{f'(z)}.$$

The dual of $f(z) = z^2$ is, up to constant, $f^*(z) = \log z$. Motivated by this observation, we define the hexagonal circle pattern $\log z$ as the dual to the circle pattern z^2:

$$\log z := (z^2)^*.$$

The corresponding constraint (46)

$$2k\frac{(z_{k+1,\ell,m} - z_{k,\ell,m})(z_{k,\ell,m} - z_{k-1,\ell,m})}{z_{k+1,\ell,m} - z_{k-1,\ell,m}} + 2\ell\frac{(z_{k,\ell+1,m} - z_{k,\ell,m})(z_{k,\ell,m} - z_{k,\ell-1,m})}{z_{k,\ell+1,m} - z_{k,\ell-1,m}} +$$
$$2m\frac{(z_{k,\ell,m+1} - z_{k,\ell,m})(z_{k,\ell,m} - z_{k,\ell,m-1})}{z_{k,\ell,m+1} - z_{k,\ell,m-1}} = 1. \tag{61}$$

can be derived as a limit $c \to +0$ (see [AB] for this limit in the square grid case). The initial data for $\log z$ are dual to the ones of z^2. In our model case of the quader $m = 0$ factorizing $q = \Delta_1/\Delta_2$ with $\Delta_1 = 1/2, \Delta_2 = e^{2i\alpha}/2$ one arrives at the following data dual to (57,58,59):

$$z_{0,0,0} = \infty,\ z_{1,0,0} = 0,\ z_{2,0,0} = \frac{1}{2},$$
$$z_{0,1,0} = i\alpha,\ z_{0,2,0} = \frac{1}{2} + i\alpha,\ z_{1,1,0} = \frac{\alpha}{2\sin\alpha}e^{i\alpha}. \tag{62}$$

In the isotropic case the circle patterns are more symmetric and can be described as mappings of Q_H.

Definition 8.10 *The isotropic hexagonal circle patterns* z^2 *and* $\log z$ *are the mappings* $Q_H \to \mathbb{C}$ *with the cross-ratios of all elementary quadrilaterals equal*[7] *to* $e^{-2\pi i/3}$. *The mapping* z^2 *is determined by the constraint (51) with* $c = 2$ *and the initial data*

$$z_{0,0,0} = \infty,\ z_{1,0,0} = 0,\ z_{0,0,-1} = 0,\ z_{0,1,0} = 0,$$
$$z_{2,0,0} = 1,\ z_{0,0,-2} = e^{2\pi i/3},\ z_{0,2,0} = e^{4\pi i/3},$$
$$z_{1,0,-1} = \frac{3\sqrt{3}}{2\pi}e^{\pi i/3},\ z_{0,1,-1} = -\frac{3\sqrt{3}}{2\pi},\ z_{1,1,0} = \frac{3\sqrt{3}}{4\pi}e^{2\pi i/3}.$$

[7] The first point in the cross-ratio is a circle center and the quadrilaterals are positively oriented.

For $\log z$ *the corresponding constraint is (61) and the initial data are*

$$z_{0,0,0} = \infty, \; z_{1,0,0} = 0, \; z_{0,0,-1} = \frac{\pi}{3}i, \; z_{0,1,0} = \frac{2\pi}{3}i,$$
$$z_{2,0,0} = \frac{1}{2}, \; z_{0,0,-2} = \frac{1}{2} + \frac{\pi}{3}i, \; z_{0,2,0} = \frac{1}{2} + \frac{2\pi}{3}i,$$
$$z_{1,0,-1} = \frac{\pi}{6}(\frac{1}{\sqrt{3}} + i), \; z_{0,1,-1} = \frac{\pi}{6}(\frac{1}{\sqrt{3}} + 3i), \; z_{1,1,0} = \frac{\pi}{3}(-\frac{1}{\sqrt{3}} + i).$$

The isotropic hexagonal circle patterns z^2 and $\log z$ are shown in Figure 13.

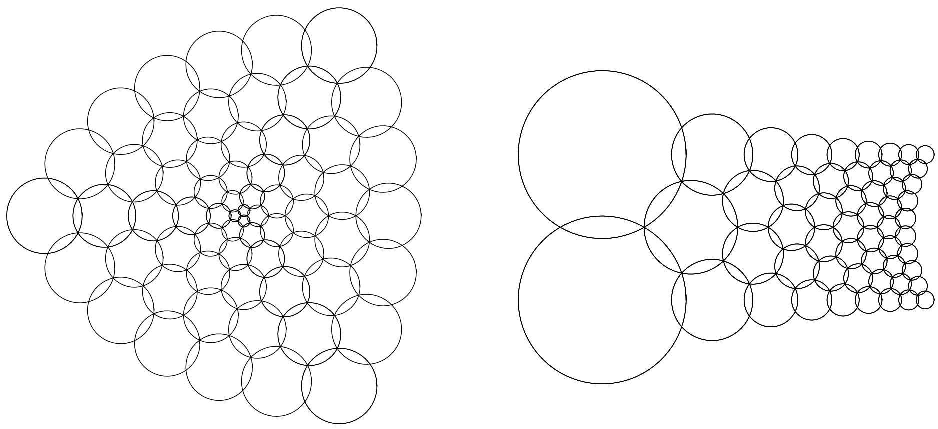

Figure 13: Isotropic circle patterns z^2 and $\log z$.

Starting with $z^c, \; c \in (0, 2]$ one can easily define z^c for arbitrary c by applying some simple transformations of hexagonal circle patterns. The construction here is the same as for Schramm's patterns (see Section 6 of [AB] for details). Applying the inversion of the complex plane $z \mapsto 1/z$ to the circle pattern $z^c, \; c \in (0, 2]$ one obtains a circle pattern satisfying the constraint with $-c$. It is natural to call it the hexagonal circle pattern $z^{-c}, \; c \in (0, 2]$. Constructing the dual circle pattern we arrive at a natural definition of z^{2+c}. Intertwining the inversion and the dualization described above, one constructs circle patterns z^c for any c.

In particular, inverting and then dualizing $z = k + \ell\omega + m\omega^2$ with $\Delta_1 = -3, \Delta_2 = -3\omega^2, \Delta_3 = -3\omega$ we obtain the circle pattern corresponding to z^3:

$$z_{k,\ell,m} = (k + \ell\omega + m\omega^2)^3 - (k + \ell + m).$$

Note that this is the central extension corresponding to $P_\infty = 0$. The points with even $k + \ell + m$ can be replaced by the Euclidean centers of the circles. As it is shown in Section 6 of [AB] the replacement $P_\infty = 0$ by $P_\infty = \infty$ preserves the constraint (51).

9 Concluding remarks

We restricted ourself to the analysis of geometric and algebraic properties of hexagonal circle patterns leaving the approximation problem beyond the scope of this paper. The convergence of circle patterns with the combinatorics of the square grid to the Riemann mapping is proven by

Schramm in [S]. We expect that his result can be extended to the hexagonal circle patterns defined in this paper.

The entire circle pattern erf$z$ also found in [S] remains to be rather mysterious. We were unable to find its analogue in the hexagonal case, and thus, have no counter-examples to the Doyle conjecture for hexagonal circle patterns with constant angles. It seems that Schramm's erf$z$ is a feature of the square grid combinatorics.

The construction of the hexagonal circle pattern analogs of z^c and $\log z$ in Section 8 was based on the extension of the corresponding integrable system to the lattice $\mathbb{Z}^3$. Theorem 8.5 claims that in this way one obtains a circle pattern labelled by three independent indices k, ℓ, m. Fixing one of the indices, say $m = m_0$, one obtains a Schramm's circle pattern with the combinatorics of the square grid. In the same way the restriction of this "three-dimensional" pattern to the sublattice $\mid k + \ell + m - n_0 \mid \leq 1$ with some fixed $n_0 \in \mathbb{Z}$ yields a hexagonal circle pattern. In Section 8 we have defined circle patterns z^c on the sublattices $m_0 = 0$ and $n_0 = 0$. In the same way the sublattices $m_0 \neq 0$ and $n_0 \neq 0$ can be interpreted as the circle pattern analogs of the analytic function $(z + a)^c$, $a \neq 0$ with the square and hexagonal grids combinatorics respectively.

It is unknown, whether the theory of integrable systems can be applied to hexagonal circle packings. As we mentioned already in the Introduction, the underdevelopment of the theory of integrable systems on the lattices different from $\mathbb{Z}^n$ may be a reason for this. We hope that integrable systems on the hexagonal and related to it lattices appeared in this paper and in [BHS] as well as the the experience of integration of hexagonal circle patterns in general will be helpful for the progress in investigation of hexagonal circle packings.

A Appendix. A Lax representation for the conformal description.

We will now give a Lax representation for equations (18). For $\mathfrak{z} \in V(\mathcal{HL})$ let $[\mathfrak{z}_i, \mathfrak{z}] \in E_i^H$ and $m_{\mathfrak{z}}$ be the Möbius transformation, that sends $(\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3)$ to $(0, 1, \infty)$. For $\mathfrak{e} = [\mathfrak{z}, \tilde{\mathfrak{z}}] \in E_i^H$ set $L_i = m_{\tilde{\mathfrak{z}}} \circ m_{\mathfrak{z}}^{-1}$. The L_i depend on $S^{(1)}$ and $T_{\mathfrak{e}}$ only. They are given in equation (63). Note that we do not need to orient the edges since $L_i L_i^{-1}$ is the identity if we normalize $\det L_i = 1$. The claim is that the closing condition when multiplying the L_i around one hexagon is equivalent to equation (18):

Theorem A.1 *Attach to each edge of E_i a matrix $L_i(T, S)$ of the form:*

$$\begin{aligned} L_1(T,S) &= \begin{pmatrix} -1 & \frac{S-1}{S} \\ \frac{T+S-1}{S-1} & 1 \end{pmatrix} \\ L_2(T,S) &= \begin{pmatrix} TS & 1-TS \\ S\frac{1+T(S-1)}{S-1} & -TS \end{pmatrix} \\ L_3(T,S) &= \begin{pmatrix} 1-S & -1+T(\frac{1}{S}-1)+S \\ -S & S-1 \end{pmatrix} \end{aligned} \tag{63}$$

then the zero curvature condition for each hexagon in $F(\mathcal{HL})$

$$L_1(T_4, S) L_2(T_5, S) L_3(T_6, S) = \rho L_3(T_3, S) L_2(T_2, S) L_1(T_1, S) \tag{64}$$

for all S is equivalent to (18).

Proof. Straight forward calculations. $\square$

Equations (63) and (64) are a Lax representation for equation (18) with the spectral parameter S.

B Appendix. Discrete equations of Toda type and cross-ratios

In the following we will discuss briefly the connection between discrete equations of Toda type and cross-ratio equations for the square grid and the dual Kagome lattice. This interrelationship holds in a more general setting, namely for discrete Toda systems on graphs. This situation will be considered in a subsequent publication.

We will start with the square grid. Let us decompose the lattice $\mathbb{Z}^2$ into two sublattices

$$\mathbb{Z}^2_k = \{(m,n) \in \mathbb{Z}^2, (m+n \mod 2) = k\}, \qquad k = 0,1.$$

Theorem B.1 *Let $z : \mathbb{Z}^2 \to \hat{\mathbb{C}}$ be a solution to the cross-ratio equation*

$$q(z_{m,n}, z_{m+1,n}, z_{m+1,n+1}, z_{m,n+1}) = q. \tag{65}$$

Then restricted to the sublattices $\mathbb{Z}^2_0$ or $\mathbb{Z}^2_1$ it satisfies the discrete equation of Toda type (16)

$$\frac{1}{z_{m,n} - z_{m+1,n+1}} + \frac{1}{z_{m,n} - z_{m-1,n-1}} = \frac{1}{z_{m,n} - z_{m+1,n-1}} + \frac{1}{z_{m,n} - z_{m-1,n+1}}. \tag{66}$$

Conversely, given a solution $z : \mathbb{Z}^2_0 \to \hat{\mathbb{C}}$ of equation (66) and arbitrary $q, w \in \hat{\mathbb{C}}$ there exists unique extension of z to the whole lattice $z : \mathbb{Z}^2 \to \hat{\mathbb{C}}$ with $z_{1,0} = w$ satisfying the cross-ratio condition (65). Moreover, restricted to the other sublattice $\mathbb{Z}^2_1$ the so defined mapping $z : \mathbb{Z}^2_1 \to \hat{\mathbb{C}}$ satisfies the same discrete equation of Toda type (66).

Proof. For any four complex numbers the cross-ratio equation can be written in a simple fraction form:

$$q(u_1, u_2, u_3, u_4) = q \Leftrightarrow \frac{1}{u_2 - u_1} - \frac{q}{u_2 - u_3} + \frac{q-1}{u_2 - u_4} = 0. \tag{67}$$

To distinguish between the two sublattices, we will denote the points belonging to $\mathbb{Z}^2_0$ with z_i while

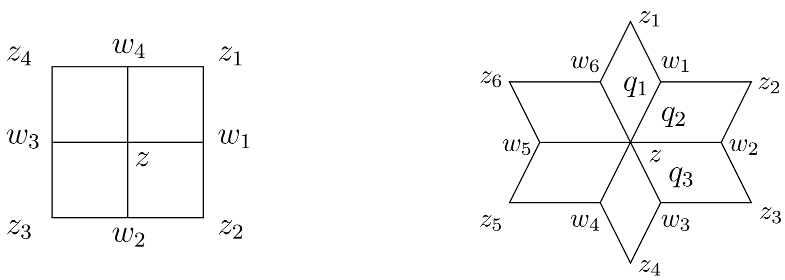

Figure 14: Nine points of $\mathbb{Z}^2$ and 13 points of $V(\mathcal{TL})$.

w_i Are the points associated with $\mathbb{Z}^2_1$. Now (67) gives four equations for a point z and its eight neighbors as shown in Fig 14:

$$\begin{aligned}
(q-1)\frac{1}{z - z_1} &= q\frac{1}{z - w_4} - \frac{1}{z - w_1}, \\
(q^{-1}-1)\frac{1}{z - z_2} &= q^{-1}\frac{1}{z - w_1} - \frac{1}{z - w_2}, \\
(q-1)\frac{1}{z - z_3} &= q\frac{1}{z - w_2} - \frac{1}{z - w_3}, \\
(q^{-1}-1)\frac{1}{z - z_4} &= q^{-1}\frac{1}{z - w_3} - \frac{1}{z - w_4}.
\end{aligned} \tag{68}$$

Multiplying the second and fourth equation with q and taking the sum over all four equations (68) yields equation (66). Conversely given w_1, the first three equations (68) determine w_2, w_3, and w_4 uniquely. The so determined w_3, and w_4 satify the fourth equation (68) (for any choice of w_1) if and only if (66) holds. This proves the second part of the theorem. Obviously interchanging w and z implies the claim for the other sublattice. □

Now we pass to the dual Kagome lattice shown in Fig. 10, where a similar relation holds for the discrete equation of Toda type on the hexagonal lattice. However, the symmetry between the sublattices is lost in this case.

Here we decompose $V(\mathcal{TL})$ into $V(\mathcal{HL})$ and $V(\mathcal{TL}) \setminus V(\mathcal{HL}) \cong F(\mathcal{HL})$.

Theorem B.2 *Given $q_1, q_2, q_3 \in \mathbb{C}$ with $q_1q_2q_3 = 1$ and a solution $z : V(\mathcal{TL}) \to \hat{\mathbb{C}}$ of the cross-ratio equations*

$$\begin{aligned} q(z_{k,\ell,m-1}, z_{k,\ell,m}, z_{k,\ell+1,m}, z_{k,\ell+1,m-1}) &= q_1 \\ q(z_{k+1,\ell,m}, z_{k,\ell,m}, z_{k,\ell,m-1}, z_{k+1,\ell,m-1}) &= q_2 \\ q(z_{k,\ell-1,m}, z_{k,\ell,m}, z_{k+1,\ell,m}, z_{k+1,\ell-1,m}) &= q_3 \end{aligned} \tag{69}$$

then

1. *restricted to $F(\mathcal{HL})$ the solution z satisfies the discrete equation of Toda type (13) on the hexagonal lattice*
$$\sum_{k=1}^{3} A_k \left(\frac{1}{z - z_k} + \frac{1}{z - z_{k+3}} \right) = 0 \tag{70}$$
2. *restricted to $V(\mathcal{HL})$ the solution z satisfies*
$$\sum_{k=1}^{3} A_k \frac{1}{z - z_k} = 0 \tag{71}$$

where $A_i = \Delta_{i+2} - \Delta_{i+1}$ with Δ_i defined through $q_i = \frac{\Delta_{i+2}}{\Delta_{i+1}}$.

Conversely given $q_1, q_2, q_3 \in \mathbb{C}$ with $q_1q_2q_3 = 1$, a solution z to (70) on $F(\mathcal{HL})$ and $z_{0,0,0}$, there is a unique extension $z : V(\mathcal{TL}) \to \hat{\mathbb{C}}$ satisfying (69).

Given $q_1, q_2, q_3 \in \mathbb{C}$ with $q_1q_2q_3 = 1$, a solution z to (71) on $V(\mathcal{HL})$ and $z_{1,0,0}$, there is a unique extension $z : V(\mathcal{TL}) \to \hat{\mathbb{C}}$ satisfying (69).

Proof. First (67) shows immediately that (71) is equivalent to having constant S as described in Section 5.

Again we will distinguish the sublattices notationally by denoting points associated with elements of $V(\mathcal{HL})$ with w_i and points from $F(\mathcal{HL})$ with z_i. If the cross-ratios and neighboring points of a $z \in F(\mathcal{HL})$ are labelled as shown in Fig. 14, equation (67) gives 6 equations:

$$\begin{aligned} (q_1 - 1)\frac{1}{z - z_1} &= q_1\frac{1}{z - w_6} - \frac{1}{z - w_1} \\ (q_1 - 1)\frac{1}{z - z_4} &= q_1\frac{1}{z - w_3} - \frac{1}{z - w_4} \\ (q_2 - 1)\frac{1}{z - z_2} &= q_2\frac{1}{z - w_1} - \frac{1}{z - w_2} \\ (q_2 - 1)\frac{1}{z - z_5} &= q_2\frac{1}{z - w_4} - \frac{1}{z - w_5} \\ (q_3 - 1)\frac{1}{z - z_3} &= q_3\frac{1}{z - w_2} - \frac{1}{z - w_3} \\ (q_3 - 1)\frac{1}{z - z_6} &= q_3\frac{1}{z - w_5} - \frac{1}{z - w_6} \end{aligned} \tag{72}$$

To proof the first statement we take a linear combination of the equations (72). Namely a the first two plus b times the second two plus c times the third two. It is easy to see that there is a choice for a, b, and c that makes the right hand side vanish if and only if $q_1 q_2 q_3 = 1$. If $q_1 = \frac{\Delta_3}{\Delta_2}, q_2 = \frac{\Delta_1}{\Delta_3}$, and $q_3 = \frac{\Delta_2}{\Delta_1}$ choose $a = \Delta_2, b = \Delta_3$, and $c = \Delta_1$. The remaining equation is (70).

To proof the third statement we note that given w_1, we can compute w_2 through w_6 from equations (72) 2. to 6. but the "closing condition" equation (72) 1. is then equivalent to (70).

The proofs of the second and fourth statements are literally the same if we choose $z_i = z_{i+3}$ and $w_i = w_{i+3}$. □

C Appendix. Proofs of Theorems of Section 8

Proof of Theorem 8.1

Let $\Phi_{k,\ell,m}(\mu)$ be a solution to (43,46) with some μ-independent matrices $C_{k,\ell,m}, B^{(n)}_{k,\ell,m}$. The determinant identity

$$\det \Phi_{k,\ell,m}(\mu) = (1-\mu\Delta_1)^k (1-\mu\Delta_2)^\ell (1-\mu\Delta_3)^m \det \Phi_{0,0,0}(\mu)$$

implies

$$\operatorname{tr} \mathcal{A}_{k,\ell,m}(\mu) = \frac{k}{\mu - 1/\Delta_1} + \frac{\ell}{\mu - 1/\Delta_2} + \frac{m}{\mu - 1/\Delta_3} + a(\mu)$$

with $a(\mu)$ independent of k, ℓ, m. Without loss of generality one can assume $a(\mu) = 0$, i.e.

$$\operatorname{tr} B^{(1)}_{k,\ell,m} = k, \operatorname{tr} B^{(2)}_{k,\ell,m} = \ell, \operatorname{tr} B^{(3)}_{k,\ell,m} = m.$$

This can be achieved by the change $\Phi \mapsto \exp(-1/2 \int a(\mu) d\mu)\Phi$.

The compatibility conditions of (48) and (49) read

$$\begin{aligned} \frac{d}{d\mu}\mathcal{L}^{(1)} &= \mathcal{A}_{k+1,\ell,m}\mathcal{L}^{(1)} - \mathcal{L}^{(1)}\mathcal{A}_{k,\ell,m} \\ \frac{d}{d\mu}\mathcal{L}^{(2)} &= \mathcal{A}_{k,\ell+1,m}\mathcal{L}^{(2)} - \mathcal{L}^{(2)}\mathcal{A}_{k,\ell,m} \\ \frac{d}{d\mu}\mathcal{L}^{(3)} &= \mathcal{A}_{k,\ell,m+1}\mathcal{L}^{(3)} - \mathcal{L}^{(3)}\mathcal{A}_{k,\ell,m} \end{aligned} \tag{73}$$

The principal parts of these equations in $\mu = 1/\Delta_1$ imply

$$\begin{aligned} B^{(1)}_{k+1,\ell,m}\begin{pmatrix} 1 & f_1 \\ \frac{1}{f_1} & 1 \end{pmatrix} &= \begin{pmatrix} 1 & f_1 \\ \frac{1}{f_1} & 1 \end{pmatrix} B^{(1)}_{k,\ell,m}, \quad f_1 = z_{k+1,\ell,m} - z_{k,\ell,m}, \\ B^{(1)}_{k,\ell+1,m}\begin{pmatrix} 1 & f_2 \\ \frac{\Delta_2}{f_2\Delta_1} & 1 \end{pmatrix} &= \begin{pmatrix} 1 & f_2 \\ \frac{\Delta_2}{f_2\Delta_1} & 1 \end{pmatrix} B^{(1)}_{k,\ell,m}, \quad f_2 = z_{k,\ell+1,m} - z_{k,\ell,m}, \\ B^{(1)}_{k,\ell,m+1}\begin{pmatrix} 1 & f_3 \\ \frac{\Delta_3}{f_3\Delta_1} & 1 \end{pmatrix} &= \begin{pmatrix} 1 & f_3 \\ \frac{\Delta_3}{f_3\Delta_1} & 1 \end{pmatrix} B^{(1)}_{k,\ell,m}, \quad f_3 = z_{k,\ell,m+1} - z_{k,\ell,m}. \end{aligned}$$

The solution with $\operatorname{tr} B^{(1)}_{k,\ell,m} = k$ is

$$B^{(1)}_{k,\ell,m} = \frac{k - a_1}{z_{k+1,\ell,m} - z_{k-1,\ell,m}} \begin{pmatrix} z_{k+1,\ell,m} - z_{k,\ell,m} & (z_{k+1,\ell,m} - z_{k,\ell,m})(z_{k,\ell,m} - z_{k-1,\ell,m}) \\ 1 & z_{k,\ell,m} - z_{k-1,\ell,m} \end{pmatrix} + \frac{a_1}{2} I.$$

The same computation yields the formulas of Theorem 8.1 for $B^{(2)}_{k,\ell,m}$ and $B^{(3)}_{k,\ell,m}$.

To derive a formula for the coefficient $C_{k,\ell,m}$ let us compare $\Phi_{k,\ell,m}(\mu)$ with the solution $\Psi_{k,\ell,m}(\lambda)$ in Theorem 7.3, more exactly with its extension to the lattice $\mathbb{Z}^3$:

$$\begin{aligned}\Psi_{k+1,\ell,m}(\lambda) &= L^{(1)}(\lambda)\Psi_{k,\ell,m},\\ \Psi_{k,\ell+1,m}(\lambda) &= L^{(2)}(\lambda)\Psi_{k,\ell,m},\\ \Psi_{k,\ell,m+1}(\lambda) &= L^{(3)}(\lambda)\Psi_{k,\ell,m},\end{aligned}$$

normalized by $\Psi_{0,0,0}(\lambda)=I$. Here the matrices $L^{(n)}$ are given by (42)

$$L^{(n)}(\lambda)=(1-\lambda^2\Delta_n)^{-1/2}\begin{pmatrix}1 & \lambda f_n\\ \lambda\Delta_n/f_n & 1\end{pmatrix}.$$

Consider

$$\tilde\Psi=h(\lambda)\begin{pmatrix}1/\sqrt{\lambda} & 0\\ 0 & \sqrt{\lambda}\end{pmatrix}\Psi\begin{pmatrix}\sqrt{\lambda} & 0\\ 0 & 1/\sqrt{\lambda}\end{pmatrix}$$

with

$$h(\lambda)=(1-\lambda^2\Delta_1)^{k/2}(1-\lambda^2\Delta_2)^{\ell/2}(1-\lambda^2\Delta_3)^{m/2}.$$

So defined $\tilde\Psi$ is a function of μ. Since it satisfies the same difference equations (48) as $\Phi(\mu)$ and is normalized by

$$\tilde\Psi_{k,\ell,m}(\mu=0)=I,$$

we have

$$\Phi_{k,\ell,m}(\mu)=\tilde\Psi_{k,\ell,m}(\mu)\Phi_{0,0,0}(\mu). \tag{74}$$

Moreover $\tilde\Psi(\mu)$ is holomorphic in $\mu=0$ and due to Theorem 7.3 equal in this point to

$$\tilde\Psi_{k,\ell,m}(\mu=0)=\begin{pmatrix}1 & Z\\ 0 & 1\end{pmatrix},\qquad Z=z_{k,\ell,m}-z_{0,0,0}.$$

Taking the logarithmic derivative of (74) with respect to μ

$$\mathcal{A}_{k,\ell,m}=\frac{d\tilde\Psi_{k,\ell,m}}{d\mu}\tilde\Psi^{-1}_{k,\ell,m}+\tilde\Psi_{k,\ell,m}\mathcal{A}_{0,0,0}\tilde\Psi^{-1}_{k,\ell,m}$$

and computing its singularity at $\mu=0$ we get

$$C_{k,\ell,m}=\begin{pmatrix}1 & Z\\ 0 & 1\end{pmatrix}C_{0,0,0}\begin{pmatrix}1 & -Z\\ 0 & 1\end{pmatrix}.$$

The formula for $C_{k,\ell,m}$ in Theorem 8.1 is the general solution to this equation.

Conversely, by direct computation one can check that the compatibility conditions (73) with $\mathcal{A}_{k,\ell,m}$ computed above are equivalent to (46).

Proof of Theorem 8.2

Proof. First note that one can assume $b=0$ by applying a suitable Möbius transformation. Next by translating z one can make $z_{k,l,m}=0$ for arbitrary fixed $k,l,m\in\mathbb{Z}$ (this will change d however). Finally we can assume $d=0$ or $d=1$ since we can scale z. Now one can show the compatibility by using a computer algebra system like MATHEMATICA as follows: given the points $z_{k,l,m}, z_{k+1,l,m}, z_{k,l\pm1,m}, z_{k,l,m\pm1}$ one can compute $z_{k-1,l,m}$ using the constraint (46). With the cross-ratio equations one can now calculate all points necessary to apply the constraint for calculating $z_{k+2,l,m}$ and $z_{k,l,m+2}$ (besides $z_{k-1,l,m-1}$ one will need all points $z_{k\pm1,l\pm1,m\pm1}$). Once again using the cross-ratios one calculates all points necessary to calculate $z_{k+1,l,m+2}$ with the constraint. Finally one can check, that the cross-ratio $q(z_{k,l,m+1}, z_{k+1,l,m+1}, z_{k+1,l,m+2}, z_{k,l,m+2})$ is correct. Since the three directions are equivalent and since all initial data was arbitrary (i.e. symbolic) this suffices to show the compatibility. $\square$

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