Conference on

Lévy Processes: theory and applications

January 18-22, 1999

MaPhySto — Centre for Mathematical Physics and Stochastics

Department of Mathematical Sciences

University of Aarhus, Denmark

Foreword

This booklet contains information on the Conference on "Lévy Processes: Theory and Applications", held 18-22 January 1999 at Department of Mathematical Sciences, University of Aarhus, and organized by **MaPhySto** – Centre for Mathematical Physics and Stochastics — a research centre under the Danish National Research Foundation.

Lévy processes, introduced by Paul Lévy in the thirties and forties have for decades been subject to intensive theoretical studies. But up until recently the general Lévy process were widely regarded to be of little applied relevance. However, a variety of recent developments has changed this perception rather dramatically. More and more applications of such processes are found. The understanding of the nature of these processes is also still increasing, and at least two monographs on the theory of Lévy processes, by Bertoin and by Sato, as well as other material of a similar character, have recently been published or are about to appear.

The time therefore seemed ripe for an international conference devoted to this area of research, and, to our delight, the interest in the conference well exceeded our expectations. The present collection of extended abstracts of the talks presented at the conference shows that the field of Lévy processes is in a prosperous state with ramifications in many different directions and steady expansion of the central body of the theory.

We are grateful to the participants for their willingness to produce these extended abstracts, and we hope that many readers will find them useful and will feel some of the excitement for the developments that was present among the participants at the conference.

Aarhus in April 1999

Ole E. Barndorff-Nielsen
Svend Erik Graversen
Thomas Mikosch

1 Contents

2 Conference Program

Monday January 18

09.00-10.00 Registration and coffee/tea

Chairman: Philip Protter

10.00-10.50 **Jean Bertoin:**
Structure of shocks in Burgers turbulence with stable noise initial data.

11.20-12.10 **Stéphane Jaffard:**
Multifractal analysis of the Lévy-Chentsov fields.

12.30-14.00 Lunch

Chairman: Bernt Øksendal

14.00-14.30 **Michael Braverman:**
The supremum of Lévy processes with light tail.

14.40-15.10 **Laurence Marsalle:**
Slow points of local times.

15.20-15.50 **Ernst Eberlein:**
Term structure models driven by general Lévy processes.

Coffee/tea

16.10-16.40 **Neil Shephard:**
Incorporation of a leverage effect in a stochastic volatility model.

16.50-17.20 **Adam Jakubowski:**
Asymptotic error in approximation of Lévy processes by discretizations.

Tuesday January 19

Chairman: Neil Shephard

9.00-9.50 **Bernt Øksendal:**
White noise generalizations of the Clark-Ocone theorem.

10.00-10.50 **Francesco Mainardi:**
Three types of random walks for approximation of Lévy-Feller diffusion processes.

Coffee/tea

11.20-12.10 **Francois Bardou:**
Cooling gases with Lévy flights: using the generalized central limit theorem in physics.

12.30-14.00 Lunch

Chairman: Ernst Eberlein

14.00-14.30 **Agnés Sulem:**
Optimal portfolio and consumption in a jump diffusion market.

14.40-15.10 **Goran Peskir:**
On the concept of market force:
From Smoluchowski's approximation to Burgers' equation.

15.20-15.50 **Hanspeter Schmidli:**
Distribution of the first ladder height of a stationary risk process perturbed by Lévy motion.

COFFEE/TEA

16.10-16.40 **Philip Protter:**
Martingale Representation: formulas and robustness.

16.50-17.20 **Bent Jørgensen:**
Tweedie models: a generalization of stable distributions.

Wednesday January 20

Chairman: Maria Emilia Caballero

9.00-9.50 **Jan Rosiński:**
Series expansions without compensation for infinitely divisible processes.

10.00-10.50 **Gennady Samorodnitsky:**
Certain probabilistic aspects of semistable laws.

COFFEE/TEA

11.20-12.10 **Michael Marcus:**
Local times and other continuous additive functionals of Lévy processes.

12.30-14.00 LUNCH

In the afternoon: SIGHT-SEEING/EXCURSION

Thursday January 21

Chairman: Jean Jacod

9.00-9.50 **David Applebaum:**
Lévy Processes in Lie groups and manifolds.

10.00-10.50 **Philippe Carmona:**
Asymptotic behaviour of exponential functionals of Lévy Processes.

COFFEE/TEA

11.20-12.10 **Ken-iti Sato:**
Recurrence and transience of Lévy processes and some processes with nonstationary independent increments.

12.30-14.00 LUNCH

Chairman: Uwe Küchler

14.00-14.30 **Zbigniew Jurek:**
Different aspects of selfdecomposability.

14.40-15.10 **Makoto Maejima:**
Semi-selfdecomposability and semi-selfsimilarity.

15.20-15.50 **A. Reza Soltani:**
Multiparameter Lévy stable noise and a selection procedure.

COFFEE/TEA

16.10-16.40 **Niels Jacob:**
The symbol of a Markov process.

16.50-17.20 **René L. Schilling:**
Lévy processes and function spaces.

Friday January 22

Chairman: Jean Bertoin

9.00-9.50 **Pio Andrea Zanzotto:**
On stochastic differential equations driven by Cauchy process and the other α-stable motions.

10.00-10.50 **Maria Emilia Caballero:**
Regularity of the Cauchy principal value of some Lévy processes.

COFFEE/TEA

11.20-12.10 **Uwe Küchler:**
On inverse local times, spectral measures and life-time distributions of one-dimensional diffusions.

12.30-14.00 LUNCH

Chairman: Thomas Mikosch

14.00-14.30 **Ole E. Barndorff-Nielsen:**
Approximate H-sssi processes.

14.40-15.10 **Albert Shiryaev:**
Around the extensions of P. Lévy's distributional theorem.

15.20-15.50 CONCLUDING REMARKS (DIRECTED BY KEN-ITI SATO)

3 List of participants

David Applebaum
Department of Mathematics, Statistics and Operational Research
The Nottingham Trent University
Burton Street
Nottingham Ng1 4BU, England
Email: dba@euler.ntu.ac.uk

Søren Asmussen
Department of Mathematical Statistics
University of Lund
Box 118
S-221 00 Lund, Sweden
Email: asmus@maths.lth.se

Francois Bardou
Department of Physics
University of Newcastle
Newcastle Upon Tyne, NE1 7RU
United Kingdom
Email: Francois.Bardou@newcastle.ac.uk

Ole E. Barndorff-Nielsen
MaPhySto
Dept. of Mathematical Sciences
University of Aarhus
8000 Aarhus C, Denmark
Email: oebn@imf.au.dk

Fred Espen Benth
Department of Mathematical Sciences
Theoretical Statistics
University of Aarhus
DK-8000 Aarhus C, Denmark
Email: fredb@imf.au.dk

Jean Bertoin
Laboratoire de Probabilités
Université Pierre et Marie Curie
4 Place Jussieu
F-75252 Paris Cedex 05, France
Email: jbe@ccr.jussieu.fr

N. H. Bingham
Statistics Department
Birkbeck College (University of London)
Malet Street
London WC1E 7HX, England
Email: n.bingham@statistics.bbk.ac.uk

Preben Blæsild
Department of Mathematical Sciences
Theoretical Statistics
University of Aarhus
DK-8000 Aarhus C, Denmark
Email: preben@imf.au.dk

Michael Braverman
Department of Mathematics
Ben Gurion University of the Negev
Beér Sheva 84105
Israel
Email: braver@math.bgu.ac.il

Maria Emilia Caballero
Instituto de Matemáticas - UNAM
Circuito Exterior
Ciudad Universitaria
04510 - México, D.F., México
Email: emilia@servidor.unam.mx

Philippe Carmona
Laboratoire de Statistique et Probabilités
Université Paul Sabatier
118 Route de Narbonne
F-31062 Toulouse Cedex, France
Email: carmona@cict.fr

Richard M. Dudley
MIT, Department of Mathematics
Room 2-245
Cambridge, MA 02139
U.S.A.
Email: rmd@math.mit.edu

Ernst Eberlein
Institut für Mathematische Stochastik
Albert-Ludwigs-Universität Freiburg
Hebelstrasse 27

D-79104 Freiburg, Germany
Email: eberlein@levy.mathematik.uni-freiburg.de

Nils Christian Framstad
Dept. of Mathematics
University of Oslo
P. O. Box 1053 Blindern
N–0316 Oslo, Norway
Email: ncf@math.uio.no

Svend Erik Graversen
Department of Mathematical Sciences
Theoretical Statistics
University of Aarhus
DK-8000 Aarhus C, Denmark
Email: matseg@imf.au.dk

Dyana Harrelsen
School of Mathematics
Georgia Institute of Technology
Atlanta, Georgia 30332-0205
U.S.A.
Email: drice@math.gatech.edu

John Hawkes
Department of Mathematics
University College of Swansea
Swansea
Wales, U. K
Email:

Christian Houdré
School of Mathematics
Georgia Institute of Technology
Atlanta, Georgia 30332-0205
U.S.A.
Email: houdre@math.gatech.edu

Jørgen Hoffmann-Jørgensen
Department of Mathematical Sciences
Theoretical Statistics
University of Aarhus
DK-8000 Aarhus C, Denmark
Email: hoff@imf.au.dk

Bjarne Højgaard

Department of Mathematics
Aalborg University
Fredrik Bajers Vej 7
9220 Aalborg Ø, Denmark
Email: malik@math.auc.dk

Martin Jacobsen
Department of Theoretical Statistics
University of Copenhagen
Universitetsparken 5
2100 Copenhagen Ø, Denmark
Email: martin@math.ku.dk

Niels Jacob
Fakultät für Informatik
Universität der Bundeswehr München
D-85577 Neubiberg
Germany
Email: jacob@informatik.unibw-muenchen.de

Jean Jacod
Laboratoire de Probabilités
Université Pierre et Marie Curie
4 Place Jussieu
F-75252 Paris Cedex 05, France
Email: jj@ccr.jussieu.fr

Stéphane Jaffard
Université Paris XII Val de Marne
61 av. du Gén. de Gaulle
F-96010 Géteil Cedex
France
Email: Stephane.Jaffard@cmla.ens-cachan.fr

Adam Jakubowski
Faculty of Mathematics
Nicholas Copernicus University
ul. Chopina 12/18
PL–87–100 Toruń, Poland
Email: adjakubo@mat.uni.torun.pl

Eva Vedel Jensen
Department of Mathematical Sciences
Theoretical Statistics

University of Aarhus
DK-8000 Aarhus C, Denmark
Email: eva@imf.au.dk

Jens Ledet Jensen
Department of Mathematical Sciences
Theoretical Statistics
University of Aarhus
DK-8000 Aarhus C, Denmark
Email: jlj@imf.au.dk

Zbigniew J. Jurek
Institute of Mathematics
University of Wroclaw
Pl. Grunwaldzki 2/4
P-50-384 Wroclaw, Poland
Email: zjjurek@math.uni.wroc.pl

Bent Jørgensen
Department of Statistics and Demography
Odense University
Campusvej 55
DK-5230 Odense M, Denmark
Email: bentj@statdem.ou.dk

Uwe Küchler
Mathematik
Humboldt-Universität
PSF 1297
D-10099 Berlin, Germany
Email: kuechler@mathematik.hu-berlin.de

Arne Løkka
Dept. of Mathematics
University of Oslo
P. O. Box 1053 Blindern
N–0316 Oslo, Norway
Email: alokka@math.uio.no

Makoto Maejima
School of OR and IE
Cornell University, Rhodes Hall
Ithaca, NY 14853-3801
U.S.A.

Email: mmaejima@orie.cornell.edu

Francesco Mainardi
Department of Physics
University of Bologna
Via Irnerio 46
I-40126 Bologna, Italy
Email: Francesco.Mainardi@bo.infn.it

Michael Marcus
Department of Mathematics
City College of CUNY
New York, NY 10031
U.S.A.
Email: mbmarcus@earthlink.net

Laurence Marsalle
Département de Mathématiques
Université Paris XII-Val de Marne
61 avenue du Général de Gaulle
F-94010 Creteil Cedex, France
Email: marsalle@proba.jussieu.fr

Thomas Mikosch
RUG Groningen
Faculty of Mathematics and Physics
PO Box 800
NL-9700 AV Groningen, The Netherlands
Email: T.Mikosch@math.rug.nl

Klaus Mølmer
Institute of Physics
University of Aarhus
DK-8000 Aarhus C
Denmark
Email: moelmer@dfi.aau.dk

Elisa Nicolato
Department of Mathematical Sciences
Theoretical Statistics
University of Aarhus
DK-8000 Aarhus C, Denmark
Email: elisa@imf.au.dk

Rimas Norvaiša
Institute of Mathematics and Informatics
Akademijos 4
2600 Vilnius
Lithuania
Email: norvaisa@ktl.mii.lt

Jan Pedersen
Department of Mathematical Sciences
Theoretical Statistics
University of Aarhus
DK-8000 Aarhus C, Denmark
Email: jan@imf.au.dk

Goran Peskir
Department of Mathematical Sciences
Theoretical Statistics
University of Aarhus
DK-8000 Aarhus C, Denmark
Email: goran@imf.au.dk

Karsten Prause
Institut fur Mathematische Stochastik
Universitt Freiburg
Eckerstrasse 1
D-79104 Freiburg, Germany
Email: prause@fdm.uni-freiburg.de

Philip Protter
Department of Mathematics
Purdue University
1395 Mathematical Sciences Building
West Lafayette, Indiana 47907-1395, U.S.A.
Email: Philip.Protter@sophia.inria.fr

Jan Rosiński
Department of Mathematics, 121 Ayres Hall
University of Tennessee
Knoxville, TN 37996-1300
U.S.A.
Email: rosinski@math.utk.edu

Francesco Russo
Université Paris XIII, Institut Galilée

Département de Mathématiques
Av. Jean-Baptiste Clément
F-93430 Villetaneuse, France
Email: russo@math.univ-paris13.fr

Gennady Samorodnitsky
Cornell University
School of OR and Industrial Engineering
206 Rhodes Hall
Ithaca, NY 14853-3801, U.S.A.
Email: gennady@orie.cornell.edu

Ken-iti Sato
Hachimanayama 1101-5-103
Tenpaku-ku
Nagoya 468-0074
Japan
Email: ksato@info.human.nagoya-u.ac.jp

Rene Schilling
The Nottingham Trent University
Mathematics Department
Burton Street
Nottingham NG1 4BU, U.K.
Email: rene.schilling@ntu.ac.uk

Hanspeter Schmidli
Department of Mathematical Sciences
Theoretical Statistics
University of Aarhus
DK-8000 Aarhus C, Denmark
Email: schmidli@imf.au.dk

Neil Shephard
Nuffield College
Oxford OX1 1NF
England
Email: neil.shephard@nuffield.oxford.ac.uk

Albert Shiryaev
Steklov Mathematical Institute
Gubkina str. 8
117966 Moscow
Russia

Email: shiryaev@mi.ras.ru

A. Reza Soltani
College of Sciences
Shiraz University
Shiraz
Iran
Email: soltani@sun01.susc.ac.ir

Agnes Sulem
INRIA
Domaine de Voluceau-Rocquencourt
B.P. 105
F–78153 Le Chesnay Cedex, France
Email: Agnes.Sulem@inria.fr

Michael Sørensen
Department of Theoretical Statistics
University of Copenhagen
Universitetsparken 5
2100 Copenhagen Ø, Denmark
Email: michael@math.ku.dk

Juha Vuolle-Apiala
Department of Mathematics
University of Helsinki
Finland
Email: Juha.Voulle-Apiala@helsinki.fi

Pio Andrea Zanzotto
Dipartimento di Matematica
Universitá di Pisa
Via F. Buonarroti
2-56127 Pisa, Italy
Email: zanzotto@dm.unipi.it

Bernt Øksendal
Department of Mathematics
University of Oslo
B.P. 1053 Blindern
N-0316 Oslo 3, Norway
Email: oksendal@math.uio.no

4 Extended abstracts

On the following pages you will find (in alphabetical order after the author who presented the work) the (extended and/or revised) abstracts submitted to the organizers.

LEVY PROCESSES IN LIE GROUPS AND MANIFOLDS

DAVID APPLEBAUM

1. Introduction

For the last twenty years there has been an explosive development in the subject of stochastic differential geometry which has been based mainly around the notion of Brownian motion in a Riemannian manifold . Of course, Brownian motion is a natural object to look at on a manifold as its generator is the Laplace-Beltrami operator which carries a lot of geometric information. However from the Euclidean space point of view it is just one member of the class of Levy processes and so it is natural to ask the question

Does it make sense to talk of a
"Levy process on a Riemannian manifold" ?

We'll answer this question in three stages, the first of which is

1.1. **Levy Processes in Lie Groups.** Let G be a finite dimensional Lie group with identity e. A Levy process X(t) taking values in G is simply a stochastically continuous process with stationary and independent (left) increments such that X(0) = e (a.s). Of course we are adopting a convention here. We could equally well take increments on the right however it can be shown that if X(t) is a left Levy process then its inverse is a right Levy process.

These processes were first studied in a very important paper by G.Hunt in 1956 ([1]) who characterised their infinitesimal generators as a sum of a second order differential operator manufactured from the Lie algebra basis and a suitably renormalised integral operator expressing translations averaged using a Levy measure. Several obscure points in Hunt's proof were clarified by Ramaswami ([2]) and these were incorporated into the seminal treatise of Heyer ([3]) (see Chapter 4).

More recently, using martingale methods, H.Kunita and the present author ([4]) were able to show that every Levy process on G could be obtained as the solution of a stochastic differential equation (SDE) driven by a Brownian motion on the Lie algebra and an independent Poisson random measure on the group. Using these ideas, Kunita was able to develop a theory of stable processes on simply connected nilpotent Lie groups ([5]).

In [6], it was shown that an analogue of the Levy-Ito decomposition holds for Levy processes in Lie groups i.e. each such process can be realised as the almost-sure limit of a sequence of Brownian motions (with drift) interlaced with the jumps from a sequence of compound Poisson process. This generalised an earlier result of Gangolli ([7]) in the spherically symmetric case (see below).

Stochastic flows of diffeomorphisms arising from SDEs driven by Levy processes are called Levy flows. These are essentially Levy processes on the infinite-dimensional Lie group Diff(M). For work on these see [4] and references therein.

In passing, it should be noted that Levy processes in general locally compact groups are also of interest. In particular, there has recently been some activity in the totally disconnected case ([8], [9])

1.2. **Levy Processes in Symmetric Spaces.** A Riemannian manifold M is a globally symmetric space if every point is an isolated fixed point of an involutive isometry. All such manifolds can be realised (up to diffeomorphism) as homogeneous spaces $M = G/K$ where G is a connected Lie group and K is a compact subgroup. In this case it is natural to define a Levy process on M as the image of a Levy process on G under the canonical surjection. The most interesting class of such processes are those which are spherically symmetric i.e. their laws are invariant under the left-action of K on M and so are projections of processes in G whose laws are bi-invariant under K. Harmonic analysis can be used to investigate these processes via the spherical transform of Harish-Chandra. This was first appreciated by Gangolli who investigated these processes in two ground-breaking papers ([7], [10]) where he established a Levy-Khinchine formula for the spherical transform.

In [6], using the stochastic differential equation techniques of [4] the author was able to simplify and extend Gangolli's work. It should be emphasised that the spherical transform is a very powerful tool which allows us to generalise many of the features of the Euclidean case, for example, subordination preserves the class of spherically symmetric Levy processes ([11]).

The general form of spherically symmetric Levy processes is that of a Brownian motion on M (without drift) interlaced with jumps along geodesics and this gives us a strong hint as to how we should proceed in more general manifolds.

2. Levy Processes in Riemannian Manifolds

We begin by recalling the case of Brownian motion in a Riemannian manifold M.

Although the idea of a process with "increments" breaks down this can be defined to be a Markov process B(t) whose infinitesimal generator is the Laplace-Beltrami operator Δ and this clearly generalises the

Euclidean case. If the manifold fails to be parallelisable, then Δ cannot be written globally as the sum of squares of smooth vector fields and so B(t) cannot be obtained as the solution of a globally defined SDE. Using the fact that on the bundle of orthonormal frames over M the horizontal Laplacian projects onto Δ, Eels and Elworthy were able to obtain B(t) as the projection of the solution to an SDE in O(M) driven by the canonical horizontal vector field (see e.g. [12]).

A simplistic attempt to imitate this construction for Levy processes was attempted in [13] and found not to work in general. In fact there is no obstacle to constructing a "canonical horizontal Levy process" in O(M). The difficulty arises through the projection which is not a homogeneous Markov process and depends on the choice of initial frame in O(M). To overcome this difficulty A.Estrade and the present author defined an *isotropic Levy process* in M to be one whose horizontal lift is the solution to an SDE driven by a spherically symmetric Levy process in Euclidean space ([14]). In this case we obtain a process which can be realised as a Brownian motion on M interlaced with jumps along geodesics. This then generalises the case of spherically symmetric processes on symmetric spaces. Furthermore the process has infinite lifetime (a.s) when the manifold is compact.

So we have a positive answer to our original question. However note that we are unable to include a global drift in our processes.

Current research by the author is focussing on Levy processes in principal fibre bundles where it is hoped to make some contact with gauge theory, perhaps via stochastic quantisation.

3. References

[1] G.A.Hunt, Semigroups of measures on Lie groups, Trans. Amer Math Soc 81, 264 (1956)

[2] S.Ramaswami, Semigroups of measures on Lie groups, J.Indian Math Soc 38, 175 (1974)

[3] H.Heyer, Probability Measures on Locally Compact Groups, Springer-Verlag (1977)

[4] D.Applebaum, H.Kunita, Levy flows on manifolds and Levy processes in Lie groups, J.Math. Kyoto Univ 33, 1003 (1993)

[5] H.Kunita, Stable Levy processes on nilpotent Lie groups in Stochastic Analysis in Infinite Dimensional Spaces, Pitman Res Notes in Math Ser A 167 (1994)

[6] D.Applebaum, Compound Poisson processes and Levy processes in groups and symmetric spaces, NTU preprint (1997)

[7] R.Gangolli, Sample functions of certain differential processes on symmetric spaces , Pacific J.Math 15, 477 (1965)

[8] S.N.Evans, Local properties of Levy processes on a totally disconnected group, J.Theor Prob 2, 209 (1989)

[9] D.Aldous, S.N.Evans, Dirichlet forms on totally disconnected spaces and bipartite Markov chains, Berkeley preprint (1998)
[10] R.Gangolli, Isotropic infinitely divisible measures on symmetric spaces, Acta Math 111, 213 (1964)
[11] D.Applebaum, On the subordination of spherically symmetric Levy processes in Lie groups, NTU preprint (1998)
[12] K.D.Elworthy, Geometric aspects of diffusions in manifolds, in Ecole d'Ete de Probabilites de Saint-Flour XVII , Springer LNM 1362, 276 (1988)
[13] D.Applebaum, A horizontal Levy process on the bundle of orthonormal frames over a Riemannian manifold, Seminaire de Probabilites XXIX, Springer LNM 1613,166 (1995)
[14] D.Applebaum, A.Estrade, Isotropic Levy processes on Riemannian manifolds, NTU preprint (1998)

Department of Mathematics, Statistics and Operational Research, The Nottingham Trent University
E-mail address: dba@maths.ntu.ac.uk

Cooling gases with Lévy flights: using the generalized central limit theorem in physics

François Bardou*

Institut de Physique et de Chimie des Matériaux de Strasbourg, 23 rue du Loess, F-67037 STRASBOURG Cedex, France

In the last ten years, the generalized central limit theorem established by Paul Lévy in the thirties has been found more and more relevant in physics. Physicists call 'Lévy flights' random walks for which the probability density of the jump lenghts x decays as $1/x^{1+\alpha}$ with $\alpha < 2$ for large x. We give here a glimpse of Lévy flights in physics through two examples, without going into technical details. We first introduce a simple toy model, the Arrhenius cascade. We then present an important physical process, subrecoil laser cooling of atomic gases, in which Lévy flights play an essential role.

Contents

*Email: bardou@morgane.u-strasbg.fr. Presently visitor at : Department of Physics, University of Newcastle, Newcastle-upon-Tyne NE1 7RU, United Kingdom.

1 Introduction

The 'usual' central limit theorem (CLT) is an essential tool in physics and in other sciences. Indeed, one often knows the probability density $f(x)$ of a quantity x associated with single events. Then, one wants to derive the probability density $f(X_N)$ for a sum X_N

$$X_N = \sum_{i=1}^{N} x_i \tag{1}$$

of a large number N of such events, considered as independent[1]. In simple terms, the usual CLT tells that the density of $(X_N - N\langle x\rangle)/\sqrt{N}$ tends to a gaussian distribution at large N and that this distribution is determined only by the average value $\langle x\rangle$ and the second moment $\langle x^2\rangle$, provided that these quantities are finite[2]. The condition for applying the usual CLT (finiteness of $\langle x^2\rangle$) is so frequently satisfied that most physicists implicitely believe that this theorem applies universally.

However, physical phenomena can exhibit statistical properties that are beyond the usual CLT. In particular, densities $f(x)$ with power law tails:

$$f(x) \sim \frac{1}{x^{1+\alpha}} \quad \text{for} \quad x \to \infty \tag{2}$$

(with $\alpha > 0$ to ensure normalizability) are simple laws that tend to appear frequently. If $\alpha > 2$, $\langle x^2\rangle$ is finite and the usual CLT applies. On the contrary, if $\alpha \leq 2$, $\langle x^2\rangle$ diverges[3] and the usual CLT does not apply. If $\alpha \leq 1$, even $\langle x\rangle$ diverges. As stated by the 'generalized' CLT, if $0 < \alpha < 2$, the density of X_N (properly renormalized) still tends to a stable law, which is not a gaussian but a Lévy law. After Mandelbrot, we call 'Lévy flight' a random walk in which the probability density of jump lengths is given by Eq. (2) with $\alpha < 2$.

The generalized CLT was already available in the thirties but, surprisingly, it has had a limited influence on physics for a long time. Most physicists were certainly not aware of it and those who were aware probably doubted that infinite average values $\langle x\rangle$ or second moments $\langle x^2\rangle$ could make sense in a real phenomenon. Note that few cases which come under the generalized CLT were known, but they remained isolated cases (such as the density of first return times τ in one dimension which decays as $1/\tau^{3/2}$ for large τ's).

In the recent years, it has been more and more recognized that the generalized CLT could shed an interesting light on many physical processes: random walks in solutions of micelles [OBL90], turbulent and chaotic transport [SZK93, SWS93], glassy dynamics [Shl88, BoD95], diffusion of spectral lines in disordered solids [ZuK94], thermodynamics [TLS95, Tsa95, ZaA95], granular flows [BoC97], laser trapped ions [MEZ96, KSW97] ... For reviews, see [Shl88, BoG90, KSZ96, Tsa97].

The interest of the physics community for the generalized CLT seems to be stimulated by two important arguments.

First, the phenomena obeying only the generalized form of the CLT, i.e. those with asymptotic power laws for $f(x)$ with $\alpha < 2$, exhibit a statistical behaviour which is

[1] For simplicity, any sum of independent events is called a 'Lévy sum' below.
[2] If $\langle x^2\rangle$ is finite, we say that $f(x)$ is a 'narrow' probability density.
[3] We then say that $f(x)$ is a 'broad' probability density.

markedly different from the behaviour of the phenomena obeying the usual CLT. It is thus important to identify whether a physical process comes under the generalized form of the CLT (in the first place to avoid the use of natural but irrelevant concepts, such as the average value, derived from the usual CLT). This is illustrated in section 2 with the simple model of the Arrhenius cascade.

Second, the generalized CLT provides an efficient tool for the quantitative study of some physical problems. This is illustrated in section 3 with a Lévy flight theory of laser cooling of atomic gases. In this case, it is worth noting that the ideas derived from the generalized CLT have had practical consequences, leading to more efficient cooling strategies and to record low temperatures.

Finally, the generalized CLT also provides a useful qualitative insight for some random walks even when it is not strictly valid. This is discussed briefly in section 4.

Note that this contribution presents the point of view of a physicist and, as such, might not be rigorous on all mathematical aspects.

2 The Arrhenius cascade

We present the model of the Arrhenius cascade in section 2.1, which is shown to exhibit an unexpected statistical behaviour in section 2.2, analyzed with the generalized CLT in section 2.3. This toy model presents generic effects of the generalized CLT in physics.

2.1 The model

We consider a physical system placed in the potential landscape schematized in figure 1 and submitted to thermal fluctuations. This model is called here the Arrhenius cascade. It is inspired from studies of disordered systems, like glasses relaxing towards low energy states [Shl88, BoD95], which obey similar equations.

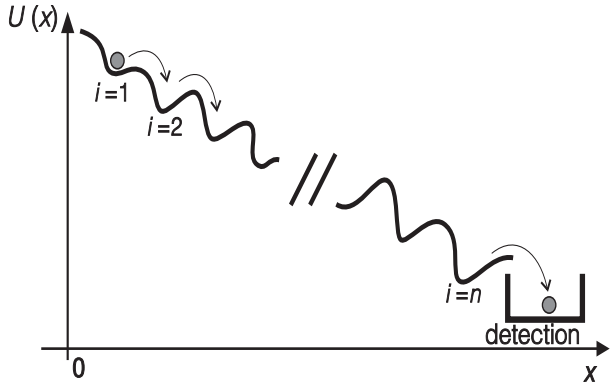


Figure 1. The Arrhenius cascade.

The potential $U(x)$ is a tilted random 'washboard'. It presents n local minima or 'wells', labelled by i, separated from the next minimum $i+1$ by a potential 'hill' of random height U_i (the U_i's are independent variables). We assume an exponential probability density

$f(U)$ for the hills U_i:

$$f(U) = \frac{1}{U_0}\, e^{-U/U_0}\,, \tag{3}$$

where U_0 is the average height of the potential hills. The variable x may be a position coordinate or any coordinate of the system.

At any time, the physical system is trapped in one of the local potential wells. Due to thermal fluctuations, the system performs sudden jumps from one well to the other one downwards. The global tilt of the potential hill $U(x)$ is large enough to prevent the system from performing upward jumps. Therefore, the system can only cascade downwards. The trapping time τ_i in the well i is related to U_i by the Arrhenius law[4]:

$$\tau_i = \tau_0\, e^{U_i/(k_{\rm B}T)}\,, \tag{4}$$

where τ_0 is a characteristic time, $k_{\rm B}$ is the Boltzmann constant and T is the temperature.

We consider a gedanken experiment in which the experimentalist wants to know the average trapping time in a single well, but can only measure the time t_n needed to go through all the n wells of the system:

$$t_n = \sum_{i=1}^{n} \tau_i \cdot \tag{5}$$

To reduce the measurement uncertainty, he can repeat m times his measurement of t_n. We assume that, between each measurement, the realization of the disorder changes, i.e. the numbers $U_1, ...U_n$ change (and thus $\tau_1, ...\tau_n$ change accordingly). His estimation $\tau_{\rm exp}(N)$ of the average trapping time will therefore be:

$$\tau_{\rm exp}(N) = \frac{1}{N}\sum_{j=1}^{m} t_n = \frac{1}{N}\sum_{i=1}^{N} \tau_i\ , \tag{6}$$

where $N = m \times n$ is the total number of explored wells and the τ_i's are independent random variables defined by Eq. (3) and (4).

2.2 Behaviour of the Arrhenius cascade

Simulated measurements of $\tau_{\rm exp}(N)$ are represented in figure 2 for two different temperatures. For a temperature $T = 3U_0/k_{\rm B}$, $\tau_{\rm exp}(N)$ converges nicely to the average value $\langle x \rangle$ when N increases, after exhibiting reasonable fluctuations at small N. This is the expected, standard behaviour.

For a lower temperature $T = 0.5U_0/k_{\rm B}$, the behaviour of $\tau_{\rm exp}(N)$ is markedly different. The measured $\tau_{\rm exp}(N)$'s do not seem to converge towards any constant value but rather to diverge in a very fluctuating way with increasing N. A detailed analysis would reveal that the t_n's also fluctuate very much from one measurement to the other. This unusual statistical behaviour would puzzle most experimentalists: large fluctuations and irreproducibility are usually considered as the indication of a problem in the experimental procedure, arising from poorly controlled parameters. But here, one would find that the experimental setup works apparently well [5].

[4] In a realistic model, τ_i is not deterministically fixed by U_i and the expression (4) gives only the average value of τ_i. Taking into account the fluctuations of τ_i for a given U_i would not change qualitatively the discussion presented here.

[5] A closely related situation has recently appeared in a quantum tunneling problem. See section 4.

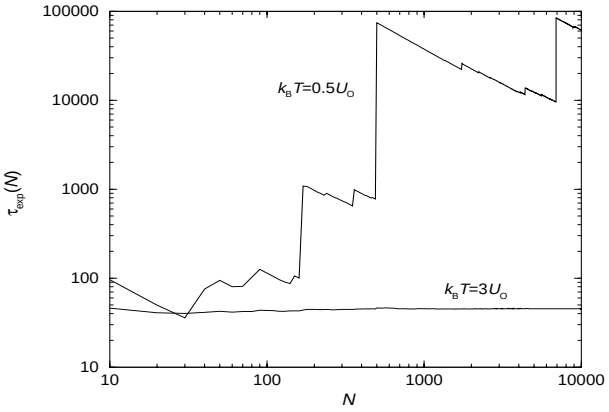


Figure 2. Simulated measurements of the estimated $\tau_{\exp}(N)$. We have chosen $n = 10$ and $\tau_0 = 1$. The quantity $\tau_{\exp}(N)$ for the case $k_{\mathrm{B}}T = 3U_0$ has been multiplied by 30 to be more visible.

2.3 Application of the generalized CLT

Let us see how the generalized CLT can shed light on the previous observations. We first easily calculate the probability density $f(\tau)$ of the trapping times. It is given by the relation $f(\tau)\mathrm{d}\tau = f(U)\mathrm{d}U$, which leads immediately to

$$f(\tau) = \alpha \, \frac{\tau_0^{\alpha}}{\tau^{1+\alpha}} \quad \text{with} \quad \alpha = \frac{k_{\mathrm{B}}T}{U_0} \cdot \tag{7}$$

Having a power law for $f(\tau)$, the probability densities of $\tau_{\exp}(N)$ at large N are provided for any α by the generalized CLT.

In the high temperature case ($T = 3U_0/k_{\mathrm{B}}$), we have $\alpha = 3$ which is larger than 2. Thus $\langle \tau^2 \rangle$ is finite and the usual CLT applies: $\tau_{\exp}(N)$ is gaussianly distributed at large N and tends to $\langle \tau \rangle$. This agrees with the observations (figure 2) and does not need further discussion.

In the low temperature case ($T = 0.5U_0/k_{\mathrm{B}}$), on the contrary, we have $\alpha = 0.5$ which is smaller than 1. The usual CLT does not hold anymore and the specific features of the generalized CLT will play a crucial role. The generalized CLT tells us that we should consider the quantity $Z_N = \sum_{i=1}^{N} \tau_i / N^{1/\alpha}$ and that the density $f(Z_N)$ tends to a Lévy law $L_\alpha(Z_N)$ at large N. This theorem has several important physical consequences: scaling of the Lévy sums, domination of the Lévy sums by a single term and large fluctuations of the Lévy sums (most of these consequences are presented in [BoG90]). We only treat here the case $\alpha < 1$ (and not $1 \leq \alpha \leq 2$) for which the consequences of the generalized CLT depart most strongly from the ones of the usual CLT. These consequences are:

- The most probable value of the sum $T_N = \sum_{i=1}^{N} \tau_i$ *scales* as

$$T_N \sim N^{1/\alpha}, \tag{8}$$

  and not as N which is more usual. Practically, this implies that the time t_n spent in an Arrhenius cascade of n wells does not scale with the size n of the cascade, but more

rapidly due to the generalized CLT[6]. Similarly, the experimental value $\tau_{\rm exp}(N) = \frac{1}{m\times n}\sum_{j=1}^{m} t_n$ does not tend to a constant for a large number of measurements m but diverges with m, as $m^{-1+1/\alpha}$. This explains the observed behaviour in figure 2. Thus, the size of the system and the number of measurements play a non trivial role in the measured values, an unusual situation in physics.

- The notion of average value is irrelevant here since $\langle\tau\rangle = \infty$. One can somehow replace it by the notion of *typical* value, i.e. most probable value. When $\alpha < 1$, the typical terms x_i of a Lévy sum $X_N = \sum_{i=1}^{N} x_i$ are not all of the same order (as they are when $f(x)$ is a narrow density) but present a hierarchical structure. In particular, *the typical largest term $x_{\rm max}$ of the sum can be shown to be of the same order of magnitude as the sum itself*

$$X_N = \sum_{i=1}^{N} x_i \simeq x_{\rm max}\,, \tag{9}$$

however large N might be (Eq. (9) is valid within prefactors that do not depend on N). This domination of a sum by a single term, or by a small number of terms, is a signature of Lévy statistics in a physical problem (see figure 3 below).

It can be used cleverly in physical experiments as a 'Lévy microscope': by measuring a macroscopic quantity $X_N = \sum_{i=1}^{N} x_i$, one may obtain an easy access to a microscopic information, $x_{\rm max}$, while the direct measurement of $x_{\rm max}$ (i.e. in the Arrhenius cascade, the direct study of a single well) can be physically impossible. Such advantageous use of the statistical domination of a single term has been made implicitly, for instance in studies of quantum tunneling [RoB84].

- *A Lévy sum $X_N = \sum_{i=1}^{N} x_i$ fluctuates as much as a single term, when $\alpha < 1$.* This is a direct consequence of the domination of the sum by a single term. It can also be seen as a consequence of the fact that the tail of the Lévy law $L_\alpha(Z_N)$, which determines the fluctuations, decays as $1/Z_N^{1/\alpha}$, exactly as the tail of $f(x)$. This explains the highly fluctuating $\tau_{\rm exp}(N)$ obtained in figure 2 as being intrinsically due to the type of involved statistics and not to some technical experimental problem. Thus, fluctuations do not vanish as usual with the increase of the size N of the statistical sample. The sums X_N retain an intrinsically large irreproducibility. This is in contradiction with a traditional motivation for applying statistical methods: to go beyond the irreproducibility of individual events in order to obtain quasi-perfect reproducibility for large ensembles of events. However, the generalized CLT still allows for some predictibility in the statistical sense, since it predicts the stable form of the probability density of Z_N.

  Physics has incorporated two new types of randomness during this century: quantum uncertainty and deterministic chaos. It seems to us that the non-averaging out of fluctuations in Lévy flights can also be recognized as an important type of randomness[7].

[6] Such 'anomalous' scaling with the system size appears in some complex phenomena like phase transitions near a critical point. What is striking here is to obtain such scaling in a very simple problem.

[7] J.P. Bouchaud speaks of a 'science of irreproducible results'. B. Mandelbrot uses the term 'wild randomness'.

3 Subrecoil laser cooling of atomic gases

In the Arrhenius cascade (section 2), a sum of N independent terms was directly measured and the generalized CLT could be applied directly to analyze the results. In this section, we proceed a step further by studying a richer physical problem, called subrecoil laser cooling. In this case, Lévy sums —and their properties dictated by the generalized CLT— play an essential role, although they are not measured directly.

We introduce subrecoil laser cooling in section 3.1. In section 3.2, we show that power law densities of time variables appear, which implies the non-ergodicity of the process. In section 3.3, the generalized CLT is used to get some insight on the cooling efficiency. In section 3.4, quantitative predictions are derived, using the 'sprinkling density'. In section 3.5, we indicate how the insight provided by the generalized CLT enables to optimize the cooling strategy.

The starting point of the approach presented here has been presented in [BBE94] and [Bar95]. A detailed description of the theory will appear in [BBA99].

3.1 Subrecoil laser cooling

Laser cooling of atomic gases is based on the momentum exchanges between photons and atoms. In standard (*not* subrecoil) laser cooling, laser configurations and atomic transitions are carefully chosen so that these momentum exchanges lead to a friction force. This friction force damps the thermal atomic momenta p, thereby reducing the momentum spread (standard deviation) δp of the atomic gas, which is equivalent to reducing the effective temperature T defined by

$$k_{\mathrm{B}}T = \delta p^2/M \tag{10}$$

where k_{B} is the Boltzmann constant and M is the mass of the atoms. Temperatures commonly achieved in the last ten years are in the range of a few microkelvins, 8 orders of magnitude below room temperature. This has opened exciting new possibilities for atomic and quantum physics and has been a key ingredient in the realization, in 1995, of a new state of matter called Bose-Einstein condensate. The 1997 Nobel prize of physics was attributed to S. Chu, C. Cohen-Tannoudji and W. Phillips for their contributions to laser cooling.

Standard laser cooling mechanisms are fundamentally limited to temperatures larger than the so-called 'recoil temperature'. Indeed, among the momentum exchanges between atoms and photons, some —the ones due to spontaneous emission— occur in a random direction. Each spontaneous emission of a photon by an atom thus results in an uncontrollable random recoil of the atomic momentum $\vec{p}$ by a quantity $\hbar\vec{k}$, where $\hbar\vec{k}$ is the momentum of a single photon. Therefore, the standard deviation δp of the atoms is expected to be always larger than $\hbar k$. This implies (cf. Eq. (10)) laser cooling temperatures T larger than the recoil temperature defined by $T_{\mathrm{R}} = (\hbar k)^2/(k_{\mathrm{B}}M)$. The recoil temperature T_{R} is on the order of one microkelvin for the configurations frequently used.

To sum up, the randomness of spontaneous emission, which is essential for the cooling since it provides a dissipative contribution to the atomic evolution, is also harmful to the cooling since it implies a limit temperature. As spontaneous emission of photons by atoms placed in laser light seems unavoidable, the recoil temperature was for some time considered as an absolute limit for laser cooling.

Subrecoil laser cooling, i.e. $T < T_{\mathrm{R}}$, is however possible. Indeed, although spontaneous emission of photons is an intrinsically random quantum process, it can be partly controlled. The key idea is to create a spontaneous emission rate $R(p)$ which depends on the atomic momentum p (figure 3) and which vanishes at $p = 0$. This was first proposed and realised in 1988 [AAK88, AAK89] using a nice quantum effect called a 'dark resonance', because a resonance occurs at $p = 0$ which prevents the spontaneous emission. Today, record low temperatures reached experimentally with dark resonances approach $T_{\mathrm{R}}/1000$, which corresponds to a few nanokelvins only.

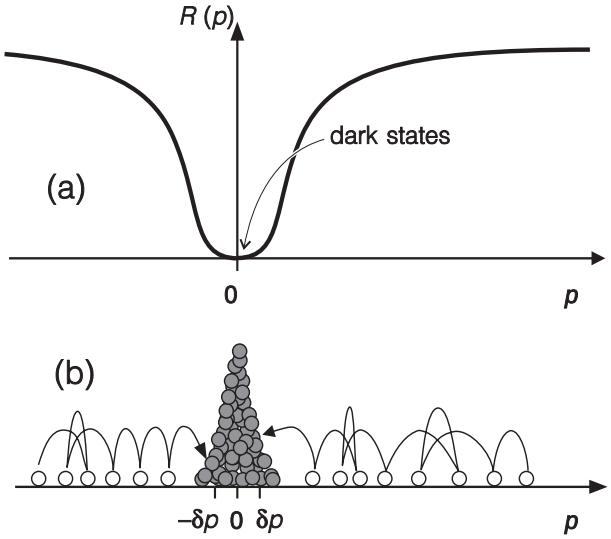


Figure 3. Principle of subrecoil cooling. a) The spontaneous emission rate $R(p)$ vanishes at momentum $p = 0$. b) The atoms perform a momentum random walk and accumulate in the vicinity of $p = 0$.

One can follow the evolution of an atom with an initially non-zero momentum p. The spontaneous emission rate[8] $R(p)$ being large, the atom will spontaneously emit photons[9] and therefore its momentum will change in a random way. This random walk will eventually lead by chance the atom in the vicinity of $p = 0$ where the spontaneous emission rate is very small (see figure 3). There, the atom stops exchanging momentum with photons and it remains so-to-speak 'trapped' in what is called a 'dark state'. The time $\tau(p)$ of residence at momentum p is

$$\tau(p) = \frac{1}{R(p)} \ , \tag{11}$$

the time interval between two spontaneous emissions. If this residence time is long enough, the atom keeps the same small momentum till the end of the experiment. If not, it emits a spontaneous photon, which starts a new momentum diffusion process and gives to the atom another chance to reach the vicinity of $p = 0$. Thus atoms accumulate in the vicinity

[8] The spontaneous emission rate $R(p)$ can be simply seen as a diffusion coefficient that has the peculiarity of varying with the momentum p.

[9] Before each spontaneous emission, the atom absorbs a laser photon. The recoil effects of photon absorption, which is a deterministic process, are not essential here and are therefore ignored.

of $p = 0$ in long-lived states: a cooling effect occurs. This cooling relies on a random walk of the atomic momentum, unlike standard laser cooling which rests on friction forces.

The most important question for a cooling process is to determine the typical momentum δp at the end of the random walk or, equivalently the temperature T (cf. Eq. (10)). It is difficult to answer it with the usual analytical or numerical methods of atomic physics[10] because very different momentum and time scales are present in the problem. A conjecture was proposed in 1988 in which the interaction time θ, defined as the time the atoms interact with the lasers, plays a crucial role. Consider an atom reaching a momentum p such that the residence time $\tau(p)$ is larger than the interaction time θ. This atom will thus keep its momentum p till the end of the experiment and will be detected with this momentum p. The conjecture consists in assuming that *only* the atoms such that

$$\tau(p) \geq \theta \tag{12}$$

keep their momentum till the end of the experiment. Obviously, this can not be strictly true: some atoms will reach a small momentum p after a significant time t has elapsed from the beginning of the interaction with the lasers so that, for them, the condition to stay at momentum p would rather be $\tau(p) \geq \theta - t$. But let us assume that condition (12) is the relevant criterion for the trapping of atoms. This predicts that a momentum peak will form with a width δp_θ given by

$$\tau(\delta p_\theta) \simeq \theta \cdot \tag{13}$$

Moreover, it can be shown that the residence time $\tau(p)$ varies as

$$\tau(p) \propto 1/p^2 \cdot \tag{14}$$

Introducing this relation into Eq. (13) gives the conjectured momentum scale δp_θ which is reached after an interaction time θ:

$$\delta p_\theta \propto \frac{1}{\sqrt{\theta}} \tag{15}$$

or, for the temperature $T_\theta \propto (\delta p_\theta)^2$ (cf. Eq. (10)),

$$T_\theta \propto \frac{1}{\theta} \cdot \tag{16}$$

This result is both interesting and surprising. Interestingly, it predicts that the temperature can be reduced towards lower and lower values when the interaction time θ is increased. The recoil temperature limit, which arises in standard laser cooling from spontaneous emission, is no more a limit here. Here indeed, spontaneous emission is present to create a random walk that brings the atoms to $p \simeq 0$, but spontaneous emission stops when the atoms reach a small enough momentum. Surprisingly, there does not seem to be *any* limit for the cooling.

[10] However, in one particular case, analytical solutions based on the usual methods have been found [AlK96, SSY97]. Numerical approaches have also been developped [CBA91] using a new type of quantum simulations.

This has motivated a series of experiments with longer and longer interaction times θ. The recoil limit was first overcome in 1988, reaching $T_\theta \simeq T_\mathrm{R}/2$ [AAK88]. Longer interaction times allowed to reach $T_\theta \simeq T_\mathrm{R}/40$ in 1994 [BSL94, Bar95] and $T_\theta \simeq T_\mathrm{R}/800$ in 1997 [SHK97], establishing a new temperature record each time. Moreover, these experiments agree well with the conjectured temperature dependence of Eq. (16).

However, one obviously needs a better understanding of what determines the temperature T_θ. A related question is the proportion of cooled atoms: can a random walk with no driving force lead to an accumulation of all the atoms in the vicinity of $p = 0$? Is there rather only a small proportion of cooled atoms? How does this fraction vary with the interaction time θ and with the number of dimensions of the random walk? As we will see below, random walk techniques and the generalized CLT provide answers to these questions.

3.2 Trapping time densities and non-ergodicity

Recently developped quantum simulations [DCM92, DZR92, Car93, CBA91] allow to follow the momentum random walk of a single atom in the process of subrecoil cooling [CBA91, BBE94, Bar95]. An example of such a random walk is represented in figure 4. We see how the random evolution of the atomic momentum sometimes leads to $p \simeq 0$ states where the atom remains for a long time because the spontaneous emission rate $R(p) = 1/\tau(p)$ vanishes in $p = 0$: this is the principle of subrecoil cooling at work.

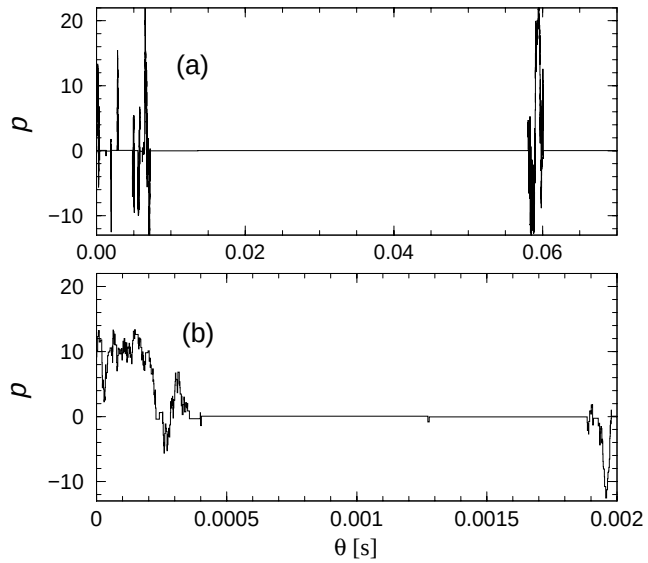


Figure 4. (a) Example of a momentum random walk resulting from a Monte-Carlo simulation of subrecoil cooling of metastable helium atoms. The unit of atomic momentum p is the momentum $\hbar k$ of the photons. The zoom (b) of the beginning of the time evolution is statistically analogous to the evolution at large scale, a fractal property typical of a Lévy flight.

More importantly, figure 4 presents an interesting statistical feature that triggered the Lévy flight approach of laser cooling. The *single* largest residence time τ_max amounts to 70 % of the total time θ while the atom has occupied 4000 different momentum states

during this total time. Thus a single event dominates a sum of a large number of events, which is an indication of a Lévy flight.

Is there really a Lévy flight in the problem? Let us estimate the probability density $f(\tau)$ of trapping times τ, defined as the times spent by an atom in an interval $[-p_{\rm trap}; +p_{\rm trap}]$ called the "trap" of size $p_{\rm trap}$ smaller than the size $\hbar k$ of a random momentum step[11]. The probability density $\rho(p)$ for an atom reaching the trap to fall in a state of momentum p can then be considered as independent of p (in one dimension):

$$\rho(p) \simeq \frac{1}{2p_{\rm trap}} \cdot \tag{17}$$

The trapping time density $f(\tau)$, given by $f(\tau)\mathrm{d}\tau = \rho(p)\mathrm{d}p$, is easily obtained using Eq. (14):

$$f(\tau) \propto \frac{1}{\tau^{1+\alpha}} \quad \text{with} \quad \alpha = \frac{1}{2} \cdot \tag{18}$$

Thus, if we consider as a first step that the interaction time θ is the sum of trapping times τ_i with the density (18)[12], the time θ is indeed a Lévy sum, whose behaviour is dictated by the generalized CLT with $\alpha < 1$. We have a Lévy flight in time, which immediately accounts for the domination of a single trapping event in figure 4 (see section 2.3).

There is a deep physical consequence of this Lévy flight, *the absence of ergodicity*. The ergodic hypothesis, an important ingredient in statistical physics, is the assumption that time averaging of a physical quantity yields the same result as ensemble averaging. Time averaging requires following a particle over a time much larger than all characteristic times of the problem. This is impossible here. Indeed, as the time θ gets larger, larger trapping time scales (up to θ) appear and the time averaging procedure does not converge. This is reflected in the fact that we have a Lévy flight on a time variable τ[13], with infinite average trapping times. Thus, subrecoil cooling is a non-ergodic process. The non-ergodicity is associated to the absence of cooling limits. The cooling goes on for ever because larger and larger trapping times $\tau(p)$, corresponding to lower and lower momenta p, can be reached with increasing θ.

3.3 Trapping, recycling and the generalized CLT

The history of an atom over a time θ_N can be seen a series of N trapping times τ_i interrupted by N times $\hat{\tau}_i$ spent out of the trap[14]. The times $\hat{\tau}_i$ are the usual 'first return times'. We also call them 'recycling times' because the atoms coming out of the trap are given another opportunity to reach the trap, they are 'recycled'.

Thus, the interaction time θ_N writes as

$$\theta_N = \tau_1 + \hat{\tau}_1 + ... + \tau_N + \hat{\tau}_N = T_N + \hat{T}_N, \tag{19}$$

[11] Under these conditions, the trapping times $\tau(p)$ are simply the residence times $\tau(p)$ of Eq. (11) and Eq. (14) in the region $[-p_{\rm trap}; +p_{\rm trap}]$.

[12] We neglect here the times $\hat{\tau}_i$ spent outside the trap. These will be taken into account in the following section.

[13] Thus, the same non-ergodic properties occur for the Arrhenius cascade at law temperatures (see section 2).

[14] Note that the initial problem is a momentum random walk, which we treat efficiently by considering the associated random walk in time, a standard method for these problems.

where

$$T_N = \sum_{i=1}^{N} \tau_i \tag{20}$$

is the total trapping time, and

$$\hat{T}_N = \sum_{i=1}^{N} \hat{\tau}_i \tag{21}$$

is the total recycling time. Both T_N and $\hat{T}_N$ are sums of independent variables. The application of the generalized CLT to these sums gives in a simple way a qualitative answer for the proportion of cooled atoms, as we discuss now.

Consider first the case in which the spontaneous emission rate $R(p)$ tends to a non-vanishing constant at large p. Then, at large p, we have a standard random walk with a constant diffusion rate. For a 1D problem, the probability density $\hat{f}(\hat{\tau})$ of first return times $\hat{\tau}$ is known to decay at large $\hat{\tau}$ as

$$\hat{f}(\hat{\tau}) \propto \frac{1}{\hat{\tau}^{1+\hat{\alpha}}} \quad \text{with} \quad \hat{\alpha} = \frac{1}{2} \cdot \tag{22}$$

It thus decays exactly in the same way as $f(\tau)$. According to the generalized CLT, for large N's, the sums T_N and $\hat{T}_N$ behave as

$$T_N \sim N^{1/\alpha} = N^2 \ , \tag{23}$$

$$\hat{T}_N \sim N^{1/\hat{\alpha}} = N^2 \cdot \tag{24}$$

Therefore, for long times (cf. large N's), one has $T_N \sim \hat{T}_N$: the atoms spend a finite fraction of their time in the trap and a finite fraction outside the trap. We thus expect the proportion of cooled atoms to tend to a constant, strictly between 0 and 1. More elaborate calculations confirm this non-trivial result.

Consider now the case in which a friction mechanism is added to prevent the atoms to diffuse to too large momenta p. This friction confines the momentum diffusion in a finite zone. In this case, $\hat{f}(\hat{\tau})$ is a narrow probability density with a finite average value. Thus, according to the usual CLT:

$$\hat{T}_N \simeq N \langle \hat{\tau} \rangle \cdot \tag{25}$$

Comparing this to Eq. (23) which is still valid here, one has

$$T_N \gg \hat{T}_N \tag{26}$$

for large N. This implies that all the atoms will be cooled, which is again confirmed by more elaborate calculations.

More complicated cases can be considered by including diffusion in 2 or 3 dimensions or by including the 'Doppler effect' which modifies the rate $R(p)$ at large p. In each case, the generalized CLT provides the asymptotic behaviours of the sums T_N and $\hat{T}_N$ which yield immediately the qualitative asymptotic proportion of cooled atoms.

3.4 Momentum distribution

Up to now, we have presented mostly qualitative results. We want to sketch here how some quantitative results are obtained.

The main features of the cooling process are given by the momentum distribution (probability density) $\mathcal{P}(p,\theta)$ of trapped atoms at time θ. This momentum distribution $\mathcal{P}(p,\theta)$ writes as an integral over the times t_l at which the atoms enter the trap for the last time:

$$\mathcal{P}(p,\theta) = \rho(p) \int_0^\theta \mathrm{d}t_l S_\mathrm{R}(t_l)\psi(\theta - t_l|p) \cdot \tag{27}$$

The quantity $\rho(p)$ is the probability density for an atom entering the trap to reach the momentum p (in one dimension, we have seen in Eq. (17) that $\rho(p) = 1/(2p_\mathrm{trap})$). The quantity $S_\mathrm{R}(t)$ called here the 'sprinkling density' is the probability density for an atom to return into the trap at time t, regardless of the number of times the atom has entered the trap before. The quantity $\psi(\tau|p) = \int_\tau^\infty f(\tau')\mathrm{d}\tau'$ is the probability that an atom remains in the trap during a time longer than τ (where $f(\tau)$ is the trapping time density defined in section 3.2).

The momentum distribution can be calculated explicitly. For instance, in a simple 1D model with infinite $\langle\tau\rangle$ and finite $\langle\hat{\tau}\rangle$, one obtains

$$\mathcal{P}(p,\theta) = h(\theta)\mathcal{G}(Ap\sqrt{\theta}) \tag{28}$$

where $h(\theta) \propto \sqrt{\theta}$ is the height of the cooled peak at $p = 0$, A is a constant. The function $\mathcal{G}(q)$ is given by $\mathcal{G}(q) = 1$ for $q \leq 1$ and by $\mathcal{G}(q) = 1 - (1 - q^{-2})^{1/\alpha}$ for $q \geq 1$. The width δp_θ of $\mathcal{P}(p,\theta)$ decays as $1/\sqrt{\theta}$, which proves the 1988 conjecture of a temperature decrease without any fundamental limit (cf. Eq. (16)). This calculation can also be done for more complicated cases in any dimension where it is very useful. For instance, one can study the influence of the exponent β in the spontaneous emission rate $R(p) \propto p^\beta$, as described in the next section.

The key point to obtain the momentum distribution is the calculation of the sprinkling density $S_\mathrm{R}(t)$. The sprinkling density is obtained relatively easily with a Laplace transform[15]. The result is interesting. If $\langle\tau\rangle$ and $\langle\hat{\tau}\rangle$ are finite, then $S_\mathrm{R}(t)$ tends to a constant at large times. This is an expected 'ergodic' result: the rate of return events is asymptotically constant[16]. On the contrary, if $\langle\tau\rangle$ or $\langle\hat{\tau}\rangle$ is infinite, then $S_\mathrm{R}(t)$ decays to 0 at large times. This is a signature of non-ergodicity: at large times, the density of return events go to 0 because the longer and longer τ_i's or $\hat{\tau}_i$'s which tend to appear slow down the diffusion. Such a process has a 'history': the measurement of $S_\mathrm{R}(t)$ at any time tells when the diffusion has started.

3.5 Optimizing laser cooling with the generalized CLT

A remarkable outcome of the usual CLT is that the statistical behaviour of Lévy sums X_N at large N is determined *only* by two parameters, $\langle x\rangle$ and $\langle x^2\rangle$. Thus, the detailed features of $f(x)$ can be forgotten if one is interested only in the large N properties of the Lévy sums. Similarly, with the generalized CLT in the cases $\alpha \leq 2$, only the asymptotic power

[15] In fact, the generalized CLT is not explicitly used in the derivation of Eq. (28).

[16] In a Poisson process, this rate is constant at any time scale.

law behaviour of $f(x)$ is relevant to determine the behaviour of Lévy sums at large N. For a positive variable x, for instance, this power law is described by two parameters only, the exponent α and the prefactor of the power law. This can provide a useful insight when confronted to a complex physical phenomenon with many parameters: the generalized CLT shows that the many physical parameters combine into only two relevant statistical parameters.

Such an insight has been applied in practice to improve a subrecoil laser cooling mechanism called Raman cooling [KaC92]. Raman cooling, like the dark resonance cooling described in section 3.1, rests on a p-dependent spontaneous emission rate $R(p)$ analogous to the one in figure 3a. The main difference is that the rate $R(p)$ results from the superposition of pulses that can be chosen nearly arbitrarily. This gives flexibility to this mechanism and makes it a good case study for cooling optimization. On the other hand, the large number of parameters ($\simeq$ 30 for the initially used sequence of pulses) to be optimized makes it necessary to find simplifying guidelines.

By carefully using the generalized CLT, we have proposed a new very simple sequence of pulses [RBB95]: it relies on 4 pulses only (compared to 14 initially); the shape of the pulses is the simplest possible while the initially used pulses were sophisticated Blackman pulses.

The results are eloquent. With the initially used sequence of 14 pulses, the temperature T_θ varied as $T_\theta \propto 1/\sqrt{\theta}$ with the interaction time θ. With the new sequence of 4 pulses, the new shape (which changes the exponent β of the rate $R(p) \simeq p^\beta$ from $\beta = 4$ to $\beta = 2$) leads to $T_\theta \propto 1/\theta$, a much faster cooling. Moreover, with this new shape, if the pulses parameters (width and position) are adapted to the considered interaction time θ, one obtains an even faster cooling $T_\theta \propto 1/\theta^{4/3}$ [RBB95, Rei96].

These predictions have been successfully tested experimentally and led to record low temperatures (2.8 ± 0.5 nK) for a cesium gas. This shows how the generalized CLT can have significant practical consequences.

4 Imperfect Lévy flights

We have presented in sections 2 and 3 two examples where the generalized CLT applied perfectly. However, there are many physical cases where the generalized CLT is useful although the conditions to apply it are not, strictly speaking, mathematically fulfilled. This may occur either because the asymptotic decay of $f(x)$ is not purely a power law or because $f(x)$ is a truncated power law.

Let us first discuss the *truncation problem*[17], i.e. the cases in which $f(x)$ decays as $1/x^{1+\alpha}$ for $x < x_0$ and decays more rapidly for $x > x_0$ so that $\langle x \rangle$ and $\langle x^2 \rangle$ are finite. In the mathematical sense, the usual CLT applies. However, due to the power law tail, *the convergence to the asymptotic gaussian for the probability density of the Lévy sums can be extremely slow*, being reached for N typically of 10^3 or larger [MaS94], while in most cases for which the usual CLT applies, the approximate convergence to a gaussian is obtained

[17] In section 3, the sums $T_N = \sum_{i=1}^{N} \tau_i$ were limited by the available interaction time θ which is also a kind of truncation. However, this *truncation of the sum itself* by an experimental parameter (here the interaction time, in other cases the system size) does not prevent the appearance of all the important effects of the generalized CLT; on the contrary, the fact that the truncation value, however large it may be, has an effect on the measured value is a signature of the generalized CLT. The truncations dealt with in section 4 bear on the density $f(x)$ itself and imply a departure from the generalized CLT.

very rapidly, with typically $N \simeq 4-5$. As, in practice, one often deals with sums of a moderately large number N of terms, the behaviour of the Lévy sums is often dictated by the Lévy laws for relevant N values, while the gaussian behaviour is recovered only for irrelevantly large N values.

Second, there are *broad probability densities which decay only approximately as power laws*. An example is provided by broad lognormal distributions, which have of course a finite second moment. They can be rewritten as power laws $1/x^{1+\alpha(x)}$ with a logarithmically varying exponent $\alpha(x)$. If the logarithmic part of $\alpha(x)$ is small enough, then the generalized CLT gives at least some qualitative guidelines for the behaviour of the Lévy sums. We have used such guidelines to study the tunneling of electrons through a thin layer of insulator, a problem which has both basic and applied interests. The striking finding related to the generalized CLT has been that the typical current density varies by more than 200 *depending on the scale* at which it is measured [Bar97, DBB98, DHB98] (see also [LaB93]), while the typical current density should be scale independent if there were no tails in the probability density of the current.

Acknowledgements

The Institut de Physique et de Chimie des Matériaux de Strasbourg is Unité Mixte de Recherches 7504 of Centre National de la Recherche Scientifique and of Université Louis Pasteur. The section 3 is mostly based on a long paper (to be published) in collaboration with J.P. Bouchaud (Saclay), A. Aspect (Orsay) and C. Cohen-Tannoudji (Paris). I also thank all the researchers with whom I collaborated on the quoted papers.

References

[AAK88] A. Aspect, E. Arimondo, R. Kaiser, N. Vansteenkiste, and C. Cohen-Tannoudji, *Laser Cooling below the One-Photon Recoil Energy by Velocity-Selective Coherent Population Trapping*, Phys. Rev. Lett. **61**, 826-829 (1988).

[AAK89] A. Aspect, E. Arimondo, R. Kaiser, N. Vansteenkiste and C. Cohen-Tannoudji, *Laser cooling below the one-photon recoil energy by velocity-selective coherent population trapping : theoretical analysis*, J. Opt. Soc. Am. B **6**, 2112-2124 (1989).

[AlK96] V. A. Alekseev and D. D. Krylova, *Fraction number of trapped atoms and velocity distribution function in sub-recoil laser cooling scheme*, Optics Communications **124**, 568-578 (1996).

[Bar95] F. Bardou, *Ph. D. Thesis*, University of Paris XI Orsay, chapters IV and V (1995).

[Bar97] F. Bardou, *Rare events in quantum tunneling*, Europhys. Lett. **39**, 239-244 (1997).

[BBA99] F. Bardou, J.-P. Bouchaud, A. Aspect, and C. Cohen-Tannoudji, *Non-ergodic cooling: subrecoil laser cooling and Lévy statistics*, to be published.

[BBE94] F. Bardou, J.-P. Bouchaud, O. Emile, A. Aspect and C. Cohen-Tannoudji, *Subrecoil Laser Cooling and Lévy Flights*, Phys. Rev. Lett. **72**, 203-206 (1994).

[BoC97] M. Boguna and A. Corral, *Long-Tailed Trapping Times and Lévy Flights in a Self-Organized Critical Granular System*, Phys. Rev. Lett. **78**, 4950-4953 (1997).

[BoD95] J.P. Bouchaud and D.S. Dean, *Aging in Parisi's Tree*, J. Phys. I (France) **5**, (1995) 265-286.

[BoG90] J.P. Bouchaud and A. Georges, *Anomalous diffusion in disordered media: statistical mechanisms, models and physical applications*, Phys. Rep. **195**, 127-293 (1990).

[BSL94] F. Bardou, B. Saubaméa, J. Lawall, K. Shimizu, O. Emile, C. Westbrook, A. Aspect and C. Cohen-Tannoudji, *Sub-recoil laser cooling with precooled atoms*, C. R. Acad. Sci. Paris **318**, Série II, 877-885 (1994).

[Car93] H. Carmichael, *An Open Systems Approach to Quantum Optics*, Springer-Verlag (1993).

[CBA91] C. Cohen-Tannoudji, F. Bardou, and A. Aspect, *Review of fundamental processes in laser cooling*, Proceedings of Laser Spectroscopy X (Font-Romeu, 1991), edited by M. Ducloy, E. Giacobino, and G. Camy, 3-14 (World Scientific, Singapore, 1992).

[DBB98] V. Da Costa, F. Bardou, C. Béal, Y. Henry, J.P. Bucher, and K. Ounadjela, *Nanometric cartography of tunnel current in metal-oxide junctions*, J. Appl. Phys. **83**, 6703-6705 (1998).

[DCM92] J. Dalibard, Y. Castin, and K. Mølmer, *Wave-Function Approach to Dissipative Processes in Quantum Optics*, Phys. Rev. Lett. **68**, 580-583 (1992).

[DHB98] V. Da Costa, Y. Henry, F. Bardou, and K. Ounadjela, *Experimental evidence and consequences of rare events in quantum tunneling*, submitted (1998).

[DZR92] R. Dum, P. Zoller, abd R. Ritsch, *Monte-Carlo simulation of the atomic master equation for spontaneous emission*, Phys. Rev. A **45**, 4879-4887 (1992).

[KaC92] M. Kasevich and S. Chu, *Laser Cooling below a Photon Recoil with Three-Level Atoms*, Phys. Rev. Lett. **69**, 1741-1744 (1992).

[KSW97] H. Katori, S. Schlipf, and H. Walther, *Anomalous Dynamics of a Single Ion in an Optical Lattice*, Phys. Rev. Lett. **79**, 2221-2224 (1997).

[KSZ96] J. Klafter, M.F. Shlesinger and G. Zumofen, *Beyond Brownian Motion*, Physics Today **49**, 33-39 (February 1996).

[LaB93] F. Ladieu and J.P. Bouchaud, *Conductance statistics in small GaAs:Si wires at low temperatures. I. Theoretical analysis: truncated quantum fluctuations in insulating wires*, J. Phys. I (France) **3**, 2311-2320 (1993).

[MaS94] R.N. Mantegna and H.E. Stanley, *Stochastic Process with Ultraslow Convergence to a Gaussian: The Truncated Lévy Flights*, Phys. Rev. Lett. **73**, 2946-2949 (1994). M.F. Stanley, *Comment*, Phys. Rev. Lett. **74**, 4959 (1995).

[MEZ96] S. Marksteiner, K. Ellinger, and P. Zoller, *Anomalous diffusion and Lévy walks in optical lattices*, Phys. Rev. A **53**, 3409-3430 (1996).

[OBL90] A. Ott, J.P. Bouchaud, D. Langevin, and W. Urbach, *Anomalous Diffusion in "Living Polymers": a Genuine Lévy Flight?*, Phys. Rev . Lett. **65**, 2201-2204 (1990).

[RBB95] J. Reichel, F. Bardou, M. Ben Dahan, E. Peik, S. Rand, C. Salomon and C. Cohen-Tannoudji, *Raman Cooling of Cesium below 3 nK: New Approach Inspired by Lévy Flight Statistics*, Phys. Rev. Lett. **75**, 4575-4578 (1995).

[Rei96] J. Reichel, *PhD thesis*, University of Paris VI (1996).

[RoB84] C.T. Rogers and R.A. Buhrman, *Composition of* $1/f$ *Noise in Metal-Insulator-Metal Tunnel Junctions*, Phys. Rev. Lett. **53**, 1272-1275 (1984).

[SHK97] B. Saubaméa, T.W. Hijmans, S. Kulin, E. Rasel, E Peik, M. Leduc, and C. Cohen-Tannoudji, *Direct measurement of the spatial correlation function of ultracold atoms*, Phys. Rev. Lett. **79**, 3146-3149 (1997).

[Shl88] M.F. Shlesinger, *Fractal time in condensed matter*, Ann. Rev. Phys. Chem. **39**, 269-290 (1988).

[SWS93] T.H. Solomon, E.R. Weeks, and H.L. Swinney, *Observation of Anomalous Diffusion and Lévy Flights in a Two-Dimensional Rotating Flow*, Phys. Rev. Lett. **71**, 3975-3978 (1993).

[SZF95] M.F. Shlesinger, G.M. Zaslavsky, and U. Frisch (Eds), *Lévy flights and related topics in physics*, Springer-Verlag (Berlin), (1995).

[SZK93] M.F. Shlesinger, G.M. Zaslavsky, and J. Klafter, *Strange kinetics*, Nature **363**, 31-37 (1993).

[SSY97] S. Schaufler, W. P. Schleich, and V. P. Yakovlev, *Scaling and asymptotic laws in subrecoil laser cooling*, Europhys. Lett. **39**, 383-388 (1997).

[TLS95] C. Tsallis, S.V.F. Lévy, A.M.C. Souza, and R. Maynard, *Statistical-Mechanical Foundation of the Ubiquity of Lévy Distributions in Physics*, Phys. Rev. Lett. **75**, 3589-3593 (1995).

[Tsa95] C. Tsallis, *Non-extensive thermostatistics: brief review and comment*, Physica A **221**, 277-290 (1995).

[Tsa97] C. Tsallis, *Lévy distributions*, Physics World, 42-45 (july 1997).

[ZaA95] D.H. Zanette and P.A. Alemany, *Thermodynamics of anomalous diffusion*, Phys. Rev. Lett. **75**, 366-369 (1995).

[ZuK94] G. Zumofen and J. Klafter, *Spectral random walk of a single molecule*, Chem. Phys. Lett. **219**, 303-309 (1994).

Superposition of Ornstein - Uhlenbeck Type Processes

Ole E. Barndorff-Nielsen
MaPhySto[1]
Aarhus University

Abstract

A class of superpositions of Ornstein-Uhlenbeck type processes is constructed, in terms of integrals with respect to independently scattered random measures. Under specified conditions the resulting processes exhibit long range dependence. By integration the superpositions yield cumulative processes with stationary increments, and integration with respect to processes of the latter type is defined. A limiting procedure results in processes that, in the case of square integrability, are second order selfsimilar with stationary increments. Certain other of the limiting processes are stable and selfsimilar with stationary increments.

1 Introduction

In studying observational processes that show significant dependence over long time periods a possible approach is to try to model the process or processes at hand by means of superposition of independent processes with short range dependence.

Cox (1984), in a review of the roles of long range dependence and selfsimilarity in statistics, introduced, on a heuristic basis, a method for construction of processes with long range dependence by weighted integration of processes with short range dependence. In Cox (1991) this was applied in a study of the relations of nonlinearity and time irreversibility to long range dependence. A somewhat similar, rigorously based, method was proposed in Barndorff-Nielsen, Jensen and Sørensen (1990) and there applied to the modelling of velocity fields in stationary turbulence, cf. also Barndorff-Nielsen, Jensen and Sørensen (1993, 1998).

Recent work on modelling observational series of financial assets have described log price processes as following a diffusion type model where the squared diffusion coefficient itself obeys a stochastic differential equation and

[1]MaPhySto - Centre for Mathematical Physics and Stochastics, funded by a grant from The Danish National Research Foundation

constitutes a stationary process. In extension of this, weighted sums of such stationary processes were used in order to capture the timewise dependencies in the price developments that are an essential feature of the financial markets. More specifically, superposition of independent Ornstein-Uhlenbeck type processes have provided flexible and analytically tractable parametric models. The integrated squared volatility process equals the quadratic variation of the log price process and plays an essential role in the analysis and applications of the models. See Barndorff-Nielsen (1998b) and Barndorff-Nielsen and Shephard (1998a,b,c).

It should also be noted that questions of moduli of continuity and large increments of infinite sums of classical, i.e. Gaussian, Ornstein-Uhlenbeck processes have been discussed in papers by Csáki, Csörgõ, Lin and Révesz (1991) and Lin (1995). See also Walsh (1981).

These developments have motivated the present study of superposition of Ornstein-Uhlenbeck type processes and their integrals, based on the theory of independently scattered random measures. An overall aim is to develop flexible classes of processes that incorporate long range dependence and selfsimilarity-like properties and are capable, furthermore, of describing some of the other key distributional features of typical data in finance, turbulence and other fields. We note that, in several respects, the class of strictly selfsimilar processes is too limited in scope for such modelling purposes. In particular, they cannot simultaneously show semiheavy tailed behaviour for short time lags and close to Gaussian behaviour for large time lags, such as do typical observational series from both finance and turbulence.

We recall that a stationary process $x = \{x(t)\}_{t\in\mathbf{R}}$ is said to exhibit long range dependence if the correlation function r of x behaves as

$$r(u) \sim L(u)u^{-2\bar{H}}$$

for $u \to \infty$ and where L is a slowly varying function and $\bar{H} \in (0, \frac{1}{2})$. Throughout we shall write $H = 1 - \bar{H}$ and we assume that $H \in (0, 1]$. When x is long range dependent the cumulative process, x^* say, derived from x is approximately second order selfsimilar, see for instance Cox (1984) or Barndorff-Nielsen, Jensen and Sørensen (1990).

A process $x^* = \{x^*(t)\}_{0\leq t}$ is selfsimilar with exponent H if

$$\{x^*(ct)\}_{t\in\mathbf{R}_+} \stackrel{\mathcal{L}}{=} c^H\{x^*(t)\}_{t\in\mathbf{R}_+}$$

for all $c > 0$. In that case one says that x^* is H-ss, and if, moreover, x^* has stationary increments we write H-sssi. For a comprehensive discussion of selfsimilarity, see Samorodnitsky and Taqqu (1994).

An H-ss process whose increments are stationary to second order (at least) will be referred to as an H-sssi$_2$ process. A class of such processes, driven by bivariate Lévy processes, is discussed in Barndorff-Nielsen and Pérez-Abreu (1998).

Further, if a process has stationary increments and is square integrable with the same type of covariance function as if it was selfsimilar we write H-ss$_2$si.

In the sequel we shall use the following notation for cumulant and Laplace transforms of a random variate x:

$$\mathrm{C}\{\zeta \ddagger x\} = \log \mathrm{E}\{e^{i\zeta x}\}$$

$$\bar{\mathrm{K}}\{u \ddagger x\} = \log \mathrm{E}\{e^{-ux}\}$$

For instance, if x is a random variable of the form $x = \sigma\varepsilon$ where σ and ε are independent with ε standard normal and σ positive (a form of key importance in finance) then

$$\mathrm{C}\{\zeta \ddagger x\} = \bar{\mathrm{K}}\{\zeta^2/2 \ddagger \sigma^2\}$$

Section 2 summarizes results on Lévy processes, selfdecomposability, Ornstein-Uhlenbeck type processes, and independently scattered random measures, needed in the subsequent sections. In Section 3 a class of superpositions, in terms of integrals, of Ornstein-Uhlenbeck type processes is introduced; under certain conditions the resulting processes will exhibit long range dependence. By integration the superpositions yield cumulative processes with stationary increments and these are investigated in Section 4. Integration of real functions with respect to the cumulative processes is considered in Section 5. A limiting procedure, discussed in Section 6, results in processes that, in the case of square integrability, are second order selfsimilar with stationary increments, i.e. H-ss$_2$si. Certain other of the limiting processes are stable and (strictly) selfsimilar with stationary increments.

References

[1] Barndorff-Nielsen, O.E. (1997): Normal inverse Gaussian distributions and stochastic volatility modelling. *Scand. J. Statist.* **24**, 1-14.

[2] Barndorff-Nielsen, O.E. (1998a): Probability and statistics: self-decomposability, finance and turbulence. In Accardi, L. and Heyde, C.C. (Eds.): *Probability Towards 2000*. Proceedings of a Symposium held 2-5 October 1995 at Columbia University. New York: Springer-Verlag. Pp. 47-57.

[3] Barndorff-Nielsen, O.E. (1998b): Processes of normal inverse Gaussian type. *Finance and Stochastics* **2**, 41-68.

[4] Barndorff-Nielsen, O.E., Jensen, J.L. and Sørensen, M. (1990): Parametric modelling of turbulence. *Phil. Trans. Roy. Soc. Lond.* **332**, 439-455.

[5] Barndorff-Nielsen, O.E., Jensen, J.L. and Sørensen, M. (1993): A statistical model for the streamwise component of a turbulent wind field. *Ann. Geophys.* **11**, 99-103.

[6] Barndorff-Nielsen, O.E., Jensen, J.L. and Sørensen, M. (1998): Some stationary processes in discrete and continuous time. *Adv. Appl. Prob.* (To appear.)

[7] Barndorff-Nielsen, O.E. and Pérez-Abreu, V. (1998): Stationary and selfsimilar processes driven by Lévy processes. Research Report 1, April 1998. MaPhySto, Aarhus University. (Submitted.)

[8] Barndorff-Nielsen, O.E. and Shephard, N. (1998a): Aggregation and model construction for volatility models. CAF - Centre for Analytical Finance, Aarhus. Working Paper Series No. 10.

[9] Barndorff-Nielsen, O.E. and Shephard, N. (1998b): Incorporation of a leverage effect in a stochastic volatility model. Research Report No. 18. MaPhySto.

[10] Barndorff-Nielsen, O.E. and Shephard, N. (1998c): Ornstein-Uhlenbeck volatility models. In preparation.

[11] Cox, D.R. (1984): Long range dependence: a review. In *Statistics: an appraisal* (Ed.: H.A. David and H.T. David). Iowa State University Press.

[12] Cox, D.R. (1991): Long range dependence, nonlinearity and time irreversibility. *J. Time Series Analysis* **12**, 329-335.

[13] Csáki, E., Csörgõ, M., Lin, Z.Y. and Révész, P. (1991): On infinite series of independent Ornstein - Uhlenbeck processes. *Stoch. Proc. Appl.* **39**, 25-44.

[14] Jurek, Z.J. and Mason, J.D. (1993): *Operator-Limit Distributions in Probability Theory*. New York: Wiley.

[15] Kwapień, S. and Woyczyński, N.A. (1992): *Random Series and Stochastic Integrals: Single and Multiple.* Boston: Birkhäuser.

[16] Lin, Z.Y. (1995): On large increments of infinite series of Ornstein-Uhlenbeck processes. *Stoch. Proc. Appl.* **60**, 161-169.

[17] Rajput, B. and Rosinski, J. (1989): Spectral representations of infinitely divisible processes. *Probab. Th. Rel. Fields* **82,** 451-487.

[18] Samorodnitsky, G. and Taqqu, M.S. (1994): *Stable Non-Gaussian Random Processes.* New York: Chapman and Hall.

[19] Walsh, J.B. (1981): A stochastic model of neural response. *Adv. Appl. Prob.* **13**, 231-281.

Structure of shocks in Burgers turbulence with stable noise initial data

Jean Bertoin

Laboratoire de Probabilités
Université Pierre et Marie Curie
4, Place Jussieu
F-75252 Paris Cedex 05, France

Burgers has introduced the equation

$$\partial_t u + \partial_x \left(u^2/2 \right) \, = \, \varepsilon \partial_{xx}^2 u$$

as a simple model of hydrodynamic turbulence for compressible fluids, where the parameter $\varepsilon > 0$ describes the viscosity of the fluid, and the solution is meant to represent the velocity of a fluid particle located at x at time t. Roughly, the dynamic of the system of particles corresponds to completely inelastic shocks, in the sense that if two (clumps of) particles collide at a given time, then they form a larger clump of particles in such a way that mass and momentum are preserved. Although it is known that this is not an accurate model for turbulence, Burgers equation is still widely used in physical problems such as for instance the study of shock wave formation in compressible fluids, or that of the formation of large clusters in the universe, or also as a simplified version of more elaborate models of turbulence (e.g. the Navier-Stokes equation). To present this work with some mathematical rigor, we first review some classical material on the Burgers turbulence that can be found for instance in [5] or [6].

1 Some basic features on Burgers equation

Given an initial velocity, Burgers equation with viscosity $\varepsilon > 0$ possesses a unique solution u_ε, and that u_ε converges as $\varepsilon \to 0+$ to a solution $u_0 = u$ to the inviscid equation, which is usually referred to as the Hopf-Cole (or entropic) solution. The Hopf-Cole solution has a simple expression in terms of potential functions. If we introduce ψ by $u = -\partial_x \psi$, then the potential at time t is expressed in terms of the Legendre transform of the function $a \to \psi(a,0) - a^2/2t$:

$$\psi(x,t) \, = \, \sup_{a \in \mathbb{R}} \left\{ \psi(a,0) - \frac{(x-a)^2}{2t} \right\} . \tag{1}$$

Of course, we implicitly supposed that

$$\psi(a,0) \, = \, o(a^2) \qquad \text{as } |a| \to \infty \, , \tag{2}$$

so that the quantity in (1) is finite. Note also that the formula (1) makes sense whenever the initial velocity $u(\cdot,0) = -\partial_x \psi(\cdot,0)$ is the derivative (in the sense of Schwartz) of a function. To that end, we shall merely assume that the initial potential $\psi(\cdot,0)$ has only discontinuities of the first kind, i.e. there exists left and right limits at each point; and it will then be convenient to work with the version that is right-continuous.

For the sake of simplicity, we shall focus on time $t = 1$ in the sequel. Of course, our results are valid at any positive time; this can be easily checked by a simple scaling argument.

The structure of the shocks in Burgers turbulence is conveniently described in terms of the inverse Lagrangian function which we now introduce. We denote by $a(x)$ is the largest location a at which the supremum in (1) is reached, i.e.

$$a(x) \,=\, \sup \left\{ a \in \mathbb{R} : \sup_{b \geq a} \left(\psi(b,0) - \frac{(x-b)^2}{2} \right) = \sup_{b \in \mathbb{R}} \left(\psi(b,0) - \frac{(x-b)^2}{2} \right) \right\} .$$

We stress that the inverse Lagrangian function $x \to a(x)$ is right-continuous and increasing. Its right-continuous inverse $a \to x(a)$, which is given by

$$x(a) \,=\, \inf \{ y \in \mathbb{R} : a(y) > a \} \ ,$$

is called the Lagrangian function; alternatively, it can be viewed as the (right) derivative of the convex hull of the function $a \to -\psi(a,0) + a^2/2$. From the point of view of hydrodynamic turbulence, the Lagrangian function describes the position at time 1 of the fluid particle initially located at a.

We see that if a discontinuity of the inverse Lagrangian function occurs at some point x, i.e.

$$\lim_{y \to x-} a(y) \,:=\, a(x-) \,<\, a(x) \,,$$

then the Lagrangian function is constant on the interval $[a(x-), a(x))$, which means that at time 1, there is a clump located at x which is formed by all the particles that were initially in the interval $[a(x-), a(x))$. Similarly, if the inverse Lagrangian function stays constant on some interval $[x, y)$, then the Lagrangian function never takes values in the open interval (x, y), which means that at time 1, there are no fluid particle in (x, y).

This motivates the following definition. We first introduce the closed range of the inverse Lagrangian function,

$$A \,=\, \{ y = a(x) \text{ or } y = a(x-) \text{ for some } x \in \mathbb{R} \} \ .$$

The open set $\mathbb{R} - A$ has a canonical decomposition into disjoint open intervals of the type $((a(x-), a(x))$; their closures $[a(x-), a(x)]$ are called the shock intervals. A Lagrangian shock point is a point that belongs to some shock interval. A Lagrangian regular point is a point in A that is isolated neither to its left nor to its right in A. We thus have a natural partition of $\mathbb{R}$ into the set of Lagrangian regular points and the set of Lagrangian shock points. From the point of view of hydrodynamic turbulence for compressible fluids, a Lagrangian shock point (respectively, a Lagrangian regular point) represents the initial location of a particle that belongs to some clump at time 1 (respectively, that has not been involved in the shocks induced by the turbulence before time 1).

One says that the shock structure is discrete if A is a discrete set. This means that there are only finitely many shock intervals in a given compact set and there exists no Lagrangian regular points. Finally one calls (x, y) is a rarefaction interval if the inverse Lagragian function stays constant on $[x, y)$.

2 Stable noise initial data

There is an abundant literature on the inviscid Burgers equation with random initial data. An interesting problem in this field is to obtain qualitative results on the shock structure. Sinai [6] has proven that when the initial velocity is given by a Brownian motion, then the set of Lagrangian regular points has Hausdorff dimension $1/2$ and that there are no rarefaction intervals. When the initial velocity is a Gaussian white noise, Avellaneda and E [1] have shown that the shock structure is discrete, in the sense that at time $t > 0$, there are no Lagrangian regular points and only finitely many clumps of particles are left in a given compact set. Quite recently, numerical simulations led Janicki and Woyczynski [4] to the conjecture that when the initial velocity is a stable Lévy process of index $\alpha \in (1, 2]$, the Hausdorff dimension of Lagrangian regular points is $1/\alpha$ (this conjecture has been proven mathematically in [2] when the Lévy process has no positive jumps).

We consider here the case when the initial velocity is given by a stable Lévy noise. Specifically, if we introduce the initial potential $\psi(\cdot, 0)$, which is formally defined by $\partial_x \psi(x, 0) = -u(x, 0)$, then the process $\psi(\cdot, 0)$ has independent and homogeneous increments and its one-dimensional distributions are stable laws with index $\alpha \in (1/2, 2]$. This situation naturally appears as limit in a large class of renormalized potentials, see [3].

Here are our main results in this setting.

Theorem 1

Suppose that initial potential $\psi(\cdot, 0)$ is a stable Lévy process with index $\alpha \in (1/2, 2]$. If either

(i) $\alpha \in (1/2, 1)$ *and $\psi(\cdot, 0)$ is completely asymmetric (i.e. has monotone paths)*

or

(ii) $$\alpha \in (1, 2]\,,$$

then the shock structure is discrete a.s. Otherwise (i.e. if $\alpha \in (1/2, 1]$ and the noise is not completely asymmetric) the probability that there exists Lagrangian regular points is one, but the probability that a fixed point (say, 0) is regular is zero.

Informally, Theorem 1 suggests that for $\alpha > 1$, the shocks induced by Burgers turbulence are numerous and strong enough to involve every single fluid particle at any time $t > 0$ and to create only finitely many clusters on any given compact interval. For $\alpha \in (1/2, 1]$, the initial data is not as rough. However in the completely asymmetric case, the monotonicity of the initial potential implies that all the particles are moving in the same direction, and this explains why again the shock structure is discrete. On the other hand, when the noise is not completely

asymmetric, the monotonicity is lost and thanks to compensations that occur when clumps of particles with opposite velocity collide, some exceptional particles are not involved in the turbulence.

Theorem 2

Suppose that initial potential $\psi(\cdot, 0)$ is a Cauchy process. Then with probability one there are no rarefaction intervals.

The absence of rarefaction intervals means that the Lagrangian function $a \to x(a)$ is continuous. On the other hand, it only increases on the set of Lagrangian regular points, and it follows from Theorem 1 and Tonelli's theorem that the latter has Lebesgue measure zero a.s. In the terminology used by Sinai [6], one says that the Lagrangian function is a complete devil staircase.

In the case $\alpha \in (1/2, 1]$, the proofs essentially rely on known sample path properties of stable Lévy processes which have been obtained in the 70's by Fristedt, Hawkes, Monrad and Silverstein. The argument to establish that the shock structure is discrete when $\alpha \in (1, 2]$ is less direct; it requires some material on fluctuation theory for Lévy processes.

References

[1] M. Avellaneda and W. E: Statistical properties of shocks in Burgers turbulence. *Commun. Math. Phys.* **172** (1995), 13-38.

[2] J. Bertoin: The inviscid Burgers equation with Brownian initial velocity, *Comm. Math. Phys.* **193** (1998), 397-406.

[3] J. Bertoin: Large deviation estimates in Burgers turbulence with stable noise initial data, *J. Stat. Phys.* **91** No. 3/4 (1998), 655-667.

[4] A. W. Janicki and W. A. Woyczynski: Hausdorff dimension of regular points in stochastic flows with Lévy α-stable initial data. *J. Stat. Phys.* **86** (1997), 277-299.

[5] Z. S. She, E. Aurell and U. Frisch: The inviscid Burgers equation with initial data of Brownian type. *Commun. Math. Phys.* **148** (1992), 623-641.

[6] Ya. Sinai: Statistics of shocks in solution of inviscid Burgers equation. *Commun. Math. Phys.* **148** (1992), 601-621.

THE SUPREMUM OF LEVY PROCESS WITH LIGHT TAIL

MICHAEL BRAVERMAN

Let $X(t), 0 < t < 1$, be a Lévy process. We consider the problem about tail behavior of the $sup\{X(t) : 0 < t < 1\}$. For Brownian motion the famous Lévy's result states that the distribution of the supremum is the same as the distribution of $|X(1)|$. Later some authors proved that under some additional conditions the tail $P(\{supX(t) : 0 < t < 1\} > x)$ is equivalent to the right tail of $X(1)$ (S.Berman (1986), E.Willekens (1987), M.Marcus (1987), J.Rosinski and G.Samorodnitsky (1993), J.Albin (1993), M.Braverman and G.Samorodnitsky (1995), M.Braverman (1997)). The main request is that the tail of $X(1)$ is to be heavy enough.

Here we consider the same question for Lévy processes with "light" tails, which means that for the corresponding Lévy measure L, the right tail of the convolution $L * L$ is heavier then the right tail of L. We show that if $X(t)$ is a Poisson process without drift and with such a tail, then the mentioned above relation holds. An example is given for which the tails of supremun and of $X(1)$ are incomparable.

Ben Gurion University, Israel
E-mail address: braver@CS.bgu.ac.il

Construction of continuous processes with arbitrary p-variation

Jean Bertoin

Laboratoire de Probabilités, tour 56

Université Pierre et Marie Curie

4 place Jussieu

75252 Paris Cedex 05

France

María Emilia Caballero

Instituto de Matemáticas, UNAM

Circuito Exterior, C.U.

04510 D.F. México

México

We present an abridged version whithout proofs of several results related to *the Cauchy principal value* of a Lévy process focusing on an application that leads to the construction of continuous processes with arbitrary p-variation.

We will work with a recurrent real valued Lévy process $(X_t : t \geq 0)$ with no negative jumps and gaussian coefficient zero. Let ψ denote its laplace exponent,

$$\mathbb{E}^0 (\exp -\lambda X_t) = \exp (t\psi(\lambda)), \qquad t, \lambda \geq 0 \tag{1}$$

We know that the *recurrence* of X is equivalent to

$$\lim_{\lambda \to 0} \frac{\psi(\lambda)}{\lambda} = 0 \tag{2}$$

This enables us to have the following Lévy-Khinchine formula for ψ:

$$\psi(\lambda) = \int_0^\infty \left(e^{-\lambda x} - 1 + \lambda x\right) \Lambda(dx) \tag{3}$$

with Λ the Lévy measure of X on $(0,\infty)$ such that $\int_0^\infty \left(l \wedge x^2\right) \Lambda(dx) < \infty$.

We shall also assume that

$$\int^\infty \frac{d\lambda}{\psi(x)} < \infty\,. \tag{4}$$

This condition implies that $\lim_{x\to\infty} \frac{\psi(\lambda)}{\lambda} = \infty$, since ψ is increasing. This last condition on ψ is a necessary and sufficient condition for 0 to be regular for itself, and this allows us to define the local time at level zero as the occupation density at 0:

$$L_t^\circ = \lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{\{|X_s|<\varepsilon\}} ds\,.$$

The inverse $\sigma(t) = \inf\{s \geq 0 \,:\, L_s^\circ > t\}\quad t \geq 0$, is a subordinator. All these results can be found in [2].

Using Itô's theory of excursions we introduce the excursion process corresponding to the local time just specified, $(e_s\colon s \geq 0)$ defined by

$e_s\colon [0,\delta_s] \to \mathbb{R}$

$e_s(t) = X_{\sigma(s^-)+t} \qquad\qquad t \in [0,\delta_s]$

where δ_s is the duration of the excursion.

We also use the fact that, in this case, the characteristic measure n of the excursions away from 0 of X is supported by the set of excursions that first stay negative then jump across 0 and then they stay positive until they return to 0. (See [1]).

Let

$$\begin{aligned} g_t &= \sup\{s < t\colon X_s = 0\} = \sigma(L_{t^-}) \\ d_t &= \inf\{s > t\colon X_s = 0\} = \sigma(L_t) \end{aligned}$$

so that $[g_t, d_t]$ is an excursion interval.

Construction of the process:

The following results can be found in [3].

1. The condition (4) is a necessary and sufficient condition for the absolute convergence of

$$\Delta_s := \int_{\sigma(s^-)}^{\sigma(s)} \frac{dt}{X_t} = \int_0^{\delta_s} \frac{dt}{e_s(t)}$$

2. $(\Delta_s : s \geq 0)$ is a Poisson point process with characteristic measure $x^{-2}dx$.

   Let

$$\begin{aligned} \Delta_s^+ &:= \int_0^{\delta_s} \mathbf{1}_{\{e_s(t)>0\}} \frac{dt}{e_s(t)} = \int_{\gamma_s}^{\sigma(s)} \frac{dt}{X_t} \\ \Delta_s^- &:= \int_0^{\delta_s} \mathbf{1}_{\{e_s(t)<0\}} \frac{dt}{e_s(t)} = \int_{\sigma(s^-)}^{\gamma(s)} \frac{dt}{X_t} \,. \end{aligned}$$

   Here $\gamma_s = \inf\{u > \sigma(s^-) : X_u > 0\}$ is the time when process jumps across zero.

   Then $(\Delta_s^+ : s \geq 0)$ and $(\Delta_s^- : s \geq 0)$ are also Poisson point processes and if μ^+, μ^- are the corresponding characteristic measures we deduce their distributions from Theorem 3.4 in [3], and we get:

$$\mu^+([t,\infty)) = \mu^-([t,\infty)) = \frac{\psi(\varphi(t))}{\varphi(t)} \qquad t > 0$$

   where φ is the inverse function of $f : (0,\infty) \to (0,\infty)$, $f(t) := \displaystyle\int_t^\infty \frac{d\xi}{\psi(\xi)}$ which under our hypothesis is a bijection .

We follow [4] in order to define $(C_t : t \geq 0)$ as

$$\begin{aligned} C_t &:= C_{g_t} + \int_{g_t}^t \frac{ds}{X_s} = C_{\sigma(L_t^-)} + \int_{g_t}^t \frac{ds}{X_s} \\ &= C_{d_t} - \int_t^{d_t} \frac{ds}{X_s} = C_{\sigma(L_t)} - \int_t^{d_t} \frac{ds}{X_s} \end{aligned}$$

where

$$\begin{aligned} C_{\sigma(s)} &= \lim_{\varepsilon \to 0^+} \sum_{0 \leq t \leq s} \mathbf{1}_{\{|\Delta_t|>\varepsilon\}} \Delta_t \qquad s \geq 0 \\ C_{\sigma(s)} &= \lim_{\varepsilon \to 0^+} \sum_{0 \leq t \leq s} \mathbf{1}_{\{|\Delta_t|>\varepsilon} \Delta_t \qquad s \geq 0 \,. \end{aligned}$$

In [3] we find the following result:

$\big(C_{\sigma(s)}\colon s \geq 0\big)$ is a Cauchy process with parameter π.

This is a remarkable fact since it agrees with the similar result in the following cases:

(i) X is a Brownian Motion (see [5])

(ii) X is a Symmetric Lévy process (see [6])

(iii) X is a Lévy process with suficiently smooth local times (see [2])

Observe that in these three cases the construction the Cauchy principal value of X is based on the existence and smothness of $(L(t,x)\colon t \geq 0, x \in \mathbb{R})$ on the space variable. The local times of X in our case do not have this condition and that is why we give this alternative construction of C, which can be done thanks to the results in [3].

Regularity properties of $(C_t\colon t \geq 0)$**.**

(1) $(C_t\colon t \geq 0)$ is a continuous process.

(2) For every $T > 0$, we have the following inequality for the p-variation of C.

$$\begin{aligned} m_p\left(\sum_{s\leq T}(\Delta_s^+) + \sum_{s\leq T}(\Delta_s^-)^p\right) &\leq V^p_{\sigma(T)}(C) \\ &\leq M_p\left(V_T^p\left(C_{\sigma(\cdot)}\right) + \sum_{s\leq T}\left(\Delta_s^+\right)^p + \sum_{s\leq T}\left(\Delta_s^-\right)^p\right) \end{aligned}$$

where m_p, M_p are constants which depend only on p.

Recall that, $V_t^p(h) = \sup_\tau V_t^p\,[h,\tau] = \sup_\tau \{\sum_{i=1}^n |h(t_i) - h(t_{i-1})|^p\}$ where $h\colon [0,t] \to \mathbb{R}$ is a function and τ runs over all partitions of $[0,t]$).

Since the p variation of a Cauchy process is well known (it is finitefor any $p > 1$), the variation of C will only depend on the p variation of the Poisson point process $(\Delta_t^+\colon t \geq 0)$ and $(\Delta_t^-\colon t \geq 0)$. These facts allows us to prove.

(3) Theorem. *For any $p > 1$, $T > 0$,*

(a) *If* $\int_{a.s.}^{\infty} (f(x))^{p-1} \frac{dx}{x} < \infty$ *then C has finite p-variation on $[0,T]$ a.s.*

(a) *If* $\int_{a.s.}^{\infty} (f(x))^{p-1} \frac{dx}{x} = \infty$ *then C has infinite p-variation on $[0,T]$*

For the proofs of these results and for examples see [4]

References

[1] J. Bertoin. *An extension of Pitman's Theorem for spectrally positive Lévy Process.* Ann. Prob. (**20**), (1992) 1464–1483.

[2] J. Bertoin. *Lévy Process.* Cambridge Univ. Press. 1996.

[3] J. Bertoin. *Cauchy principal value of local times of a Lévy process with no negative jumps via continuous branching processes.* Electronic Journal of Probability No. 2, 1997.

[4] J. Bertoin, M.E. Caballero. *Regularity of the Cauchy principal values of some Lévy processes.* Bull. Sc. Math. **123**, (1999), 47–58.

[5] Ph. Biane, M. Yor. *Valeurs principales associées aux temps locaux browniens*, Bull. Sc. Math. **111** (1987), 23–101.

[6] P.J. Fitzsimmons, R.K. Getoor. *On the distributions of the Hilbert transform of the local time of a symmetric Lévy process.* Ann. Prob. **20** (1992), 1484–1497.

[7] P.J. Fitzsimmons, R.K. Getoor. *Limit theorems and variation properties for functional derivaties of the local time of a stable process.* Ann. Institut Henri Poincaré **28**, (1992), 311–333.

[8] K. Yamada. *Principal values of Brownian local times and related topics.* In: *Itô's stochastic calculus and probability theory*, M. Fukushima, N. Ikeda, H. Kunita and S. Watanabe, Ed. Berlin-Springer (1996).

Exponential functionals of Lévy processes

P. Carmona, F. Petit and M. Yor

What is a perpetuity ?

Assume that

- at each time period you are given one euro ;
- the value at time n of one euro given at time $n+1$, is X_n.

Then, the perpetuity is defined to be the the value at time $t=0$ of everything you receive

$$Y = 1 + X_1 + X_1X_2 + \cdots = \sum_n \prod_{i \le n} X_i \,.$$

The continuous time analogue is, for iid X_n's, the exponential functional

$$A_\infty = \int_0^\infty e^{\xi_s}\, ds$$

In a more general model you are given B_i at time $t=i$ so that

$$Y = \sum_n B_n \prod_{i \le n} X_i \,.$$

The continuous time analogue is, for ξ, η independent Lévy processes,

$$A_\infty(\xi,\eta) = \int_0^\infty e^{\xi_s}\, d\eta_s .$$

We shall give three ways of determining the law of A_∞.

1. The law of A_∞ is the invariant probability law of a Markov process.
2. Moments formulae

$$\mathbb{E}\big[A_\infty^\lambda\big] = \frac{\lambda}{\phi(\lambda)}\mathbb{E}\big[A_\infty^{\lambda-1}\big], \qquad \text{with } \mathbb{E}\big[e^{\lambda\xi_t}\big] = e^{-t\phi(\lambda)}$$

3. Martingale methods.

The perpetuity equation

Let T be a finite stopping time. Then $\bar\xi_t = \xi_{t+T} - \xi_t$, $\bar\eta_t = \eta_{t+T} - \eta_t$ are Lévy processes independent of $\mathcal{F}_T$. Since

$$A_\infty(\xi,\eta) = A_T(\xi,\eta) + e^{\xi_T} A_\infty(\bar\xi,\bar\eta),$$

therefore A_∞ is a solution of the perpetuity equation

$$X \stackrel{d}{=} UX + V \qquad X \perp (U, V).$$

An extensive study of this equation has been done by Verwaat [10]. One open problem is to prove, for solutions of this equation, an analogue of Jurek's [5] random integral representation of self decomposable laws:

$$X = \int_0^\infty e^{-s}\, d\eta_s \qquad \mathbb{E}[\log(1 + |\eta_1|)] < +\infty.$$

The underlying Markov process

Define, for $x \in \mathbb{R}$,

$$V_t^x(\xi) = e^{\xi_t}\left(x + \int_0^t e^{-\xi_s}\, ds\right).$$

If ξ is a semimartingale then V is the solution of the stochastic differential equation

$$dV_t = dt + V_t d\xi_t\,.$$

In the general case, V is still an homogeneous Markov process. Furthermore, as a consequence of the time reversal property of Lévy processes, we have

1. for all $t > 0$: $V_t \stackrel{d}{=} e^{\xi_t} x + A_t$.
2. If a.s. $\xi_t \to -\infty$ and $A_\infty < \infty$, then the law of A_∞ is the invariant probability law of V.
3. The generator of V is given by

$$L^V f(x) = f'(x) + L^\xi(f_x)(0) \qquad \text{with} f_x(u) = f(xe^u)\,,\ f \in \mathcal{S}\,,$$

where $\mathcal{S}$ denotes the Schwarz space of rapidly decreasing functions.

Theorem 1. *If ξ is nice, then A_∞ has a density k solution of*

$$-k'(x) + \frac{1}{x} L^{-\xi}(\tilde{k})(\log x) = 0 \qquad x > 0,\ \tilde{k}(u) = e^u k(e^u)\,.$$

Examples

1. **Dufresne [3] identity** For Brownian motion with drift

$$\xi_t = \sigma B_t - ct \qquad (c > 0)$$

k is the solution of the differential equation

$$-\frac{\sigma^2}{2}\frac{d}{dx}(x^2 k(x)) + ((\frac{\sigma^2}{2} - c) + 1)k(x) = 0$$

and so A_∞ has the law of the inverse of a Gamma random variable

$$A_\infty \stackrel{d}{=} \frac{2}{\sigma^2 Z_{2c/\sigma^2}} \qquad Z_a \sim \gamma(a)$$

2. The opposite of a subordinator with drift :
$\xi_t = -(ct + \tau_t^+)$ with : $c + \int_0^\infty \bar{\nu}_+(x)\,dx > 0$. k satisfies the integral equation

$$(1-cx)k(x) = \int_x^\infty \bar{\nu}_+(\log(u/x))\,k(u)du\,.$$

We consider the case $\bar{\nu}_+(x) = ae^{-bx}$ where $a > 0$, $b > 0$ and $\mathbb{E}[-\xi_1] = c + (a/b) > 0$.

$$A_\infty \stackrel{d}{=} \begin{cases} \frac{1}{c} Z_{b+1,\,a/c} & \text{if } c > 0\ ; \\ \frac{1}{a} Z_{b+1} & \text{if } c = 0\ ; \\ \frac{1}{|c|} \frac{Z_{b+1}}{Z_{(a/|c|)-b}} & \text{if } c < 0, \end{cases}$$

$$\mathbb{P}(Z_{\alpha,\beta} \in dx) = \frac{dx}{B(\alpha,\beta)}\, x^{\alpha-1}\,(1-x)^{\beta-1}\,1_{(0<x<1)}\,.$$

The generalized moving average processes

This model is inspired by works of Novikov [7], Samorodnitsky and Taqqu [9]. Given two independent Lévy processes ξ and η we define

$$V_t = V_t^x(\xi,\eta) = e^{\xi_t}(x + \int_0^t e^{-\xi_s}\,d\eta_s)\,.$$

When ξ and η are semimartingales, the process V is the unique solution of the stochastic differential equation:

$$dV_t = d\eta_t + V_t d\xi_t\,.$$

V is an homogeneous Markov process whose semi group is characterized by

$$\mathbb{E}_x\left[e^{iuV_t}\right] = \mathbb{E}\left[\exp(iuxe^{\xi_t} + \int_0^t \psi(ue^{\xi_s})\,ds)\right]$$

$$\text{with} \quad \mathbb{E}\left[e^{iu\eta_t}\right] = e^{t\psi(u)}$$

Furthermore,

1. for every fixed time t : $V_t \stackrel{d}{=} e^{\xi_t}x + A_t$.
2. If a.s. $\xi_t \to -\infty$ and $A_\infty < \infty$, then the law of A_∞ is the invariant probability law of V.
3. The generator of V is given on $\mathcal{S}$ by

$$L^V f(x) = L^\eta f(x) + L^\xi(f_x)(0) \quad \text{with } f_x(u) = f(xe^u)\,.$$

Theorem 2. *If ξ and η are nice, then A_∞ has a density k solution of*

$$L^{-\eta}k(x) + \frac{1}{x}L^{-\xi}(\tilde{k})(\log x) = 0 \qquad x > 0,$$

where $\tilde{k}(u) = e^u k(e^u)$.

Example: Paulsen [8] W^η, W^ξ being two independent Brownian motions, we let:

$$\xi_t = -rt + \sigma_\xi W_t^\xi, \qquad \eta_t = pt + \sigma_\eta W_t^\eta, \qquad (r > \sigma_\xi^2).$$

Then,

$$k(x) = K(\sigma_\eta + \sigma_\xi x^2)^{-(1/2 + r/\sigma_\xi^2)} \exp\left(\frac{2p}{\sigma_\xi \sigma_\eta} \arctan(\frac{\sigma_\xi x}{\sigma_\eta})\right)$$

Generalized Ornstein-Uhlenbeck processes

We think that they were introduced by Hadjiev [4]. If $(X_t; t \geq 0)$is an H self similar Markov process and ξ an independent Lévy process, then

$$U_t = e^{H\xi_t} X\left(\int_0^t e^{-\xi_s}\, ds\right)$$

is an homogeneous Markov process.

When do we have Ornstein-Uhlenbeck $\stackrel{d}{=}$ Moving Average ?

Theorem 3. *If $(\eta_t\,; t \geq 0)$ is an α-stable Lévy process then the two following Markov processes have the same law.*

$$U_t = e^{\frac{1}{\alpha}\xi_t}(x + \eta(\int_0^t e^{-\xi_s}\, ds))$$

$$V_t = e^{\frac{1}{\alpha}\xi_t}(x + \int_0^t e^{-\frac{1}{\alpha}\xi_{s-}}\, d\eta_s)$$

The moments formulas

Theorem 4. *For $\lambda \geq 1$ or $0 < \lambda < 1$ and $\phi(\lambda) > 0$*

$$\mathbb{E}\left[A_\infty^\lambda\right] = \frac{\lambda}{\phi(\lambda)} \mathbb{E}\left[A_\infty^{\lambda-1}\right], \qquad \textit{with } \mathbb{E}\left[e^{\lambda\xi_t}\right] = e^{-t\phi(\lambda)}$$

If $\phi'(0+) > 0$, then

$$\mathbb{E}\left[A_\infty^{-1}\right] = \phi'(0+) (= -\mathbb{E}[\xi_1]).$$

To give an idea of the proof, we establish first that

$$\mathbb{E}\left[A_t^\lambda\right] = \lambda \int_0^t e^{-v\phi(\lambda)} \mathbb{E}\left[A_{t-v}^{\lambda-1}\right] dv$$

and then we let t go to $+\infty$.

Theorem 5. *If $-\xi$ is a subordinator, then the law of A_∞ is determined by its moments*

$$\mathbb{E}[A_\infty^n] = \frac{n!}{\prod_{1 \leq j \leq n} \phi(j)}$$

$$\mathbb{E}\left[e^{\alpha A_\infty}\right] < +\infty, \qquad \textit{for } 0 < \alpha < \phi(+\infty)$$

The Martingale Approach (Nilsen and Paulsen [6])

Theorem 6. *Under nice assumptions the characteristic function* $f(u) = \mathbb{E}\left[e^{iuA_\infty}\right]$ *is a solution of*

$$L^\xi f_u(0) + f(u)\psi(u) = 0$$
$$f_u(x) = f(ue^x) \qquad \mathbb{E}\left[e^{iu\eta_t}\right] = e^{t\psi(u)}$$

Example *Let* $(B_t\,; t \geq 0)$ *be a Brownian motion and* $(N_t\,; t \geq 0)$ *be an independent Poisson process. We let* $\xi_t = \sigma B_t - rt$, $\eta_t = \sum_{i=!}^{N_t} S_i$, *where* S_i *are iid of law* $exp(\mu)$. *Then*

$$A_\infty \stackrel{d}{=} \frac{X}{Y}, \quad X \stackrel{d}{=} \gamma(b,\mu), \quad Y \stackrel{d}{=} \beta(a, 1+b)$$

where $a = 2r/\sigma^2$, $b = 2/a(\sqrt{1+\lambda a} - 1)$, X *and* Y *are independent.*

References

[1] Ph. Carmona, *Some complements on : On the distribution and asymptotic results for exponential functionals of Lévy processes*, Unpublished, sep 96.

[2] Ph. Carmona, F. Petit, and M. Yor, *On the distribution and asymptotic results for exponential functionals of Lévy processes*, Exponential functionals and principal values related to Brownian motion, Biblioteca de la Revista Matematica Ibero-Americana, 1998.

[3] D. Dufresne, *The distribution of a perpetuity, with application to risk theory and pension funding*, Scandinavian Actuarial Journal (1990), 39–79.

[4] D.I. Hadjiev, *The first passage problem for generalized Ornstein-Uhlenbeck processes with non-positive jumps*, Séminaire de Probabilités XIX, Lecture Notes in Mathematics 1123, Springer, 1985.

[5] Z.J. Jurek, *Random integral representations for classes of limit distributions similar to Lévy class* L_0, Probability Theory and Related Fields **78** (1988), 473–490.

[6] T. Nilsen and J. Paulsen, *On the distribution of a randomly discounted compound Poisson process*, Stochastic Processes and their Applications **61** (1996), 305–310.

[7] A.A. Novikov, *On the first passage time of an autoregressive process over a level and an application to a disorder problem*, Theory Probab. Appl. **35** (1987), 269–279.

[8] J. Paulsen, *Risk theory in a stochastic economic environment*, Stochastic Processes and their Applications **46** (1993), 327–361.

[9] G. Samorodnitsky and M. S. Taqqu, *Stable Non-Gaussian random processes*, Stochastic Modelling, Chapman and Hall, New York, London, 1994.

[10] W. Vervaat, *On a stochastic difference equation and a representation of non negative infinitely divisible random variables*, AAP **11** (1979), 750–783.

TERM STRUCTURE MODELS DRIVEN BY GENERAL LÉVY PROCESSES

ERNST EBERLEIN AND SEBASTIAN RAIBLE

Empirical investigations ([2], [3]) showed that purely discontinuous Lévy processes allow more realistic modeling of stock returns. This motivates a similar generalization of Gaussian term structure models. We present a new class of bond price models that can be driven by a wide range of Lévy processes (L_t). We do not just replace Brownian motion (B_t) in the standard diffusion model for the price of a zero coupon bond with maturity T

$$dP(t,T) \;=\; P(t,T)(r(t)dt + \sigma(t,T)\; dB_t)$$

by (L_t). As the result we would get the Doléans-Dade exponential. Instead we introduce (L_t) in the solution of this equation, which can be written in the form

$$P(t,T) \;=\; P(0,T)\cdot\left(\int_0^t r(s)ds\right)\frac{\exp\left(\int_0^t \sigma(s,T)dB_s\right)}{E\left[\exp\left(\int_0^t \sigma(s,T)dB_s\right)\right]}$$

This is consistent with martingale modeling. In order to guarantee existence of the corresponding moment generating functions, which appear in the denominator, the only restriction one needs is an integrability assumption for the tails of the Lévy measure of $\mathcal{L}(L_1)$.

After some analysis we get the derived forward rate process $f(t,T)$ in the form

$$f(t,T) \;=\; f(0,T) + \int_0^t \theta'(\sigma(s,T))\sigma_2(s,T)ds - \int_0^t \sigma_2(s,T)dL_s.$$

Here $\theta(u) = \log(E[\exp(uL_1)])$ denotes the log of the moment generating function of $\mathcal{L}(L_1)$ and σ_2 the partial derivative of σ with respect to maturity. If $r(t) = f(t,t)$ denotes the short rate process we finally get the following representation for the bond price process

$$P(t,T) \;=\; P(0,T)\cdot\exp\left[\int_0^t (r(s)-\theta(\sigma(s,T)))\,ds+\int_0^t \sigma(s,T)dL_s\right].$$

By Ito's formula for semimartingales this process satisfies the following stochastic differential equation

$$\begin{aligned} dP(t,T) \;=\; P(t-,T)\cdot\Big(r(t)dt+\Big(\frac{c}{2}\sigma^2(t,T)-\theta(\sigma(t,T))\Big)\,dt \\ +\sigma(t,T)dL_t+\big(e^{\sigma(t,T)\Delta L_t}-1-\sigma(t,T)\Delta L_t\big)\Big) \end{aligned}$$

In [1] Björk, di Masi, Kabanov and Runggaldier introduce a general semimartingale model for the term structure. Its forward rate dynamics is given by

$$f(t,T) \;=\; f(0,T)+\int_0^t \alpha(s,T)ds+\int_0^t \sigma(s,T)dB_s+\delta(s,x,T)*(\mu-\nu).$$

Here μ is a σ-finite random measure with absolutely continuous compensator
$\nu(\omega,dt,dx)=\lambda_t(\omega,x)dt$ and α,σ and δ satisfy certain measurability and integrability conditions. Since the Lévy processes considered above are special semimartingales, by using the canonical representation

$$L_t \;=\; \overline{\alpha}t+\overline{\sigma}B_t+x*(\mu^L-\nu^L)$$

one sees that our model fits into their framework. In particular our model satisfies the generalized Heath-Jarrow-Morton drift condition derived in [1].

Under additional assumptions we characterize volatility structures $\sigma(t,T)$ which lead to Markovian short rates. The key property here is a factorization of the partial derivative of the volatility of the form

$$\sigma_2(t,T) \;=\; \tau(t)\cdot\zeta(T)$$

for C^1-functions τ and ζ. In the Markovian case the forward rates turn out to be deterministic functions of the current short rate $r(t)$. If σ is stationary the only candidates which remain are the Vasiček and the Ho-Lee structure. For the Vasiček volatility structure the short rate process is mean-reverting and satisfies a stochastic differential equation of the form

$$dr(t) \;=\; a(\varrho(t)-r(t))dt-\widehat{\sigma}dL_s.$$

The mean ϱ is a deterministic process and a and $\widehat{\sigma}$ are the parameters of the Vasiček structure.

Numerically we investigate the case where the driving process is a hyperbolic Lévy motion (L_t). Hyperbolic forward rates turn out to be higher than Gaussian rates. Finally we price options on bonds. Since the underlying measure is a martingale measure which is unique, by construction no change of measure is necessary here. The resulting option prices, plotted as a function of the forward price-strike ratio, show a characteristic, W-shaped deviation from option prices in the Gaussian model. Similarly structured deviations were observed, when hyperbolic stock option prices were compared to standard Black-Scholes prices.

References

[1] Björk, T.; di Masi, G.; Kabanov, Y.; Runggaldier, W.: Towards a general theory of bond markets. Finance and Stochastics 1, 141-174 (1997)

[2] Eberlein, E.; Keller, U.: Hyperbolic distributions in finance. Bernoulli 1, 281-299 (1995)

[3] Eberlein, E.; Keller, U.; Prause, K.: New insights into smile, mispricing and value at risk: the hyperbolic model. Journal of Business 71, 371-406 (1998)

[4] Eberlein, E.: Grundideen moderner Finanzmathematik. Mitteilungen der DMV 3/98, 10-20 (1998)

[5] Eberlein, E.; Raible, S.: Term structure models driven by general Lévy processes. Mathematical Finance 9, 31-54 (1999)

Institut für Mathematische Stochastik and, Center for Data Analysis and Model Building (FDM), Universität Freiburg, Germany

E-mail address: eberlein@levy.mathematik.uni-freiburg.de

INFERENCE ON LÉVY MEASURE PARAMETERS: LOCAL ASYMPTOTICS IN NEIGHBOURHOODS OF d-DIMENSIONAL STABLE PROCESSES

REINHARD HÖPFNER

Lévy-measures of d-dimensional stable processes $X = (X_t)_{t\geq 0}$ are σ-finite measures $\Lambda_{\alpha,\Phi}$ on $E{:=}\mathbb{R}^d\backslash\{0\}$, with Borel-$\sigma$-field $\mathcal{E}$, of form

$$\Lambda_{\alpha,\Phi}(\{\, x{\in}E : |x|{>}r\,, \frac{x}{|x|}{\in}A\,\}) \;=\; r^{-\alpha}\Phi(A) \tag{1}$$

for all $r{>}0$, $A{\in}\mathcal{S}$, where Φ is some finite measure on the unit sphere S of $\mathbb{R}^d$, with Borel-σ-field $\mathcal{S}$, and $0{<}\alpha{<}2$ is a stability parameter (Lévy 1954). More generally, consider Lévy processes X where α, Φ as above appear only asymptotically in the Lévy measure Λ (e.g., Greenwood and Resnick 1978):

$$\Lambda(\{\, x{\in}E : |x|{>}r\,, \frac{x}{|x|}{\in}A\,\}) \;\sim\; r^{-\alpha}\Phi(A) \;\text{ as }\; r{\to}\infty \tag{2}$$

for all $A{\in}\mathcal{S}$. Assume that stability parameter α, total mass $\xi{:=}\Phi(S)$ of Φ and angular distribution $\theta(A){:=}\xi^{-1}\Phi(A)$, $A{\in}\mathcal{S}$, are unknown and are to be estimated from observation of all big jumps of the process which have occurred over some long time interval. The statistical problem is the following. We do know estimators for (α,ξ,θ) in the stable process context (1) which are efficient in a certain asymptotic sense (cf. LeCam 1969, Hájek 1970, LeCam and Yang 1990). Do their properties carry over to neighbourhood models of type (2) where (α,ξ,θ) are only tail parameters, and which conditions are needed to ensure that a neighbourhood model (2) is sufficiently close (in a sense of Hellinger distance) to (1) in order to inherit its statistical properties ?

Observing big jumps of X over some long time interval means the following. Let $\mu^X(dt,dx)$ denote the point process of jumps of X which is Poisson random measure on $(0,\infty){\times}E$ with intensity $dt\,\Lambda_{\alpha,\Phi}(dx)$ under (1), and $dt\,\Lambda(dx)$ under (2). Then we observe $\mu^X(dt,dx)$ in a window of type $[0,T]{\times}\{x{\in}E{:}|x|{\geq}Y\}$, for suitably defined Y, T. We are interested in asymptotics, so $Y{=}Y_n$ and $T{=}T_n$ increase to ∞. One of these bounds is always deterministic, either the spatial threshold Y_n or the

Editors note: R. Höpfner was unfortunately unable to attend the Conference. We reproduce here the abstract of his planned talk.

observation time T_n, the other is then random and defined sequentially such that the window $K_n := [0,T_n]\times\{x\in E:|x|\geq Y_n\}$ contains exactly n points of $\mu^X(dt,dx)$. We write $P^n_{\alpha,\xi,\theta}$, P^n_Λ for the law of the restricted point process $1_{K_n}(s,x)\mu^X(dt,dx)$ under $\Lambda_{\alpha,\Phi}$, Λ, and

$$d_n(\Lambda) = H^2(P^n_\Lambda, P^n_{\alpha,\xi,\theta}) \tag{3}$$

for their squared Hellinger distance in case (2).

Disintegrate a measure Λ on $(E,\mathcal{E})$ meeting (2) into its radial component $G(dr)$ and a conditional law $K(r,ds)$ of its angular component. If $\Lambda{=}\Lambda_{\alpha,\Phi}$, $K(r,ds){=}\theta(ds)$ is clearly independent of r, and we write $G_{\alpha,\xi}(dr)$ for the radial component $\xi\alpha x^{-\alpha-1}dx$. Under (2), let $(L^{G/G_{\alpha,\xi}}, N^{G/G_{\alpha,\xi}})$ denote a Lebesgue decomposition of G with respect to $G_{\alpha,\xi}$, and write $h_\Lambda(r)$ for the squared Hellinger distance between $K(r,\cdot)$ and θ. We say that Λ admits (α,ξ,θ) as tail parameters if

$$\eta^2_\Lambda(r) := \max\{\, h_\Lambda(r),\, |\sqrt{L^{G/G_{\alpha,\xi}}(r)}-1|^2,\, r^\alpha G(N^{G/G_{\alpha,\xi}}\cap[r,\infty))\,\}$$

vanishes as $r{\to}\infty$. Recall that a function $\psi:(0,\infty){\to}(0,\infty)$ varies regularly at ∞ with index η, $\eta{\in}\mathbb{R}$, if $\psi(u)=l(u)\,u^\eta$ for some $l(\cdot)$ varying slowly at ∞, see Bingham, Goldie and Teugels (1987).

Definition: Consider a function $\psi:(0,\infty){\to}(0,\infty)$ varying regularly at ∞ with negative index, and a family $\mathcal{T}$ of measures (2) which admit tail parameters. Write $\mathcal{Q}$ for the family of corresponding $\Lambda_{\alpha,\Phi}$, with (α,ξ,θ) ranging over the class of tail parameters arising in $\mathcal{T}$. Then $\mathcal{T}$ is called a ψ-neighbourhood of $\mathcal{Q}$ if the following holds, for fixed constants $1{\leq}M,M'{<}\infty$:

$$\eta_\Lambda(r) \leq M\,\psi(r) \;\text{ Lebesgue}{-}\text{a.s. on }\; [M',\infty)\,,\;\; \forall\,\Lambda{\in}\mathcal{T}$$

in case of windows $(K_n)_n$ with deterministic observation time $(T_n)_n$, and

$$\eta_\Lambda(r) \leq M\,\psi(r^\alpha) \;\text{ Lebesgue}{-}\text{a.s. on }\; [M',\infty)\,,\;\; \forall\,\Lambda{\in}\mathcal{T}$$

in case of deterministic spatial thresholds $(Y_n)_n$.

The following is our main result; it compares $\mathcal{E}_n := \{P^n_\Lambda : \Lambda{\in}\mathcal{T}\}$, the statistical experiment of type (2) at stage n, to its 'stable' counterpart $\mathcal{F}_n := \{P^n_{\alpha,\xi,\theta} : \Lambda_{\alpha,\Phi}{\in}\mathcal{Q}\}$ in terms of Hellinger distances (3). Below we say that $\mathcal{T}$ has tail parameters bounded away from 0 and ∞ if all tail parameters (α,ξ) corresponding to measures Λ in $\mathcal{T}$ are contained in some compact subset of $(0,\infty){\times}(0,\infty)$. The Hill estimator sequence

for α in $(\mathcal{E}_n)_n$ or $(\mathcal{F}_n)_n$ is

$$\widehat{\alpha}_n \;=\; [\frac{1}{n}\sum_{j=1}^{n}(\log R_{n,j} - \log R_{n,n})]^{-1}$$

where $R_{n,1}{>}R_{n,2}{>}...{>}R_{n,n}$ is the decreasing rearrangement of absolute values $\{|Z_j| : 1{\leq}j{\leq}n\}$ coming from the n points $(S_j, Z_j)_{1\leq j\leq n}$ of $\mu^X(ds, dz)$ in K_n. In the classical iid setting of extreme value theory, this estimator has been extensively studied since Hill (1975), Hall (1982), Hall and Welsh (1985).

Theorem 1: Assume $Y_n{\to}\infty$ if $(Y_n)_n$ is deterministic, and $\frac{T_n}{n}{\to}\infty$ in case where $(T_n)_n$ is deterministic. Assume that $\mathcal{T}$ is a ψ-neighbourhood of $\mathcal{Q}$.

a) The following chain of inclusions *i)*$\Rightarrow$*ii)*$\Rightarrow$*iii)* holds:
i) $(K_n)_n$ tends to ∞ fast enough, i.e.: one has $\sqrt{n}\psi(Y_n) \to 0$ if $(Y_n)_n$ is deterministic, and $\sqrt{n}\psi(\frac{T_n}{n}) \to 0$ if $(T_n)_n$ is deterministic, as $n{\to}\infty$;
ii) $(\mathcal{E}_n)_n$ is an accompanying sequence for $(\mathcal{F}_n)_n$ in the sense that

$$\sup\{d_n(\Lambda) : \Lambda{\in}\mathcal{T}_0\} \;\to\; 0 \;\text{ as } n{\to}\infty$$

for every subset $\mathcal{T}_0{\subset}\mathcal{T}$ having tail parameters bounded away from 0 and ∞;
iii) the Hill sequence $(\hat{\alpha}_n)_n$ is asymptotically normal in $(\mathcal{E}_n)_n$:

$$\forall\ \Lambda{\in}\mathcal{T}\ ,\ \mathcal{L}(\ \frac{\sqrt{n}}{\alpha}(\widehat{\alpha}_n{-}\alpha)\,|\,P^n_\Lambda\ )\ \to\ \mathcal{N}(0,1)\ \text{ as } n{\to}\infty\ .$$

b) Equivalence of *i)*-*iii)* holds if there are measures Λ in the family $\mathcal{T}$ related to their tail parameters in the following way:

$$G([r,\infty)) = (1{+}c\psi(r){+}o(\psi(r)))\ \xi\, r^{-\alpha}\ \text{ as }\ r{\to}\infty$$

if $(Y_n)_n$ is deterministic, for some $c{\neq}0$, respectively

$$G([r,\infty)) = (1{+}c\psi(r^\alpha){+}o(\psi(r^\alpha)))\ \xi\, r^{-\alpha}\ \text{ as }\ r{\to}\infty$$

in case where $(T_n)_n$ is deterministic.

As a consequence, results for tail parameter estimation available in $(\mathcal{F}_n)_n$ (strong consistency, asymptotic normality, efficiency within classes of regular estimators, notions of regularity, ..., see Höpfner and Jacod (1994), Höpfner (1997), or Marohn (1995) for corresponding results in classical iid extreme value theory) immediately carry over to $(\mathcal{E}_n)_n$.

References:

Bingham, N.H., Goldie, C.M., Teugels, J.L. (1987). **Regular Variation**. Cambridge University Press, Cambridge.

Greenwood, P., Resnick, S. (1979). A bivariate stable characterization and domains of attraction. J. Multivar. Analysis **9**, 206-221.

Hájek, J. (1970). A characterization of limiting distributions for regular estimators. Z. Wahrscheinlichkeitsth. Verw. Geb. **14**, 323-330.

Hall, P. (1982). On some simple estimates of an exponent of regular variation. J. Royal Statist. Soc. B **44**, 37-42.

Hall, P., Welsh, A.H. (1985). Adaptive estimates of parameters of regular variation. Ann. Statist. **13**, 331-341.

Hill, B.M. (1975). A simple general approach to inference about the tail of a distribution. Ann. Math. Statist. **3**, 1163-1174.

Höpfner, R. (1997). Two comments on parameter estimation in stable processes. Math. Methods of Statistics **6**, 125-134.

Höpfner, R., Jacod, J. (1994). Some remarks on the joint estimation of the index and the scale parameter for stable processes. In: P. Mandl and M. Huskova (Eds.): **Asymptotic Statistics**. Proceedings of the Fifth Prague Symposium 1993, pp. 273-284. Physica-Verlag, Heidelberg.

LeCam, L. (1969). **Théorie asymptotique de la décision statistique**. Les Presses de l'Université de Montréal, Montréal.

LeCam, L., Yang, G. (1990). **Asymptotics in statistics**. Springer, New York.

Lévy, P. (1954). **Théorie de l'addition des variables aléatoires**. 2nd éd.. Gauthier-Villars, Paris.

Marohn, F. (1995). Contributions to a local approach in extreme value statistics. Habilitationsschrift, Universität Eichstätt.

Universität Paderborn
E-mail address: hoepfner@plato.uni-paderborn.de

THE SYMBOL OF A MARKOV PROCESS

NIELS JACOB

Let us start with a Lévy process $(Y_t)_{t\geq 0}$ with state space $\mathbf{R^n}$. We denote its corresponding semigroup by $(S_t)_{t\geq 0}$, i. e.

$$S_t u(x) = E^x(u(Y_t)) = \int_{\mathbf{R}^n} u(x+y)\mu_t(dy), \tag{1}$$

where $(\mu_t)_{t\geq 0}$ is the convolution semigroup on $\mathbf{R}^n$ which determines $(Y_t)_{t\geq 0}$. The Fourier transforms $\hat{\mu}_t$ are given by

$$\hat{\mu}_t(\xi) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{-ix\cdot\xi}\mu_t(dy) = (2\pi)^{-n/2} e^{-t\,\psi(\xi)}, \tag{2}$$

where $\psi : \mathbf{R}^n \to \mathbf{C}$ is a continuous negative definite function, i. e. we have the Lévy-Khinchin representation

$$\psi(\xi) = c + i\ d\cdot\xi + Q(\xi) + \int\limits_{\mathbf{R^n}\setminus\{0\}} \left(1 - e^{-i\,y\cdot\xi} - \frac{i\,y\cdot\xi}{1+|y|^2}\right)\frac{1+|y|^2}{|y|^2}\nu(dy) \tag{3}$$

with $c \geq 0, d \in \mathbf{R}^n$, a positive semidefinite, symmetric quadratic form Q and a finite Borel measure ν on $\mathbf{R}^n \setminus \{0\}$. The convolution theorem yields for $u \in S(\mathbf{R}^n)$ (for simplicity)

$$S_t u(x) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{ix\cdot\xi} e^{-t\psi(\xi)} \hat{u}(\xi) d\xi, \tag{4}$$

and for the generator of $(S_t)_{t\geq 0}$ (with respect to the sup-norm) we find

$$Au(x) = -\psi(\mathbf{D})u(x) = -(2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{ix\cdot\xi}\psi(\xi)\hat{u}(\xi)dy. \tag{5}$$

Thus both, the operators $S_t, t \geq 0$, and A are in the language of analysis pseudo differential operators with symbols

$$\lambda_t(\xi) = e^{-ix\cdot\xi} E^x(e^{iY_t\cdot\xi}) = E^0(e^{iY_t\cdot\xi}) = e^{-t\psi(\xi)} \tag{6}$$

and

$$-\psi(\xi) = \frac{d}{dt}\lambda_t(\xi)|_{t=0}, \tag{7}$$

respectively. The formulas (6) and (7) say that the symbols λ_t and $-\psi$ can be obtained directly from the process $(Y_t)_{t\geq 0}$ with obvious interpretation: λ_t is nothing but the characteristic function of the random variable Y_t and ψ is the characteristic exponent of the Lévy process $(Y_t)_{t\geq 0}$. It is a well known fact that $\lambda_t, t \geq 0$, and especially ψ are very useful to describe properties of the corresponding Lévy process $(Y_t)_{t\geq 0}$. In fact Bochner's approach, see [2], to these processes goes systematically along this line.

Let us mention some properties of $(Y_t)_{t\geq 0}$ determined by the characteristic exponent: conservativeness, transience, recurrence, Hausdorff dimensions of sample paths, local and gobal regularity properties of sample paths, ...

We refer to the existing literature, especially to the monographs by J. Bertoin [1] and K.-I. Sato [21] for a comprehensive discussion of Lévy processes. Using the Lévy-Khinchin formula (3) we may obtain also a pathwise decomposition of the process $(Y_t)_{t\geq 0}$. For example in the one-dimensional, conservative case, i. e. $\psi(0) = 0$, we have

$$Y_t = \gamma' t + B_t + \int\limits_{\{|x|<1\}} x\,\{N_t(\cdot, dx) - t\,\widetilde{\nu}\,(dx)\} + \sum_{0<s\leq t} \Delta\, Y_s \chi_{\{|\Delta Y_s|\geq 1\}}, \tag{8}$$

with a drift coefficient γ', a Brownian motion $(B_t)_{t\geq 0}$, the Lévy measure $\widetilde{\nu}$ and a suitable kernel $N_t(\cdot, dy)$. We refer to Ph. Protter [20] for a discussion of the formula (8). The Lévy decomposition (8) opens the gate to a stochastic analysis, i.e. to an analysis on the path space of a Lévy process, while concentrating on the characteristic exponent emphasis more the potential theoretical aspect in the theory of Markov processes. Both aspects are complementary, neither compete nor do exclude each other.

Now let us look on a general Markov process $(X_t)_{t\geq 0}$ with state space $\mathbf{R}^n$. For simplicity let us assume $(X_t)_{t\geq 0}$ to be a Feller process. A lot of work is done on the stochastic analysis of these processes, especially by J. Jacod, see [12] and [13], and the references therein. Our aim is to show that there is a natural analytic approach to these processes analogous to the approach to Lévy processes by characteristic exponents. The following considerations are taken from our paper [10], we refer also to R.L. Schilling's paper [24] who could reduce some of our earlier assumptions.

In order not to overload this report, we are a little unprecise in stating exact technical conditions on the processes under consideration.

By definition $(X_t)_{t\geq 0}$ is a Feller process with state space $\mathbf{R}^n$ if the semigroup

$$T_t u(x) = E^x(u(X_t)) \tag{9}$$

is a Feller semigroup, i. e. a positivity preserving strongly continuous contraction semigroup on $\mathbf{C}_\infty(\mathbf{R}^n)$. Defining the function

$$\lambda_t(x,\xi) = E^x(e^{i(X_t - x)\cdot\xi}), \tag{10}$$

a straightforward calculation leads (let us say on $S(\mathbf{R}^n)$) to

$$T_t u(x) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{\,ix\cdot\xi} \lambda_t\,(x,\xi)\,\hat{u}\,(\xi)\,d\,\xi. \tag{11}$$

Thus, T_t, $t \geq 0$, is a pseudo differential operator and its symbol $\lambda_t(x,\xi)$ is given by the family of the characteristic functions of $(X_t, P^x)_{x\in\mathbf{R}^n}$. Further, a more involved calculation shows for the generator A of $(T_t)_{t\geq 0}$ (again on $S(\mathbf{R}^n)$) that it is a pseudo differential operator

$$\begin{aligned} Au(x) &= -(2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{\,ix\,\cdot\,\xi}\, \frac{d}{dt}\lambda_t\,(x,\xi)|_{t=0}\,\hat{u}(\xi)\,d\,\xi \qquad (12)\\ &= -(2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{\,ix\,\cdot\,\xi}\, q(x,\xi)\hat{u}\,(\xi) d\,\xi \\ &=: -q(x,\mathbf{D})\,u\,(x), \end{aligned}$$

with symbol

$$-q(x,\xi) = \frac{d}{dt}\lambda_{\,t}\,(x,\xi)|_{t=0} \tag{13}$$

Clearly, $q(x,\xi)$ cannot be arbitrary. While the regularity of $x \to q(x,\xi)$ is determined by some mapping properties of $(T_t)_{t\geq 0}$ (or equivalently of A), the function $\xi \to q(x,\xi)$ must be a continuous negative definite function, which can be experssed by the formula

$$
\begin{aligned}
q(x,\xi) = \;& c(x) + i\,d\,(x)\cdot\xi + \sum_{k,\,l=1}^{n} a_{kl}(x)\,\xi_k\,\xi_l \\
& + \int\limits_{\mathbf{R}^n\setminus\{0\}} \left(1 - e^{-i\,y\cdot\xi} - \frac{i\;y\cdot\xi}{1+|y|^2}\right)\frac{1+|y|^2}{|y|^2}\,\mu(x,dy)
\end{aligned}
\tag{14}
$$

with (say continuous) functions $c \geq 0, d$, and $a_{kl} = a_{lk}$ such that $\sum\limits_{k,\,l=1}^{n} a_{kl}(x)\,\xi_k\xi_l \geq 0$, and an appropriate finite kernel $\mu(x,dy)$.

Now, first note that $(X_t)_{t\geq 0}$ determines $q(x,\xi)$ uniquely. But since $-q(x,\mathbf{D})$ determines $(X_t)_{t\geq 0}$ uniquely and is itself determined by $q(x,\xi)$, we have to conclude that $q(x,\xi)$ determines uniquely the process $(X_t)_{t\geq 0}$. Note further that if $(X_t)_{t\geq 0}$ is a Lévy process, $q(x,\xi)$ will be x-independent, thus $q(x,\xi) = \psi(\xi)$ is nothing but the characterstic exponent of this Lévy process. Thus $q(x,\xi)$ generalizes the concept of the characteristic exponent to processes which have not independent and stationary increments. In the analysis of pseudo differential operators $-q(x,\xi)$ is called the symbol of $-q(x,\mathbf{D})$. For this reason we call $q(x,\xi)$ **the symbol of the process** $(X_t)_{t\geq 0}$.

Now let us collect some results which show that we may derive properties of the process $(X_t)_{t\geq 0}$ from its symbol $q(x,\xi)$.

1. (R. L. Schilling [22] and [26]) Let $q(x,\xi)$ be the symbol of a Feller process $(X_t)_{t\geq 0}$ and $\psi(\xi)$ be the symbol (characteristic exponent) of a Lévy process $(Y_t)_{t\geq 0}$.

Suppose further that $|q(x_i\xi)| \leq c(1+\psi(\xi))$ holds for all $\xi \in \mathbf{R}^n$. Denote by β the upper index of the Lévy process $(Y_t)_{t\geq 0}$, i. e.

$$
\beta = inf\{\lambda > 0;\, \lim_{|\xi|\to\infty} \frac{|\psi_1(\xi)|}{|\xi|^\lambda} = 0\}
\tag{15}
$$

Then we have for every Borel set $E \subset [0,1]$ $P^x - a.s.$.

$$
dim_H\{X_t(w); t\in E\} \leq \beta\, dim_H E.
\tag{16}
$$

2. (R. L. Schilling [25]) Let $(X_t)_{t\geq 0}$ be a Feller process with symbol $q(x,\xi)$ and set

$$
\beta_0 := sup\,\{\lambda \geq 0;\, \lim_{|\xi|\to 0} \frac{sup_{\,y\in\mathbf{R^n}}|\,q(y,\xi)|}{|\xi|^\lambda} = 0\}
\tag{17}
$$

and

$$\beta_\infty^x := inf\{\lambda > 0;\ \lim_{|\xi|\to\infty} \frac{sup_{\,|x-y|\le\frac{2}{|\xi|}}\,|\,q(x,\xi)|}{|\xi|^\lambda} = 0\}. \tag{18}$$

.
Then $P^x - a.s.$ we have for the components $(X_t^{(j)})_{t\ge 0}$ of $(X_t)_{t\ge 0}$

$$(t \to X_{x\vee 0}(w))^{(j)} \in \mathbf{B}^s_{pq}(\mathbf{R}, (1+|\cdot|^2)^{-\mu/2}), \tag{19}$$

where $s \in \mathbf{R}, p, q \le \infty$ such that $s > (\frac{1}{p}-1)_+, \mu > \frac{1}{p} + \frac{1}{\beta_0}$ and $s(p\vee q\vee\beta_\infty^x) < 1$, or $s > (\frac{1}{p}-1)_+, \mu > \frac{1}{p}+\frac{1}{p_0}, q = \infty$ and $s(p\vee\beta_\infty^x) < 1$. Here $\mathbf{B}^s_{pq}(\mathbf{R}, (1+|\cdot|^2)^{-\mu/2})$ denotes a weighted Besov space. Analogous results do hold for weighted Triebel-Lizorkin spaces.

3. (W. Hoh [6]) Let $\psi : \mathbf{R}^n \to \mathbf{R}$ be a continuous negative definite function such that $\psi(0) = 0$ and define

$$A_\psi(\varrho) := \sup_{|\xi|\le\frac{1}{\varrho}} \psi(\xi)\ , \varrho > 0. \tag{20}$$

Further let $q : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ be a continuous function such that $\xi \to q(x,\xi)$ is negative definite and satisfying $q(x,0) = 0$.
If

$$q(x,\xi) \le \mathbf{C}\frac{1}{A_\psi(|x|)}\psi(\xi) \text{ for } |x| \ge 1, \tag{21}$$

then for any initial distribution $\mu \in \mathcal{M}_1(\mathbf{R^n})$ there is a solution to the martingal problem for $-q(x,D)$.

These results taken from our former student's work show the usefulness of the concept of the symbol of a Feller process. Finally let us mention that there are now a lot of results proving that under certain assumptions we may start with a continuous function $q : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ such that $\xi \to q(x,\xi)$ is negative definite and construct the Feller process having $q(x,\xi)$ as symbol. We refer to the deep papers of W. Hoh [3] - [6] and earlier work of the author [7] - [9], and [11] as well as the achievements of our Japanese colleagues K. Kikuchi, T. Komatsu, A. Negoro and M. Tsuchiya.

References

[1] Bertoin, J., *Lévy processes.* Cambridge University Press, Cambridge 1996.

[2] Bochner, S., *Harmonic analysis and the theory of probability.* University of California Press, Berkeley 1955.

[3] Hoh, W., *The martingale problem for a class of pseudo differential operators.* Math. Ann. 300 (1994) 121-147.

[4] Hoh, W., *Pseudo differential operators with negative definite symbols and the martingale problem.* Stoch. and Stoch. Rep. 55 (1995) 225-252.
[5] Hoh, W., *A symbolic calculus for pseudo differential operators generating Feller semigroups.* Osaka J. Math. (in press)
[6] Hoh, W., *Pseudo differential operators generating Markov processes.* Habilitationsschrift, Universität Bielefeld, 1998.
[7] Jacob, N., *Feller semigroups, Dirichlet forms, and pseudo-differential operators.* Forum Math. 4 (1992) 433-446.
[8] Jacob, N., *A class of Feller semigroups generated by pseudo-differential operators.* Math. Z. 215 (1994) 151-166.
[9] Jacob, N., *Pseudo-differential operators and Markov processes.* Akademie Verlag, Berlin 1996.
[10] Jacob, N., *Characteristic functions and symbols in the theory of Feller processes.* Potential Anal. 8 (1998) 61-68.
[11] Jacob, N., and H.-G. Leopold, *Pseudo-differential operators with variable order of differentiation generating Feller semigroups.* Integr. Equat. Oper. Th. 17 (1993) 544-553.
[12] Jacod, J., *Calcul stochastique et problèmes de martingales.* Springer Verlag, Berlin 1979.
[13] Jacod, J., and A. N. Shiryaev, *Limit theorems for stochastic processes.* Springer Verlag, Berlin 1987.
[14] Kikuchi, K., and A. Negoro, *On Markov processes generated by pseudodifferential operators of variable order.* Osaka J. Math. 34 (1997) 319-335.
[15] Komatsu, T., *Pseudo-differential operators and Markov processes.* J. Math. Soc. Japan 36 (1984) 387-418.
[16] Komatsu, T., *Uniform estimates for fundamental solutions associated with non-local Dirichlet forms.* Osaka J. Math. 32 (1995) 833-860.
[17] Komatsu, T., *On stable-like processes.* In: Proc. 7th Japan-Russian-Symp-Prob. Th. Math. Statistics 1995, World Scientific, Singapore 1996.
[18] Negoro, A., *Stable-like processes: Construction of the transition density and the behavior of sample paths near $t = 0$.* Osaka J. Math. 31(1994) 189-214.
[19] Negoro, A., and M. Tsuchiya, *Stochastic processes and semigroups associated with degenerate Lévy generating operators.* Stoch. and Stoch. Rep. 26 (1989) 29-61.
[20] Protter, Ph., *Stochastic integration and differential equations.* Springer Verlag, Berlin 1990.
[21] Sato, K.-J., *Additive Processes (in Japanese) Kinokuniya, Tokyo 1990.*
[22] Schilling, R. L., *Feller processes generated by pseudo-differential operators: On the Hausdorff dimension of their sample paths.* J. Theoretical Probab. 11 (1998) 303-330.
[23] Schilling, R. L., *Conservativeness and extensions of Feller semigroups.* Positivity 2 (1998) 239-256.
[24] Schilling, R. L., *Conservativeness of semigroups by pseudo differential operators.* Potential Anal. 9 (1998) 91-104.
[25] Schilling, R. L., *Function spaces as path spaces of Feller processes.* Math. Nachr. (in press).
[26] Schilling, R. L., *Growth and Hölder conditions for the sample paths of Feller processes.* Probab. Theory Relat. Fields 112 (1998) 565-611.

Universität der Bundeswehr München, Fakultät für Informatik, Institut für Theoretische Informatik und Mathematik, Werner-Heisenberg-Weg 39, 85577 Neubiberg, Germany

E-mail address: jacob@laplace.informatik.unibw-muenchen.de

Multifractal analysis of Lévy processes and Lévy-Chentsov random fields

Stéphane Jaffard*

Let us first recall some basic definitions concerning multifractal functions. The starting point is the definition of the *pointwise*Hölder regularity criterium $C^l(t_0)$. Let $t_0 \in \mathbb{R}^d$ and let l be a positive real number. A function $f(t) : \mathbb{R}^d \to \mathbb{R}$ is $C^l(t_0)$ if there exists a constant $C > 0$ and a polynomial P_{t_0} of degree at most $[l]$ such that in a neighbourhood of t_0

$$|f(t) - P_{t_0}(t)| \leq C|t - t_0|^l;$$

therefore, this definition is local and involves no uniform regularity. The Hölder exponent of f at t_0 is

$$h_f(t_0) = \sup\{l : f \in C^l(t_0)\}$$

(note that the Hölder exponent is not sensitive to logarithmic corrections in the modulus of continuity).

The *multifractal analysis* is concerned in the study of the (usually fractal) sets S_h where a function f has a given Hölder exponent h and in particular in the determination of the Hausdorff dimension $d(h)$ of S_h. The function $d(h)$ is called the *spectrum of singularities* of f. The notion of 'multifractal functions' was first introduced by physicists in the context of fully developed turbulence, see [4]. It is now used in several applications such as finance, traffic data analysis, image analysis...

The Brownian motion is an example of Lévy process that can be qualified as *monofractal*: Indeed, the Hölder exponent of the Brownian motion is

*Département de Mathématiques, Université Paris XII, Faculté des Sciences et Technologie, 61 Av. du Gal. de Gaulle, 94010 Créteil Cedex, France

everywhere 1/2 (the variations of its regularity are only of a logarithmic order of magnitude). This example is not typical: we will see that the other Lévy processes are multifractal provided that their Lévy measure is neither too small nor too large near zero. Furthermore almost every sample path has the same spectrum of singularities which depends precisely on the growth of the Lévy measure near the origin.

When the Lévy measure $\pi(dx)$ of a Lévy process X_t satisfies $\pi(\mathbb{R}^d) = +\infty$, its growth near the origin can be estimated using the upper index

$$\beta = \inf\{\gamma \geq 0 : \int_{|x|\leq 1} |x|^\gamma \pi(dx) < \infty\}.$$

This index was introduced by R. Blumenthal and R. Getoor in [2]. W. Pruitt in [9] showed that the Hölder exponent of Lévy processes (without Brownian component) at $t = 0$ is $1/\beta$.

Let

$$\begin{aligned} d_\beta(h) &= \beta h \quad \text{if} \quad h \in [0, 1/\beta] \\ &= -\infty \quad \text{else,} \end{aligned}$$

and

$$C_j = \int_{2^{-j-1}\leq|x|\leq 2^{-j}} \pi(dx);$$

the exponent β can also be defined using the C_j's by

$$\beta = \sup\left(0, \limsup_{j\to\infty} \frac{\log C_j}{j \log 2}\right).$$

The following theorem is proved in [5].

Theorem 1 *Let X_t be a Lévy process of Lévy measure $\pi(dx)$ satisfying $\beta > 0$ and*

$$\sum 2^{-j}\sqrt{C_j \log(1 + C_j)} < \infty. \tag{1}$$

If X_t has no Brownian component, the spectrum of singularities of almost every sample path of X_t is $d_\beta(h)$.

Remarks:

1. Condition (1) fails only when $\beta = 2$; thus all Lévy processes of upper index β such that $0 < \beta < 2$ satisfy the assumptions of Theorem 1, and in particular all stable Lévy processes are covered by this theorem.

2. In [8], S. Orey and S. J. Taylor proved that if X_t is a *stable symetric* Lévy process, the Hausdorff dimension of the set of points where the Hölder exponent of X_t is *at most* h is βh. Note however that their method cannot give regularity results at these points.

3. The assertion expressed in the theorem is stronger than stating that, for each h, $d(h)$ has almost surely a given value, which would not be sufficient to determine the spectrum of singularities of almost every sample path.

4. The almost everywhere Hölder exponent of Lévy processes without Brownian component is $1/\beta$, see [9], which of course agrees with the theorem (case where $h = 1/\beta$).

5. Many results have been proved concerning the fractal nature of the *range* of Lévy processes, see for instance [10], or [6] for references concerning 'Lévy flights', or [1] for results concerning the range of subordinators.

Our next interest is to generalize these results to Lévy-Chentsov random fields. These fields are a very natural multi-dimensional generalization of Lévy processes since they have the following simple geometric property: Their traces on half-lines are Lévy processes (up to a constant). More precisely, if $(X_t)_{t\in\mathbb{R}^d}$ is a Lévy-Chentsov field, for any a and b in $\mathbb{R}^d$, the function defined on $\mathbb{R}^+$ by $u \to X_{au+b} - X_b$ is a Lévy process. Furthermore, if X_t is a Lévy-Chentsov field and if ϕ is an affine transformation, $X_{\phi(t)}$ is a Lévy-Chentsov field.

The first example was given Paul Lévy's construction of the several dimensional Brownian process. The next step was taken by Chentsov who gave an hyperplane based construction of this Brownian field. This idea was generalized by Mori, Sato et al., see [7] or [11] for instance, who gave the general construction of a Lévy Chentsov field as sums of Poisson processe with jumps on hyperplanes. These fields have properties that are very similar to Lévy processes; for instance, if they have no Brownian component, their characteristic function satisfies

$$\mathbb{E}(e^{i\lambda X_t}) = \exp\left(\int_{S^{d-1}}\int_{\mathbf{R}} \langle t \mid \theta\rangle \left(e^{i\lambda x} - 1 - i\lambda x 1_{|x|<1}\right)\nu(d\theta, dx)\right); \quad (2)$$

S^{d-1} is the $d-1$ dimensional unit sphere and $\nu(d\theta, dx)$ is the Lévy measure of X_t, i.e. a positive Radon measure satisfying

$$\int (1 \wedge |x|^2)\nu(S^{d-1} \times dx) < \infty. \tag{3}$$

The growth of the Lévy measure near the origin can be estimated using the upper index

$$\beta = \inf\{\gamma \geq 0 : \int_{|x|\leq 1} |x|^\gamma \nu(S^{d-1} \times dx) < \infty\}.$$

Condition (3) implies that $0 \leq \beta \leq 2$.

Let us briefly describe the construction of these fields as sums of Poisson processes having jumps on hyperplanes. Such a Poisson process is defined as follows.

One constructs first a Poisson hyperplane process associated with a finite measure $m(d\theta)$ on S^{d-1} by considering a Poisson point process on $\mathbb{R} \times S^{d-1}$ of Poisson measure $dr \times m(d\theta)$ and by associating to each point (r, θ) the hyperplane orthogonal to the direction θ and at distance r from the origin. If $\nu(d\theta, dx)$ is finite, the Poisson hyperplan process is obtained by first constructing the Poisson hyperplane process of measure $m(d\theta) = \int_x \nu(d\theta, dx)$ and then by making the process jump on this hyperplane, the size of the jump being chosen with the probability $\pi_\theta(dx)$ such that the following conditioning equality holds

$$\int\int \pi_\theta(dx)m(d\theta) = \int\int \nu(d\theta, dx).$$

The Lévy Chentsov fields on $\mathbb{R}^d$ are then constructed by splitting the Lévy measure on domains $2^{-j} \leq x < 2^{-j+1}$ and by summing up independent copies of the corresponding Poisson hyperplane processes obtained with these restricted (hence finite) Lévy measures.

Let

$$\begin{aligned} d'_\beta(h) &= \beta h + d - 1 \quad \text{if} \quad h \in [0, 1/\beta] \\ &= -\infty \qquad\qquad\quad \text{else;} \end{aligned}$$

The following theorem is proved in [3].

Theorem 2 *Let X_t be a Lévy Chentsov field with no Brownian part, taking values in $\mathbb{R}$, of Lévy measure $\nu(d\theta, dx)$ satisfying $0 < \beta < 2$. The spectrum of singularities of almost every sample path of X_t is $d'_\beta(h)$.*

References

[1] J. Bertoin *Lévy processes.* Cambridge University Press (1996).

[2] R. Blumenthal and R. Getoor *Sample functions of stochastic processes with stationary independent increments.* J. Math. Mech. 10 p.493-516 (1961).

[3] S. Cohen and S. Jaffard *Multifractal analysis of Lévy Chentsov random fields* (in preparation).

[4] U. Frisch and G. Parisi *Fully developped turbulence and intermittency* . Proc. Int. Summer school Phys. Enrico Fermi, pp.84-88 North Holland (1985).

[5] S. Jaffard *The multifractal nature of Lévy processes.* To appear in Prob. Th. Rel. Fields

[6] B. Mandelbrot *Les objets fractals* Flammarion (1995).

[7] T. Mori *Representation of linearly additive random fields* . Prob. Th. Rel. Fields v.92 pp.91-115 (1992).

[8] S. Orey and S. J. Taylor *How often on a Brownian path does the law of the iterated logarithm fail?* Proc. London Math. Soc. (3) 28 p.174-192 (1974).

[9] W. Pruitt *The growth of random walks and Lévy processes* Annals of Probability 9, n.6 p.948-956 (1981).

[10] W. Pruitt and S. J.Taylor *Packing and covering indices for a general Lévy processes* Annals of Probability 24, n.2 p.971-986 (1996).

[11] Y. Sato *Structure of Lévy measures of stable random fields of Chentsov type.* Prob. Math. Stat. v.13 pp.165-176 (1992).

On the asymptotic error in approximation of Lévy processes by discretizations[1]

Adam Jakubowski
Nicholas Copernicus University

Extended abstract

Let $\{X_t\}_{t\in[0,1]}$, $X_0 = 0$, be a Lévy process with nontrivial jump part and characteristic function $\phi(\lambda)$:

$$E\exp(i\lambda X_t) = \phi(\lambda)^t, \quad t \in [0,1]. \tag{1}$$

Let $\{X_t^n\}_{t\in[0,1]}$ be a discretization of $\{X_t\}$:

$$X_t^n = X_{[nt]/n} (= X_{k/n} \quad \text{if} \quad k/n \le t < (k+1)/n), \tag{2}$$

and let

$$Y_t^n = X_t - X_t^n \tag{3}$$

be "the error of the approximation".

It is well-known that

$$X^n \to X$$

in Skorokhod's J_1-topology, but *not* in the uniform topology. It is also easy to show that

$$Y_t^n \to 0 \quad \text{a.s.}$$

for aech $t \in [0,1]$, while due to the "incompatible" location of jumps of X and X^n's

$$Y^n \not\to 0$$

in J_1-topology. By the same reason, if $\alpha_n \to \infty$ then the sequence

$$U_n = \alpha_n \sup_{0\le t\le 1} |Y^n(t)|$$

is not bounded in probability.

It follows that in general there is no "rate of convergence" of Y_n's to 0, in the sense that there is no $\alpha_n \to \infty$ such that the sequence $\alpha_n \cdot Y_n$ converges in distribution or at least remains tight with respect to some suitable (and interesting) topology on $\mathbb{D}([0,1])$.

[1]Research supported by Komitet Badań Naukowych under Grant PB 483/P03/97/12, and completed while the author was visiting Université Paris VI

The crucial observation for the present paper is that "smoothing" by means of integration gives

$$Z_t^n = \int_0^t Y^n(s)\,ds \to 0 \quad \text{a.s. and uniformly in } t, \tag{4}$$

and that the scaled processes $n \cdot Z^n$ converge in the sense of finite dimensional distributions:

$$n \cdot Z^n \underset{f.d.}{\longrightarrow} Z^0 \tag{5}$$

where Z^0 is a *Lévy process* with characteristic function given by the formula

$$E\exp(i\lambda Z_t^0) = \exp(t \cdot \int_0^1 \log \phi(\lambda s)\,ds). \tag{6}$$

The marginal laws of Z^0 belong to so called class $\mathcal{U}$ of infinitely divisible laws. We refer to Jurek (1985) for definitions, properties and further references.

The trajectories of $n \cdot Z^n$ are continuous, while Z^0 has discontinuous trajectories. Hence it is impossible that $n \cdot Z^n$ converges to Z^0 functionally when $\mathbb{D}([0,1])$ is equipped with Skorokhod's J_1 topology. If we weaken the topology then we are able to strengthen the finite dimensional convergence (5) to functional convergence.

Theorem

$$n \cdot Z^n \underset{S}{\longrightarrow} Z^0 \tag{7}$$

Here S stands for the topology formally defined in Jakubowski (1997). The topology S arises naturally in limit theorems for stochastic integrals (see Jakubowski (1996)) and has good continuity properties with respect to the Skorokhod Problem for convex domains. Since S is not commonly known we shall summarize its properties.

- Write $x_n \longrightarrow_S x_0$ if for every $\varepsilon > 0$ one can find functions $v_{n,\varepsilon}$ of bounded variation on $[0,1]$, which are ε-uniformly close to x_n's and weakly-$*$ convergent:

$$\sup_{t\in[0,1]} |x_n(t) - v_{n,\varepsilon}(t)| \le \varepsilon,\ n = 0,1,2,\ldots,$$

$$v_{n,\varepsilon} \underset{w}{\longrightarrow} v_{0,\varepsilon},\ \text{as } n \to \infty.$$

- The topology S is sequential (i.e. defined in terms of convergent sequences $x_n \longrightarrow_S x_0$) and cannot be metricized.

- $K \subset \mathbb{D}([0,1])$ is relatively S-compact if, and only if, the following conditions hold.

$$\sup_{x\in K}\sup_{t\in[0,1]}|x(t)| \le C_K < +\infty.$$

$$\sup_{x\in K} N_\eta(x) \le C_\eta < +\infty, \quad \eta > 0,$$

  where $N_\eta(x)$ is the number of η-oscillations of x: $N_\eta(x) \ge k$ iff one can find numbers $0 \le t_1 \le t_2 \le \ldots \le t_{2k-1} \le t_{2k} \le 1$ such that $|x(t_{2i-1}) - x(t_{2i})| > \eta$, $i = 1, 2, \ldots, k$.

- If $x_n \longrightarrow_S x_0$, then $x_n(t) \to x_0(t)$ for each t except for a countable set.

- If $x_n(t) \to x_0(t)$ for each t in a dense set containing 0 and 1 and $\{x_n\}$ is S-relatively compact then $x_n \longrightarrow_S x_0$ (not true for the convergence in measure!).

- There exists a countable family of S-continuous functions which separate points in $\mathbb{D}([0,1])$.

- The σ-field $\mathcal{B}_S$ of Borel subsets for S coincides with the usual σ-field generated by projections (or evaluations) on $\mathbb{D}([0,1])$: $\mathcal{B}_S = \sigma(\pi_t : t \in [0,1])$.

- The set $\mathcal{P}(\mathbb{D}([0,1]), S)$ of *S-tight* probability measures is exactly the set of distributions of stochastic processes with trajectories in $\mathbb{D}([0,1])$.

- S is weaker than Skorohod's M_1 and J_1 topologies. Since J_1 is Polish, S is Lusin in the sense of Fernique. But we do not know whether it is (completely) regular.

Let us emphasize that $I\!D([0,1])$ equipped with S-topology has equally good properties as Polish spaces: there exists the a.s. Skorokhod representation and both the direct and the converse Prohorov's theorems are valid. However, it should be pointed out that topology S is advantageous when we are interested in existence problems (this is typical for *weak* topologies). In problems of the type considered above its usefulness is moderate. For example the functional $I\!D \ni x \mapsto \sup_{t\in[0,1]}|x(t)|$ is only lower semicontinuous in *all* $x \in I\!D$. On the contrary, this functional is continuous in topology J_1 in each $x \in I\!D$ for which $x(-1) = x(1)$. The same property holds for another topology, M_1, also introduced by Skorokhod (1956), which seems to be more suitable here. For a long time it was neglected (see, however, Kasahara and Kotani (1979) and Avram and Taqqu (1992)). Recently there seems to be growing interest in this topology for it is expected to be applicable in continuous approximations of Lévy processes arising in models of heavy traffic networks - see Konstantopoulos and Lin (1998). For completness we list below basic properties of topology M_1.

- A sequence x_n converges to x_0 in M_1-topology ($x_n \longrightarrow_{M_1} x_0$) if one can find continuous parametrizations $(\theta_n(s), \tau_n(s))$, $s \in [0,1]$, $n = 0, 1, 2, \ldots$, of (extended) graphs of x_n's such that

$$\lim_{n\to\infty} \sup_{s\in[0,1]} |\theta_n(s) - \theta_0(s)| + |\tau_n(s) - \tau_0(s)| = 0 \tag{8}$$

- The topology M_1 is metric and separable. It is weaker than the usual topology J_1.

- $K \subset \mathbb{D}([0,1])$ is relatively M_1-compact if, and only if, the following conditions hold.

$$\sup_{x\in K} \sup_{t\in[0,1]} |x(t)| \le C_K < +\infty.$$

$$\lim_{\varepsilon\searrow 0} \sup_{x\in K} \Delta_{M_1}(\varepsilon, x) = 0,$$

  where

$$\Delta_{M_1}(\delta, x) = \sup_{0\le s<t<u<s+\varepsilon)\wedge 1} H(x(s), x(t), x(u))$$

  and for $y_1, y_2, y_3 \in \mathbb{R}^d$, $H(y_1, y_2, y_3)$ is the distance of y_2 from the segment $[y_1, y_3]$.

- If $x_n \longrightarrow_S x_0$, then $x_n(t) \to x_0(t)$ in each point of continuity of x_0 .

- The σ-field $\mathcal{B}_S$ of Borel subsets for S coincides with the usual σ-field generated by projections (or evaluations) on $\mathbb{D}([0,1])$: $\mathcal{B}_S = \sigma(\pi_t : t \in [0,1])$.

- The set $\mathcal{P}(\mathbb{D}([0,1]), S)$ of *S-tight* probability measures is exactly the set of distributions of stochastic processes with trajectories in $\mathbb{D}([0,1])$.

We conjecture that our Theorem can be improved to

Conjecture

$$n \cdot Z^n \underset{M_1}{\longrightarrow} Z^0 \tag{9}$$

Finally, let us notice that our result complements in some sense the considerations contained in Jacod and Protter (1998).

References

1. Avram, F. and Taqqu, M., Weak convergence of sums of moving averages in the α-stable domain of attraction. *Ann. Probab.*, **20 (1992)**, 483–503.

2. Jacod. J. and Protter, Ph., Asymptotic error distributions for the Euler method for stochastic differential equations, *Ann. Probab.*, **26 (1998)**, 267–307.

3. Jakubowski, A., Convergence in various topologies for stochastic integrals driven by semimartingales, *Ann. Probab.*, **24 (1996)**, 2141–2153.

4. Jakubowski, A. A non-Skorohod topology on the Skorohod space, *Electr. J. Probab.*, **2 (1997)**, No 4, 1–21.

5. Jurek, Z., Relations between the s-selfdecomposable and selfdecomposable measures, *Ann. Probab.*, **13 (1985)**, 592–608.

6. Kasahara, Y. and Kotani, S., On limit processes for a class of additive functionals of recurrent diffusion processes, *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, **49 (1979)**, 133-153.

7. Konstantopoulos, T. and Lin, S., Macroscopic models for long-range dependent network traffic. *Queueing Systems Theory Appl.*, **28 (1998)**, 215–243.

8. Skorokhod, A.V., Limit theorems for stochastic processes, *Theory Probab. Appl.*, **1 (1956)**, 261–290.

DIFFERENT ASPECTS OF SELFDECOMPOSABILITY

Zbigniew J. Jurek (University of Wrocław, Poland)

A SELFDECOMPOSABLE rv X (or its probability distribution or its characteristic function ϕ) is defined as a limit of normalized partial sums of independent rv. Of course, stable rv S are selfdecomposable – they correspond to i.i.d. summands in the above partial sums. On the other hand, all selfdecomposable rv are infinitely divisible (denoted by ID). Letting L be the class of all selfdecomposable rv we get inclusions $S \subset L \subset ID$. Usually the first characterization of selfdecomposability is as follows:

$$\phi \in L \quad \text{iff} \quad \forall(0 < c < 1)\exists(char.f.\psi_c) \;\; \phi(\cdot) = \phi(c\cdot)\psi_c(\cdot). \tag{1}$$

In terms of rv we have

$$X \in L \quad \text{iff} \quad \forall(0 < c < 1)\exists(X_c \sim X) \quad X \stackrel{d}{=} cX + X_c, \tag{2}$$

where "$\stackrel{d}{=}$" means equality in distribution and "$\sim$" means stochastic independence . "Solving" equation (2) one gets

$$X \in L \quad \text{iff} \quad X \stackrel{d}{=} \int_{(0,\infty)} e^{-s}dY(s), \text{ with } E\{\log(1 + |Y(1)|)\} < \infty, \tag{3}$$

where $Y(\cdot)$ is a Lévy process called the background driving Lévy process; in short: Y is the BDLP of X or X is driven by Y. Cf. [14], [15], [16], [18].

A. Identification of BDLP's. Class L is quite rich and contains among others: gamma $\gamma_{\alpha,\lambda}$, t-Student, F-Fisher, log-normal, inverse Gaussian, Barndorff-Nielsen generalized hyperbolic, generalized gamma, etc., cf. [15], [16], [3]. However, finding explicitly their BDLP's might be rather difficult. Stable laws are driven by stable processes ([18], Thm 4.4.2.), compound Poisson processes are BDLP for gamma ([14], [15]), but BDLP for normal inverse Gaussian is a sum of three independent Lévy processes ([2], [3], Thm 3.1). Similarly, for inverse Gaussian distribution the driving process is a sum

of inverse Gaussian Lévy process and compound Poisson process; [2], p.18. [Proposition 1.1. in [15] shows that a class L distribution may appear in its own BDLP]. For BDLP identification purposes it might also be useful to observe that from (3) follows

$$\log \phi(t) = \int_0^t \log \psi(u) \frac{du}{u}, \quad t \neq 0, \tag{4}$$

where ϕ is char.f. of X and ψ is char.f. of $Y(1)$. Hence we conclude that

$$\psi(t) = \exp[t(\log \phi(t))']. \tag{5}$$

In particular, class L char.f. are differentiable for $t \neq 0$ and $\psi \in ID$ with finite logarithmic moment.

B. Class $\mathcal{L}$ Ising models. De Coninck (1984) had observed that in some classical Ising models of ferromagnetism the partition function is of the form of an infinite product of $(1 + a_n^2 t^2)$. Thus

$$\phi(t) = \prod_{n=1}^{\infty} (1 + a_n^2 t^2)^{-1} \in L \tag{6}$$

and this corresponds to a random series $\sum_n a_n \eta_n$, where (η_n) are i.i.d. Laplace/double exponential distributions. Later, in [7] *class $\mathcal{L}$ Ising models* are defined as those for which the free energy function

$$g(\beta, h) :\equiv -\beta^{-1} \lim_{\Lambda \uparrow \mathbb{Z}^d} |\Lambda|^{-1} \log Z_\Lambda(\beta, \beta h) \quad \text{exists}, \tag{7}$$

and

$$\phi(\beta, h) \equiv \phi(h) := \exp[\beta(g(\beta, h) - g(\beta, 0)] \in L. \tag{8}$$

Here $Z_\Lambda(\beta, \beta h)$ is the partition function corresponding to a Hamiltonian with pairwise interaction and h is the external field and β is the inverse of a temperature. Using the BDLP corresponding to ϕ, in (8), one gets formulae for spontaneous magnetization, critical exponents, etc., for class $\mathcal{L}$ Ising models. Also, from the Schoenberg Thm, we get new inequalities for Ursell functions $u_n(h) := -\beta \partial_h^n g(\beta, h)$. An open problem is to give an intrinsic characterization of those class L distribution that arise from *infinite* series of Laplace

distributions (as in (6)), their sums and week limits. Also let us quote here Remark 3.1. from [7], that $Y_\beta(1)$, which is from the BDLP for $\phi(\beta,\cdot)$, satisfies a weak law of large numbers for $\beta < \beta_c :=$ critical temperature.

C. Perpetuities and autoregression. Let (A,B), (A_n,B_n), $n>1$, be i.i.d. random vectors in $\mathbb{R}^2$ and define a stochastic difference equation

$$Z_{n+1} = A_n Z_n + B_n, \quad n = 1,2,\dots \ . \tag{9}$$

Putting $Z_0 = 0$ and assuming that (Z_n) converges to Z we get the distributional equation

$$Z \stackrel{d}{=} AZ + B, \tag{10}$$

i.e. Z is a fixed-point of a random affine mapping $x \mapsto Ax + B$. Laws of Z satsifying (10) are called *perpetuities*. Iterating (9) we easily obtain that

$$Z \stackrel{d}{=} \sum_{k=1}^{\infty} B_k A_1 A_2 \dots A_{k-1}, \tag{11}$$

and in actuarial science (insurance mathematics) it represents "the present value of a permanent commitment to make a payment annually into the future forever". Cf. [8], [9]. Using the BDLP of a selfdecomposable X one can prove that

$$X \stackrel{d}{=} \int_{(0,\infty)} e^{-s} dY(s) = e^{-\tau} \int_{(0,\infty)} e^{-s} dY(s+\tau) + \int_{(0,\tau]} e^{-s} dY(s), \tag{12}$$

for any stopping time τ with respect to $\mathcal{F}_t := \sigma(Y(s) : s \le t)$; cf. [15], Corollary 2. Since the second integral in (12) has the same law as the first one we conclude :

$$\textit{all selfdecomposable distributions are perpetuities.} \tag{13}$$

(Of course here it is important that A in (13) is a random variable.) In [15] there is a method how to "generate" the innovation process in autoregressions with $A_n = c =$ constant.

D. Dirichlet series and gamma distributions. In the analytic number theory an important role is played by so called generalized Dirichlet series. These are series of the form

$$w(z) := \sum_n a_n \exp[-\lambda_n z], \quad z \in \mathbb{C} \tag{14}$$

with $\mathbb{R} \ni \lambda_n \to \infty$; cf. [1], Chapter 8. A special case of [14] was already noticed in [15], Remark 2.

For $\gamma_{\alpha,\lambda}$ a rv with density $\lambda^\alpha/, (\alpha)x^{\alpha-1}e^{-\lambda x}$, we have that its characteristic function $\widehat{\gamma}_{\alpha,\lambda}(t)$ is of the form

$$\widehat{\gamma}_{\alpha,\lambda}(t) = (1 - it/\lambda)^{-\alpha} = \exp\left[\int_{(0,\infty)} (e^{itx} - 1)\alpha e^{-\lambda x}/x dx\right], \quad t \in \mathbb{R}. \tag{15}$$

Hence for a series $\sum_n \overset{\circ}{\gamma}_{\alpha_n,\lambda_n}$ of independent and symmetrized gamma rv, if it converges a.s., to say G, then $G \in L$ and its char.f. is

$$\widehat{G}(t) = \prod_{n=1}^{\infty}(1 + t^2/\lambda_n^2)^{-\alpha_n} = \exp\left[\int_{\mathbb{R}\setminus\{0\}} (\cos tx - 1)\left(\sum_{n=1}^{\infty} \alpha_n e^{-\lambda_n|x|}/|x|\right) dx\right]. \tag{16}$$

This means that in the Lévy-Khintchine formula for G its Lévy spectral measure M (describing the average number of jumps) has density given by generalized Dirichlet series

$$dM(x) = \sum_{n=1}^{\infty} \alpha_n e^{-\lambda_n|x|}/|x| dx, \quad x \neq 0. \tag{17}$$

On the other hand, since $G \in L$ it has a BDLP Y such that $Y(1)$ has Lévy spectral measure N given by

$$dN(x) = \sum_{n=1}^{\infty} \alpha_n \lambda_n e^{-\lambda_n|x|} dx. \tag{18}$$

In particular, if

$$\sum_n Var[\overset{\circ}{\gamma}_{\alpha_n,\lambda_n}] = 2\sum_n \frac{\alpha_n}{\lambda_n^2} < \infty,, \tag{19}$$

then by Kolomogorov's Thm, the random series of symmetrized gamma rv converges a.s. (in distribution, in probability), i.e. G is well defined. Consequently the Dirichlet series in (17) and (18) are summable. Conversely, if the Dirichlet series converges and (17) or (18) are Lévy spectral measures then the random series of gamma rv converges. This "stochastic" approach allow to view some of Dirichlet series via the BDLP corresponding to sums of

random variables. Of course, the most interesting examples are those where the sums are given explicitly.

E. $\mathcal{C}$-decomposabilty and cocycle equations. The property of being a selfdecomposable char.f. means that the factorization property (1) holds for all $c \in (0,1)$. For a closed multiplicative semigroup $\mathcal{C}$ (in the unit interval) one may say that a char.f. ϕ is $\mathcal{C}$-decomposable if

$$\forall (c \in \mathcal{C})\ \exists (char.f. \psi_c)\ \ \phi(\cdot) = \phi(c\cdot)\psi_c(\cdot). \tag{20}$$

[Note that in this case ψ_c is not necessarily ID, as opposed to the case $\mathcal{C} = (0,1)$ and (1).] Ref. [5], [19], [20] are devoted to this line of investigations. It seems that there are not too many explicitly given $\mathcal{C}$-decomposable distributions. Also it is an open problem to find a random integral representation of $\mathcal{C}$-decomosable measures analogous to the formula (3).

From (20), assuming that $\psi_c(t) \neq 0$, $c \in \mathcal{C}$, $t \in \mathbb{R}$, we get the following equation:

$$\psi_{c_1 \cdot c_2}(t) = \psi_{c_1}(c_2 t)\psi_{c_2}(t), \quad \text{for all } c_1, c_2 \in \mathcal{C}, t \in \mathbb{R}. \tag{21}$$

Here we recognize a cocycle equation with values in probability measures. Can cohomology methods be used to "solve" (21)?. In [12] a special case was solved. Furthermore, in [10] a slightly different equation than (21) was solved using random integrals.

CONCLUDING REMARKS

(a) Many of the above results or questions have natural extensions for Banach space valued rv.

(b) W. Hazod, H. Kunita, G. Pap and others have generalized selfdecomposability to the Lie group case. Those results are not mentioned here but they also consitute a new aspect of selfdecomposability.

(c) Relations to statistical physics, analytic number theory and cohomology algebra may have impact on those areas of mathematics. But at the same time they may provide new tools for better understanding of the probabilistic notion of selfdecomposability.

REFERENCES

[1] T. M. Apostol (1976). *Modular functions and Dirichlet series in number theory.* Springer-Verlag, New York, Heidelberg, Berlin.

[2] O. E. Barndorff-Nielsen (1996). Processes of normal inverse Gaussian type. *Finance and Stochastics* 2, 41-68.
[3] O. E. Barndorff-Nielsen (1998). Probability and Statistics: selfdecomposability, finance and turbulence. In Accardi, L. and Heyde, C.C. (Eds.): *"Probability towards 2000"* New York: Springer-Verlag.
[4] L. Bondesson (1992). Generalized gamma convolutions and related classes of distribution nad densities. *Lecture Notes in Statistics*, vol. 76, Springer-Verlag, New York.
[5] J. Bunge (1997). Nested clases of C-decomposable laws. *Ann. Probab.* 25, pp. 215-229.
[6] J. De Coninck (1984). Infinitely divisible distributions functions of class L and the Lee-Yang Theorem. *Comm. Math. Phys.* 96, pp. 371-385.
[7] J. De Coninck and Z. J. Jurek (1998). Lee-Yang models, selfdecomposability and negative-definite functions. *Preprint.*
[8] D. Dufrense (1996). On the stochastic equation L(X)=L[B(X+C)] and a property of gamma distributions. *Bernoulli* 2, pp. 287-291.
[9] Ch. M. Goldie and R. Grubel (1996). Perpetuities with thin tails. *Adv. Appl. Probab.* 28, pp. 463-480.
[10] K. H. Hofmann and Z. J. Jurek (1996). Some analytic semigroups occurring in probability theory. *J. Theor. Probab.* 10, pp. 745-763.
[11] A. Janicki and A. Weron (1994). *Simulation and chaotic behaviour of p-stable stochastic processes.* Marcel Dekker, New York.
[12] Z. J. Jurek (1988). Remarks concerning the theory of operator-limit distributions. *Bull. Pol. Acad.: Math.* 36, pp. 307-313.
[13] Z. J. Jurek (1995). (Lie) Semigroups and probability: a survey. In: *Semigroups in Algebra, Geometry and Analysis*; K. H. Hofmann, J. D. Lawson, E. Vinberg Eds. DeGruyter Expositions in Mathematics, pp. 355-365.
[14] Z. J. Jurek (1996). Sereies of independent exponential random variables. In: *Proc. 7th Japan-Russia Symposium in Probabilty and Math. Statistics: Tokyo, July 1995*; Y.V.Prokhorov, A.N. Shiryaev, S. Watanabe, M. Fukushima Eds. World Scientific, pp. 174-182.
[15] Z. J. Jurek (1997). Selfdecomposability: an exception or a rule? *Annales Univer. M. Curie-Skłodowska, Lublin-Polonia*, vol. 51, Sectio A, pp. 93-107.
[16] Z. J. Jurek (1998). Selfdecomposabilty , perpetuity laws and stopping times. *Preprint.*
[17] Z. J. Jurek and J. D. Mason (1993). *Operator limit distributions in probability theory.* J. Wiley & Sons, New York.

[18] M. Maejima and Y. Naito (1997). Semi-selfdecomposable distributions and a new class of limit theorems. *Research Report*, Keio University, Japan.
[19] K. Sato and M. Maejima (1997). Semi-selfsimilar processes, *Research Report*, Keio University, Japan.

Author's address:
Institute of Mathematics
The University of Wrocław
pl. Grunwaldzki 2/4
50-384 Wrocław, Poland
E-mail address: zjjurek@math.uni.wroc.pl

Tweedie Models: a Generalization of Stable Distributions

Bent Jørgensen*

April 20, 1999

The class of Tweedie exponential dispersion models were introduced by Tweedie (1984) as a useful class of statistical models, which later found applications in for example insurance (Jørgensen and de Souza, 1994) and many other areas. See Jørgensen (1997) for background on exponential dispersion models and a survey of Tweedie models.

Let us denote the Tweedie models by $Tw_p(\mu, \sigma^2)$, where $\mu > 0$ is the mean (the domain is $\mathbf{R}$ when $p = 0$), $\sigma^2 > 0$ is the dispersion parameter, and p is a shape parameter. To make the connection with stable distributions, let us introduce the parameter $\alpha = (p-2)/(p-1)$. For $\alpha \in (0, 2]$, the corresponding Tweedie model is the natural exponential family generated by either an extreme stable or positive stable distribution with index α, the stable case corresponding to either $\mu = \infty$ or 0. However, the Tweedie family contains further cases, namely the gamma distribution ($p = 2$ or $\alpha = 0$), a class of compound Poisson distributions ($\alpha < 0$ or $p \in (1, 2)$) and the Poisson distribution ($p = 1$). In the latter case, $Tw_1(\mu, \sigma^2)$ corresponds to the distribution $\sigma^2 Po(\mu/\sigma^2)$, a scaled Poisson distribution.

In which sense do the Tweedie distributions generalize the (positive and extreme) stable distributions? In order to explain this, let us introduce two properties of the Tweedie models. First, we have the scale transformation property:

$$cTw_p(\mu, \sigma^2) = Tw_p(c\mu, c^{2-p}\sigma^2) \qquad \text{for} \qquad c > 0.$$

Second, if $X_1, \ldots, X_n$ are independent and identically distributed $Tw_p(\mu, \sigma^2)$, then the sample average $\overline{X}$ has distribution $Tw_p(\mu, n^{-1}\sigma^2)$, the reproductive property.

*Department of Statistics and Demography, University of Southern Denmark at Odense, Campusvej 55, DK-5230 Odense M, Denmark, bentj@statdem.ou.dk

Combining these properties, and taking $c = n^{1/(2-p)}$, we obtain

$$n^{1/(2-p)}\overline{X} \sim Tw_p(n^{1/(2-p)}\mu, \sigma^2). \tag{1}$$

The left-hand side of (1) is $n^{-1/\alpha}$ times the sum of the Xs, which shows that (1) generalizes the definition of the strictly stable distributions.

For μ finite and non-zero, the property (1) is not so useful, because of the dependence of the mean on n. But if we now take μ to be a function of n,

$$\mu = \mu(n) = \mu_0 n^{-1/(2-p)},$$

say, the right-hand side of (1) is fixed and equal to $Tw_p(\mu_0, \sigma^2)$. In other words, we now have a distributional invariance involving the average of independent and identically distributed random variables, and this invariance has a domain of attraction reached for n tending to infinity, as first shown by Jørgensen, Martínez and Tsao (1994), although their formulation was slightly different. We call this result the *Tweedie Convergence Theorem*.

A possible interpretation of the theorem is that the following two operations commute: 1. generating a natural exponential family and 2. convergence of the standardized sum. This result that was also conjectured by Wentzell, and has now been made precise and proved by Vinogradov (1998). A connection with critical regimes in large deviation theory, with applications to risk theory, was explored by Jørgensen and Vinogradov (1997).

An interesting aspect of the Tweedie convergence theorem is that the two extremes $\alpha = 2$ and $\alpha = -\infty$ correspond to respectively the classical central limit theorem and Bortkiewicz's Poisson convergence theorem, thereby tying together these two famous results within a single framework.

The Tweedie models, being infinitely divisible, generate Lévy processes, which deserve special attention because of the properties of the Tweedie models, and because of several important special cases, ranging from Brownian motion to the Poisson process. These processes have a property that generalizes the idea of long-range dependence. Also, Jørgensen and Martínez (1996) have shown that the canonical measures corresponding to the Tweedie family are proportional to the gamma densities. Furthermore, Jørgensen and Martínez (1997) have shown a kind of Tauber theory for infinitely divisible exponential dispersion models, linking the asymptotic behaviour of the variance function of the model to that of the Lévy measure.

References

[1] Jørgensen, B. (1997). The Theory of Dispersion Models. London: Chapman & Hall.

[2] Jørgensen, B. and Martínez, J.R. (1996). The Lévy-Khinchine representation of the Tweedie family. Brazilian J. Probab. Statist. 10, 225-233.

[3] Jørgensen, B. and Martínez, J.R. (1997). Tauber theory for infinitely divisible variance functions. Bernoulli 3, 213–224.

[4] Jørgensen, B. and Vinogradov, V. (1997). Domains of attraction to Tweedie exponential dispersion models and critical regimes in large deviation theory, with applications to risk theory. Technical Report #8, Department of Mathematics and Computer Science, University of Northern British Columbia.

[5] Jørgensen, B., Martínez, J.R. and Tsao, M. (1994). Asymptotic behaviour of the variance function. Scand. J. Statist. 21, 223–243.

[6] Jørgensen, B. and de Souza, M.C.P. (1994). Fitting Tweedie's compound Poisson model to insurance claim data. Scand. Actuarial J. 1, 69–93.

[7] Tweedie, M.C.K. (1984). An index which distinguishes between some important exponential families. In: Statistics: Applications and New Directions. Proceedings of the Indian Statistical Institute Golden Jubilee International Conference (eds. J.K. Ghosh and J. Roy), pp. 579–604. Calcutta: Indian Statistical Institute.

[8] Vinogradov (1998). On a conjecture of B. Jørgensen and A.D. Wentzell: from extreme stable laws to Tweedie exponential dispersion models. Technical Report #9, Department of Mathematics and Computer Science, University of Northern British Columbia.

On inverse local times, spectral measures and life-time distributions of one-dimensional diffusions

Extended Abstract

Uwe Küchler, Humboldt University Berlin

Suppose $X = (\Omega, \mathcal{F}, (\mathcal{F}_t), X_t, \zeta, P_x)$ is a (regular) diffusion on $[0, L)$ with lifetime ζ, speed measure $m(\cdot)$ and scale $s(\cdot)$ in the sense of Ito, McKean (1974). Assume $s(0) = m(0-) = 0, s(L-) = \infty$ and let 0 be a killing, elastic killing or reflecting boundary. (The following notions and properties can easily be extended to gap diffusions.)

The infinitesimale generator A of X in $L_2(m)$ can be given by the restriction of the generalized second order differential operator $D_m D_s$ to

$$D_A := \{f \in L_2(m) | D_m D_s f \in L_2(m), a \cdot D_s^- f(0) = f(0)\},$$

where $a \in [0, \infty]$ is fixed. (The number a is connected with the killing rate of X at zero.)
Because of $s(L-) = \infty$ we have for $a < \infty$ that

$$P_x(\zeta < \infty, X_{\zeta-} = 0) \equiv 1.$$

For every complex λ and $a \in [0, \infty]$ let φ_a be the solution of

$$D_m D_s \Phi + \lambda \Phi = 0$$

satisfying the boundary conditions

$$\begin{aligned} \varphi_a(0, \lambda) &= 1, \quad D_s^- \varphi_a(0, \lambda) = \frac{1}{a} \text{ if } a \in (0, \infty], \\ \varphi_0(0, \lambda) &= 0, \quad D_s^- \varphi_0(0, \lambda) = 1. \end{aligned}$$

Define

$$\chi(x, \lambda) := \varphi_\infty(x, \lambda) \int_x^L \varphi_\infty^{-2}(u, \lambda) s(du), \quad x \in [0, L), \lambda \in K_- := K \backslash [0, \infty).$$

(K denotes the set of complex numbers.)

The function $G(\cdot)$ given by

$$G(\lambda) := \chi(0,\lambda), \quad \lambda \in K_-$$

is called the *characteristic function* of (m,s) (more precisely, of $m \circ s^{-1}$). By assumption we have

$$G(0) := \lim_{\lambda \uparrow 0} G(\lambda) = s(L-) = \infty.$$

Moreover, it holds

$$\chi(x,\lambda) = G(\lambda)\varphi_\infty(x,\lambda) - \varphi_0(x,\lambda), \quad x \in [0,L), \lambda \in K_-.$$

The Laplace transform of the first hitting time σ_y of y satisfies

$$E_x \exp(\lambda \sigma_y) = \Phi(y,\lambda)/\Phi(x,\lambda), \quad \lambda < 0, x, y \in [0,L)$$

with

$$\Phi = \varphi_a \text{ if } x \le y \text{ and } \Phi = \chi \text{ if } x \ge y.$$

We have $P_x(\zeta < \infty, X_{\zeta-} = 0) = 1$ if $0 \le a < \infty$ and $P_x(\zeta \equiv \infty) = 1$ if $a = \infty$. Moreover, it holds for $0 < a \le \infty$

$$E_x \exp[\lambda\zeta] = \frac{\chi(x,y)}{\chi(0,\lambda)} \cdot \frac{\frac{1}{a}}{\frac{1}{a} + \frac{1}{G(\lambda)}} \quad x \in [0,L), \lambda < 0.$$

For every $a \in (0,\infty]$ the inverse local time $l^{-1}(t,0)$ is a Lévy-process with

$$E_0 \exp[\lambda l^{-1}(t,0)] = \exp[-t\Big[\frac{1}{a} + \frac{1}{G(\lambda)}\Big]), \quad \lambda < 0.$$

For every $a \in [0,\infty]$ there exists the *transition density* $p^{(a)}(t,x,y)$:

$$P_x(X_t \in dy) = p^{(a)}(t,x,y)m(dy), \quad t > 0, x, y \in [0,L).$$

Define the *resolvent kernel*

$$r_\lambda^{(a)}(x,y) := \int_0^\infty e^{\lambda t} p^{(a)}(t,x,y)dt, \quad \lambda \in K_-, x, y \in [0,L).$$

It holds

$$r_\lambda^{(a)}(x,y) = \frac{\varphi^{(a)}(x \wedge y, \lambda)\chi(x \vee y, \lambda)}{W(\lambda)} \quad , \quad x, y \in [0,L), \lambda < 0$$

where $W(\lambda)$ denotes the Wronskian of $\varphi^{(a)}$ and χ.

For every $a \in [0,\infty]$ the *spectral measure* $\tau = \tau_a$ of $D_m D_s|_{D_A}$ is defined to be a measure on $[0,\infty)$ (on $(0,\infty)$ if $a < \infty$) with

$$r_\lambda^{(a)}(x,y) = \int_0^\infty \frac{\varphi_a(x,u)\varphi_a(y,u)}{u-\lambda}\tau_a(du), \quad \lambda < 0; x, y \in [0, L).$$

It exists and is uniquely determined. Moreover, it holds

$$\left(\frac{1}{a}+\frac{1}{G(\lambda)}\right)^{-1} = r_\lambda^{(a)}(0,0) = \int_0^\infty \frac{\tau_a(du)}{u-\lambda}, \lambda < 0, a \in (0,\infty] \qquad (*)$$

and

$$\frac{1}{G(\lambda)} = \int_0^\infty \left(\frac{1}{u} - \frac{1}{u-\lambda}\right)\tau_0(du).$$

We have

$$G(\lambda) = r_\lambda^{(\infty)}(0,0) = \int_0^\infty \frac{\tau_\infty(du)}{u-\lambda}, \quad \lambda < 0,$$

and using $G(0-) = \infty$ we get from $(*)$

$$\int_0^\infty \frac{\tau_a(du)}{u} = a, \quad a \in (0,\infty].$$

Now it follows that for $0 < a < \infty$

$$P_0(\zeta \in dt) = \frac{1}{a}\int_0^\infty u \exp[-ut]\frac{\tau_a(du)}{u}dt, \quad t > 0 \qquad (**)$$

(mixed exponential distribution).

Let $l = l(t,0)$ be the *local time of X at zero*. The inverse local time $l^{-1}(t,0), t \geq 0$, is a process with independend stationary increments having the Lévy-measure

$$\nu(du) = \int_0^\infty e^{ul}\tau_0(dl)du, \quad u > 0.$$

We have

$$E_0 \exp[\lambda l^{-1}(t,0)] = \exp[-\frac{t}{a}] \cdot \exp\Big[-t\int_0^\infty (1-e^{u\lambda})\nu(du)\Big], \quad t \geq 0.$$

In the following fix an $a \in (0,\infty)$. Then it holds $P_x(\zeta < \infty, X_\zeta - = 0) \equiv 1$ and for every $\vartheta < 0$

$$h^{(\vartheta)}(x) := E_x \exp[\vartheta\zeta], \quad x \in [0, L)$$

defines an ϑ-exessive function. The $h^{(\vartheta)}$-transformed diffusion $X^{(\vartheta)}$ on $[0, L)$ can be described by the scale $ds^{(\vartheta)}(x) = \frac{G^2(\vartheta)}{\chi^2(x,\vartheta)} \cdot ds(x)$, the speed measure $dm^{(\vartheta)}(x) = \frac{x^2(x,\vartheta)}{G^2(\vartheta)} dm(x)$ and the boundary coefficient at zero $a_\vartheta = \left[\frac{1}{a} + \frac{1}{G(\vartheta)}\right]^{-1}$. Its characteristic function $G^{(\vartheta)}(\lambda)$ satisfies

$$\frac{1}{G^{(\vartheta)}(\lambda)} = \frac{1}{G(\lambda + \vartheta)} - \frac{1}{G(\vartheta)}, \quad \lambda \in K_-.$$

For the spectral measure $\tau_{a_\vartheta}^{(\vartheta)}$ it holds

$$\tau_{a_\vartheta}^{(\vartheta)}(du) = \tau_a(du + \{\vartheta\}).$$

Theorem *Fix an $a \in (0,\infty)$. For the family $(X^{(\vartheta)}, \vartheta \le 0)$ of diffusions the following properties hold:*

(i) Assume $0 \le y < x < L$. Then the first hitting time distributions

$$(P_x^{(\vartheta)}(\sigma_y \in dt)), \quad \vartheta \le 0$$

form an exponential familiy of distributions:

$$P_x^{(\vartheta)}(\sigma_y \in dt) = \frac{\exp(\vartheta t) P_x(\sigma_y \in dt)}{\int_0^\infty \exp(\vartheta s) P_x(\sigma_y \in ds)}$$

(ii) For every $x \in [0, L)$ the life-time distributions

$$(P_x^{(\vartheta)}(\zeta \in dt), \vartheta \le 0)$$

form an exponential family of distributions:

$$P_x^{(\vartheta)}(\zeta \in dt) = \frac{\exp(\vartheta t) P_x(\zeta \in dt)}{h^{(\vartheta)}(x)}$$

where $P_x(\zeta \in dt)$ is given by $(**)$.

If $x = 0$ then we have

$$P_0^{(\vartheta)}(\zeta \in dt) = \frac{1}{a_\vartheta} \int_0^\infty u e^{ut} \frac{\tau_a(du + \{\vartheta\})}{u} dt, t > 0, \quad \vartheta \le 0,$$

with $\frac{1}{a_\vartheta} = \frac{1}{a} + \frac{1}{G(\vartheta)}$ (mixed exponential distribution).

(iii) The inverse local times $(l_\vartheta^{-1}(t,0), t \geq 0)$ *of* $X^{(\vartheta)}$ *at zero form an exponential family of increasing processes with independent stationary increments, and their Lévy-measures* ν_ϑ *are given by*

$$\nu_\vartheta(du) = e^{\vartheta u}\nu(du) = e^{\vartheta u}\int_0^\infty e^{-us}\tau_0(ds)du.$$

The process $(l_\vartheta^{-1}(t,0)), t \geq 0$ *is killed with constant killing rate* a_ϑ.

References

Ito, K.; McKean, H.P. (1974): Diffusion processes and their Sample Paths, 2nd Printing, Springer, Berlin, reprint 1996.

Knight, F.B. (1981): Characterization of the Lévy measures of inverse local times of gap diffusion, Progress in Prob. Statist. 1, Birkhäuser, Boston, Mass.

Küchler, U.; Neumann, K. (1991): An extension of Krein's inverse spectral theorem to strings with nonreflecting left boundaries, Lectures Notes in Math., 1485 (Séminaire des Probabilités XXV), 354-373

Küchler, U. (1994): On life-time-distributions of some one-dimensional diffusions and related exponential families, Preprint 94-9, Humboldt-Universität Berlin, Fachbereich Mathematik

SEMI-SELFDECOMPOSABILITY AND SEMI-SELFSIMILARITY

MAKOTO MAEJIMA
Keio University

In this talk, we first discuss the class of *semi-selfdecomposable* distributions as a natural extension of that of selfdecomposable distributions, and its nested subclasses. Then we introduce a new notion of *semi-selfsimilarity* of stochastic processes. We conclude the talk with marginal and joint distributions of semi-selfsimilar processes with *independent increments*, and we see how the class of semi-selfdecomposable distributions and its nested subclasses are connected to distributions of semi-selfsimilar processes with independent increments.

This talk is based on several recent works jointly done by Y. Naito, K. Sato and T. Watanabe ([MN98], [MS99], [MSW98], [MSW99a], [MSW99b] and [MSW99c]).

1. The class of semi-selfdecomposable distributions and its nested subclasses

Stable distributions are characterized as limiting distributions of normalized partial sums of independent and identically distributed random variables. Self-decomposable distributions are natural extensions of stable distributions and are given by limiting distributions of normalized partial sums of independent random variables, which are not necessarily identically distributed but satisfy the infinitesimal condition.

On the other hand, semi-stable distributions have also been well studied as extensions of stable distributions. As is well known, semi-stable distributions are characterized as limiting distributions of some geometrically increasing subsequences of normalized partial sums of independent

and identically distributed random variables. However, although selfdecomposable distributions are natural extensions of stable distributions, it seems that the distributions extending semi-stable distributions in the same way have not been well recognized. We start by defining semi-selfdecomposable distributions as limiting distributions of *some subsequences* of normalized partial sums of independent random variables, which are not necessarily identically distributed but satisfy the infinitesimal condition.

We use the following notation. $\mathcal{P}(\mathbf{R}^d)$ is the class of all probability distributions on $\mathbf{R}^d$, $I(\mathbf{R}^d)$ is the class of all infinitely divisible distributions on $\mathbf{R}^d$, $L(\mathbf{R}^d)$ is the class of all selfdecomposable distributions on $\mathbf{R}^d$, $\widehat{\mu}(z)$ is the characteristic function of $\mu \in \mathcal{P}(\mathbf{R}^d)$, $\mathcal{L}(X)$ is the law of X, $\langle\ ,\ \rangle$ is the Euclidean inner product in $\mathbf{R}^d$, $|\cdot|$ is the norm induced by $\langle\ ,\ \rangle$ in $\mathbf{R}^d$, $S = \{x \in \mathbf{R}^d : |x| = 1\}$ and $D = \{x \in \mathbf{R}^d : |x| \leq 1\}$.

Most of results in this section are proved in [MN98]. We first introduce a way of making a new class of limiting distributions derived from a class of distributions,

Definition 1.1. Let $C \subset \mathcal{P}(\mathbf{R}^d)$ and $0 < b < 1$. $\mu \in \mathcal{P}(\mathbf{R}^d)$ is said to belong to the class $K(C, b)$ if there exist independent $\mathbf{R}^d$-valued random vectors $\{X_j\}, a_n > 0, \uparrow \infty, c_n \in \mathbf{R}^d, \{k_n\} \subset \mathbf{N}, k_n \uparrow \infty$ such that

$$\lim_{n\to\infty} a_n/a_{n+1} = b,$$

$$\mathcal{L}(X_j) \in C,$$

$$\mathcal{L}\left(a_n^{-1}\sum_{j=1}^{k_n} X_j + c_n\right) \to \mu, \tag{1.1}$$

$$\lim_{n\to\infty} \max_{1\leq j\leq k_n} P\{|a_n^{-1}X_j| > \varepsilon\} = 0, \quad \forall \varepsilon > 0. \tag{1.2}$$

Remark 1.1.

(i) When (1.2) is satisfied, we say that random variables $\{a_n^{-1}X_j, 1 \leq j \leq k_n, n = 1, 2, \cdots\}$ satisfy the infinitesimal condition.

(ii) If we take $k_n = n$, (then automatically $b = 1$), then the above class is turned out to be the one which Sato [S80] studied.

(iii) Although we are dealing with $\mathbf{R}^d$-valued random variables $\{X_j\}$, the normalization in (1.1) is scalar. The normalization by linear operators might be natural in higher dimensions. Generalization to the linear operator normalization is discussed in [MSW98] and [MSW99a].

Theorem 1.1. $K(C, b) \subset I(\mathbf{R}^d)$.

Proof. Obvious from Definition 1.1.

Definition 1.2. $C \subset \mathcal{P}(\mathbf{R}^d)$ is said to be completely closed if C is closed under weak convergence, convolution and type equivalence. Here C is said to be closed under type equivalence if $\mathcal{L}(X) \in C$ implies $\mathcal{L}(aX + c) \in C$ for any $a > 0$ and $c \in \mathbf{R}^d$.

Theorem 1.2. Let $0 < b < 1$ and suppose that C is completely closed. Then $K(C, b) \subset C$.

Proof. Easy from the definitions.

Definition 1.3. For each $b \in (0, 1)$, define $L_0(b) = K(\mathcal{P}(\mathbf{R}^d), b)$. If μ belongs to $L_0(b)$ for some $b \in (0, 1)$, it is called semi-selfdecomposable.

Theorem 1.3. $L(\mathbf{R}^d) = \bigcap_{0<b<1} L_0(b)$.

Proof. It is known that $\mu \in L(\mathbf{R}^d)$ if and only if for any $b \in (0, 1)$, there exists $\rho_b \in I(\mathbf{R}^d)$ such that $\widehat{\mu}(z) = \widehat{\mu}(bz)\widehat{\rho}_b(z), \forall z \in \mathbf{R}^d$, (see [S80]). Hence, th4 statement is obvious from Theorem 1.4 below.

We now give the first characterization of the class $L_0(b)$ in terms of *decomposability*.

Theorem 1.4 ([MN98])**.** A necessary and sufficient condition for that $\mu \in L_0(b)$ is that there exists $\rho_0 \in I(\mathbf{R}^d)$ such that $\widehat{\mu}(z) = \widehat{\mu}(bz)\widehat{\rho}_0(z), \forall z \in \mathbf{R}^d$.

We next give the second characterization of $L_0(b)$. Any $\mu \in L_0(b)$ is in-

finitely divisible and its characteristic function $\widehat{\mu}(z)$ has the following Lévy-Khintchine representation:

$$\widehat{\mu}(z) = \exp\left\{ i\langle z, \gamma\rangle - \frac{1}{2}\langle Az, z\rangle + \int_{\mathbf{R}^d\setminus\{0\}} g(z,x)\nu(dx) \right\},$$

where ν is the Lévy measure of μ, A is the Gaussian covariance matrix, $\gamma \in \mathbf{R}^d$ and $g(z,x) = e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle 1_D(x)$.

Theorem 1.5 ([MN98])**.** Let $0 < b < 1$. A necessary and sufficient condition for that $\mu \in L_0(b)$ is that $\mu \in I(\mathbf{R}^d)$ and its Lévy measure is $\nu \equiv 0$ or

$$\nu(EB) = -\int_B \lambda(d\xi) \int_E dF_\xi(r), \quad E \in \mathcal{B}((0,\infty)), B \in \mathcal{B}(S),$$

where λ is a probability measure on S, for each $r > 0$, $F_\xi(r)$ is ξ-measurable, and for each $\xi \in S$, $F_\xi(r)$ is right continuous and nonincreasing, $\lim_{r\to\infty} F_\xi(r) = 0$,

$$F_\xi(b(r+\delta)) - F_\xi(br) \le F_\xi(r+\delta) - F_\xi(r) \le 0$$

for every $\delta > 0$ and $r > 0$, and for each $\xi \in S$

$$0 < -\int_0^\infty (1 \wedge r^2) dF_\xi(r) = \int_{\mathbf{R}^d\setminus\{0\}} (1 \wedge \|x\|^2)\nu(dx) =: K < \infty.$$

Here the value K is independent of ξ. This representation is unique in the following sense. If $\nu \not\equiv 0$ and two pairs (λ, F_ξ) and $(\widetilde{\lambda}, \widetilde{F}_\xi)$ satisfy the above conditions, then $\lambda = \widetilde{\lambda}$ and $F_\xi = \widetilde{F}_\xi$ for λ-a.e. ξ. (We call F_ξ, uniquely determined in this sense, the F-function of $\mu \in L_0(b)$.)

We now define the nested subclasses of $L_0(b)$.

Definition 1.4. For each $b \in (0,1)$, define

$$L_m(b) = K(L_{m-1}(b), b), \quad m = 1, 2, \cdots, \quad \text{and} \quad L_\infty(b) = \bigcap_{m=0}^{\infty} L_m(b).$$

Theorem 1.6. (Nested classes.) Let $0 < b < 1$. The we have

$$I(\mathbf{R}^d) \supset L_0(b) \supset \cdots \supset L_m(b) \supset \cdots \supset L_\infty(b),$$

and all inclusions are strict.

Proof. Easy from Definition 1.4 and Theorem 1.2.

Theorem 1.7 ([MN98])**.** (Characterization of $L_m(b)$ by decomposability.) Let $0 < b < 1$ and $m = 1, 2, \cdots, \infty$.

(i) A necessary and sufficient condition for that $\mu \in L_m(b)$ is that there exists $\rho_m \in L_{m-1}(b)$ such that $\widehat{\mu}(z) = \widehat{\mu}(bz)\widehat{\rho}_m(z), \forall z \in \mathbf{R}^d$, where $m - 1 = \infty$ when $m = \infty$.

(ii) $L_\infty(b) = K(L_\infty(b), b)$, and furthermore $L_\infty(b)$ is the largest class among the classes which are invariant under the $K(\cdot, b)$-operation.

We next give a characterization of $L_m(b)$ in terms of F-function of $\mu \in L_0(b)$. We need one more definition.

Definition 1.5. Let $0 < b < 1$. For a function $F : (0, \infty) \to \mathbf{R}$, define $\mathcal{E}_b F(s) = F(bs) - F(s)$ and its m-th iteration $\mathcal{E}_b^m F(s) = \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} F(b^j s)$. Also define, for $\delta > 0$, $\Delta_\delta f(s) = f(s+\delta) - f(s)$. Then we say that F has *the property* (m, b) if $\Delta_\delta \mathcal{E}_b^j F(s) \leq 0, 1 \leq \forall j \leq m, \forall s > 0, \forall \delta > 0$. When F has the property (m, b) for any $m \geq 1$, then we say that it has *the property* (∞, b).

Theorem 1.8 ([MN98])**.** Let $0 < b < 1$ and $m = 0, 1, 2, \cdots, \infty$. A necessary and sufficient condition for that $\mu \in L_m(b)$ is that $\mu \in L_0(b)$, and if $\nu \not\equiv 0$, then the F-function, F_ξ, of μ has the property $(m+1, b)$ for λ-a.e. ξ.

The relationship between semi-selfdecomposability and semi-stability is the following.

Theorem 1.9 ([MN98])**.** Let μ is semi-stable, namely suppose that for some $a, b \in (0, 1)$ and $c \in \mathbf{R}^d$, $\widehat{\mu}(z)^a = \widehat{\mu}(bz)e^{i\langle z, c\rangle}, \forall z \in \mathbf{R}^d$. Then $\mu \in L_\infty(b)$.

Hence if we define, for each $b \in (0, 1)$, $SS(b)$ as the set of all $\mu \in I(\mathbf{R}^d)$ such that $\widehat{\mu}(z)^a = \widehat{\mu}(bz)e^{i\langle c, z\rangle}$ for some $a \in (0, 1)$ and $c \in \mathbf{R}^d$, then we have

$$I(\mathbf{R}^d) \supset L_0(b) \supset \cdots \supset L_m(b) \supset \cdots \supset L_\infty(b) \supset SS(b).$$

Another important characterization of $L_\infty(b)$ connected to $SS(b)$ is the following.

Theorem 1.10 ([MSW98])**.** For each $b \in (0,1)$, $L_\infty(b)$ is the closure of $SS(b)$ under the operations of type equivalence, convolution and raising to the t-th convolution for every $t > 0$, and weak convergence.

Let us define classes $\widetilde{L}_m(b), m = 0, 1, 2, ..., \infty$ in the same as we have defined $L_m(b)$ but without infinitesimal condition (1.2). Loève ([Lo45]) (for $m = 0$) and Bunge ([Bu97]) (for general m) studied these classes. They are also nested classes namely

$$\mathcal{P}(\mathbf{R}^d) \supset \widetilde{L}_0(b) \supset \cdots \supset \widetilde{L}_m(b) \supset \cdots \supset \widetilde{L}_\infty(b).$$

Trivially from their definitions, $\widetilde{L}_m(b) \supset L_m(b)$. But, Bunge ([Bu97]) showed that $\widetilde{L}_m(b) \cap (I(\mathbf{R}^d))^c \neq \emptyset$, for finite m, when $d = 1$. Namely, $\widetilde{L}_m(b)$ is strictly bigger than $L_m(b)$ for finite m. On the other hand, he also showed that $\widetilde{L}_\infty \subset I(\mathbf{R}^d)$, when $d = 1$. We can also show the same assertions for the case $d \geq 2$ ([MSW99a]). Then a natural question arises, how $\widetilde{L}_\infty(b)$ is related to $L_m(b)$. The answer is the following.

Theorem 1.11 ([MSW99a])**.** $L_\infty(b) = \widetilde{L}_\infty(b)$.

2. Semi-selfsimilar processes

An $\mathbf{R}^d$-valued Lévy process $\{X(t), t \geq 0\}$ with strictly α-semi-stable marginal distribution at each t is, in general, not selfsimilar, but it has the following property: For some $a \in (0,1) \cup (1,\infty)$,

$$\{X(at), t \geq 0\} \stackrel{d}{=} \{a^{1/\alpha} X(t), t \geq 0\}, \tag{2.1}$$

where $\stackrel{d}{=}$ denotes the equality in all finite-dimensional distributions. Here, by a Lévy process, we mean a stochastically continuous process starting at the origin with independent and stationary increments.

Being motivated by the property (2.1), we introduce a new notion of *semi-selfsimilarity* as follows.

Definition 2.1. An $\mathbf{R}^d$-valued stochastic process $\{X(t), t \geq 0\}$ is said to be *semi-selfsimilar* if there exist $a \in (0,1) \cup (1,\infty)$ and $b > 0$ such that

$$\{X(at), t \geq 0\} \stackrel{d}{=} \{bX(t), t \geq 0\}. \tag{2.2}$$

We can define semi-selfsimilar processes allowing a drift function on the right hand side of (2.2), which we call wide-sense semi-selfsimilar. The results in this section remain true for wide-sense semi-selfsimilar processes. (For details, see [MS99].)

Recall that $\{X(t)\}$ is said to be *selfsimilar* if, for every $a > 0$, there is $b = b(a) > 0$ satisfying (2.2). Thus the notion of semi-selfsimilarity extends that of selfsimilarity. Besides semi-stable Lévy processes, processes with property (2.2) are found in the literature about diffusions on Sierpinski gaskets (Example 2.1 below).

One of the important results on selfsimilar processes is the existence of exponent. Namely, if $\{X(t)\}$ is selfsimilar, we know that there exists an H, exponent of the selfsimilarity, so that b in (2.2) has the form $b = a^H$. (See [La62].) The first theorem in this section is that the same conclusion remains true for semi-selfsimilar processes.

Definition 2.2. An $\mathbf{R}^d$-valued random variable X is called degenerate if it is a constant a. s. An $\mathbf{R}^d$-valued process $\{X(t)\}$ is called trivial if $X(t)$ is degenerate for every t.

Theorem 2.1. Let $\{X(t),\, t \geq 0\}$ be a nontrivial semi-selfsimilar $\mathbf{R}^d$-valued process such that it is stochastically continuous at $t = 0$. Then the following statements hold.
(i) ([MSW99b]) There exists a unique $H \geq 0$ such that if $a > 0$ and $b > 0$ satisfy (2.2), then $b = a^H$. $H > 0$ if and only if $X(0) = 0$ a. s.

(ii) ([MS99]) Suppose furthermore that $\{X(t)\}$ is stochastically continuous at any $t \geq 0$. Let Γ be the set of $a > 0$ such that there is $b > 0$ satisfying (2.2). Then $\Gamma \cap (1, \infty)$ is nonempty. Denote the infimum of $\Gamma \cap (1, \infty)$ by a_0. If $a_0 > 1$, then $\Gamma = \{{a_0}^n : n \in \mathbf{Z}\}$, and $\{X(t)\}$ is *not* selfsimilar. If $a_0 = 1$, then $\Gamma = (0, \infty)$, and $\{X(t)\}$ is *actually* selfsimilar.

The real number H is called exponent of the semi-selfsimilar process. In order to signify it, we call $\{X(t)\}$ H-semi-selfsimilar. It is worthwhile to remark that (ii) of Theorem 2.1 implies that if a stochastically continuous process $\{X(t)\}$ satisfies (2.2) for some a_1 and a_2 such that $\log a_1 / \log a_2$ is irrational, then it is selfsimilar.

As is well known, selfsimilar processes are realized as scaling limits of stochastic processes. We show that semi-selfsimilar processes are characterized as limiting processes of *some subsequences*

of usually normalized processes.

Theorem 2.2 ([MS99])**.**

(i) Suppose that an $\mathbf{R}^d$-valued process $\{X(t), t \geq 0\}$ is stochastically continuous at $t = 0$. Suppose that there exist another $\mathbf{R}^d$-valued process $\{Y(t), t \geq 0\}$, $0 < b_n \uparrow \infty$, $0 < a_n \uparrow \infty$ such that, for some $a > 1$,

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = a, \tag{2.3}$$

$$\frac{1}{b_n} \{Y(a_{n+1}t) - Y(a \cdot a_n t)\} \to 0 \quad \text{in probability,} \tag{2.4}$$

$$\left\{ \frac{1}{b_n} Y(a_n t), t \geq 0 \right\} \stackrel{d}{\Rightarrow} \{X(t), t \geq 0\},$$

where $\stackrel{d}{\Rightarrow}$ denotes the convergence in all finite-dimensional distributions. Suppose further that there exists $t_0 > 0$ such that $X(t_0)$ and $X(at_0)$ are nondegenerate. Then $\{X(t)\}$ is H-semi-selfsimilar with some $H > 0$.

(ii) Conversely, if $\{X(t)\}$ is nontrivial, H-semi-selfsimilar with $H > 0$, and stochastically continuous at $t = 0$, then $\{X(t)\}$ is such a limit.

We remark that if $a_n = a^n$ with $a > 1$, then (2.3) and (2.4) are automatically satisfied.

Example 2.1. The diffusions on Sierpinski gaskets $\{X(t)\}$ on $\mathbf{R}^d$ are constructed as

$$\left\{ \frac{1}{2^n} Y((d+3)^n t) \right\} \stackrel{d}{\Rightarrow} \{X(t)\}$$

for some $\{Y(t)\}$ and $X(t)$ is shown to be nondegenerate for $t > 0$. (See [K87], [G87] and [BaP88].) Hence by Theorem 2.2, $\{X(t)\}$ is semi-selfsimilar. We can see that the exponent H of $\{X(t)\}$ is $\log 2/\log(d+3)$.

The connection between the semi-selfsimilarity and the semi-stability of Lévy processes is as follows.

Theorem 2.3 ([MS99])**.** Let $\{X(t), t \geq 0\}$ be an $\mathbf{R}^d$-valued Lévy process. Then $\{X(t)\}$ is semi-selfsimilar if and only if $\mathcal{L}(X(1))$ is strictly semi-stable.

3. Marginal and joint distributions of semi-selfsimilar processes with independent increments

Theorem 3.1 ([MS99])**.** Suppose that $\{X(t), t \geq 0\}$ is a nontrivial, stochastically continuous, H-semi-selfsimilar $\mathbf{R}^d$-valued process with independent increments, where $H > 0$. Choose $a \in \Gamma \cap (1, \infty)$. Then $\mathcal{L}(X(t))$ is semi-selfdecomposable for any $t \geq 0$. Actually it belongs to the class $L_0(a^{-H})$. For any $t > 0$, $\mathcal{L}(X(t))$ is nondegenerate.

Theorem 3.2 ([MS99])**.** Let $a > 1$ and $H > 0$. Suppose that μ is semi-selfdecomposable and $\mu \in L_0(a^{-H})$. Then there exists a nontrivial, stochastically continuous, H-semi-selfsimilar $\mathbf{R}^d$-valued process $\{X(t), t \geq 0\}$ with independent increments such that $a \in \Gamma \cap (1, \infty)$ and $\mathcal{L}(X(1)) = \mu$

A process $\{X(t)\}$ constructed in Theorem 3.2 is not unique. However, if we are given a set of semi-selfdecomposable distributions $\{\mu_t, 1 \leq t < a\}$ with certain conditions, then there exists uniquely in law such a semi-selfsimilar

process $\{X(t)\}$ satisfying $\mathcal{L}(X(t)) = \mu_t, 1 \le t < a$.

We now give conditions for the joint distributions of semi-selfsimilar processes with independent increments to be semi-selfdecomposable, and further, conditions for them to belong to the subclasses L_m.

Theorem 3.3 ([MSW99c])**.** Let $\{X(t), t \ge 0\}$ and H be the same as in Theorem 3.1 and let

$a_0 = \inf \Gamma \cap (1, \infty) > 1$. Let m be a positive integer or ∞. Then the following two statements are equivalent. We understand $m - 1 = \infty$ if $m = \infty$.

(i) $\mathcal{L}(X(t)) \in L_m(a_0^{-H}, \mathbf{R}^d), \quad \forall t \ge 0.$

(ii) $\mathcal{L}((X(u_1 t), \cdots, X(u_n t))) \in L_{m-1}(a_0^{-H}, \mathbf{R}^{nd}), \quad \forall n, \forall t \ge 0, \forall u_1, \cdots, u_n \in \Gamma.$

Let us compare the situation with the case of selfsimilar processes. The class of selfdecomposable distributions on $\mathbf{R}^d$ is denoted by $L_0(\mathbf{R}^d)$. A sequence of its subclasses $L_m(\mathbf{R}^d), m = 0, 1, \cdots, \infty$, is studied in [U72], [U73], [S80] and others. A description of the classes is as follows. A distribution $\mu \in \mathcal{P}(\mathbf{R}^d)$ belongs to $L_0(\mathbf{R}^d)$ if and only if, for any $b \in (0, 1)$, there is $\rho_b \in \mathcal{P}(\mathbf{R}^d)$ such that

$$\widehat{\mu}(z) = \widehat{\mu}(bz)\widehat{\rho}_b(z), \quad \forall z \in \mathbf{R}^d. \tag{3.1}$$

Let m be a positive integer. A distribution $\mu \in \mathcal{P}(\mathbf{R}^d)$ belongs to $L_m(\mathbf{R}^d)$ if and only if $\mu \in L_0(\mathbf{R}^d)$ and, for every $b \in (0, 1), \rho_b$ in (3.1) belongs to $L_{m-1}(\mathbf{R}^d)$. The class $L_\infty(\mathbf{R}^d)$ is the intersection of the classes $L_m(\mathbf{R}^d), m = 0, 1, \cdots$. Thus we have $I(\mathbf{R}^d) \supset L_0(\mathbf{R}^d) \supset L_1(\mathbf{R}^d) \supset \cdots \supset L_\infty(\mathbf{R}^d)$.

Theorem 3.4 ([MSW99c]**.** Let $\{X(t), t \ge 0\}$ be a stochastically continuous, selfsimilar $\mathbf{R}^d$-valued process with independent increments and $X(0) = 0$ a.s. Let m be a positive integer or ∞. Then the following statements are equivalent.

(i) $\mathcal{L}(X(t)) \in L_m(\mathbf{R}^d), \quad \forall t \ge 0.$

(ii) $\mathcal{L}((X(t_1), \cdots, X(t_n))) \in L_{m-1}(\mathbf{R}^{nd}), \quad \forall n, \ \forall t_1, \cdots, t_n \ge 0.$

Comparing Theorems 3.3 and 3.4, one might ask whether in Theorem 3.3

the condition (i) implies that all $n \times d$-dimensional joint distributions of $\{X(t)\}$ are in $L_{m-1}(a_0^{-H}, \mathbf{R}^{nd})$, without any restriction on choosing $t_1, ..., t_n$. However, the answer is negative.

Example 3.1 ([MSW99c]). For $d = 1$, we can construct a semi-selfsimilar process with independent increments $\{X(t)\}$ such that $\mathcal{L}(X(t)) \in L_1(b, \mathbf{R}), 1 < t < a_0$, but $\mathcal{L}((X(t), X(1))) \notin L_0(b, \mathbf{R}^2), 1 < t < 1 + \varepsilon$, for some small $\varepsilon > 0$.

References

[BaP88] Barlow, M.T. and Perkins, E.A. (1988) . Brownian motion on the Sierpinski gasket. *Probab. Th. Rel. Fields* **79**, 543–623.

[Bu97] Bunge, J. (1997). Nested classes of C-decomposable laws. *Ann. Probab.* **25** 215–229.

[G87] Goldstein, S. (1987) . Random walks and diffusions on fractals. In : Kesten, H. (ed.) *Percolation Theory and Ergodic Theory of Infinite Particle Systems*, IMA Vol. Math. Appl. **8**, 121–128. Springer.

[K87] Kusuoka, S. (1987) . A diffusion process on a fractal. In : Itô, K. and Ikeda, N. (eds.) *Probabilistic Methods in Mathematical Physics*, Proceedings Taniguchi Symposium, Katata 1985, 251–274. Kinokuniya-North Holland.

[La62] Lamperti, J. (1962) . Semi-stable stochastic processes. *Trans. Amer. Math. Soc.* **104**, 62–78.

[Lo45] Loève, M. (1945). Nouvelles classes de lois limites. *Bull. Soc. Math. France* **73** 107–126. (In French.)

[MN98] Maejima, M. and Naito, Y. (1998). Semi-selfdecomposable distributions and a new class of limit theorems. *Probab. Th. Rel. Fields* **112**, 13–31

[MS99] Maejima, M. and Sato, K. (1999). Semi-selfsimilar processes. To appear in *J. Theor. Probab.*

[MSW98] Maejima, M., Sato, K. and Watanabe, T. (1998). Completely operator

semi-selfdecomposable distributions. Preprint.

[MSW99a] Maejima, M., Sato, K. and Watanabe, T. (1999). Operator semi-selfdecomposability, (C, Q)-decomposability and related nested classes. To appear in *Tokyo J. Math.*

[MSW99b] Maejima, M., Sato, K. and Watanabe, T. (1999). Exponents of semi-selfsimilar processes. Preprint.

[MSW99c] Maejima, M., Sato, K. and Watanabe, T. (1999). Distributions of selfsimilar and semiselfsimilar processes with independent increments. Preprint.

[S80] Sato, K. (1980). Class L of multivariate distributions and its subclasses. *J. Multivar. Anal.* **10** 207–232.

[U72] Urbanik, K. (1972). Slowly varying sequences of random variables. *Bull. Acad. Pol. Sci. Sér. Sci. Math. Astronom. Phys.* **20** 679–682.

[U73] Urbanik, K. (1973). Limit laws for sequences of normed sums satisfying some stability conditions. In : Karishniah, P.R. (ed.) *Multivariate Analysis-III*, 225–237, Academic Press.

Department of Mathematics, Keio University
3-14-1, Hiyoshi, Kohoku-ku, Yokohama 223-8522, Japan
(*E-mail address*: maejima@math.keio.ac.jp)

Conference on Lévy Processes: Theory and Applications, 18-22 January 1999
MaPhySto Centre, Department of Mathematics, University of Aarhus, Denmark

Feller fractional diffusion and Lévy stable motions

Rudolf GORENFLO [1] and Francesco MAINARDI [2]
[1] Dept. of Mathematics and Computer Science, Free University of Berlin
Arnimallee 2-6, D-14195 Berlin, Germany
e-mail: gorenflo@math.fu-berlin.de
[2] Department of Physics, University of Bologna
Via Irnerio 46, I-40126 Bologna, Italy
e-mail: mainardi@bo.infn.it

Abstract – Fractional calculus allows to generalize the standard (linear and one dimensional) diffusion equation by replacing the second-order space derivative by a derivative of fractional order. If this is taken as the pseudo-differential operator introduced by Feller in 1952 the fundamental solution of the resulting diffusion equation is a probability density evolving in time and stable in the sense of Lévy. Like the standard diffusion equation through its fundamental solution, the Gaussian density, yields the well-known Brownian motion, so the Feller diffusion equation yields the so called Lévy stable motions, whose increments are independent and stably distributed. We show how to approximate each of these motions by a discrete-time, discrete-space random walk model, which is based on an integer-valued random variable lying in the domain of attraction of the corresponding Lévy probability distribution.

Keywords – Fractional calculus, diffusion equation, stable distributions, random-walk.

1. Introduction

The purposes of this lecture are to outline the role of generalized diffusion equations of fractional order in generating the probability density functions (*pdf*'s) of the Lévy stable distributions and to construct some models of random walks, discrete in space and time, related to Lévy stable motions.

For the standard diffusion equation

$$\frac{\partial}{\partial t}\,u(x,t) = \frac{\partial^2}{\partial x^2}\,u(x,t)\,, \quad -\infty < x < +\infty\,, \quad t \geq 0\,, \tag{1.1}$$

it is well known that the fundamental solution (the Green function) of the Cauchy problem provides the *pdf* of the Gaussian or normal distribution with variance proportional to time. Feller (1952) considered the problem of generating all the Lévy stable *pdf*'s (satisfying his special parameterization) evolving in time, through the Green function of the Cauchy problem for a generalized diffusion equation. Feller's essential idea is to replace the second-order space derivative of the standard diffusion equation

with a special pseudo-differential operator. In our notation the corresponding stable densities are denoted as $g_\alpha(x,t;\theta)$, whose spatial Fourier transform (the characteristic functions) read

$$\widehat{g_\alpha}(\kappa,t;\theta) = \int_{-\infty}^{+\infty} \mathrm{e}^{\,i\kappa x}\, g_\alpha(x,t;\theta)\, dx = \exp\left(-t|\kappa|^\alpha\, \mathrm{e}^{\,i(\mathrm{sign}\,\kappa)\theta\pi/2}\right), \tag{1.2}$$

where x, $\kappa \in \mathbb{R}$, $t > 0$. The two relevant parameters, α, called the *index of stability*, and θ (related to the asymmetry), improperly referred to as the *skewness*, are real numbers subject to the conditions

$$0 < \alpha \le 2\,; \qquad |\theta| \le \begin{cases} \alpha\,, & \text{if} \quad 0 < \alpha \le 1\,, \\ 2-\alpha\,, & \text{if} \quad 1 < \alpha \le 2\,, \end{cases} \tag{1.3}$$

whereas time t plays the role of a *scale parameter*. For $\alpha = 2$ and $\alpha = 1$ (with $\theta = 0$) we recover the standard Gaussian and Cauchy *pdf*'s

$$g_2(x,t;0) = \frac{1}{2\sqrt{\pi}}\, t^{-1/2} \exp\left(-\frac{x^2}{4t}\right), \quad g_1(x,t;0) = \frac{1}{\pi}\, \frac{t}{x^2+t^2}\,. \tag{1.4}$$

By introducing the *similarity variable* $x\, t^{-1/\alpha}$, we can write $g_\alpha(x,t;\theta) = t^{-1/\alpha}\, p_\alpha(x\, t^{-1/\alpha};\theta)$, where $p_\alpha(x;\theta)$ is the stable *pdf* at $t = 1$.

The specific Feller form of the characteristic function (1.2) allows us to easily recognize $g_\alpha(x,t;\theta)$ as the Green function of the Cauchy problem

$$\frac{\partial}{\partial t}\, u(x,t) = D^\alpha_\theta\, u(x,t)\,, \quad u(x,0) = \delta(x)\,, \quad x \in \mathbb{R}\,, \quad t > 0\,, \tag{1.5}$$

where with δ denotes the Dirac generalized function and D^α_θ is the Feller pseudo-differential operator acting with respect to the space variable x, with symbol

$$\widehat{D^\alpha_\theta} = -|\kappa|^\alpha\, \mathrm{e}^{\,i(\mathrm{sign}\,\kappa)\theta\pi/2}\,. \tag{1.6}$$

Let us recall that a generic pseudo-differential operator A, acting with respect to the variable $x \in \mathbb{R}$, is defined through its Fourier representation, namely $\int_{-\infty}^{+\infty} \mathrm{e}^{\,i\kappa x}\, A\,\phi(x)\, dx = \widehat{A}(\kappa)\,\widehat{\phi}(\kappa)$, where $\phi(x)$ denotes a sufficiently well-behaved function in $\mathbb{R}$, and $\widehat{A}(\kappa)$ is referred to as symbol of A, given as $\hat{A}(\kappa) = \left(A\, \mathrm{e}^{\,-i\kappa x}\right)\, \mathrm{e}^{\,+i\kappa x}$. The n-th derivative operator $D^n = \frac{d^n}{dx^n}$ is a special case with symbol $\widehat{D^n} = (-i\kappa)^n$.

Honouring both Lévy and Feller for their essential contributions, see *e.g.* Lévy (1924, 1925, 1954), Feller (1952, 1971), we call the pseudo-differential operator D^α_θ *Feller fractional derivative* of order α and the process described by (1.5) *Lévy-Feller diffusion.*

Recently, Gorenflo and Mainardi (1998a, 1998b, 1999) have revised Feller's original arguments by interpreting (1.5) as a space-fractional diffusion equation (of *order* α and *skewness* θ) and have provided a variety of related random walk models, discrete in space and time, which by properly scaled transition to vanishing space and time steps converge to the corresponding continuous Markovian processes. In other words the discrete probability distributions generated by the random walk models have been proved to belong to the domain of attraction of the corresponding stable distribution.

Here we would like to summarize some results of our theory and exhibit a few numerical case studies for random walks related to Lévy stable motions.

2. The Feller fractional derivative

We now give a brief account of the Feller fractional derivative. Then, starting from its explicit expression we shall provide a discretized approximation useful to obtain our random walk models. For details we refer the reader to our original works.

The explicit expression of the Feller pseudo-differential operator D^α_θ acting on a function $\phi(x)$ can be obtained if we first define the Weyl fractional derivatives $D^\alpha_\pm$ and the Hilbert transform operator H .

For the Weyl fractional derivatives we have

$$D^\alpha_\pm \, \phi(x) = \begin{cases} \pm (D^1 \, I^{1-\alpha}_\pm) \, \phi(x) \, , & \text{if} \quad 0 < \alpha < 1 \, , \\ (D^2 \, I^{2-\alpha}_\pm) \, \phi(x) \, , & \text{if} \quad 1 < \alpha < 2 \, , \end{cases} \tag{2.1}$$

where $I^\beta_\pm$ denote the Weyl fractional integrals of order $\beta > 0$, which are defined as

$$\begin{cases} I^\beta_+ \, \phi(x) = \dfrac{1}{, (\beta)} \, \displaystyle\int_{-\infty}^{x} (x-\xi)^{\beta-1} \, \phi(\xi) \, d\xi \, , \\ I^\beta_- \, \phi(x) = \dfrac{1}{, (\alpha)} \, \displaystyle\int_{x}^{+\infty} (\xi - x)^{\beta-1} \, \phi(\xi) \, d\xi \, . \end{cases} \tag{2.2}$$

For continuity we get

$$D^1_\pm = \pm D^1 \, , \qquad D^2_\pm = D^2 \, . \tag{2.3}$$

For the Hilbert transform we have

$$H \, \phi(x) = \frac{1}{\pi} \, \int_{-\infty}^{+\infty} \frac{\phi(\xi)}{x-\xi} \, d\xi = \frac{1}{\pi} \, \int_{-\infty}^{+\infty} \frac{\phi(x-\xi)}{\xi} \, d\xi \, , \tag{2.4}$$

the integral understood in the Cauchy principal value sense. Incidentally, we note that $H^{-1} = -H$, so, in view of this peculiar property, the Hilbert transform operator is by some authors defined with the opposite sign in the kernel, *i.e.* writing $\xi - x$ instead of $x - \xi$ in the first integral in (2.4).

We then can prove that the Feller fractional derivatives turns out to be

$$D^\alpha_\theta \, \phi(x) = \begin{cases} - \left[c_+(\alpha,\theta) \, D^\alpha_+ + c_-(\alpha,\theta) \, D^\alpha_- \right] \, \phi(x) \, , & \text{if} \;\; \alpha \neq 1 \, , \\ \left[\cos(\theta\pi/2) \, (D^1 \, H) + \, \sin(\theta\pi/2) \, D^1 \right] \, \phi(x) \, , & \text{if} \;\; \alpha = 1 \, , \end{cases} \tag{2.5}$$

where

$$c_+(\alpha;\theta) = \frac{\sin[(\alpha-\theta)\pi/2]}{\sin(\alpha\pi)} \, , \qquad c_-(\alpha;\theta) = \frac{\sin[(\alpha+\theta)\pi/2]}{\sin(\alpha\pi)} \, . \tag{2.6}$$

Note that the case $\alpha = 2$ is obtained from the passage to the limit with $\theta = 0$ for which $c_+(2,0) = c_-(2,0) = -1/2$, and $D^2_0 = D^2$.

Henceforth, for ease of notation, we shall omit the arguments of the coefficients $c_+ = c_+(\alpha,\theta)$ and $c_- = c_-(\alpha,\theta)$. We have

$$c_\pm \begin{cases} \geq 0\,, & \text{if } 0<\alpha<1\,, \\ \leq 0\,, & \text{if } 1<\alpha\leq 2\,, \end{cases} \qquad c_+ + c_- = \frac{\cos(\theta\pi/2)}{\cos(\alpha\pi/2)} \begin{cases} >0\,, & \text{if } 0<\alpha<1\,, \\ <0\,, & \text{if } 1<\alpha\leq 2\,. \end{cases} \tag{2.7}$$

In the symmetric case ($\theta = 0$) the Feller fractional derivative reduces to

$$D_0^\alpha\,\phi(x) = \begin{cases} -\,\dfrac{D_+^\alpha + D_-^\alpha}{2\cos(\alpha\pi/2)}\,\phi(x)\,, & \text{if } \alpha\neq 1\,, \\ -\,(D^1\,H)\,\phi(x)\,, & \text{if } \alpha = 1\,. \end{cases} \tag{2.8}$$

We can verify the validity of (2.5-6) [and (2.8)] by playing with symbols of the operators and, consequently, by proving the identity (1.6). For this purpose it suffices to note that

$$\widehat{D_\pm^\alpha} = (\mp i\kappa)^\alpha = |\kappa|^\alpha\,\mathrm{e}^{\,\mp i\,(\mathrm{sign}\,\kappa)\,\alpha\pi/2}\,, \qquad \widehat{H} = i\,(\mathrm{sign}\,\kappa)\,. \tag{2.9}$$

It is interesting to note that for $\alpha = 1$ the generalized diffusion equation (1.5) reduces to

$$\frac{\partial}{\partial t}\,u(x,t) = \frac{\partial}{\partial x}\left[\cos(\theta\pi/2)\,\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{u(\xi,t)}{\xi - x}\,d\xi + \sin(\theta\pi/2)\,u(x,t)\right]\,, \tag{2.10}$$

and the corresponding stable *pdf* (its fundamental solution) turns out to be

$$g_1(x,t;\theta) = \frac{1}{\pi}\,\frac{t\,\cos(\theta\pi/2)}{[x + t\,\sin(\theta\pi/2)]^2 + [t\,\cos(\theta\pi/2)]^2}\,, \quad |\theta|<1\,. \tag{2.11}$$

We note that in the extremal cases $\theta = \pm 1$ we get $D^1_{\pm 1} = \pm D^1$, so (2.10) degenerates into kinematic (*i.e.* first-order) wave equations and the corresponding stable densities (2.11) reduce to $g_1(x,t;\pm 1) = \delta(x \pm t)$.

3. Outline of the random walk approach

In this section we define a random variable Y assuming only integers as values, its probability distribution depending on three parameters α, θ, proper of the stable *pdf*'s as seen in (1.3), and μ, a scale parameter, to be introduced later. By aid of this random variable we define a random walk on an equidistant grid $\{jh\,|\,j\in Z\}$ with a space-step $h>0$. We show that after introduction of a suitable time-step $\tau>0$ this random walk admits an interpretation as an explicit difference scheme, convergent and stable in the limit as $h\to 0$ in a sense to be seen later, for the Cauchy problem (1.5).

Let Y be an integer-valued random variable and let the random variables $Y_1, Y_2, Y_3, \ldots$ be *independent identically distributed*, all having their probability distribution common with Y .

We define a spatial-temporal grid $\{(x_j\,,\,t_n)\,|\,j\in Z\,,\ n\in\mathbb{N}_0\}$ by $x_j=x_j(h)=j\,h$, $t_n=t_n(\tau)=n\,\tau$, where $h>0$ and $\tau>0$. Then we consider the sequence of random variables

$$S_n=hY_1+hY_2+\ldots+hY_n,\quad n\in\mathbb{N}\,, \tag{3.1}$$

with (for convenience) $S_0=0$, and interpret it as follows. A particle, sitting in $x=x_0=0$ at time $t=t_0=0$ finds itself at a later instant $t=t_n$ in point $x=S_n$ which is an integer multiple of h. We recognize the $p_k=P(Y=k)$ (for $k\in Z$) as *transition probabilities*: p_k is the probability of a particle jumping from a point $x_j=S_n$ to a point $x_{j+k}=S_{n+1}$ as time proceeds from t_n to t_{n+1}. All p_k are non-negative, and their sum equals 1.

The probability $y_j(t_n)$ of sojourn of our particle in point x_j at instant t_n obeys the *transition law*

$$y_j(t_{n+1})=\sum_{k=-\infty}^{+\infty}p_k\,y_{j-k}(t_n)\,,\quad y_j(0)=\delta_{j\,0}\,,\qquad j\in Z\,,\quad n\in\mathbb{N}_0\,. \tag{3.2}$$

which has the form of a discrete convolution. Hence, introducing *generating functions*

$$\widetilde{p}(z)=\sum_{j=-\infty}^{+\infty}p_j z^j,\quad \widetilde{y}_n(z)=\sum_{j=-\infty}^{+\infty}y_j(t_n)z^j, \tag{3.3}$$

we obtain

$$\widetilde{y}_n(z)=\widetilde{y}_0(z)\cdot[\widetilde{p}(z)]^n=[\widetilde{p}(z)]^n\,,\ n\in\mathbb{N}_0\,. \tag{3.4}$$

The power series in (3.3) and (3.4) are absolutely and uniformly convergent on $|z|=1$ and assume the value 1 at $z=1$. Putting $z=\mathrm{e}^{i\kappa h}$, $\kappa\in\mathbb{R}$, and observing $z^j=\mathrm{e}^{i\kappa(jh)}=\mathrm{e}^{i\kappa x_j}$, we recognize $\widehat{p}(\kappa;h)=\widetilde{p}(\mathrm{e}^{i\kappa h})$ and $\widehat{y}(\kappa,t_n;h)=\widetilde{y}_n(\mathrm{e}^{i\kappa h})$ as *characteristic functions* of the random variables $h\,Y$ and S_n, respectively.

Our aim is to approximate the Lévy-Feller diffusion process, which is governed by the evolution equation (1.5), arbitrarily well. To this purpose we introduce a strictly monotonic *scaling relation* $\tau=\sigma(h)\to 0$ as $h\to 0$. We will fix $t>0$ and let h (and likewise τ) go to zero over such values that always $n=t/\tau=t/\sigma(h)$ is a positive integer. Then we have the equivalences

$$n\to\infty\iff h\to 0\iff\tau\to 0\,,$$

and h depends on τ, finally on n, so that $h=h(n)$.

Replacing h by $h(n)$ in (3.1) we obtain a sequence of random variables X_n with characteristic functions $\widehat{y}(\kappa,t_n;h)=[\widetilde{p}(\mathrm{e}^{i\kappa h})]^n$ (note that now $t_n=t$ is fixed). Invoking the *Continuity Theorem*, see *e.g.* Lukacs (1960), Th. 3.6.1, what remains to be shown is that $\widehat{y}(\kappa,t;h)\to\exp\left(-t|\kappa|^\alpha\,\mathrm{e}^{i(\mathrm{sign}\,\kappa)\theta\pi/2}\right)$ as $h\to 0$, the characteristic function of the corresponding Lévy-Feller process. For this it suffices that, for fixed $\kappa\neq 0$,

$$\log\left[\hat{y}(\kappa,t;h)\right]\equiv\frac{t}{\sigma(h)}\,\log\left[\tilde{p}(\mathrm{e}^{i\kappa h})\right]\to-t\,|\kappa|^\alpha\,\mathrm{e}^{i(\mathrm{sign}\,\kappa)\theta\pi/2}\,,\quad\text{as}\quad h\to 0\,. \tag{3.5}$$

Our random walk can be interpreted as a "difference scheme" to approximate the evolution equation (1.5), if we write (3.2) in the equivalent form, observing the scaling relation,

$$\frac{y_j(t_{n+1}) - y_j(t_n)}{\tau} = \frac{1}{\sigma(h)} \left[(p_0 - 1)\, y_j(t_n) + \sum_{k \neq 0} p_k\, y_{j-k}(t_n) \right], \tag{3.6}$$

In fact, after division by the spatial mesh-width h, the L.H.S. is the forward discrete approximation of $\frac{\partial}{\partial t} u(x,t)$, and the R.H.S can be considered as a discrete approximation to the space pseudo-differential term $D_\theta^\alpha\, u(x,t)$, provided we mean $y_j(t_n)$ to approximate $\int_{x_j - h/2}^{x_j + h/2} u(x,t_n)\, dx \approx h\, u(x_j, t_n)$ with $y_j(0) = \delta_{j\,0}$, and dispose of $\sigma(h)$ as an appropriate function of h.

4. The random walk models

From the previous Section we have learnt that, in order to construct discrete random walk models which are convergent (in distribution) to the stable *pdf*'s, the clue points are: 1) to guess a suitable *generating function* $\tilde{p}(z)$, whose coefficients of its power series expansion provide the transition probabilities, 2) to determine the corresponding *scaling relation* $\tau = \sigma(h)$ which ensures the required convergence.

In the classical case of the Gaussian distribution ($\alpha = 2$) the matter is easily treated if we remember that the corresponding density is the fundamental solution of the standard diffusion equation, which is known to be well approximated via the finite-difference equation

$$\frac{y_j(t_{n+1}) - y_j(t_n)}{\tau} = \frac{y_{j+1}(t_n) - 2y_j(t_n) + y_{j-1}(t_n)}{h^2}, \quad y_j(0) = \delta_{j\,0}\,. \tag{4.1}$$

In this case, introducing the scaling parameter $\mu = \tau/h^2$, so $\tau = \sigma(h) = \mu\, h^2$, the transition probabilities turn out to be

$$p_0 = 1 - 2\mu\,, \quad p_{\pm 1} = \mu\,, \quad p_{\pm k} = 0\,, \quad k = 2, 3 \ldots,. \tag{4.2}$$

subject to the condition $0 < \mu \leq 1/2$. Thus the generating function is

$$\widetilde{p}(z) = 1 + \mu[z - 2 + z^{-1}]\,. \tag{4.3}$$

The proof of the convergence to the Gaussian is simple since one easily finds $[t/(\mu h^2)]\, \log[\widetilde{p}(\mathrm{e}^{i\kappa h}] \to -t\kappa^2$ as $h \to 0$. The scheme (4.2) means that for approximation of the standard Gaussian process the corresponding random walk model exhibits only jumps of one step to the right or one to the left or jumps of width zero. For the stable non Gaussian processes we expect to find a non-polynomial generating function with infinitely many transition coefficients which imply the occurrence of arbitrarily large jumps. It is common practice to refer to the corresponding random walks as *Lévy flights*, to stress the relation to the Lévy stable distributions and the occurrence of large jumps.

In general the *essential idea* for meeting the clue point 1) is to provide a discrete approximation of the pseudo-differential operator D_θ^α, that we denote by ${}_hD_\theta^\alpha$, where h refers to the spatial mesh-width. If we define the spatial *translation operator* T_ξ by $T_\xi\,\phi(x) := \phi(x+\xi)$, we note that our complex variable $z = \mathrm{e}^{i\kappa h}$ can be interpreted as the symbol of the backward shift operator T_{-h}, namely $\widehat{T_{-h}} = \mathrm{e}^{+i\kappa h} = z$. Similarly we get the symbol of the forward shift operator T_{+h}, $\widehat{T_{+h}} = \mathrm{e}^{-i\kappa h} = z^{-1} = \bar{z}$. We then recognize from the considerations in §3, specially from (3.6), that the symbol $\widehat{{}_hD_\theta^\alpha}$ is related to the generating function by the identity

$$\frac{\widetilde{p}(z)-1}{\tau} = \widehat{{}_hD_\theta^\alpha}\,, \tag{4.4}$$

The identity (4.4) is easily verified for the classical case of the standard diffusion where (4.3) is valid. In fact, from

$$D^2\,\phi(x) = \lim_{h\to 0}\frac{\phi(x+h)-2\phi(x)+\phi(x-h)}{h^2}\,, \tag{4.5}$$

we recognize in the Fourier domain

$$\widehat{{}_hD^2} = h^{-2}\,(z^{-1}-2+z) = h^{-2}\,[z^{-1}\,(1-z)^2] = h^{-2}\,[z\,(1-z^{-1})^2]\,. \tag{4.6}$$

For the fractional diffusion ($\alpha \neq 2$) we may have different ways of discretization for ${}_hD_\theta^\alpha$, which lead to different generating functions and, consequently, different classes of discrete random walks. Here we limit ourselves to show the results for a single class distinguishing the cases $\alpha \neq 1$ and $\alpha = 1$.

For $\alpha \neq 1$ the starting point is the Grünwald-Letnikov scheme for the Weyl fractional derivatives on which the reader can inform himself in the treatises on fractional calculus, see *e.g.* Oldham and Spanier (1974), Samko, Kilbas and Marichev (1993), Miller and Ross (1993), Podlubny (1999) or in the review article by Gorenflo (1997). This scheme, suitably improved, leads to the generating function

$$\widetilde{p}(z) = \begin{cases} 1-\mu\,[c_+\,(1-z)^\alpha + c_-(1-z^{-1})^\alpha]\,, & \text{if } 0<\alpha<1\,, \\ 1-\mu\,[c_+\,z^{-1}\,(1-z)^\alpha) + c_-\,z\,(1-z^{-1})^\alpha)]\,, & \text{if } 1<\alpha\le 2\,, \end{cases} \tag{4.7}$$

where $\mu = \tau/h^\alpha$ and the $c_\pm$ are given by (2.6). We note that the expression (4.3) for the classical case $\alpha = 2$ is recovered taking into account that in the limit $c_+ = c_- = -1/2$.

The transition probabilities turn out to be

$$\begin{cases} p_0 = 1-\mu\,(c_+ + c_-) = 1-\mu\,\dfrac{\cos\,(\theta\pi/2)}{\cos\,(\alpha\pi/2)}\,, \\ p_{\pm k} = (-1)^{k+1}\mu\,\dbinom{\alpha}{k}\,c_\pm = \quad k = 1,2,\ldots \end{cases} \qquad 0<\alpha<1\,; \tag{4.8a}$$

$$\begin{cases} p_0 = 1 + \mu \binom{\alpha}{1} (c_+ + c_-) = 1 - \mu\,\alpha\,\dfrac{\cos(\theta\pi/2)}{|\cos(\alpha\pi/2)|}\,, \\ p_{\pm 1} = -\mu \left[\binom{\alpha}{2} c_\pm + c_\mp \right]\,, \qquad\qquad 1 < \alpha \le 2\,. \\ p_{\pm k} = (-1)^k\,\mu \binom{\alpha}{k+1} c_\pm\,,\ k = 2, 3, \ldots \end{cases} \tag{4.8b}$$

In both cases the scaling relation is $\tau = \sigma(h) = \mu\, h^\alpha$, where μ is restricted as follows

$$0 < \mu \le \begin{cases} \dfrac{\cos(\alpha\pi/2)}{\cos(\theta\pi/2)} & \text{in the case (a),} \\ \dfrac{1}{\alpha}\dfrac{|\cos(\alpha\pi/2|}{\cos(\theta\pi/2)} & \text{in the case (b).} \end{cases} \tag{4.9}$$

For the case $\alpha = 1$ we can limit ourselves to the symmetric case $\theta = 0$ for which the starting point is the discretization of the Hilbert transform operator. For $0 < |\theta| < 1$ we have to add to the symmetric random walk, weighted with $\cos(\theta\pi/2)$, a pure drift to right or to left, weighted with $\sin(\theta\pi/2)$, as we can see from (2.5). We obtain

$$\widetilde{p}(z) = 1 - \frac{\mu}{\pi}\left[(1 - z^{-1})\,\log(1 - z) + (1 - z)\,\log(1 - z^{-1})\right], \quad \alpha = 1\,,\ \theta = 0\,, \tag{4.10}$$

$$p_0 = 1 - \frac{2\mu}{\pi}\,, \quad p_{\pm k} = \frac{\mu}{\pi}\,\frac{1}{k(k+1)}\,, \quad k = 1, 2, 3, 4, \ldots\,, \quad \alpha = 1\,,\ \theta = 0\,, \tag{4.11}$$

where $0 < \mu = \tau/h \le \pi/2$.

We see how the generating function (4.10) cannot be obtained from (4.7) with $\theta = 0$ by a passage to the limit $\alpha \to 1$. Indeed, in both the limits $\alpha \to 1^-$ and $\alpha \to 1^+$ the permissible range of the scaling factor μ is vanishing. In order to get a continuous transition for $\alpha = 1$ we need to consider different approaches, but we loose the continuity as $\alpha \to 2^-$; for this we refer the interested reader to Gorenflo, De Fabritiis and Mainardi (1999).

5 Numerical results

In general the random walk models are not only valuable from the conceptual point of view for visualizing what the diffusion means but also for numerical calculations, either as Monte Carlo simulation of particle paths in a diffusion process or as discrete imitation of the process in form of redistribution (from one time level to the next) of clumps of an extensive quantity (across the spatial grid points). Our models can be used in at least three different ways: (*a*) as *finite difference schemes* for approximate calculation of symmetric stable densities; (*b*) for producing *sample paths* of individual particles performing the random walk; (*c*) for producing *histograms* of the approximate realization of the densities g_α by simulating many individual paths with the same number of time steps and making statistics of the final positions of the particles.

For numerical simulations of stable random variables different algorithms have been provided by a number of specialists, including Chambers, Mallows and Stuck (1976), Bartels (1978), Mantegna (1994), Janicki and Weron (1994), Samorodnitsky and Taqqu (1994), Nolan (1997). Our present approach for treating Lévy statistics has been carried out independently from all the above references but uniquely based on the random walk model presented here, so, as far as we know, our results are original.

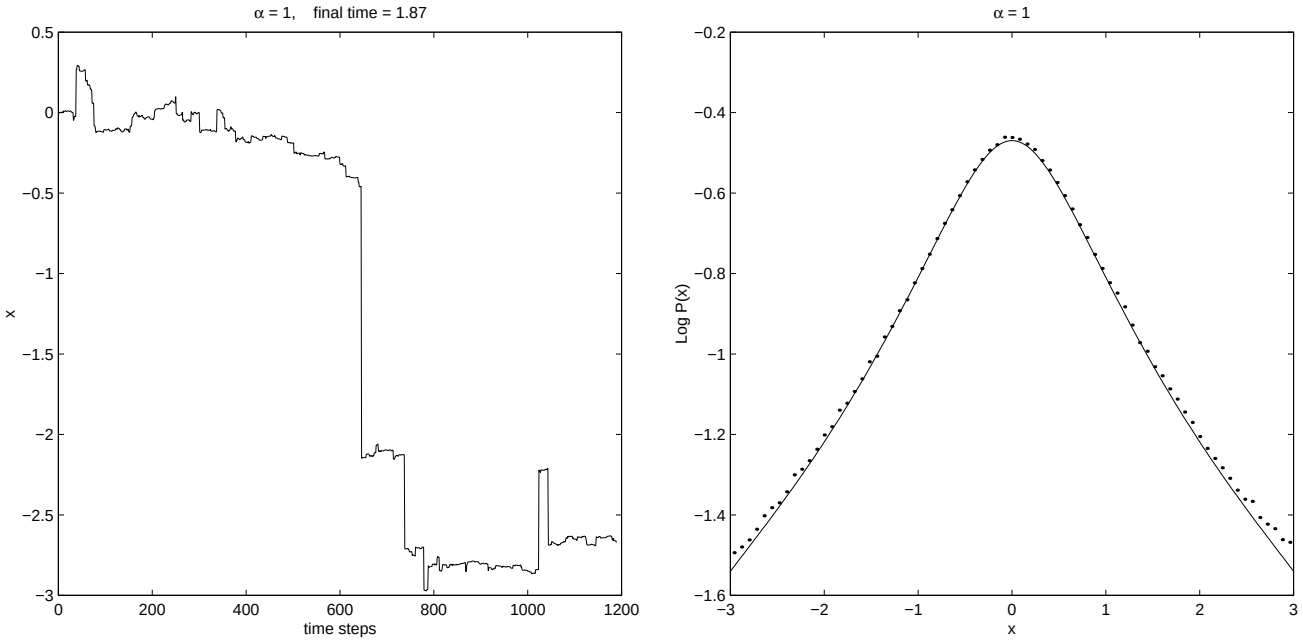


Fig. 1 Sample path (left) and histogram (right) for $\alpha = 1$. (Cauchy)

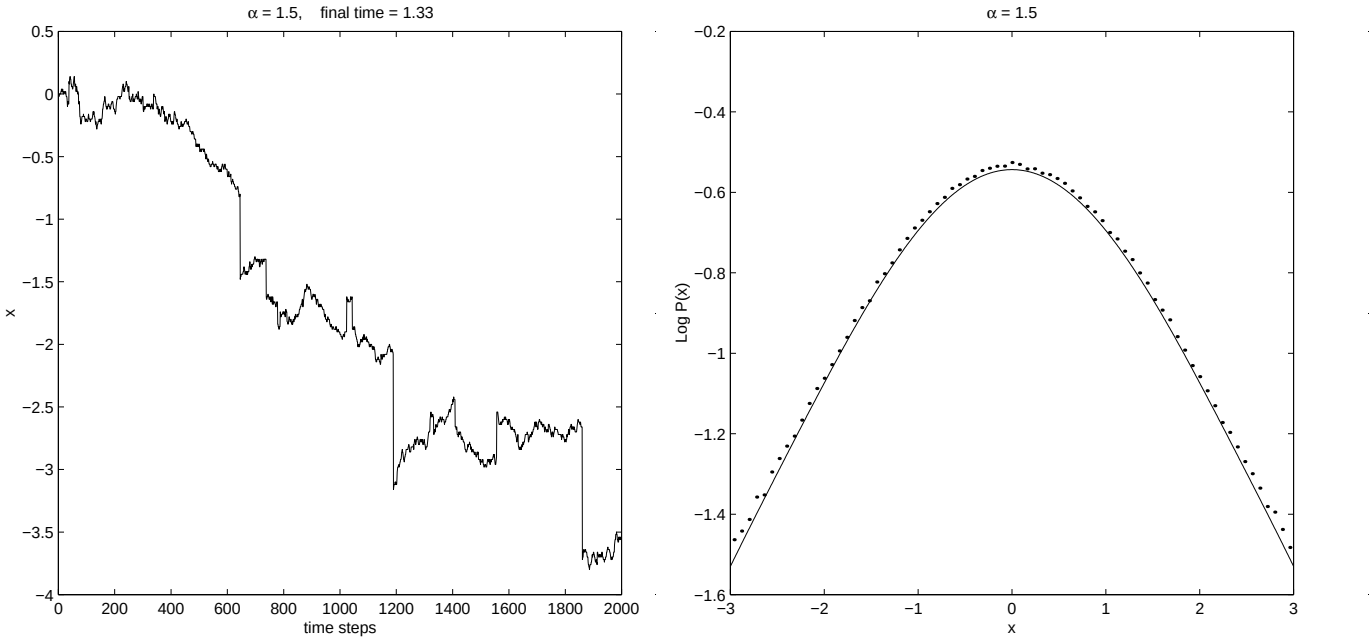


Fig. 2 Sample path (left) and histogram (right) for $\alpha = 1.5$.

Having preliminarily checked a sufficient level of accuracy for our finite difference schemes with the existing tables of stable densities, here we report some results, recently obtained by Gorenflo, De Fabritiis and Mainardi (1999), on the simulation of the sample paths and histograms corresponding to some typical values of the index of stability, namely $\alpha = 1, 1.5, 2$. In practice, in our numerical studies there is required truncation in two ways. It is impossible to simulate all infinitely many discrete probabilities, so the size of possible jumps must be limited to a maximal possible jump length. The other truncation is required if a priori one wants a definite region of space to be considered in which the walk takes place. Then, particles leaving this space have been ignored.

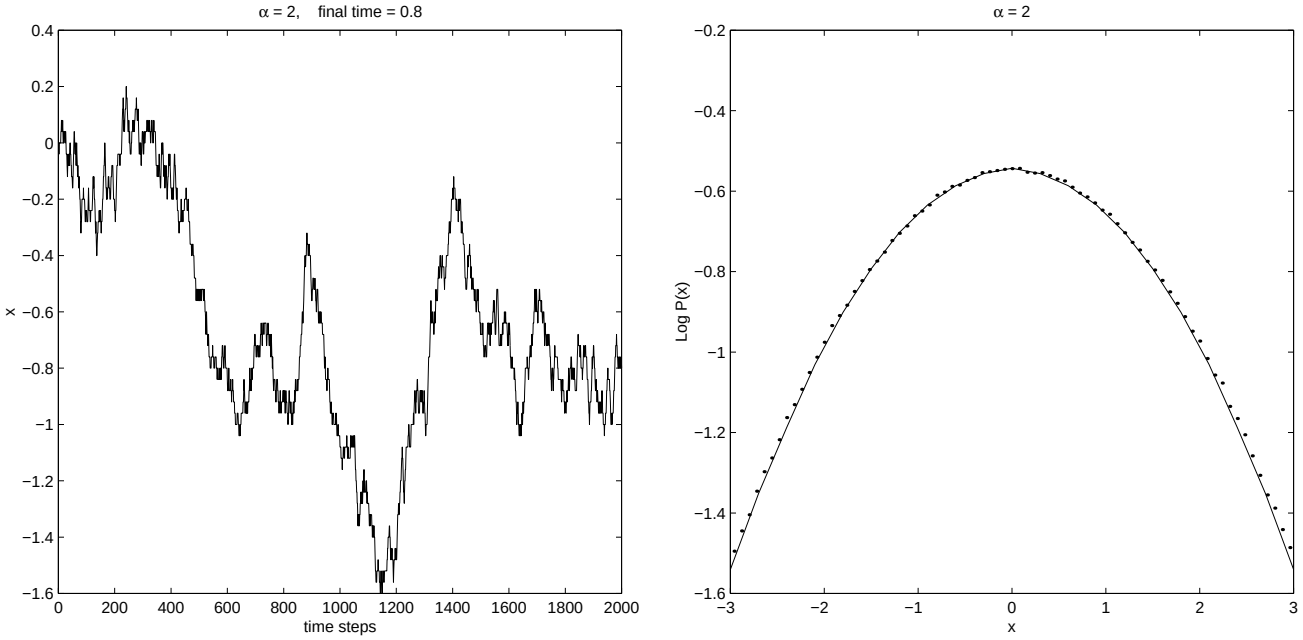


Fig. 3 Sample path (left) and histogram (right) for $\alpha = 2$. (Gauss)

Our simulations, based on one million of realizations, have been carried out in the interval $|x| \leq 4$. All the histograms refer to stable densities at $t = 1$ for $|x| \leq 3$, the space interval being reduced to avoid the border effects. The sample paths are plotted against the time steps, up to 1200 for $\alpha = 1$ and up to 2000 for $\alpha = 1.5\,, 2$, so they refer to different final times, namely $t = 1.87, 1.33, 0.8$, respectively.

The transition probabilities have been chosen from our random walk model as follows: $\alpha = 1$ from (4.11) with scaling parameter $\mu = \pi/4$; $\alpha = 1.5$ from (4.8b) with $\mu = (2/3)\cos(3\pi/4)$; for $\alpha = 2$ from (4.2) with $\mu = 1/4$.

The cases $\alpha = 1$ (Cauchy process) and $\alpha = 2$ (normal process) have been considered for a possible comparison with the standard and accurate algorithms existing in the literature, whereas $\alpha = 1.5$ has been chosen in view of possible applications in econophysics where usually the index of stability ranges from 1.4 to 1.7, see *e.g.* Mandelbrot(1997), Mantegna and Stanley (1997).

6. Conclusions

For the simulation of Markovian processes characterized by Lévy probability densities evolving in time we have presented a random walk model, discrete in space and time, by giving its transition probabilities. We have displayed preliminary results of a few numerical case studies concerning sample paths and histograms to check the efficiency of our algorithms. From the sample paths one can recognize the "wild" character of the Lévy flights with respect to the "tame" character of the Brownian motion.

We expect that our arguments can be relevant in different fields of physics including the emerging one of econophysics, where stable distributions are becoming more and more common. In statistical physics the stable distributions play a key role in the (*wonderful*) world of random walks constructed by the late Montroll and continued through his school, see *e.g.* Montroll and West (1979), Montroll and Shlesinger (1984), Klafter, Shlesinger and Zumofen (1996). Here we have pointed out their relation to space-fractional diffusion equations. However, they turn out to be related also to time-fractional diffusion equations, see *e.g.* Engler (1997), Mainardi and Tomirotti (1997) and Mainardi, Paradisi and Gorenflo (1999). Furthermore, this topic is relevant for fractal phenomena, where differential equations of fractional order are usually adopted to describe their evolution, see *e.g.* Carpinteri and Mainardi (1997), Zaslavsky (1998), Hilfer (1999), and reference therein.

References

- R. Bartels, Generating non-normal stable variates using limit theorems properties, *J. Stat. Comp. Simulation*, **7**, 199–212 (1978).
- A. Carpinteri and F. Mainardi, eds., *Fractals and Fractional Calculus in Continuum Mechanics*, CISM Courses and Lectures # 378, Springer-Verlag, Wien, 1997.
- J.M. Chambers, C.L. Mallows and B.W. Stuck, A method for simulating stable random variables, *J. Amer. Stat. Assoc.*, **71**, 340–344 (1976).
- H. Engler, Similarity solutions for a class of hyperbolic integrodifferential equations, *Differential Integral Equations*, **10**, 815-840 (1997).
- W. Feller, On a generalization of Marcel Riesz' potentials and the semi-groups generated by them, *Meddelanden Lunds Universitets Matematiska Seminarium* (Comm. Sém. Mathém. Université de Lund), Tome suppl. dédié a M. Riesz, Lund, 73-81, 1952.
- W. Feller, *An Introduction to Probability Theory and its Applications*, Vol. 2, Wiley, New York, 1971. [1-st ed., 1966]
- R. Gorenflo, Fractional calculus: some numerical methods, in A. Carpinteri and F. Mainardi, eds., *Fractals and Fractional Calculus in Continuum Mechanics* CISM Courses and Lectures # 378, Springer Verlag, Wien, 1997, pp. 277-290.
- R. Gorenflo and F. Mainardi, Random walk models for space-fractional diffusion processes, *Fractional Calculus and Applied Analysis*, **1** (2), 167-191 (1998a).
- R. Gorenflo and F. Mainardi, Fractional calculus and stable probability distributions, *Archives of Mechanics*, **50** (2), 377-388 (1998b).
- R. Gorenflo and F. Mainardi, Approximation of Lévy-Feller diffusion by random walk, *J. for Analysis and its Applications*, **18** (2), in press (1999).

- R. Gorenflo, G. De Fabritiis and F. Mainardi, Discrete random walk models for symmetric Lévy-Feller diffusion processes, *Physica A*, in press (1999).
- R. Hilfer, ed., *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 1999.
- A. Janicki and A. Weron, *Simulation and Chaotic Behavior of α-Stable Stochastic Processes*, Marcel Dekker, New York 1994.
- J. Klafter, M. F. Shlesinger and G. Zumofen, Beyond Brownian motion, *Physics Today* **49** No 2, 33–39 (1996).
- P. Lévy, Théorie des erreurs. La lois de Gauss et les lois exceptionnelles, *Bull. Soc. Math. France*, **52**, 49-84 (1924).
- P. Lévy, *Calcul des probabilités*, Gauthier-Villars, Paris, 1925.
- P. Lévy, *Théorie de l'Addition des Variables Aléatoires*, Gauthier-Villars, Paris, 1954. [1-st ed. 1937]
- E. Lukacs, *Characteristic Functions*, Griffin, London, 1960.
- F. Mainardi and M. Tomirotti, Seismic pulse propagation with constant Q and stable probability distributions, *Annali di Geofisica*, **40**, 1311-1328 (1997).
- F. Mainardi, P. Paradisi and R. Gorenflo, Probability distributions generated by fractional diffusion equations, in: J. Kertesz and I. Kondor, eds., *Econophysics: an Emerging Science* Kluwer, Dordrecht, 1999, in press, pp. 39.
- B.B. Mandelbrot, *Fractals and Scaling in Finance* Springer-Verlag, New York, 1997.
- R.N. Mantegna, Fast, accurate algorithm for numerical simulation of Lévy stable stochastic processes, *Phys. Rev. E*, **49**, 4677–4683 (1994).
- R.N. Mantegna and H.E. Stanley, Physics investigation of financial markets, in F. Mallamace and H.E. Stanley, eds., *Proceedings International School of Physics "E. Fermi", Course CXXXIV*, IOS Press, Amsterdam, 1997, pp. 473–489.
- K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.
- E.W. Montroll and M.F. Shlesinger, On the wonderful world of random walks, in J. Leibowitz and E.W. Montroll, eds., *Nonequilibrium Phenomena II: from Stochastics to Hydrodynamics*, North-Holland, Amsterdam, 1984, pp. 1-121.
- E.W. Montroll and B.J. West, On an enriched collection of stochastic processes, in E.W. Montroll and J. Leibowitz, eds., *Fluctuation Phenomena*, North-Holland, Amsterdam, 1979, pp. 61-175.
- J.P. Nolan, Numerical calculation of stable densities and distribution functions, *Comm. Statist. - Stochastic Models*, **13** (4), 759-774 (1997).
- K.B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York, 1974.
- I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
- S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Amsterdam, 1993. [Engl. Transl. from the Russian edition, 1987]
- G. Samorodnitsky and M.S. Taqqu, *Stable non-Gaussian Random Processes*, Chapman and Hall, New York ,1994.
- G.M. Zaslavsky, *Physics of Chaos in Hamiltonian Systems*, Imperial College Press and World Scientific, London and Singapore, 1998.

Local times and other continuous additive functionals of Lévy processes

Michael B. Marcus *

This talk is a survey of some of the results that Jay Rosen and I have obtained over the last ten years. These results apply to strongly symmetric Markov processes. For the purposes of this talk it is sufficient to say that these processes have symmetric transition probability density functions. Furthermore, since this is a conference on Lévy processes we will restrict our attention to symmetric Lévy processes.

Let X be a symmetric Lévy process with values in S, where S is either R^m or T^m, (the m dimensional torus) and $m = 1, 2, 3$. Let $\xi \in S$. We write the characteristic function of X in the form

$$Ee^{i\xi X_t} = e^{-t\psi(\xi)}. \tag{1}$$

Let $p_t(x, y)$ denote the transition probability density for X.

Let $x \in S$ and A be a measurable set in S. Consider the occupation measure of the set A up to time t, i.e.

$$\nu_t(A) \stackrel{def}{=} \int_0^t I_{[X_s \in A]}\, ds. \tag{2}$$

When ν_t is absolutely continuous with respect to Lebesgue measure on S, we can write this as

$$\nu_t(A) \stackrel{def}{=} \int_A L_t^x\, dx. \tag{3}$$

L_t^x is the local time of X at x up to time t. Now let $f_\epsilon(\,\cdot\,)$ be an approximate δ–function at zero. We prefer to think of L_t^x as

$$L_t^x \stackrel{def}{=} \lim_{\epsilon \to 0} \int_0^t f_\epsilon(X(s) \Leftrightarrow x)\, ds. \tag{4}$$

*Department of Mathematics, The City College of CUNY, New York, NY 10031, mbmarcus@earthlink.net, http://home.earthlink.net/~mbmarcus/. This research was supported, in part, by grants from the National Science Foundation and PSC-CUNY.

Both of these definitions can be made rigorous under certain conditions. Consider the 1-potential density of X

$$u^1(x,y) = \int_0^\infty p_t(x,y)e^{-t}\,dt. \tag{5}$$

Note that $p_t(x,y)$ is positive definite, this is a simple consequence of the Chapman–Kolmogorov equation and uses the fact that $p_t(x,y)$ is symmetric. Consequently, so is u^1.

Let $G = \{G(x), x \in S\}$ be a mean zero Gaussian process with

$$EG(x)G(y) = u^1(x,y). \tag{6}$$

The next theorem relates the continuity of the local time of X with that of G.

Theorem 0.1 *(Barlow–Hawkes [1]) $L = \{L_t^x, (x,t) \in S \times R^+\}$ is continuous almost surely if and only if $\{G(x), x \in S\}$ is continuous almost surely.*

Actually, in [1] necessary and sufficient conditions are given for the continuity of the local time process for all Lévy processes not only symmetric Lévy processes. For processes that are not symmetric these conditions can not be described in terms of Gaussian processes since u^1 in (6), as the covariance of a Gaussian process, must be symmetric. Also, in [1] Theorem 0.1 is not expressed this way. This is the way Theorem 1, in [8] is stated. The contribution of the latter theorem is that it holds for Markov processes with symmetric transition probability density functions, not just for symmetric Lévy processes. (For symmetric Lévy processes $p_t(x,y)$ is a function of $|x \Leftrightarrow y|$.). There is another important difference in the work in [1] and the work in [8]. In [1] concrete conditions for continuity are obtained which imply Theorem 0.1 as stated. In [8] the comparison between local times of Lévy processes and Gaussian process is obtained abstractly, without obtaining any conditions to verify when either class of processes is continuous. However, since necessary and sufficient conditions for the continuity of Gaussian processes are known we don't need to. This is the theme of all the work discussed in this survey. Equivalencies are obtained between properties of continuous additive functionals of Lévy processes and associated Gaussian and Gaussian chaos processes. Then concrete results about the Gaussian and Gaussian chaos processes are used to give concrete conditions for the

functionals of Lévy processes to have certain properties. In this survey the property we consider is continuity.

The major tool in all this work is an isomorphism theorem of Dynkin and various generalizations of it that we have obtained. It is not simple to describe Dynkin's theorem or how we use it. See the papers in the bibliography for details.

The class of Lévy processes with continuous local times is really quite small. A symmetric Lévy process has a local time if and only if $u^1(x,x) \stackrel{def}{=} u^1(0) < \infty$. (We also sometimes write $u^1(x \Leftrightarrow y)$ for $u^1(x,y)$.) Since

$$u^1(x,y) = \frac{2}{\pi} \int_S \frac{\cos(\xi \cdot (x \Leftrightarrow y))}{1+\psi(\xi)}\, d\xi \tag{7}$$

and $\psi(\xi) \le C(|\xi|^2 \vee 1)$, we see that $u^1(0) = \infty$ in R^m or T^m, for $m \ge 2$. Thus for a local time to exist X must take values in R^1 or T^1. Also, for stable processes in R^1, i. e. $\psi(\xi) = |\xi|^\alpha$ in (1), we must have $\alpha > 1$.

When X does not have a local time, equivalently, when $u^1(0) = \infty$, we can consider different continuous additive functionals of X. The ones we describe now may be thought of as weighted occupation measures. Let μ be a positive measure on S. Consider

$$L_t^\mu \stackrel{def}{=} \lim_{\epsilon \to 0} \int_0^t \int_S f_\epsilon(X(s) \Leftrightarrow x)\, d\mu(x)\, ds. \tag{8}$$

This limit exists in L^2 when

$$\iint (u^1(x \Leftrightarrow y))^2\, d\mu(x)\, d\mu(y) < \infty \tag{9}$$

Let $A \subset S$ and define $\mu_y(A) = \mu(A+y)$. As a generalization of the local time process we consider

$$\{L_t^{\mu_y}, (y,t) \in S \times R^+\}. \tag{10}$$

The local time process is of this form when $\mu = \delta_0$, the delta function at zero.

Just as local time processes are related to Gaussian processes, the more general class of processes in (10) are related to second order Gaussian chaos processes. Here is how such processes are defined. Let u^1 be the 1-potential of a symmetric Lévy process such that $u^1(0) = \infty$. Take a positive definite

function $u^1_\delta(x)$ such that $u^1_\delta(0) < \infty$ and $\lim_{\delta\to 0} u^1_\delta(x) = u^1(x)$. Define $G_\delta(x)$ to be a mean zero Gaussian process with covariance $EG_\delta(x)G_\delta(y) = u^1_\delta(x \Leftrightarrow y)$ and

$$H(\mu_y) = \lim_{\delta\to 0} \int :G^2_\delta(x): \, d\mu_y(x) \tag{11}$$

where $:G^2_\delta(x): \stackrel{def}{=} G^2_\delta(x) \Leftrightarrow EG^2_\delta(x)$ is the second degree Hermite polynomial in $G_\delta(x)$ with leading coefficient one. ($:G^2:$ is called the Wick square of G.)

$H(\mu_y)$ is a second order Gaussian chaos process, i.e. it is equivalent to a process of the form $\sum_{j,k}\{g_j g_k \Leftrightarrow \delta_{j,k}\}\phi_{j,k}(y)$ where $\{g_j\}$ are independent identically distributed normal random variables with mean zero and variance one. To see this write a Karhunen–Loeve expansion, $G_\delta(x) = \sum_j g_j \rho_j(x,\delta)$. Then

$$\int :G^2_\delta(x): \, d\mu_y = \sum_{j,k}\{g_j g_k \Leftrightarrow \delta_{j,k}\} \int \rho_j(x,\delta)\rho_k(x,\delta)\, d\mu_y(x) \tag{12}$$

and pass to the limit as δ goes to zero.

Let

$$\begin{aligned} d^2(x,y) &= E\left(H(\mu_x) \Leftrightarrow H(\mu_y)\right)^2 \\ &= \int\int (u^1(s \Leftrightarrow t))^2 \, d(\mu_x(s) \Leftrightarrow d\mu_y(s)) \, d(\mu_x(t) \Leftrightarrow d\mu_y(t)) \\ &= \int\int \Delta^2_{x-y,x-y}(u^1(s \Leftrightarrow t))^2 \, d\mu(s)\, d\mu(t) \end{aligned} \tag{13}$$

where

$$\Delta^2_{b,b}(u^1(s \Leftrightarrow t))^2 \stackrel{def}{=} 2u^1(s \Leftrightarrow t))^2 \Leftrightarrow u^1(s \Leftrightarrow t+b))^2 \Leftrightarrow u^1(s \Leftrightarrow t \Leftrightarrow b))^2. \tag{14}$$

Let $N_d([0,1]^m, \epsilon)$ denote the minimum number of balls of radius ϵ in the metric d that covers $[0,1]^m$.

Consider the following five assertions:

1. $\int (\log N_d([0,1]^m, \epsilon))\, d\epsilon < \infty$.
2. $\int (\log N_d([0,1]^m, \epsilon))^{1/2}\, d\epsilon < \infty$.
3. $\{H(\mu_y), y \in S\}$ is continuous almost surely.
4. $\{L^{\mu_y}_t, (y,t) \in S \times R^+\}$ is continuous almost surely.

5. $\int_S u^1(x)\, d\mu(x) < \infty$.

The next theorem is contained in [10].

Theorem 0.2 *1. $\Rightarrow$ 3. $\Rightarrow$ 4. $\Rightarrow$ 5.*

The implication 3. $\Rightarrow$ 5, is a necessary condition for the continuity of second order Gaussian chaos processes, of the type described here, that we can only obtain by going through 4., that is, by considering the related continuous additive functionals $L_t^{\mu_y}$ and using Dynkin's isomorphism theorem.

Necessary and sufficient conditions are not known for the continuity of second order Gaussian chaos processes. However, if we consider Lévy process with values in T^m, with smooth 1-potential densities and restrict our attention to chaoses associated with these processes and consider smooth measures μ, (i.e. measures on T^m with regularly behaving Fourier coefficients) we can sharpen Theorem 0.2.

Theorem 0.3 *([10]) For a large class of Lévy processes in T^m, m=1,2,3 and for smooth measures μ (in terms of $\widehat{\mu}(k)$)*

$$2. \Leftrightarrow 3. \Leftrightarrow 4. \Leftrightarrow 5. \tag{15}$$

with S replaced by T^m.

Note that given a Lévy process X on R^m one can define a Lévy process $\widetilde{X}$ on T^m by simply taking X modulo $[0, 2\pi]^m$. Equivalently, one can define transition densities for $\widetilde{X}$ as $\widetilde{p}_t(x, y) = \sum_{k=-\infty}^{\infty} p_t(x, y + 2k\pi)$, when $m = 1$ and similarly for $m > 1$.

In addition to results in [10], Theorem 0.3 uses results from [2] and [12], which show that 2. and 3. are equivalent for certain second order Gaussian chaoses on T^m and certain measures μ. We don't know how to obtain these results for R^m. (We use explicit representations of the process which we have when the processes are on T^m.) The class of processes included in Theorem 0.3 include Brownian motion and other stable processes as well as processes in their domains of attraction.

We now consider the question of n–fold self intersections of a Lévy process in R^m or T^m, $m = 1, 2$. This is done by studying their intersection local times. These random functionals "measure" the amount of self-intersections

of a stochastic process. To define an n⇔fold self-intersection local time, the natural approach is to set

$$\alpha_{n,\epsilon}(\mu,t) \stackrel{def}{=}$$

$$\int\!\!\int_{\{0\le t_1\le\cdots\le t_n\le t\}} f_\epsilon(X(t_1)\Leftrightarrow x)\prod_{j=2}^{n} f_\epsilon(X(t_j)\Leftrightarrow X(t_{j-1}))\,dt_1\cdots dt_n\,d\mu(x) \tag{16}$$

where f_ϵ is an approximate δ⇔function at zero, and take the limit as $\epsilon\to 0$. Intuitively, this gives a measure of the set of times $(t_1,\ldots,t_n)$ such that

$$X(t_1)=\cdots=X(t_n)=x$$

where the "n–multiple points" $x\in R^m$ are weighted by the measure μ. However, in general, this limit does not exist because of the effect of the integral in the neighborhood of the diagonals in $(t_1,\ldots,t_n)\in(R^m)^n$, i.e. any point in $(R^m)^n$ where $t_i=t_j$, for some $i\neq j$. The method used to compensate for this is called renormalization. One subtracts from $\alpha_{n,\epsilon}(\mu,t)$ terms involving lower order intersections $\alpha_{k,\epsilon}(\mu,t)$ for $k<n$, in such a way that a finite limit results.

For technical reasons we replace t by λ in (16), where λ is an exponential random variable with mean one, independent of X.

Let

$$u^1_\epsilon(x)=\int f_\epsilon(x\Leftrightarrow y)u^1(y)\,dy \tag{17}$$

and note that

$$u^1_\epsilon(0)=E^0_\lambda\left(\int_0^\lambda f_\epsilon(X_t)\,dt\right). \tag{18}$$

Let

$$\gamma_{n,\epsilon}(\mu)=\sum_{k=0}^{n-1}(\Leftrightarrow 1)^k\binom{n\Leftrightarrow 1}{k}(u^1_\epsilon(0))^k\alpha_{n-k,\epsilon}(\mu,\lambda). \tag{19}$$

Heuristically, one may think of $\gamma_{n,\epsilon}(\mu)$ as being equal to

$$\int\!\!\int_{\{0\le t_1\le\cdots\le t_n\le\lambda\}} f_\epsilon(X(t_1)\Leftrightarrow x)$$
$$\prod_{j=2}^{n}\left(f_\epsilon(X(t_j)\Leftrightarrow X(t_{j-1}))\Leftrightarrow\delta(t_j\Leftrightarrow t_{j-1})\right)u^1_\epsilon(0)\,dt_1\cdots dt_n\,d\mu(x). \tag{20}$$

In this formulation the δ-functions compensate for the singularities that occur when various of the t_i are close to each other.

Let $\gamma_n(\mu_x) \stackrel{def}{=} \lim_{\epsilon\to 0} \gamma_{n,\epsilon}(\mu_x)$. This limit exists in L^2 whenever

$$\iint (u^1(x \Leftrightarrow y))^{2n}\, d\mu(x)\, d\mu(y) < \infty. \tag{21}$$

Similarly to (11), define

$$H_{2n}(\mu_x) = \lim_{\delta\to 0} \int : G_\delta^{2n}(y) :\, d\mu_x(y). \tag{22}$$

This limit also exists in L^2 when (21) holds. $H_{2n}(\mu_x)$ is a $2n$–th order Gaussian chaos. The next theorem is from [5].

Theorem 0.4 *For a large class of Lévy processes in S, with m=1 or 2; if $\{H_{2n}(\mu_x), x \in S\}$ is continuous almost surely then $\{\gamma_n(\mu_x), x \in S\}$ is continuous almost surely.*

A well known sufficient condition for the continuity of $\{H_{2n}(\mu_x), x \in S\}$, in terms of metric entropy, shows that

$$\iint_{|x-y|\le\epsilon} (u^1(x \Leftrightarrow y))^{2n} \left(\log \frac{1}{|x \Leftrightarrow y|}\right)^{2n+\delta} d\mu(x)\, d\mu(y) < \infty \tag{23}$$

for any $\epsilon, \delta > 0$, is a sufficient condition for the continuity of $\{\gamma_n(\mu_x), x \in S\}$. Here u^1 is the 1–potential density of the Lévy process whose intersection local times we are studying. One sees that very little more than (21) is needed to prove continuity. This result is from [4].

There is a considerable bibliography related to these results. Here I am only giving references to the papers mentioned above and to all my work with Jay Rosen that uses Dynkin's isomorphism theorem and its generalizations to explore the relationship between functionals of Lévy processes and Gaussian chaoses. Please see the bibliographies of the papers below for additional references.

References

[1] M. Barlow. Necessary and sufficient conditions for the continuity of local time of Levy processes. *Ann. Probab.*, 16:1389–1427, 1988.

[2] M.B. Marcus. A sufficient condition for the continuity of high order Gaussian chaos processes. In *Progress in Probability*, volume 43, pages 263–276. Birkhäuser, Boston, 1998.

[3] M.B. Marcus and J. Rosen. Additive functionals of several Lévy processes, and self-intersection local times. preprint.

[4] M.B. Marcus and J. Rosen. Multiple Wick product chaos processes, to appear in *J. Theor. Probab.*

[5] M.B. Marcus and J. Rosen. Renormalized self-intersection local times and Wick power chaos processes, to appear as a volume in the Memoirs of the Amer. Math. Soc. series.

[6] M.B. Marcus and J. Rosen. Moduli of continuity of local times of strongly symmetric Markov processes via Gaussian processes. *J. Theor. Probab.*, 5:791–825, 1992.

[7] M.B. Marcus and J. Rosen. p-variation of the local times of symmetric stable processes and of Gaussian processes with stationary increments. *Ann. Probab.*, 20:1685–1713, 1992.

[8] M.B. Marcus and J. Rosen. Sample path properties of the local times of strongly symmetric Markov processes via Gaussian processes. *Ann. Probab.*, 20:1603–1684, 1992.

[9] M.B. Marcus and J. Rosen. ϕ-variation of the local times of symmetric Levy processes and stationary Gaussian processes. In *Seminar on Stochastic Processes, 1992*, volume 33 of *Progress in Probability*, pages 209–220. Birkhauser, Boston, 1993.

[10] M.B. Marcus and J. Rosen. Gaussian chaos and sample path properties of additive functionals of symmetric Markov processes. *Ann. Probab.*, 24:1130–1177, 1996.

[11] M.B. Marcus and J. Rosen. Random Fourier series and continuous additive functionals of Lévy processes on the torus. *Ann. Probab.*, 24:1178–1218, 1996.

[12] M.B. Marcus and M. Talagrand. Continuity conditions for a class of Gaussian chaos processes related to continuous additive functionals of Lévy processes. *Jour. Theoret. Prob.*, 11:157–179, 1998.

SLOW POINTS OF LOCAL TIMES

LAURENCE MARSALLE

If B denotes a real Brownian motion, it is well-known that it speed at the origin is $s \mapsto \sqrt{2s \log |\log s|}$ (law of the iterarted logarithm). Dvoretzky and Kahane proved the existence of exceptional times where the speed was $s \mapsto c\sqrt{|s|}$ for some $c > 0$, and that this speed was the minimal one. Such times are called *slow points* of Brownian motion.

We are now interested in L, local time at 0 of a real Markov process M. That is L is an increasing continuous process and roughly speaking L_t represents the amount of time spent by M at 0 till time t. For most of local times, there exists a law of the iterated logarithm which specifies the speed at the origin to be $s \mapsto \frac{\log |\log s|}{\phi(s^{-1} \log |\log s|)}$. Here ϕ denotes the Laplace exponent of the *subordinator* X, right-continuous inverse of L.

Under mild conditions on ϕ, we prove the existence of non-trivial slow points for L, and precise the minimal speed of L to be $s \mapsto c(\phi(|s|^{-1}))^{-1}$ (c some positive constant).

Département de Mathématiques, Faculté des Sciences et Technologie, Université Paris XII-Val de Marne, 61 avenue du Général de Gaulle, F-94010 Creteil Cedex, France

E-mail address: marsalle@proba.jussieu.fr

On the Concept of Market Force: From Smoluchowski's Approximation to Burgers' Equation

GORAN PESKIR*

The lectures given under the title above evolved into a paper** entitled "*Market Forces and Dynamic Asset Pricing*". This paper consists of the following sections:

1. Introduction
2. Description of the model
3. Brownian motion and Newtonian mechanics
4. Market force and the specialist's optimisation revisited
5. Solution of the investor's problem
6. Solution of the specialist's problem
7. Concluding remarks and open questions

In order to provide some insight into this material, we first give a tentative abstract of the paper and then review relevant details from Section 2. The exposition finishes with a short list of references, which is incomplete and subject to change.

Abstract

We study a dynamic model of asset pricing which is driven by two characteristic market features: the law of investor demand (e.g. 'buy low, sell high') and the law of the market institution (which codifies the trading rules under which the market operates). We demonstrate in a simple investor-specialist trading model that these features are sufficient to guarantee an equilibrium where investors' trading strategies and the specialist's rule of price adjustments are best responses to each other. The drift term appearing in the resulting equation of the asset price movement may be interpreted using Newtonian mechanics as the acceleration of a "market force".

The idea that a market price fluctuates around a "fundamental" value is classic, and the extent to which stock prices would tend to revert to their mean values over long time horisons has been the subject of long-standing attention in the finance literature. The popular model of Black and Scholes (1973) suggests that the stock price S_t follows a *geometric Brownian motion*:

$$dS_t = S_t\left(\mu\,dt + \sigma\,dW_t\right) \tag{2.1}$$

where the *drift rate* $\mu \in \mathbb{R}$ and *volatility* $\sigma > 0$ are assumed constant, and $W = (W_t)_{t\geq 0}$ is

* Centre for Mathematical Physics and Stochastics, supported by the Danish National Research Foundation.
** Jointly with J. Shorish, Department of Economics, Aarhus.

a standard Brownian motion. Clearly, this approach is an idealisation of real world phenomena. To overcome disagreements of this assumption with observation, much attention in (2.1) has been given to generalising both the σ-term (leading to *stochastic volatility* models) or to the dW_t-term (leading to *Lévy process* models). Less attention, however, has been given to the form of the μ-term, and this is one of the foci of the present work.

More specifically, and in view of the mean-reversion puzzle stated above, we focus on the *dynamical* aspect of this question: What is μ to be, where does it originate, and how is it determined? It should be emphasised that although for simplicity we leave the volatility σ constant, and the noise term equal to dW_t , *a more realistic picture will be obtained if* σ *is allowed to be random, and* dW_t *is replaced by* dL_t *where* $(L_t)_{t\geq 0}$ *is a Lévy process*. We did not want the technical complexity of these more general assumptions to obscure the clarity of the dynamical issue we concentrate upon. It seems more likely, moreover, that these two quantities are to be determined by statistical observations of the real-world stock price (cf. Barndorff-Nielsen 1998).

Our main aim in this study is to describe dynamical aspects of the stock price movement and initiate a theory which is aimed at uniting its kinematics and dynamics, and which is built upon analogies with the laws of classical mechanics. The central new concept which arises in this attempt is the concept of the *market force*.

1. *Description of the Model*. We consider a model of asset pricing which is driven by two characteristic market features: (i) *the law of investor demand* (e.g. 'buy low, sell high') and (ii) *the law of the market institution* (which codifies the trading rules under which the market operates). Thus, the market participants are: (i) *an investor* (who can be also seen as a *representative investor*, i.e. an aggregate of 'small' investors) and (ii) *a specialist* (who can be identified with the trading mechanism of the market institution). There exists a risky asset (stock) and the investor is assumed to has at his disposal a risk-free asset (bond). The bond continuously compounds at a constant interest rate $r > 0$.

The *dividend* D_t paid by the stock is assumed to evolve according to:

$$dD_t \,=\, \sigma\, D_t\, dW_t \tag{2.2}$$

where $\sigma > 0$ (volatility) and $(W_t)_{t\geq 0}$ is standard Brownian motion (a source of randomness). The *fundamental stock price* is then defined to be the expected value of all future dividends:

$$S_t^o \,=\, E\bigg(\int_t^\infty e^{-r(s-t)} D_s\, ds \;\bigm|\; \mathcal{F}_t^W \bigg) \tag{2.3}$$

where $\mathcal{F}_t^W = \sigma(\{W_s \,|\, 0 \leq s \leq t\})$ is the information set available at time t (Lucas 1978).

In deciding how to revise the stock price, the specialist faces constraints specified by the market institution (see e.g. Ait-Sahalia 1998, Madhavan and Smidt 1993). It will be assumed that the specialist adjusts the stock price through relative returns according to the following rule:

$$\frac{dS_t}{S_t} \,=\, \mu_t\, dt + \frac{dS_t^o}{S_t^o} \tag{2.6}$$

where dS_t/S_t is the relative return of the market price, dS_t^o/S_t^o is the relative return of the fundamental price, and μ_t is the drift chosen by the specialist. Thus, the specialist 'controls' the

market price through the choice of μ_t . For simplicity, we deal with *Markov controls* $\mu_t = \mu(t, S_t)$, but other treatments may also be of interest.

We note in (2.6) that $\mu_t \equiv 0$ *if and only if* $(S_t)_{t\geq 0} = (S^o_t)_{t\geq 0}$ *if and only if* there is no control exercised by the specialist, and *if and only if* there is no 'external' force (influence) exerted. In this case the price is an 'inertial equilibrium'. These facts reveal some analogy with *Newton's first law of motion* (see Section 3 for more details).

The specialist's aim is to determine an optimal $\mu^* = \mu^*(t, s)$ from this class. If we now consider the *log-price*:

$$X_t = \log\left(S_t\right) \tag{2.9}$$

it follows by Itô's formula that X_t solves:

$$dX_t = \widehat{\mu}(t, X_t)dt + \sigma dW_t \tag{2.10}$$

where $\widehat{\mu}(t, x) = \mu(t, e^x) - \sigma^2/2$, and this holds for any admissible $\mu = \mu(t, s)$. Thus, by *Smoluchowski's argument* (reviewed in Section 3), once the optimal $\mu^* = \mu^*(t, s)$ is found, we may think of it as *the acceleration of the market force* being exerted as a superposition of external influences by the market players. Thus, formally we can write:

$$\mu^*(t, s) \sim \text{the } \textit{acceleration} \text{ of the } \textit{market force}. \tag{2.11}$$

These considerations are clarified in Sections 4 and 5.

2. *The Specialist's Optimisation*. How does the specialist determine the optimal adjustment μ_t ? We suppose that given a demand function γ_t as the number of shares of the stock required by an investor, by the rule of the market institution *the specialist must take the opposite side of the trade*. That is, she must clear the market and hold $-\gamma_t$ shares of the stock. We note in passing that this rule is connected with *Newton's third law of motion* (see Section 3).

Depending upon the choice of discounting (which is analysed at some length in Section 4), we study two possible formulations of the specialist's optimisation problem. (The formulations imply markedly different consequences for the asset price process.) Setting:

$$S^*_t = e^{-rt}S_t \tag{2.13}$$

the *first specialist's formulation* is to solve:

$$\sup_{\mu} E\left(\int_t^\infty (-\gamma_s)\, dS^*_s \;\middle|\; \mathcal{F}_t\right) \tag{2.14}$$

where $\mathcal{F}_t$ represents the information set available at time t , and γ_s is an optimal investor's demand at time s (to be specified below).

The *second specialist's formulation* is to solve:

$$\sup_{\mu} E\left(\int_t^\infty e^{-rs}(-\gamma_s)\, dS_s \;\middle|\; \mathcal{F}_t\right) \tag{2.15}$$

with $\mathcal{F}_t$ as above. Thus, in this case the discounting is applied before the d-sign and not after.

In both formulations the supremum is taken over all $\mu = (\mu_s)_{s \geq t}$ from an admissible class for which (2.6) makes sense; in this work we study Markov controls $\mu_t = \mu(t, S_t)$, but other controls are also of interest.

The crucial role in the treatment of the specialist's problems (2.14) and (2.15) is played by the Markovian structure of the process (S_t, Z_t). This will enable us to reformulate problems (2.14) and (2.15) as stochastic control problems which can then be solved explicitly (Section 6).

3. *The Investor's Optimisation*. To formulate the investor's problem assume that his initial wealth is $z > 0$, and that he is free to transfer his holdings continuously in time from one investment to another without paying transaction costs. There is no restriction on borrowing or lending, and short sales are allowed. We assume that the investor has at his disposal two investment possibilities: the stock given by (2.6) above, and the risk-free bond satisfying:

$$dB_t = rB_t\,dt \tag{2.16}$$

with $B_0 = 1$. Thus $B_t = e^{rt}$ continuously compounds at the constant interest rate $r > 0$.

The fraction of investor's wealth held at time t in the stock is conveniently denoted by

$$u_t = \frac{Y_t}{X_t + Y_t} \tag{2.17}$$

where Y_t is the wealth held in the stock (may be positive or negative), and $Z_t := X_t + Y_t$ is the total wealth held both in the stock and the bond.

Given a consumption rate c_t, the investor's wealth process $Z = (Z_t)_{t \geq 0}$ is assumed to satisfy to following *budget* equation:

$$dZ_t = (1-u_t)\,r\,Z_t\,dt + u_t\,Z_t\left(\mu_t dt + \sigma\,dW_t\right) - c_t\,dt \tag{2.18}$$

where $(1-u_t)\,rZ_t\,dt$ is the fraction of wealth held in the bond, $u_t Z_t\left(\mu_t dt + \sigma dW_t\right)$ is the fraction of wealth held in the stock, and $c_t\,dt$ is the fraction of wealth consumed. By writing (2.18) in this form we are actually imposing a *self-financing* property on the strategy of the investor. (This is addressed in more detail in Section 4.)

In the sequel we will avoid dealing with *the time of bankruptcy*:

$$\tau = \inf\{\, t > 0 \mid Z_t = 0 \,\} \tag{2.19}$$

and replace it with a *transversality condition* (specified later) which will imply that at the 'end of time' the wealth must be non-negative (i.e. the investor cannot 'die' holding a debt).

Given a utility function $U = U(c)$, the investor's aim is to solve:

$$\sup_{u\,,c} E\left(\int_t^\infty e^{-\rho s} U(c_s)\,ds \;\middle|\; \mathcal{F}_t \right) \tag{2.20}$$

where $\mathcal{F}_t$ represents the information set available at time t.

The utility function of the investor is assumed to be:

$$U_\gamma(c) = \frac{c^\gamma - 1}{\gamma} \quad (0 < \gamma < 1) \tag{2.21}$$

which has an *Arrow-Pratt coefficient* of *relative risk aversion* given by $-cU''(c)/U'(c) = 1-\gamma$. We shall also deal with the *logarithmic* utility function:

$$U_0(c) = \log(c) \tag{2.22}$$

which is obtained as a limit of (2.21) for $\gamma \downarrow 0$. These utility functions will be sufficient to grasp most of the essentials offered by the model. The problem (2.20) in this case reduces to the problem posed and solved by Merton (1969).

4. *Concluding Remarks*. Thus, if it is known (from the trading rules specified by the market institution) that the stock price S_t will be driven as in (2.6) for some admissible μ_t , then the specialist-investor equilibrium is achieved as follows. The investor takes any admissible μ_t as given, and solves her optimisation problem (2.20), thus obtaining an optimal demand γ_t (which depends on μ_t). Given this demand function the specialist solves her optimisation problem (2.14) or (2.15) and obtains the optimal drift μ_t^* . As the optimal γ_t found by the investor applies to any μ_t , it will also apply to the optimal μ_t^* , thus leading to the optimal demand function γ_t^* . This procedure gives the equilibrium actions (μ_t^*, γ_t^*) , which are mutual best responses. In accordance with our considerations taken up in the next section, and as already stated in (2.11), this solution establishes a *'dynamic equilibrium'* defined by a *'market force'* with *'acceleration'* μ_t^* . This identification utilizes *Newton's second law of motion* (the principle of *'superposition'* of forces). In this cases the forces are an *action force* of the investor and a *reaction force* of the specialist (see Section 3 for more details).

Note. While we are aware of the Hodges-Carverhill-Selby result (*The Economic Journal*, 103, 1993, 395-405, and in "*Mathematics of Derivative Securities*", Cambridge Univ. Press 1997, 41-52), which in the context above reads "consistency with 'equilibrium' is achieved if the acceleration solves *Burger's equation*", we do not make any use of this interesting fact in the paper.

REFERENCES

[1] BARNDORFF-NIELSEN, O. E. (1998). Processes of normal inverse Gaussian type. *Finance Stochast.* 2 (41-68).

[2] BLACK, F. *and* SCHOLES, M. (1973). The pricing of options and corporate liabilities. *J. Political Econom.* 81 (637-654).

[3] DOOB, J. L. (1942). The Brownian movement and stochastic equations. *Ann. of Math.* 43 (351-369).

[4] EINSTEIN, A. (1905). On the motion of small particles suspended in liquids at rest required by the molecular-kinetic theory of heat. *Ann. Phys.* 17 (549-560). English translation in the book "*Einstein's miraculous year*" by Princeton Univ. Press. 1998 (85-98).

[5] LANGEVIN, P. (1908). Sur la theorie du mouvement brownien. *C. R. Acad. Sci. Paris* 146 (530-533).

[6] MERTON, R. C. (1969). Lifetime portfolio selection under uncertainity: The continuous time case. *Rev. Econom. Stud.* 51 (247-257).

[7] NELSON, E. (1967). *Dynamical Theories of Brownian Motion.* Princeton Univ. Press.

[8] SMOLUCHOWSKI, M. R. von (1906). Zur kinetischen Theorie der Brownschen Molekularbewegung und der Suspensionen. *Ann. Phys.* 21 (756-780).

[9] UHLENBECK, G. E. *and* ORNSTEIN, L. S. (1930). On the theory of Brownian motion. *Physical Review* 36 (823-841).

[10] WEIDNER, R. T. *and* SELLS, R. L. (1965). *Elementary Classical Physics. Vol I (Mechanics, Kinetic Theory, Thermodynamics).* Allyn and Bacon, Inc. Boston.

GORAN PESKIR
Institute of Mathematics, University of Aarhus
Ny Munkegade, 8000 AARHUS, DENMARK
goran@imf.au.dk
http://www.imf.au.dk/∼goran

SERIES EXPANSIONS WITHOUT COMPENSATION FOR INFINITELY DIVISIBLE PROCESSES

Jan Rosiński

University of Tennessee, Knoxville

Throughout this note $\{\Gamma_n\}_{n\geq 1}$ will stand for the sequence of the consecutive arrival times in a Poisson process of unit rate, $\{\xi_n\}_{n\geq 1}$ will denote a sequence of i.i.d. random elements in a suitable measurable space, and $\{U_n\}_{n\geq 1}$ will be a sequence of i.i.d. uniform on $[0,1]$ r.v.'s. It is assumed that the sequences $\{\Gamma_n\}_{n\geq 1}$, $\{\xi_n\}_{n\geq 1}$, and $\{U_n\}_{n\geq 1}$ are independent of each other.

1. Representations of stable Lévy processes.

Let $\{X(t), 0\leq t\leq 1\}$ be an α-stable Lévy process with the skewness parameter $\beta\in[-1,1]$, the scale parameter 1, and the shift parameter 0 ($X(1)\sim S_\alpha(1,\beta,0)$ in the notation of [ST94]). This process admits the following series expansions (see [ST94], Example 3.10.3), for $0<\alpha<1$

$$\{X(t),\ 0\leq t\leq 1\}\stackrel{d}{=}\{C_\alpha^{1/\alpha}\sum_{n=1}^{\infty}\xi_n\Gamma_n^{-1/\alpha}\mathbf{1}(U_n\leq t),\quad 0\leq t\leq 1\},\tag{1.1}$$

for $1<\alpha<2$

$$\{X(t),\ 0\leq t\leq 1\}\stackrel{d}{=}\{C_\alpha^{1/\alpha}\sum_{n=1}^{\infty}[\xi_n\Gamma_n^{-1/\alpha}\mathbf{1}(U_n\leq t)-\beta b_n^{(\alpha)}t],\quad 0\leq t\leq 1\},\tag{1.2}$$

and for $\alpha=1$,

$$\{X(t),\ 0\leq t\leq 1\}\stackrel{d}{=}\{\frac{2}{\pi}\sum_{n=1}^{\infty}[\xi_n\Gamma_n^{-1}\mathbf{1}(U_n\leq t)-\beta a_n t]+(\beta\frac{2}{\pi}\ln\frac{2}{\pi})t,\quad 0\leq t\leq 1\},\tag{1.3}$$

where ξ_n has the distribution given by

$$P(\xi_n=1)=1-P(\xi_n=-1)=\frac{1+\beta}{2},\tag{1.4}$$

and $a_n=\int_{n^{-1}}^{(n-1)^{-1}}x^{-2}\sin x\,dx$, $b_n^{(\alpha)}=\frac{\alpha}{\alpha-1}(n^{\frac{\alpha-1}{\alpha}}-(n-1)^{\frac{\alpha-1}{\alpha}})$; $C_\alpha=(\int_0^\infty x^{-\alpha}\sin x\,dx)^{-1}$.

One can view series (1.2) and (1.3) as integrals with respect a marked Poisson point process $M=\sum\delta_{(\Gamma_n,\xi_n,U_n)}$ with compensation; indeed $E[\xi_n\Gamma_n^{-1/\alpha}\mathbf{1}(U_n\leq$

$t)] \sim \beta b_n^{(\alpha)} t$ and $E[\xi_n \Gamma_n^{-1} \mathbf{1}(U_n \le t)] \sim \beta a_n t$ ([ST94], Remark 5, page 30). However, it is more desirable and simpler to work with series expansions without compensation, such as (1.1). This is obviously the case when $\beta = 0$ (symmetric α-stable processes). But it is also possible to avoid compensation when X is not totally skewed, that is $|\beta| < 1$ and $\alpha \neq 1$. Indeed, using a representation from [JW94] we have the following.

Proposition 1.1. *Let $\{X(t), 0 \le t \le 1\}$ be an α-stable Lévy process with $\alpha \neq 1$, the skewness parameter $\beta \in (-1, 1)$, the scale parameter 1, and the shift parameter 0. Then*

$$\{X(t),\ 0 \le t \le 1\} \stackrel{d}{=} \{C_\alpha^{1/\alpha} \sum_{n=1}^{\infty} \xi_n \Gamma_n^{-1/\alpha} \mathbf{1}(U_n \le t),\quad 0 \le t \le 1\}, \tag{1.5}$$

provided the distribution of ξ_n is given by

$$P(\xi_n = a) = 1 - P(\xi_n = -b) = \frac{b}{a+b} \tag{1.6}$$

where $a = \left[\frac{1+\beta}{2}((\frac{1+\beta}{1-\beta})^{1/(\alpha-1)} + 1)\right]^{1/\alpha}$ *and* $b = \left[\frac{1-\beta}{2}((\frac{1-\beta}{1+\beta})^{1/(\alpha-1)} + 1)\right]^{1/\alpha}$.

Representation (1.5) can not be extended to skewed 1-stable processes. This follows from the next lemma.

Lemma 1.2. *If $\sum_{n=1}^{\infty} \xi_n \Gamma_n^{-1}$ converges a.s., where $\{\xi_n\}$ is an arbitrary i.i.d. sequence, then $\sum_{n=1}^{\infty} \xi_n \Gamma_n^{-1}$ has a Cauchy distribution, i.e., $\alpha = 1$ and $\beta = 0$.*

Therefore, we need to look at different forms of series expansions to represent without compensation skewed 1-stable processes. We obtain the following representation.

Theorem 1.3. *Let $\{X(t),\ 0 \le t \le 1\}$ be a 1-stable Lévy processes with the skewness parameter $\beta \in (-1, 1)$, the scale parameter 1, and the shift parameter 0. Let $\{\xi_n\}$ be an i.i.d. sequence with the common probability density function of the form*

$$f(x) = \begin{cases} A(1+\beta)x^{-2} g(x^{1+\beta}), & x > 0 \\ A(1-\beta)x^{-2} g(|x|^{1-\beta}), & x < 0 \end{cases}$$

where $A > 0$ and $g : \mathbf{R}^+ \to \mathbf{R}^+$ are subject to the conditions: $g(x) > 0$ for $x > 0$ and $\int_{-\infty}^{\infty} f(x)\, dx = 1$. Then

$$\{X(t),\ 0 \le t \le 1\} \stackrel{d}{=} \{(\pi A)^{-1} \sum_{n=1}^{\infty} \xi_n \mathbf{1}(h(\xi_n) \le \Gamma_n^{-1}) \mathbf{1}(U_n \le t),\quad 0 \le t \le 1\}, \tag{1.7}$$

where $h(x) = g(x^{1+\beta})\mathbf{1}(x > 0) + g(|x|^{1-\beta})\mathbf{1}(x < 0)$. The series converges unconditionally a.s. and uniformly with respect to $t \in [0, 1]$.

Proposition 2.4 provides a wide latitude in the choice of the common distribution of ξ_n's. A counterpart to Proposition 1.1 is the following.

Theorem 1.4. *Let $\{X(t), 0 \le t \le 1\}$ be an α-stable Lévy process with $\alpha \neq 1$, the skewness parameter $\beta \in (-1, 1)$, the scale parameter 1, and the shift parameter 0. Let $\{\xi_n\}$ be an i.i.d. sequence with the common probability density function of the form*

$$f(x) = \begin{cases} A(1+\beta)x^{-\alpha-1}g((1+\beta)^{\frac{1}{1-\alpha}}x), & x > 0 \\ A(1-\beta)|x|^{-\alpha-1}g((1-\beta)^{\frac{1}{1-\alpha}}|x|), & x < 0 \end{cases}$$

where $A > 0$ and $g : \mathbf{R}^+ \to \mathbf{R}^+$ are subject to the conditions: $g(x) > 0$ for $x > 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$. Then

$$\{X(t),\ 0 \le t \le 1\} \stackrel{d}{=} \{K_\alpha \sum_{n=1}^{\infty} \xi_n \mathbf{1}(h(\xi_n) \le \Gamma_n^{-1})\mathbf{1}(U_n \le t),\ \ 0 \le t \le 1\}, \tag{1.8}$$

where $K_\alpha = (\alpha C_\alpha/(2A))^{1/\alpha}$ and $h(x) = g((1+\beta)^{\frac{1}{1-\alpha}}x)\mathbf{1}(x > 0) + g((1-\beta)^{\frac{1}{1-\alpha}}|x|)\mathbf{1}(x < 0)$. The series converges unconditionally a.s. and uniformly with respect to $t \in [0, 1]$.

Notice that the series in (1.7) and (1.8) have exactly the same form. They are obtained by a random thinning of an i.i.d. sequence $\{\xi_n\}$.

2. Random thinning of i.i.d. sequences.

The idea of representing infinitely divisible random vectors by series of randomly thinned i.i.d. sequences was introduced in [R90a]. Such series were applied to the study of sample paths of infinitely divisible processes in [R90b], [Ta92], [Ta92]. The simplest way to present this idea is a construction of a Poisson point process with a given σ-finite intensity measure ν on a measurable space S. Namely, choose any probability measure μ on S such that ν is absolutely continuous with respect to μ, and let $\{\xi_n\}$ be an i.i.d. sequence of random elements in S with the common distribution μ. Define

$$g(x) = \frac{d\nu}{d\mu}(x) \qquad x \in S. \tag{2.1}$$

Then

$$N(dx) = \sum_{n=1}^{\infty} \mathbf{1}(g(\xi_n) \ge \Gamma_n)\delta_{\xi_n}(dx) \tag{2.2}$$

is a Poisson point process on S with the intensity measure ν. Here a Poisson point process N is constructed "at once" as opposed to the usual construction based on a partition of S into sets of finite ν-measure and pasting the corresponding Poisson point processes on those sets. N is obtained by a random thinning of i.i.d. points $\{\xi_n\}$ and again, we have a wide selection of their common distribution μ.

3. Representations of Lévy processes without compensation.

Here we will only give a general result on series representations without compensation for Lévy processes. Specific methods of defining the common distribution of ξ_n's, depending on the underlying Lévy measure, and extensions to nonstationary increment processes and to infinitely divisible random measures, will be given in [Ro98].

Theorem 3.1. *Let $\{X(t),\ 0 \le t \le 1\}$ be a Lévy processes with Lévy measure ν of $\mathcal{L}(X(1))$. Suppose that either both integrals*

$$\int_0^1 x\,\nu(dx) \quad \text{and} \quad \int_{-1}^0 |x|\,\nu(dx) \tag{3.1}$$

are finite or both are infinite. Then this process admits a representation of the form

$$\{X(t),\ 0 \le t \le 1\} \stackrel{d}{=} \{\sum_{n=1}^{\infty} w(\xi_n, \Gamma_n)\mathbf{1}(U_n \le t) + ct,\quad 0 \le t \le 1\}, \tag{3.2}$$

for a suitably chosen $w : \mathbf{R} \times \mathbf{R}^+ \to \mathbf{R}$, an i.i.d. sequence $\{\xi_n\}$, and $c \in \mathbf{R}$. The series converges unconditionally a.s. and uniformly with respect to $t \in [0,1]$.

Remark 3.2. Simultaneous convergence or divergence of integrals (3.1) is also necessary for (3.2).

References

[JW94] JANICKI, A. and WERON, A. (1994), *Simulation and chaotic behavior of α-stable stochastic processes*, Marcel Dekker.

[LP80] LE PAGE, R. (1980), *Multidimentional infinitely divisible variables and processes, Part 2*, Lectures Notes in Mathematics **860**, Springer–Verlag, 279–284.

[R90a] ROSIŃSKI, J. (1990), *On series representations of infinitely divisible random vectors*, Ann. Probab. **18**, 405–430.

[R90b] ROSIŃSKI, J. (1990), *An application of series representations for zero-one laws for infinitely divisible random vectors*, Probability in Banach Spaces 7, Progress in Probability, Birkhäuser **21**, 189–199.

[Ro98] ROSIŃSKI, J. (1998), *Series expansions without compensation for infinitely divisible processes*, in preparation.

[ST94] SAMORODNITSKY, G. and TAQQU, M.S. (1994), *Non-Gaussian Stable Processes*, Chapman & Hall.

[Ta92] TALAGRAND, M. (1992), *Necessary and sufficient conditions for sample continuity of random Fourier series and of harmonic infinitely divisible processes*, Ann. Probab. **20**, 1–28.

[Ta93] TALAGRAND, M. (1993), *Regularity of infinitely divisible processes*, Ann. Probab. **21**, 362–432.

Jan Rosinski, University of Tennessee, Department of Mathematics, Knoxville, TN 37996 USA

E-mail address: rosinski@math.utk.edu

CERTAIN PROBABILISTIC ASPECTS OF SEMISTABLE LAWS

GENNADY SAMORODNITSKY

The law μ of a non-Gaussian random vector $\mathbf{X}$ in $\Re^d$ (or even in a more general space) is called semistable if it is infinitely divisible and there exist $r, b \in (0,1) \cup (1,\infty)$ and $\mathbf{c} \in \Re^d$ such that

$$\mu^{*r} = \mu(b\cdot) * \delta_{\mathbf{c}}, \tag{1.1}$$

where for positive r, μ^{*r} stands for the rth convolution power of μ, $*$ means the convolution of two measures, and $\delta_{\mathbf{c}}$ is the point mass at $\mathbf{c}$. It is well known that there exists an $\alpha \in (0,2)$ depending only on $\mathbf{X}$ such that b in (1.1) is expressed as $b = r^{-1/\alpha}$, namely

$$\mu^{*r} = \mu(r^{-1/\alpha}\cdot) * \delta_{\mathbf{c}}.$$

Thus α is a characteristic of $\mathbf{X}$ and is called the index of $\mathbf{X}$.

Suppose $\mathbf{X}$ is semistable with index α. Let , be the collection of all $r \in (0,\infty)$ for which $\mathbf{X}$ satisfies (1.1) for some $\mathbf{c}$ in $\Re^d$. Then , is a closed multiplicative subgroup of $(0,\infty)$. We will say that $\mathbf{X}$ is , –semistable index α.

A , -semistable index α random vector $\mathbf{X}$ for which , $= (0,\infty)$ (this follows automatically if , contains a sequence of r's approaching 1) is α-stable. Since α-stable random vectors, $0 < \alpha < 2$, are often viewed as heavy tailed analogs of Gaussian random vectors, the dependence structure of α-stable random vectors and processes has been extensively studied. The tails of semistable random variables are similar (even though not necessarily strictly comparable) to those of stable random variables, and since the family of α-stable laws is, from many points of view, a small subset of the family of all semistable index α laws, the latter offer higher flexibility in stochastic modeling than the former. This fact points to potential uses of semistable laws in applied probability. The first step in realizing such potential is to understand the probabilistic structures of semistable laws.

We concentrate on two issues, that are of interest in clarifying the place of semistable laws among all the infinitely divisible laws. First, we discuss the extent to which the property of semistability of a random vector in $\Re^d$ is determined by the property of semistability of its

This is a joint work with Makoto Maejima.

marginals, and then we clarify which of the semistable random variables in $\Re$ are of type G, and which of them are, in fact, sub–stable.

Cornell University, School of Operations Research and Industrial Engineering, 206 Rhodes Hall, Ithaca, NY 14853-3801, U.S.A.
E-mail address: gennady@orie.cornell.edu

RECURRENCE AND TRANSIENCE OF LÉVY PROCESSES AND SOME PROCESSES WITH NONSTATIONARY INDEPENDENT INCREMENTS

KEN-ITI SATO

In the first half of this talk we discuss criteria of recurrence and transience for Lévy processes. In the second half we introduce selfsimilar additive processes and report on the problem of recurrence and transience for those processes.

Terminology. A *Lévy process* $\{X_t,\, t \geq 0\}$ on $\mathbf{R}^d$ is defined to be a stochastically continuous process with stationary independent increments with $X_0 = 0$ and with sample functions right-continuous with left limits. Dropping the stationarity requirement for increments, we define an *additive process*. A Lévy process $\{X_t\}$ is called *recurrent* if

$$\liminf_{t\to\infty} |X_t| = 0 \quad \text{a. s.}; \tag{0.1}$$

it is called *transient* if

$$\lim_{t\to\infty} |X_t| = \infty \quad \text{a. s.} \tag{0.2}$$

An additive process $\{X_t\}$ is called *recurrent* if, for every $s \geq 0$,

$$\liminf_{t\to\infty} |X_t - X_s| = 0 \quad \text{a. s.} \tag{0.3}$$

or, equivalently,

$$P\Big[\liminf_{t\to\infty} |X_t - X_s| = 0 \text{ for every } s \geq 0\Big] = 1; \tag{0.4}$$

it is called *transient* if (0.2) holds. For a Lévy process (0.1) and (0.4) are equivalent but there are additive processes satisfying (0.1), but not (0.4).

A stochastic process $\{X_t,\, t \geq 0\}$ on $\mathbf{R}^d$ is called *selfsimilar* if, for every $a > 0$, there is $b > 0$ such that

$$\{X_{at},\, t \geq 0\} \overset{\mathrm{d}}{=} \{bX_t,\, t \geq 0\}, \tag{0.5}$$

where $\overset{\mathrm{d}}{=}$ means identity in finite-dimensional distributions.

The distribution of X is denoted by $\mathcal{L}(X)$.

An *infinitely divisible* distribution μ on $\mathbf{R}^d$ has the characteristic function $\widehat{\mu}(z)$, $z \in \mathbf{R}^d$, expressible by the *Lévy–Khintchine representation*

$$\widehat{\mu}(z) = \exp\left[-\frac{1}{2}\langle z, Az\rangle + \langle \gamma, z\rangle + \int_{\mathbf{R}^d\setminus\{0\}} (e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle 1_{\{|x|\le 1\}}(x))\,\nu(dx)\right],$$

where A is a nonnegative-definite symmetric operator, $\gamma \in \mathbf{R}^d$, and ν (called *Lévy measure*) is a measure on $\mathbf{R}^d$ with $\nu(\{0\}) = 0$ and $\int(1 \wedge |x|^2)\nu(dx) < \infty$. We call (A, ν, γ) the *generating triplet* of μ. When $\{X_t\}$ is a Lévy process, the generating triplet of $\mu = \mathcal{L}(X_1)$ is called that of $\{X_t\}$.

A probability measure μ is called *selfdecomposable* if, for every $b > 1$, there is a probability measure ρ_b such that

$$\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho}_b(z). \tag{0.6}$$

1. Recurrence and transience of Lévy processes

It is well-known that a Lévy process $\{X_t\}$ is either recurrent or transient and that they are expressible, respectively, by infiniteness of mean sojourn times on open neighborhoods of 0 or by finiteness of mean sojourn times on compact sets. They can be described by the behavior of $\log\widehat{\mu}(z)$ near the origin. The following result is a consequence of Chung and Fuchs [1], Spitzer [13], Ornstein [5], Stone [14], and Port and Stone [6].

Theorem 1.1. *Fix $\varepsilon > 0$. A Lévy process $\{X_t\}$ on $\mathbf{R}^d$ is recurrent if and only if*

$$\int_{|z|<\varepsilon} \mathrm{Re}\left(\frac{1}{-\psi(z)}\right)dz = \infty, \tag{1.1}$$

where $\psi(z) = \log\widehat{\mu}(z)$ with $\mu = \mathcal{L}(X_1)$.

This is a definitive result. We call it *Spitzer type criterion*. However, it is hard for us to decide whether (1.1) holds or not, looking at the generating triplet (A, ν, γ). To be more specific, how do the properties of the Lévy measure ν outside compact sets influence recurrence and transience? Only in symmetric case we can handle this question using the technique of Shepp [11, 12].

We restrict ourselves to one dimension, $d = 1$, in the rest of this section. If $E|X_t| < \infty$ for $t > 0$, a necessary and sufficient condition for recurrence is $EX_t = 0$. Therefore, any symmetric Lévy process with $E|X_t| < \infty$ is recurrent. Let ρ_1 and ρ_2 be measures on $\mathbf{R}$. We say that ρ_2 has an *identical tail* with ρ_1 if there is x_0 such that, for any $x \ge x_0$, $\int_{|y|>x}\rho_2(dy) = \int_{|y|>x}\rho_1(dy)$. We say that ρ_2 has a *bigger tail*

than ρ_1 if there is x_0 such that, for any $x \geq x_0$, $\int_{|y|>x} \rho_2(dy) \geq \int_{|y|>x} \rho_1(dy)$. The following three theorems are analogues of Shepp's results on random walks. Proofs are given in Sato [9].

Theorem 1.2. *Let $\{X_t\}$ and $\{Y_t\}$ be symmetric Lévy processes.*

(i) *If their Lévy measures ν_X and ν_Y satisfy*

$$\int_0^\infty x^2 |\nu_X - \nu_Y|(dx) < \infty, \tag{1.2}$$

then recurrence of $\{X_t\}$ is equivalent to that of $\{Y_t\}$. Here $|\nu_X - \nu_Y|$ is the variation of the signed measure $\nu_X - \nu_Y$.

(ii) *If ν_Y has a bigger tail than ν_X and if ν_Y is unimodal with mode 0, then transience of $\{X_t\}$ implies that of $\{Y_t\}$.*

In particular, in symmetric case, if ν_Y has an identical tail with ν_X, then the conclusion in (i) is true. The Gaussian part is irrelevant. Even without symmetry, it is plausible that, under the condition that $E|X_t| = \infty$ for $t > 0$, or under a more stringent condition, recurrence of $\{X_t\}$ is determined only by the behavior of ν_X outside any compact set.

Theorem 1.3. *For an arbitrarily given symmetric finite measure ρ on $\mathbf{R}$, there exists a recurrent symmetric Lévy process $\{X_t\}$ such that its Lévy measure ν has a bigger tail than ρ.*

This shows that, in Theorem 1.2(ii), the assumption of unimodality of ν_Y is essential.

Theorem 1.4. *Let $\{X_t\}$ be a symmetric Lévy process with Lévy measure ν. Define*

$$R(r,x) = \nu\left(\bigcup_{n=0}^{\infty} (2nr + x, 2(n+1)r - x] \cap (1,\infty)\right) \quad \text{for } r \geq x \geq 0, \tag{1.3}$$

$$N(x) = \nu\big((x \vee 1, \infty)\big) \quad \text{for } x \geq 0. \tag{1.4}$$

Let $c > 0$ be fixed. Then transience of $\{X_t\}$ is equivalent to

$$\int_c^\infty \left(\int_0^r x R(r,x)dx\right)^{-1} dr < \infty. \tag{1.5}$$

If ν is unimodal with mode 0, then transience of $\{X_t\}$ is equivalent to

$$\int_c^\infty \left(\int_0^r x N(x)dx\right)^{-1} dr < \infty. \tag{1.6}$$

Even without unimodality of ν, transience implies the condition (1.6), since $N(x) \geq R(r,x)$. But, in general, (1.6) does not imply transience, because, if it does, Theorem 1.3 is not true.

It is known that, if $\{X_t\}$ is symmetric, then unimodality of ν with mode 0 is equivalent to unimodality of the distribution of X_t with mode 0 for every t.

We add some consequences of Theorem 1.1 concerning relations between symmetric and non-symmetric Lévy processes. See [8] for the proof.

Proposition 1.5. *Let $\{X_t\}$ and $\{Y_t\}$ be independent Lévy processes. If $\{X_t\}$ is transient and symmetric, then $\{X_t + Y_t\}$ is transient.*

Proposition 1.6. *If $\{X_t\}$ is recurrent, then its symmetrization is recurrent.*

If $\{X_t\}$ is non-symmetric stable process with index 1, then it is transient but its symmetrization is recurrent.

Proposition 1.7. *Let $\{X_t\}$ and $\{Y_t\}$ be independent Lévy processes. Suppose that $\{X_t\}$ is symmetric and that*

$$\liminf_{z \downarrow 0} z^{-1} \int_0^\infty (1 - \cos zx)\, \nu_X(dx) > 0. \tag{1.7}$$

Suppose that $E|Y_t| < \infty$ for $t > 0$. If $\{X_t\}$ is recurrent, then $\{X_t + Y_t\}$ is recurrent.

The condition (1.7) is determined only by the tail of ν_X. It implies that $E|X_t| = \infty$ for $t > 0$. An example of $\{X_t\}$ is the Cauchy process. It is well-known that the Cauchy process plus a drift γt is still recurrent. Proposition 1.7 is an extension of this fact.

2. Recurrence and transience of selfsimilar additive processes

For the class of additive processes we do not have the dichotomy of recurrence and transience. Namely, there are additive processes that are neither recurrent nor transient. Let us restrict our attention to selfsimilar additive processes.

Theorem 2.1 (Sato and Yamamuro [10])**.** *If $\{X_t\}$ is a selfsimilar additive process on $\mathbf{R}^d$, then it is either recurrent or transient.*

The following fact shows that mean sojourn times are irrelevant to recurrence and transience.

Proposition 2.2. *There is a recurrent selfsimilar additive process* $\{X_t\}$ *such that* $E\big[\int_s^\infty 1_{\{|x|\le r\}}(X_t - X_s)dt\big] < \infty$ *for all* $r > 0$ *and* $s \ge 0$. *There is a transient selfsimilar additive process* $\{Y_t\}$ *such that* $E\big[\int_s^\infty 1_{\{|x|\le r\}}(Y_t - Y_s)dt\big] = \infty$ *for all* $r > 0$ *and* $s \ge 0$.

If $\{X_t\}$ is a nonzero, selfsimilar, stochastically continuous process on $\mathbf{R}^d$ with $X_0 = 0$, then it has a unique *exponent* $H > 0$, that is, for any $a > 0$, the b in (0.5) is expressed as $b = a^H$. A nonzero process is selfsimilar Lévy with exponent H if and only if it is strictly stable with index $\alpha = 1/H$. It is known that a strictly stable process with index α on $\mathbf{R}$ is recurrent if and only if $\alpha \ge 1$. But, in case of selfsimilar additive processes, the exponents have no relation with recurrence and transience. In fact, if $\{X_t\}$ is a selfsimilar additive process with exponent H, then, for any $\gamma > 0$, $\{X_{t^\gamma}\}$ is a selfsimilar additive process with exponent γH; we can freely change the exponent, not affecting recurrence or transience.

Selfsimilar additive processes correspond to selfdecomposable distributions as follows.

Theorem 2.3 (Sato [7]). *If* $\{X_t\}$ *is a selfsimilar additive process on* $\mathbf{R}^d$, *then, for any* $t \ge 0$, $\mathcal{L}(X_t)$ *is selfdecomposable. Conversely, if* μ *is a selfdecomposable distribution on* $\mathbf{R}^d$, *not being the unit mass at* 0, *then, for any* $H > 0$, *there is, uniquely in law, a selfsimilar additive process* $\{X_t^{(H)}\}$ *with exponent* H *such that* $\mathcal{L}(X_1^{(H)}) = \mu$.

A probability measure μ on $\mathbf{R}$ is selfdecomposable if and only if it is infinitely divisible and its Lévy measure ν is such that $\nu(dx) = k(x)|x|^{-1}dx$ with $k(x)$ being increasing on $(-\infty, 0)$, decreasing on $(0, \infty)$, and nonnegative.

Recurrence or transience of the process $\{X_t^{(H)}\}$ in Theorem 2.3 is determined by μ and independent of H. But we do not know any result of the type of Theorem 1.1. Given a selfdecomposable distribution μ, we have the process $\{X_t^{(H)}\}$ on one hand and the Lévy process $\{Y_t\}$ with $\mathcal{L}(Y_1) = \mu$ on the other. We are interested in the comparison of these two processes.

Example 2.4 (Sato and Yamamuro [10]). Let μ be a Gaussian on $\mathbf{R}$ with mean $\gamma \ne 0$ and variance 1. Then $Y_t = B_t + \gamma t$ and $X_t^{(H)} = B_{t^{2H}} + \gamma t^H$, where $\{B_t\}$ is the Brownian motion on $\mathbf{R}$. The process $\{Y_t\}$ is transient but $\{X_t^{(H)}\}$ is recurrent.

Example 2.5 (Sato and Yamamuro [10]). Let μ be a selfdecomposable distribution on $\mathbf{R}$ such that $\widehat{\mu}(z) = \exp\big[\int_{-\infty}^\infty (e^{izx} - 1)k(x)|x|^{-1}dx\big]$ with $0 < \int_{-\infty}^0 k(x)dx =$

$\int_0^\infty k(x)dx < \infty$, $k(0-) < \infty$, and $k(0+) < \infty$. Then $\{Y_t\}$ is recurrent, but $\{X_t^{(H)}\}$ is transient.

Theorem 2.6 (Yamamuro [15]). *If $d \geq 3$, then any nondegenerate selfsimilar additive process on $\mathbf{R}^d$ is transient.*

Theorem 2.7 (Yamamuro [15]). *Let $d = 1$. Let $\{X_t\}$ be a nondegenerate selfsimilar additive process on $\mathbf{R}$ with $\mathcal{L}(X_1) = \mu$.*

(i) *If μ has nondegenerate Gaussian part, then $\{X_t\}$ is recurrent.*

(ii) *If $\widehat{\mu}(z) = \exp\big[\int_{-\infty}^{\infty}(e^{izx} - 1)k(x)|x|^{-1}dx\big]$ with $k(0-) < \infty$ and $k(0+) < \infty$, then $\{X_t\}$ is transient.*

(iii) *If $\widehat{\mu}(z) = \exp\big[\int_0^\infty (e^{izx}-1)k(x)x^{-1}dx + i\gamma_0 z\big]$ with $\int_0^1 k(x)dx < \infty$ and $\gamma_0 < 0$, then $\{X_t\}$ is recurrent.*

(iv) *If $\widehat{\mu}(z) = \exp\big[\int_0^\infty (e^{izx}-1-izx1_{(0,1]}(x))k(x)x^{-1}dx + i\gamma z\big]$ with $\int_0^1 k(x)dx = \infty$, then $\{X_t\}$ is recurrent.*

The recurrence-transience classification of selfsimilar additive processes on $\mathbf{R}$ and on $\mathbf{R}^2$ is an open problem. An interesting example of selfsimilar additive process is the Brownian escape process of Getoor [2].

3. Remarks on generalizations

The notion of selfsimilarity is extended in a few ways: broad-sense selfsimilar, semi-selfsimilar, and broad-sense semi-selfsimilar.

A process $\{X_t\}$ on $\mathbf{R}^d$ is called *broad-sense selfsimilar* if, for any $a > 0$, there are $b > 0$ and a function $c(t)$ such that

$$\{X_{at}\} \stackrel{\mathrm{d}}{=} \{bX_t + c(t)\}. \tag{3.1}$$

A Lévy process is broad-sense selfsimilar if and only if it is a stable process. The class of stable processes is richer than that of strictly stable processes. However, broad-sense selfsimilar additive processes are reducible to selfsimilar additive processes as follows.

Theorem 3.1 (Sato [7]). *If $\{X_t\}$ is a broad-sense selfsimilar additive process on $\mathbf{R}^d$, then, there is a function $k(t)$ such that $\{X_t - k(t)\}$ is selfsimilar additive process.*

Note that if $\{X_t\}$ in the theorem above is a broad-sense selfsimilar Lévy process, the process $\{X_t - k(t)\}$ is not always a Lévy process, since $k(t)$ may not be linear.

A process $\{X_t\}$ on $\mathbf{R}^d$ is called *semi-selfsimilar* if, for *some* $a > 1$, there is $b > 0$ such that (0.5) holds. This a is called an *epoch*. Epochs are not unique, but, if $\{X_t\}$ is nonzero, semi-selfsimilar, stochastically continuous and $X_0 = 0$, then there is $H > 0$, called the *exponent*, such that $b = a^H$ for all epochs a. The notions are introduced in Maejima and Sato [4]. A Lévy process is semi-selfsimilar if and only if it is a strictly semi-stable process. Strictly semi-stable distributions are studied by Lévy himself together with stable. Semi-selfsimilar additive processes are connected to semi-selfdecomposable distributions, which are defined by Maejima and Naito [3] in the following way. A probability measure μ on $\mathbf{R}^d$ is *semi-selfdecomposable* if, for *some* $b > 1$, there is an infinitely divisible distribution ρ_b satisfying (0.6). This b is called a *span*.

Theorem 3.2 (Maejima and Sato [4]). *If $\{X_t\}$ is a nonzero semi-selfsimilar additive process on $\mathbf{R}^d$ with exponent H and with a as an epoch, then, for any $t \geq 0$, $\mathcal{L}(X_t)$ is semi-selfdecomposable having a^H as a span. Conversely, if μ is semi-selfdecomposable with b as a span and if μ is not the unit mass at 0, then, for any $H > 0$, there is a semi-selfsimilar additive process $\{X_t\}$ having H as the exponent and $b^{1/H}$ as an epoch such that $\mathcal{L}(X_1) = \mu$. This $\{X_t\}$ is not determined by μ and H uniquely in law. Given $H > 0$, it is determined uniquely in law by the system $\{\mathcal{L}(X_t) \colon 1 \leq t < b^{1/H}\}$.*

Is there the dichotomy of recurrence and transience for semi-selfsimilar additive processes? We know that the answer is affirmative for $d = 1$ (Sato and Yamamuro [10]). We do not know the answer for $d \geq 2$.

A process $\{X_t\}$ on $\mathbf{R}^d$ is called *broad-sense semi-selfsimilar* if there are $a > 1$, $b > 0$, and a function $c(t)$ satisfying (3.1). If $\{X_t\}$ is nontrivial, broad-sense semi-selfsimilar, stochastically continuous and $X_0 = 0$, then there is $H > 0$ such that $b = a^H$ whenever (3.1) is satisfied.

Theorem 3.3 (Maejima and Sato [4]). *Theorem* 3.1 *remains true if "self-similar" in the assumption and the conclusion is replaced by "semi-selfsimilar".*

References

[1] K. L. Chung and W. H. Fuchs (1951) On the distribution of values of sums of random variables, *Mem. Amer. Math. Soc.* **6**, 1–12.

[2] R. K. Getoor (1979) The Brownian escape process, *Ann. Probab.* **7**, 864–867.

[3] M. Maejima and Y. Naito (1998) Semi-selfdecomposable distributions and a new class of limit theorems, *Probab. Theory Related Fields,* **112**, 13–31.

[4] M. Maejima and K. Sato (1998) Semi-selfsimilar processes, *J. Theor. Probab.,* to appear.

[5] D. Ornstein (1969) Random walks, I and II, *Trans. Amer. Math. Soc.* **138**, 1–43 and 45–60.

[6] S. C. Port and C. J. Stone (1971) Infinitely divisible processes and their potential theory, I and II, *Ann. Inst. Fourier* **21**, Fasc. 2, 157–275 and Fasc. 4, 179–265.

[7] K. Sato (1991) Self-similar processes with independent increments, *Probab. Theory Related Fields* **89**, 285–300.

[8] K. Sato (1996) Criteria of weak and strong transience for Lévy processes, *Probability Theory and Mathematical Statistics, Proc. Seventh Japan-Russia Symp.* (eds. S. Watanabe et al., World Scientific, Singapore), 438–449.

[9] K. Sato (1997) Time evolution of Lévy processes, *Trends in Probability and Related Analysis, Proc. SAP '96* (eds. N. Kono and N.-R. Shieh, World Scientific, Singapore), 35–82.

[10] K. Sato and K. Yamamuro (1998) On selfsimilar and semi-selfsimilar processes with independent increments, *J. Korean Math. Soc.* **35**, 207–224.

[11] L. A. Shepp (1962) Symmetric random walk, *Trans. Amer. Math. Soc.* **104**, 144–153.

[12] L. A. Shepp (1964) Recurrent random walks with arbitrary large steps, *Bull. Amer. Math. Soc.* **70**, 540–542.

[13] F. Spitzer (1964) *Principles of random walk,* Van Nostrand, Princeton.

[14] C. J. Stone (1969) On the potential operator for one-dimensional recurrent random walks, *Trans. Amer. Math. Soc.* **136**, 413–426.

[15] K. Yamamuro (1998) Transience conditions for self-similar additive processes, *J. Math. Soc. Japan,* to appear.

Hachiman-yama 1101-5-103, Tenpaku-ku, Nagoya, 468-0074, Japan

E-mail address: `ksato@info.human.nagoya-u.ac.jp`

Lévy processes and function spaces

René L. Schilling

The Nottingham Trent University, Mathematics Department, Burton Street
Nottingham NG1 4BU, U.K., email: rls@maths.ntu.ac.uk

With the exception of Brownian motions, Lévy processes are typical examples of jump processes. Many aspects of Lévy processes, their potential theory, and their path behaviour are well-known. Lévy processes have discontinuous, càdlàg paths, the distribution of the jumps and their semimartingale decomposition are well understood. But *how càdlàg* are the paths really? The question of the *smoothness* of the *per se* discontinuous paths was only recently asked. The first results in this direction were the papers by Z. Ciesielski, G. Kerkyacharian, and B. Roynette [6] (for stable and Gaussian processes) and by B. Roynette [12] (for Brownian motions). These authors proposed to use a tool familiar to most analysts who have to *measure smoothness*: function spaces, in particular *Besov spaces*. Here we give a full characterization of smoothness of Lévy processes in terms of Besov and Triebel-Lizorkin spaces, based on the papers [14, 15, 16]

1. Some notions from probability theory. A *Lévy process* $\{X_t\}_{t\geq 0}$ on $\mathbb{R}^n$ is a stochastic process with stationary and independent increments that is continuous in probability. Lévy processes can be described in terms of their Fourier transforms,

$$\mathbb{E}^0\left(e^{i\langle \xi, X_t\rangle}\right) = e^{-t\psi(\xi)}, \qquad t\geq 0,\ \xi\in\mathbb{R}^n, \tag{1}$$

where the *characteristic exponent* $\psi:\mathbb{R}^n\to\mathbb{C}$ is given by the following Lévy-Khinchine-formula

$$\psi(\xi) = i\langle \ell,\xi\rangle + \langle Q\xi,\xi\rangle + \int_{y\neq 0}\left(1-e^{-i\langle \xi,y\rangle} - \frac{i\langle \xi,y\rangle}{1+|y|^2}\right)\nu(dy). \tag{2}$$

The triplet (ℓ, Q, ν) consisting of a vector $\ell\in\mathbb{R}^n$, a symmetric, positive semidefinite matrix $Q\in\mathbb{R}^{n\times n}$, and the Lévy measure ν on $\mathbb{R}^n\setminus\{0\}$, $\int_{y\neq 0}|y|^2/(1+|y|^2)\,\nu(dy)<\infty$, gives a one-to-one characterization of all possible characteristic exponents.
Since ψ gives a full characterization of the process $\{X_t\}_{t\geq 0}$, it contains all relevant information on the process, and many (probabilistic) properties of the process should be available through the (Fourier-analytic) study of the characteristic exponent. Such reasoning was the beginning of a success story dating back to S. Bochner [3, 4]. For an up-to-date account we refer to the books by J. Bertoin [2] and K. Sato [13]. For our purposes we recall the definition of certain *indices* that take into account the growth behaviour of ψ at infinity,

$$\beta = \beta_\infty = \inf\left\{\lambda>0\ :\ \lim_{|\xi|\to\infty}\frac{\psi(\xi)}{|\xi|^\lambda}=0\right\} \tag{3}$$

and at the origin

$$\beta_0 = \sup\left\{\lambda\geq 0\ :\ \lim_{|\xi|\to 0}\frac{\psi(\xi)}{|\xi|^\lambda}=0\right\};\qquad \delta_0 = \sup\left\{\lambda\geq 0\ :\ \liminf_{|\xi|\to 0}\frac{\psi(\xi)}{|\xi|^\lambda}=0\right\}. \tag{4}$$

The index β_∞ was introduced by R. Blumenthal and R. Getoor in their 1961 paper [5] in order to study Hölder properties of the paths of X_t in small time. Twenty years later, B. Pruitt proposed to study β_0 and δ_0. (His original definition was in terms of the triplet (ℓ, Q, ν). The above form can be found, e.g., in [15]). Provided that there is no dominating drift, he showed that

$$\lim_{t\to\infty} t^{-1/\lambda} \sup_{s\le t} |X_t - x| = 0 \text{ or } \infty, \qquad \text{a.s. } (\mathbb{P}^x) \tag{5}$$

according to $\lambda < \beta_0$ or $\lambda > \delta_0$. Note that this implies the bound

$$\sup_{s\le t} |X_s - x| \le c_\bullet\, (1+t)^{1/\lambda}, \quad \lambda < \beta_0, \tag{6}$$

with an a.s. finite random variable $c = c(\omega)$ depending on λ.
The limits (5) can be easily proved by a Borel-Cantelli argument and estimates of the type

$$\mathbb{P}^x(\sup_{s\le t} |X_s - x| > R) \;\le\; c_n\, t \sup_{|\epsilon|\le 1} \left|\psi\left(\frac{\epsilon}{R}\right)\right| \tag{7}$$

$$\mathbb{P}^x(\sup_{s\le t} |X_s - x| \le R) \;\le\; c_\kappa \frac{1}{t \sup_{|\epsilon|\le 1} \left|\psi\left(\frac{\epsilon}{4R\kappa}\right)\right|} \tag{8}$$

where in (8) we have to assume that $|\operatorname{Im}\psi(\xi)| \le \kappa \operatorname{Re}\psi(\xi)$, i.e. that there is no dominating deterministic drift. These estimates are implicit in [11], see also [15].

2. Function spaces. Function spaces are widely used in analysis, especially in the analysis of partial differential equations, in order to describe mapping properties of operators and smoothness of functions. Usually, smoothness of a function means *differentiability* or at least *continuity* properties which can be measured in scales of Sobolev/Bessel-potential spaces W_p^s, H_p^s or Hölder/Zygmund spaces $C^s, \mathcal{C}^s$. In the 1960s and 1970s *Besov-* B_{pq}^s and *Triebel-Lizorkin spaces* F_{pq}^s were developed that made it possible to give a unified and systematic approach to the above classical spaces. However, these new spaces have one additional feature: their scales go beyond continuity and allow us to describe the "smoothness" of, say, càdlàg functions. This is the application we have in mind here. In order to keep technicalities at a minimum, we restrict ourselves to Besov spaces.
Following H. Triebel [17, 18] we give a Fourier-analytic description of Besov spaces. Let $\{\phi_j\}_{j\in\mathbb{N}}$ be a *smooth, dyadic partition of unity*, i.e., $\phi_j \in C^\infty(\mathbb{R}^n)$, $\operatorname{supp}\phi_0 \subset \{\xi : |\xi| \le 2\}$, $\operatorname{supp}\phi_j \subset \{\xi : 2^j \le |\xi| \le 2^{j+1}\}$, $\sup_{j,\xi} |2^{j|\alpha|} D^\alpha \phi_j| < \infty$ for all $\alpha \in \mathbb{N}_0^n$ and $\sum_j \phi_j \equiv 1$. Denote by Fu the Fourier transform of u. For $0 < p, q \le \infty$, $s \in \mathbb{R}$ we set

$$\|u|B_{pq}^s(\mathbb{R}^n)\| := \left(\sum_{j=0}^\infty 2^{jsq} \left\|F^{-1}(\phi_j F u)\right\|_{L^p(\mathbb{R}^n)}^q\right)^{1/q} \tag{9}$$

(with the usual modification if $p = \infty$ and/or $q = \infty$). Clearly, for $u \in \mathcal{S}'(\mathbb{R}^n)$ (9) is well-defined and—in case it is finite—independent of the particular choice of the partition $\{\phi_j\}_{j\in\mathbb{N}}$.

Definition. Let $0 < p, q \le \infty$ and $s \in \mathbb{R}$. The *Besov space* $B^s_{pq}(\mathbb{R}^n)$ consists of all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that (9) is finite.

Equipped with the norms (9) Besov spaces are Banach spaces (quasi-Banach spaces if $p < 1$ or $q < 1$) that arise naturally as real interpolation spaces between the classical L^p-Sobolev spaces $W^m_p(\mathbb{R}^n)$. It is, therefore, not surprising that we have the following alternative description of Besov norms,

$$\|u|B^s_{pq}(\mathbb{R}^n)\| \sim \|u\|_{L^p(\mathbb{R}^n)} + \left(\int_0^\theta r^{-sq} \sup_{0<|h|\le r} \|\Delta^M_h u\|^q_{L^p(\mathbb{R}^n)} \frac{dr}{r} \right)^{1/q} \tag{10}$$

(with the usual modification for $q = \infty$) whenever $n(\frac{1}{p} - 1)_+ < s < M$, $M \in \mathbb{N}$ suitably chosen, where $\Delta^M_h u$ is the M-fold repeated difference of step h, $\Delta_h u(x) = u(x+h) - u(x)$, and $\theta > 0$ is arbitrary. The $\sim$ in (10) indicates that (10) and (9) are equivalent (quasi-) norms for the admissible scope of s and M.

Remarks. **(A)** Replacing $L^p(\mathbb{R}^n) = L^p(\mathbb{R}^n; dx)$ through the space $L^p(\mathbb{R}^n; \rho_\alpha(x)\, dx)$ with weight function $\rho_\alpha(x) = (1 + |x|^2)^{\alpha/2}$ gives polynomially *weighted Besov spaces* $B^s_{pq}(\mathbb{R}^n; \rho_\alpha)$. **(B)** A function u is said to be *locally* in $B^s_{pq}(\mathbb{R}^n; \rho_\alpha)$, if for some test-function $\phi \in C^\infty_c(\mathbb{R}^n)$ the product $\phi u \in B^s_{pq}(\mathbb{R}^n)$. **(C)** Besov spaces give a unified approach to various scales of function spaces. If, for example, $p \ge 1$ holds, then $B^s_{pp} = W^s_p$ ($s \ne$ integer, Sobolev-Slobodeckij scale "W"), $B^s_{2\,2} = H^s_2 = W^s_2$ (Bessel-potential or Liouville scale "H"), or $B^s_{\infty\infty} = \mathcal{C}^s$ (Hölder-Zygmund scale). **(D)** One has the following analogue of the Sobolev embedding theorem: $B^{s+n/p}_{pq}(\mathbb{R}^n) \subset C^s(\mathbb{R}^n)$. In particular,

$$B^t_{pq}(\mathbb{R}^n) \subset C(\mathbb{R}^n) \qquad \text{for all} \qquad t > \frac{n}{p}. \tag{11}$$

As usual "$\subset$" means continuous embedding. **(E)** The *smoothness* index s dominates the other two indices p, q in the following sense

$$B^{s+\epsilon}_{pq}(\mathbb{R}^n) \subset B^s_{pr}(\mathbb{R}^n) \qquad \text{for all} \qquad \epsilon > 0, \quad 0 < p, q, r \le \infty. \tag{12}$$

Lemma 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a compactly supported càdlàg function and write $\Delta f(x) := f(x) - \lim_{y \uparrow x} f(y)$ for its jump at x. Then*

$$\Big(\sum_{x \in \operatorname{supp} f} |\Delta f(x)|^p \Big)^{1/p} \le C_p \, \|f|B^{1/p}_{p\infty}(\mathbb{R})\| \tag{13}$$

holds true for all $0 < p < \infty$ with some $C_p > 0$ given by the norm-equivalence (10).

Proof. Choose $M = \lfloor \frac{1}{p} \rfloor + 1$ and observe that $\Delta^M_h f(x) = \sum_{k=0}^M (-1)^{M-k} \binom{M}{k} f(x + kh)$. Therefore, we get for $h < 0$

$$\lim_{h \uparrow 0} \Delta^M_h f(x) = (-1)^M f(x) + \lim_{h \uparrow 0} \sum_{k=1}^M (-1)^{M-k} \binom{M}{k} f(x + kh) = (-1)^M (f(x) - f(x-)).$$

Since f is càdlàg and compactly supported, there are for a fixed $0 < \epsilon < 1$ at most $N(\epsilon) \in \mathbb{N}_0$ jumps of size $|\Delta f(x)| > \epsilon$; denote the corresponding sites by $x_1, x_2, \ldots, x_{N(\epsilon)}$ and set $U_t(x_j) = \left[x_j, x_j + \frac{t}{2}\right)$, $t < t_0 < \theta$ so small that $f|U_t(x_j)$ is right continuous at x_j. Together with (10) we have

$$\begin{aligned} C_p \, \|f|B^{1/p}_{p\infty}(\mathbb{R})\|^p &\geq \sup_{0<t\leq\theta} \left(\frac{1}{t} \sup_{0<|h|\leq t} \int_{\mathbb{R}} |\Delta_h^M f(x)|^p \, dx \right) \\ &\geq \sup_{0<t\leq t_0/M} \left(\frac{1}{t} \sup_{-t<h\leq 0} \sum_{j=1}^{N(\epsilon)} \int_{U_t(x_j)} |\Delta_h^M f(x)|^p \, dx \right) \\ &\geq \sup_{0<t\leq t_0/M} \left(\frac{1}{2} \sum_{j=1}^{N(\epsilon)} \inf_{x\in U_t(x_j)} |\Delta_{-t}^M f(x)|^p \right) \\ &\geq \frac{1}{2} \sum_{j=1}^{N(\epsilon)} \liminf_{t\to 0} \inf_{x\in U_t(x_j)} |\Delta_{-t}^M f(x)|^p \end{aligned}$$

where the last estimate follows from Fatou's lemma. By definition, $x \in U_t(x_j)$ satisfies $x \geq x_j$, thus $f(x) \to f(x_j)$ as $x \to x_j$, whereas $x - kt \leq x_j + \frac{t}{2} - kt \leq x_j - \frac{t}{2} < x_j$, for all $k = 1, 2, \ldots, M$. Hence $f(x - kt) \to f(x_j-)$ as $t \to 0$ (and, a fortiori, $x \to x_j$). Therefore,

$$\|f|B^{1/p}_{p\infty}\|^p \geq c'_p \sum_{j=1}^{N(\epsilon)} |f(x_j) - f(x_j-)|^p$$

and the assertion follows as $\epsilon \to 0$.

3. Lévy processes with paths in Besov spaces. We can now turn back to the question: *How smooth are the paths of Lévy processes*? An answer to this question would be to identify the sample paths as elements of certain Besov spaces. Since the paths of a Lévy process grow at a polynomial rate, there is no chance that $t \mapsto X_{t\vee 0} \in B^s_{pq}(\mathbb{R})$ *globally*. The natural spaces are, therefore, either localized spaces $B^{s,\mathrm{loc}}_{pq}(\mathbb{R})$ or weighted spaces. As we have already observed,

$$\sup_{s\leq t} |X_s(\omega) - x| \leq c(\omega) \, (1 + t^2)^{1/(2\lambda)}, \qquad \text{a.s. } \mathbb{P}^x \text{ for } \quad \lambda < \beta_0,$$

and the natural spaces to look at are polynomially weighted spaces.
To show the actual embedding, we have to prove that the norms (9) are finite. For this, set $\Omega_k := \{\omega \, : \, c(\omega) < k\}$ and observe that $\lim_{k\to\infty} \mathbb{P}^x(\Omega_k) = 1$. The major technical step is now contained in the following lemma.

Lemma 2. ([16, Lemma 2.5]) *Let $\{X_t\}_{t\geq 0}$ be as above. Then*

$$\int_{\Omega_k} \sup_{|h|\leq r} \left| \left[\Delta_h^M X_{\bullet\vee 0}(\omega)\right](t) \right|^p \mathbb{P}^x(d\omega) \leq C \, r \, (1 + t^2)^{p/(2\lambda)} \tag{14}$$

for $\lambda < \beta_0$ and $p > \beta_\infty$. The constant C depends on $p, \beta_\infty, \beta_0, k, \lambda, M$.

The proof of the above lemma uses essentially the estimate (7).
We are now in a position to state our main result. See the last section for further comments.

Theorem ([14] and [16]). *Let $\{X_t\}_{t\geq 0}$ be a Lévy process with characteristic exponent ψ and indices β_0, β_∞ as given above, and let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$.*

(A) $\{t \mapsto X_{t\vee 0}\} \in L^p(\mathbb{R}; (1+t^2)^{-\mu/2})$ *almost surely* $\mathbb{P}^x$ *for all* $\mu > \frac{1}{\beta_0} + \frac{1}{p}$.

(B) $\{t \mapsto X_{t\vee 0}\} \in B^s_{pq}(\mathbb{R}; (1+t^2)^{-\mu/2})$ *almost surely* $\mathbb{P}^x$ *for all* $\mu > \frac{1}{\beta_0} + \frac{1}{p}$, $s > (\frac{1}{p} - 1)_+$ *and either* $q < \infty, s \max\{p, q, \beta_\infty\} < 1$ *or* $q = \infty, s \max\{p, \beta_\infty\} < 1$.

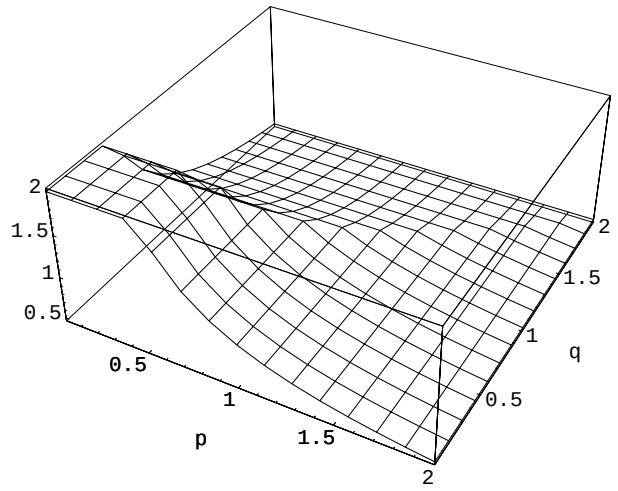
(C) $\{t \mapsto X_{t\vee 0}\} \in B^{s,\mathrm{loc}}_{pq}(\mathbb{R})$ *almost surely* $\mathbb{P}^x$ *for all* $s > (\frac{1}{p} - 1)_+$ *and either* $q < \infty, s \max\{p, q, \beta_\infty\} < 1$ *or* $q = \infty, s \max\{p, \beta_\infty\} < 1$.

(D) $\{t \mapsto X_{t\vee 0}\} \notin B^{s,\mathrm{loc}}_{pq}(\mathbb{R})$ *almost surely if either* $sp > 1$ *or* $sp = 1$ *and* $0 < q \leq 1$.

In order to prove (B) and (C) we have to check the finiteness of the norms (9). Having established Lemma 2 above, this is a straightforward calculation whenever $q \neq \infty$. The case $q = \infty$ is special, as it requires a Borel-Cantelli-trick. See [16] for details. The assertion (D) follows from embedding considerations: for $sp > 1$ Besov spaces are contained in spaces of *continuous* functions, hence cannot contain Lévy paths that are with probability 1 jump functions. The borderline case $ps = 1$ is, in general, difficult to answer. In the presence of scaling properties, e.g., for stable processes, Ciesielski, Kerkyacharian, and Roynette [6] showed a non-embedding result for $q = \infty$ and $\beta_\infty > 1$.

Corollary 1. *The assertions of the above Theorem hold for all $s \geq 0$ without the restriction $s > (\frac{1}{p} - 1)_+$.*

Proof. Use the fact that $B^s_{pq} \subset B^t_{pr}$ if $s > t$.



The graph shows the parameter s as a function of p and q. The area below the graph $s = \max\{p, q, \beta_\infty\}^{-1}$ represents the region of admissible parameters (s, p, q) for embedding into $B^{s,\mathrm{loc}}_{pq}$ with $q \neq \infty$.

For this picture we chose $\beta_\infty = 0.50$, i.e., the plateau occurs at $1/\beta_\infty = 2.00$.

Corollary 2. *Let $\{X_t\}_{t\geq 0}$ be as in the Theorem. Then*

$$\{t \mapsto X_{t\vee 0}(\omega)\} \in B^{1/p,\mathrm{loc}}_{p\infty}(\mathbb{R}) \qquad \text{almost surely } \mathbb{P}^x \text{ for all} \qquad p > \beta_\infty.$$

Proof. We refer to the proof in [16, Corollary 4.3, Case 1]. If we have the *strict* inequality $p > \beta_\infty$ and $q = \infty$, $s = 1/p$, we can find a $0 < \theta < 1$ such that $p\theta > \beta_\infty$ and $sp\theta = \theta < 1$. We can now proceed along the lines of the proof in [16]: instead of using Jensen's inequality for the finite measure $\langle t\rangle^{-p\mu}\,dt$, we use Jensen's inequality for the concave function $x \mapsto |x|^\theta$, $\theta < 1$, and the expectation $\mathbb{E}^x$. This yields $\mathbb{E}^x\left(|Y|^\theta\right) \leq \left(\mathbb{E}^x(|Y|)\right)^\theta$. The other arguments in [14] need not be changed.

Corollary 3. *Let $\{X_t\}_{t\geq 0}$ be as in the Theorem and denote by $\Delta X_t := X_t - X_{t-}$ the jump at time t. Then*

$$\sum_{t\leq 1} |\Delta X_t|^p < \infty \qquad \textit{almost surely } \mathbb{P}^x \textit{ for all} \qquad p > \beta_\infty.$$

Proof. Combine Corollary 2 and estimate (13)

Remarks. (A) The above Theorem and its Corollaries generalize to all Feller processes that are generated by pseudo-differential operators cf. [14, 16] and the notes below. **(B)** The assertions of the Theorem and Corollary 1 remain valid for (weighted) spaces of Triebel-Lizorkin type F^s_{pq}, cf. [16].

4. Concluding remarks. Some remarks on the development of the above theorem seem to be in order. Originally, assertion (C) of the theorem was proved by Ciesielski, Kerkyacharian, Roynette [6] both for certain Gaussian and symmetric α-stable Lévy processes with index $\alpha > 1$ and for $q = \infty$. Their method used essentially the scaling property of stable processes and an atomic decomposition of Besov spaces. Subsequently, V. Herren generalized this result (using the same technique) to any Lévy process with index $\beta_\infty > 1$. In [14], the present author studied a class of Feller processes that are generated by pseudo-differential operators

$$-p(x,D)u(x) = -(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\langle\xi,x\rangle}p(x,\xi)Fu(\xi)\,d\xi, \qquad u \in C_c^\infty(\mathbb{R}^n), \tag{15}$$

where $\xi \mapsto p(x,\xi)$ is given by a Lévy-Khinchine formula (depending on the parameter x)—cf. the talk of N. Jacob or [9]. (It is known that *every* Feller process whose generator has a domain containing the test functions $C_c^\infty(\mathbb{R}^n)$ is already of this type, cf. [7]; Lévy processes are exactly those processes where $p(x,\xi) = \psi(\xi)$, i.e., independent of x.) In the paper [14] assertion (C) is proved for this class of processes; as a by-product, (C) could be shown to hold for *all* Lévy processes, and the restriction $\beta_\infty > 1$ of [8] could be removed.

In [15, 16] we proved the global embedding for *Feller processes* (and, in particular, Lévy processes) as stated in the theorem above. The approach used in these papers is basically the one sketched here. The advantage of this approach is that it is more flexible and allows us to include also spaces of Triebel-Lizorkin type.

In the recently published paper [1], Orlicz spaces were used to describe the paths of Lévy processes, see also [10].

Acknowledgement. I am grateful to my colleague D. Kaye, Nottingham, for producing the 3-d plot included in this note.

References

[1] Braverman, M. and G. Samorodnitsky, Symmetric infinitely divisible processes with sample paths in Orlicz spaces and absolute continuity of infinitely divisible processes, *Stoch. Proc. Appl.* **78** (1998), 1–26.

[2] Bertoin, J., *Lévy Processes*, Cambridge University Press, Cambridge Tracts in Math. vol. **121**, Cambridge 1996.

[3] Bochner, S., Stochastic Processes, *Ann. Math.* **48** (1947), 1014–1061.

[4] Bochner, S., *Harmonic Analysis and the Theory of Probability*, University of California Press, California Monographs in Math. Sci., Berkeley, CA 1955.

[5] Blumenthal, R. M. and R. K. Getoor, Sample Functions of Stochastic Processes with Stationary Independent Increments, *J. Math. Mech.* **10** (1961), 493–516.

[6] Ciesielski, Z., Kerkyacharian, G. et B. Roynette, Quelques espaces fonctionnels associés à des processus gaussiens, *Studia Math.* **107** (1993), 171–204.

[7] Courrège, Ph., *Sur la forme intégro-différentielle des opérateurs de C_K^∞ dans C satisfaisant au principe du maximum*, Sém. Théorie du Potentiel (1965/66) exposé 2, 38 pp.

[8] Herren, V., Lévy type processes and Besov spaces, *Potential Analysis* **7** (1997), 689–704.

[9] Jacob, N., *Pseudo-differential operators and Markov processes*, Akademie Verlag, Mathematical Research vol. **94**, Berlin 1996.

[10] Norvaiša, R. and G. Samorodnitsky, Stable processes with sample paths in Orlicz spaces, *Ann. Probab.* **22** (1994), 1904–1929.

[11] Pruitt, W. E., The growth of random walks and Lévy processes, *Ann. Probab.* **9** (1981), 948–956.

[12] Roynette, B., Mouvement brownien et espaces de Besov, *Stoch. and Stoch. Rep.* **43** (1993), 221–260.

[13] Sato, K., *Additive Processes*, monograph to be published by Cambridge University Press, 1999.

[14] Schilling, R. L., On Feller processes with sample paths in Besov spaces, *Math. Ann.*, **309** (1997), 663–675.

[15] Schilling, R. L., Growth and Hölder conditions for the sample paths of Feller processes, *Probab. Theor. Rel. Fields* **112** (1998), 565–611.

[16] Schilling, R. L., Function Spaces as Path Spaces of Feller Processes, to appear in *Math. Nachr.*

[17] Triebel, H., *Theory of Function Spaces*, Birkhäuser, Monographs in Mathematics vol. **78**, Basel 1983.

[18] Triebel, H., *Theory of Function Spaces II*, Birkhäuser, Monographs in Mathematics vol. **84**, Basel 1992.

DISTRIBUTION OF THE FIRST LADDER HEIGHT OF A STATIONARY RISK PROCESS PERTURBED BY LÉVY MOTION

HANSPETER SCHMIDLI

Consider a stationary and ergodic marked point process (smpp) $\mathcal{M} = (\sigma_i, U_i, M_i)$ on a probability space $(\Omega, \mathcal{F}, P)$ with event times $\cdots < \sigma_{-1} < \sigma_0 \le 0 < \sigma_1 < \sigma_2 < \cdots$ and marks $(U_i, M_i) \in (0,\infty) \times E$. Here E is a Polish space and M_i is interpreted as an environmental variable. Let

$$N_t = \begin{cases} \sum_{i=1}^{\infty} \mathbb{1}_{0<\sigma_i \le t}, & \text{if } t > 0, \\ -\sum_{i=0}^{\infty} \mathbb{1}_{t<\sigma_{-i} \le 0}, & \text{if } t \le 0. \end{cases}$$

We call $\mathcal{M}$ a compound Poisson model if $E = \{0\}$, i.e. there are no environmental marks, and $\ldots, \sigma_{-1} - \sigma_{-2}, \sigma_0 - \sigma_{-1}, -\sigma_0, \sigma_1, \sigma_2 - \sigma_1, \ldots$ are iid exponentially distributed random variables and (U_i) is an iid sequence of positive random variables independent of (σ_i). In this paper we consider the process

$$S_t = \sum_{i=1}^{N_t} U_i - t + \eta B_t \tag{1}$$

where $\eta > 0$ is some constant and (B_t) is a zero-mean Lévy process with no downward jumps.

Let $\lambda = E[N_1]$ and $\mu = \lambda^{-1} E[\sum_{i=1}^{N_1} U_i]$ be be the intensity of the claim arrivals and the mean value of a typical claim. We assume the net profit condition $\rho = \lambda\mu \le 1$. This implies that $\liminf_{t\to\infty} S_t = -\infty$. Let $m_t = \sup\{S_s : 0 \le s < t\}$, $\tau_+ = \inf\{\sigma_k : k > 0,\ S_{\sigma_k} > m_{\sigma_k}\}$, $L_c = m_{\tau_+}$ and $L_d = S_{\tau_+} - L_c$. τ_+ is then the first time where a jump of the unperturbed model ($\eta = 0$) leads to a new maximum of the process (S_t). We will call τ_+ the first (modified) ladder epoch. Let G denote the distribution function with density $\mu^{-1} P[U_i > \cdot]$ and by H the distribution function of $\sup(\eta B_t - t : t \ge 0)$. Note that G is also the distribution of L_d in the unperturbed case. If (B_t) is a Brownian motion and $\mathcal{M}$ is a compound Poisson model Dufresne and Gerber (1991) showed and that $P[\tau_+ < \infty] = \rho$ and that L_c has distribution H. Moreover, given $\tau_+ < \infty$, L_c and L_d are independent with distribution functions H and G, respectively. Hence it follows

that

$$P\Big[\sup_{t\geq 0} S_t \leq u\Big] = (1-\rho)\sum_{n=0}^{\infty} \rho^n (G^{*n} * H^{*(n+1)})(u)\,. \tag{2}$$

Furrer (1998) proved that (2) also holds for perturbation by an α-stable Lévy motion with no downward jumps as long as $\mathcal{M}$ is a compound Poisson process. His approach did, however, not show whether or not H and G still can be interpreted as the distribution functions of L_c and L_d. We will show that, for any zero-mean Lévy process with no downward jumps and any smmp $P[\tau_+ < \infty] = \rho$ and that L_c has distribution H. Given $\tau_+ < \infty$, L_c and L_d are independent with distribution functions H and G, respectively. This again proves the result of Dufresne and Gerber (1991) and Furrer (1998) in the case of compound Poisson process, because this process always is in its stationary state after a jump.

Department of Theoretical Statistics, Departments of Mathematical Sciences, University of Aarhus, DK-8000 Aarhus C, Denmark

E-mail address: schmidli@imf.au.dk

Incorporation of a Leverage Effect in a Stochastic Volatility Model

Ole E. Barndorff-Nielsen
The Centre for Mathematical Physics and Stochastics (MaPhySto), University of Aarhus, Ny Munkegade, DK-8000 Aarhus C, Denmark. *
oebn@imf.au.dk

and Neil Shephard
Nuffield College, Oxford OX1 1NF, UK.
neil.shephard@nuf.ox.ac.uk

1 Introduction

A class of stochastic volatility models, discussed in Barndorff-Nielsen and Shephard (1998), incorporates a number of the well-established common features of observational series of financial assets, in particular series of stock prices and of exchange rates. One fairly important such stylized feature, the so called leverage effect, was however not covered. This is where negative return sequences are associated with increases in the volatility of stock returns. Such asymmetries are not usually observed for exchange rates.

The leverage effect was studied in some early work by Black (1976), while it motivated the introduction of the EGARCH model of Nelson (1991) and the threshold ARCH model of Glosten, Jagannathan, and Runkle (1993). An economic theory behind such effects is discussed by Campbell and Kyle (1993).

In the present note we indicate a way of extending the type of models referred to so that they reflect the leverage effect, and we calculate a few of the consequences. Only the simplest, one-dimensional, version of the models will be considered here.

2 Incorporating leverage

2.1 Model construction

As a model for the log price process of, for instance, a stock we consider a stochastic process $x^*(t)$, $0 \leq t < \infty$, defined by a stochastic differential equation

$$\mathrm{d}x^*(t) = \left\{\mu + \beta\sigma^2(t)\right\}\mathrm{d}t + \sigma(t)\mathrm{d}w(t) + \rho\mathrm{d}\bar{z}(\lambda t) \tag{1}$$

where w is the Wiener process, $\bar{z}$ is the centered version of a Lévy process $\grave{z}$ (that is $\bar{z}(t) = \grave{z}(t) - \mathrm{E}\grave{z}(t)$), and σ is a stationary process defined, also in terms of $\grave{z}$, by the stochastic differential equation

$$\mathrm{d}\sigma^2(t) = -\lambda\sigma^2(t)\mathrm{d}t + \mathrm{d}\grave{z}(\lambda t) \tag{2}$$

with $0 < \lambda < \infty$. This volatility process is thus of Ornstein-Uhlenbeck (OU) type although it will not have Gaussian increments. The process $\grave{z}$ is a homogeneous Lévy process (so it has independent and stationary increments) with positive increments (also termed a subordinator). As it is the driving process for the OU process we call it a background driving Lévy process

*MaPhySto is funded by the Danish National Research foundation

(BDLP). The filtration determined by w and $\grave{z}$ jointly will be denoted by $\mathcal{F} = \{\mathcal{F}_t : 0 \leq t < \infty\}$. Throughout we will assume $\mu = \beta = 0$ for simplicity of exposition as these terms raise no new issues.

Although we have focused on the simplest OU volatility process, our model extends to where volatility follows a weighted sum of independent Ornstein-Uhlenbeck processes with different persistence rates. That is

$$\sigma^2(t) = \sum_{j=1}^{m} w_j^* \sigma_j^2(t), \qquad \text{where} \qquad \sum_{j=1}^{m} w_j^* = 1,$$

with

$$\mathrm{d}\sigma_j^2(t) = -\lambda_j \sigma_j^2(t)\mathrm{d}t + \mathrm{d}\grave{z}_j(\lambda_j t),$$

where the $\{\grave{z}_j(t)\}$ are independent (not necessarily identically distributed) BDLPs. Hence some of the components of the volatility may represent short term variation in the process while others represent long term movements. In such a case we would have a process for the price of the type

$$\mathrm{d}x^*(t) = \left\{\mu + \beta\sigma^2(t)\right\}\mathrm{d}t + \sigma(t)\mathrm{d}w(t) + \sum_{j=1}^{m} \rho_j \mathrm{d}\bar{z}_j(\lambda_j t),$$

where the leverage effect could be different for the various components of volatility. As this type of extension will raise no new technical issues we will not deal with it in this note and instead focus on the case where $m = 1$ and so the model is made up of (1) and (2).

For $\rho = 0$ this setup reduces to the elemental version of the models studied in the earlier paper (Barndorff-Nielsen and Shephard (1998)). If ρ is negative a positive (infinitesimal) increment $\mathrm{d}\grave{z}(\lambda t)$ in the volatility process $\sigma^2(t)$ will have a negative effect on the stock price. This expresses, at least qualitatively, the stylized leverage effect — that negative returns are associated with increases in observed volatility. Notice however that there is no feedback from the x^* process to the volatility process σ^2 in our model - the innovations from $\grave{z}$ affects x^* and σ^2 simultaneously. Hence this model differs from the EGARCH and threshold ARCH models of leverage referenced above. However, it is in keeping with the non-symmetrical stochastic volatility models of leverage previously discussed in the literature (see, for example, the review in Ghysels, Harvey, and Renault (1996)). A typical example of that style of models is where

$$\mathrm{d}\log\sigma^2(t) = -\lambda\left\{\log\sigma^2(t) - \mu\right\}\mathrm{d}t + \varkappa\mathrm{d}s(t),$$

a geometric Gaussian Ornstein-Uhlenbeck process, whose increments of the standard Brownian motion $s(t)$ are correlated with those of $w(t)$. Our construction is mathematically more tractable.

2.2 Volatility and the BDLP

Denoting the cumulants and moments of $\grave{z}(1)$ (when they exist) respectively by $\grave{\kappa}_m$ and $\grave{\mu}_m$ $(m = 1, 2, ...)$ we have

$$\bar{z}(t) = \grave{z}(t) - t\grave{\kappa}_1 = \grave{z}(t) - t\xi \tag{3}$$

where for brevity we have written $\xi = \grave{\kappa}_1$. For some purposes it maybe helpful to note that $\bar{z}(t)$ is bounded from below and that the (marginal) cumulants of $\sigma^2(t)$ follow directly from the cumulants of $\grave{z}(1)$. In particular, if we write the cumulants of σ^2 as $\acute{\kappa}_m$ $(m = 1, 2, ...)$ then it follows (see the proof in Barndorff-Nielsen (1998)) that

$$\grave{\kappa}_m = m\acute{\kappa}_m, \qquad m = 1, 2, ...$$

It will be helpful later to establish the notation that the corresponding cumulant generating functions will be written as $\acute{k}(\theta) = \log\left[\mathrm{E}\exp\left\{\theta\sigma^2(t)\right\}\right]$ and $\grave{k}(\theta) = \log\left\{\mathrm{E}\exp\left(\theta\grave{z}(1)\right)\right\}$ for $\sigma^2(t)$

and $\grave{z}(1)$ respectively. Indeed they are related by the fundamental equality (Barndorff-Nielsen (1998))

$$\acute{k}(\theta) = \int_0^\infty \grave{k}(\theta e^{-s})\mathrm{d}s, \tag{4}$$

which can be reexpressed as

$$\grave{k}(\theta) = \theta \acute{k}'(\theta) \tag{5}$$

(where $\acute{k}'(\theta) = \theta \mathrm{d}\acute{k}(\theta)/\mathrm{d}\theta$). The common feature of this notation is the BDLP objects, $\grave{z}(1)$, have graves over them while the volatility itself , $\sigma^2(t)$, have acutes. This style of notation will be maintained throughout this note. Important special cases of this are

$$\begin{aligned} \grave{\kappa}_1 &= \mathrm{E}\{\sigma^2(t)\} = \acute{\kappa}_1 = \xi \quad \text{and} \\ \grave{\kappa}_2 &= 2\mathrm{Var}\{\sigma^2(t)\} = 2\acute{\kappa}_2 = 2\omega^2. \end{aligned}$$

Now, let

$$x_0^*(t) = \int_0^t \sigma(s)\mathrm{d}w(s), \tag{6}$$

which is the log-price process minus the leverage effect. The solution of the equation (1) is then

$$x^*(t) = x_0^*(t) + \rho \bar{z}(\lambda t) \tag{7}$$

2.3 Moments of returns

As in the previous paper, we shall determine some of the properties of the increments over time spans of length Δ of the model process x^*. Thus, let

$$\begin{aligned} y_n &= x^*(\Delta n) - x^*\left\{\Delta(n-1)\right\} \\ &= y_{0n} + \rho \bar{z}_n \end{aligned}$$

where

$$\bar{z}_n = \grave{z}(\lambda\Delta n) - \grave{z}\left\{\lambda\Delta(n-1)\right\} - \lambda\Delta\xi$$

and

$$y_{0n} = x_0^*(\Delta n) - x_0^*\left\{\Delta(n-1)\right\}. \tag{8}$$

The implication is that

$$y_n|\sigma_n^2, \bar{z}_n \sim N(\rho\bar{z}_n, \sigma_n^2), \tag{9}$$

where integrated volatility influences the distribution of returns through

$$\sigma_n^2 = \sigma^{2*}(\Delta n) - \sigma^{2*}\left\{\Delta(n-1)\right\} \tag{10}$$

and

$$\sigma^{2*}(t) = \int_0^t \sigma^2(u)\mathrm{d}u. \tag{11}$$

We will write

$$\grave{z}_n = \grave{z}(\lambda\Delta n) - \grave{z}\left\{\lambda\Delta(n-1)\right\}, \tag{12}$$

and

$$\acute{z}_n = \sigma^2\left(\Delta n\right) - \sigma^2\left\{\Delta\left(n-1\right)\right\}.$$

Then an important implication (Barndorff-Nielsen and Shephard (1998)) of these constructions is that

$$\sigma_n^2 = \lambda^{-1}\left(\grave{z}_n - \acute{z}_n\right). \tag{13}$$

It is this linear structure which will allow us to perform a number of analytic calculations which are not possible for other models.

The formula for the returns (9) is informative for it shows that the effect of the leverage is to shift the distribution. If the volatility innovations are unexpected large and ρ is negative, then the mean return will be negative. Hence negative returns are associated with increases in volatility. Likewise small innovations in the volatility process will happen at the same time as positive returns.

We now define

$$\bar{z}_n = \grave{z}_n - \lambda\Delta\xi \tag{14}$$

$$\bar{\sigma}_n^2 = \sigma_n^2 - \mathrm{E}\left(\sigma_n^2\right) = \sigma_n^2 - \Delta\xi. \tag{15}$$

Further, let

$$\bar{\sigma}^2(t) = \sigma^2(t) - \xi.$$

After these preparations we have that

$$\mathrm{E}\{y_n^2\} = \lambda\Delta(\xi + \rho^2\grave{\kappa}_2) \tag{16}$$

and

$$\mathrm{Var}\{y_n^2\} = \mathrm{Var}\{y_{0n}^2\} + 2\rho^2\{2\lambda^2\Delta^2\xi\grave{\kappa}_2 + 3(e^{-\lambda\Delta} - 1 + \lambda\Delta)\grave{\mu}_3\} + \rho^4\lambda\Delta(\grave{\kappa}_4 + 2\lambda\Delta\grave{\kappa}_2^2), \tag{17}$$

where the expression for

$$\begin{aligned}\mathrm{Var}\{y_{0n}^2\} &= 6\mathrm{Var}\left\{\sigma^2(t)\right\}\lambda^{-2}\left(e^{-\lambda\Delta} - 1 + \lambda\Delta\right) + 2\Delta^2\xi^2 \\ &= 3\grave{\kappa}_2\lambda^{-2}\left(e^{-\lambda\Delta} - 1 + \lambda\Delta\right) + 2\Delta^2\xi^2\end{aligned}$$

was derived in Barndorff-Nielsen and Shephard (1998).

Likewise, for $s = 1, 2, ...$ we derive that

$$\mathrm{E}\{y_n y_{n+s}\} = 0, \tag{18}$$

$$\mathrm{Cov}\left(y_n, y_{n+s}^2\right) = \mathrm{E}\{y_n y_{n+s}^2\} = \rho\grave{\kappa}_2(1 - e^{-\lambda\Delta})^2\exp\{-\lambda\Delta(s-1)\} \tag{19}$$

$$\mathrm{Cov}(y_n^2, y_{n+s}^2) = \mathrm{Cov}(y_{0n}^2, y_{0n+s}^2) + \rho^2(1 - e^{-\lambda\Delta})^2\exp\{-\lambda\Delta(s-1)\}\,\grave{\mu}_3 \tag{20}$$

where the expression

$$\begin{aligned}\mathrm{Cov}(y_{0n}^2, y_{0n+s}^2) &= \mathrm{Var}\left\{\sigma^2(t)\right\}\lambda^{-2}(1 - e^{-\lambda\Delta})^2\exp\{-\lambda\Delta(s-1)\} \\ &= \frac{\grave{\kappa}_2}{2\lambda^2}(1 - e^{-\lambda\Delta})^2\exp\{-\lambda\Delta(s-1)\}\end{aligned}$$

is given in Barndorff-Nielsen and Shephard (1998). We note then that there is some simplification as

$$\mathrm{Cov}(y_n^2, y_{n+s}^2) = \left(\frac{\grave{\kappa}_2}{2\lambda^2} + \rho^2\grave{\mu}_3\right)(1 - e^{-\lambda\Delta})^2\exp\{-\lambda\Delta(s-1)\}.$$

Only the formulae (17), (19) and (20) require some steps of calculation, which we place in the appendix.

The effect of the leverage on the dynamic properties of discrete time returns is made quite clear by these formulae. First both $\mathrm{E}\left(y_n y_{n+s}^2\right)$ and $\mathrm{Cov}(y_n^2, y_{n+s}^2)$ damp down exponentially with the lag length. In other words, the leverage effect diminishes exponentially with s. The effect of the leverage on the covariance between the squares is to increase (decrease) volatility clustering if the BDLP is positively (negatively) skewed. In practice we would expect the BDLP to be highly positively skewed.

We should note that exactly the same dynamic structure was found by Sentana (1991) in his work on the discrete time quadratic ARCH model (QARCH). Hence we can think of the QARCH model as a discrete time representation of our continuous time leverage model, generalising the unleveraged result associated with the work of Drost and Nijman (1993) and Drost and Werker (1996).

The simplicity of the effect of the leverage term means that we can still compute analytically the spectrum of squared returns. We may write this as

$$\begin{aligned} f(\psi) &= \sum_{s=-\infty}^{\infty} \mathrm{cor}\{y_n^2 y_{n+s}^2\} \cos(s\psi) \\ &= 1 - c\phi^{-1} + c\phi^{-1} a(\psi;\phi) \end{aligned}$$

where $\phi = \exp(-\lambda\Delta)$,

$$a(\psi;\phi) = \frac{1-\phi^2}{1-2\phi\cos\psi+\phi^2}$$

and

$$c = \frac{(1-e^{-\lambda\Delta})^2(\lambda^{-2}\omega^2 + \rho^2\grave{\mu}_3)}{\mathrm{Var}\{y_n^2\}},$$

which generalises the previous result of Barndorff-Nielsen and Shephard (1998).

3 Rest of the paper

In the rest of the paper we discuss extensions of this work and focus on designing simulation methods to enable us to sample return sequences from these types of processes.

References

Barndorff-Nielsen, O. E. (1998). Superposition of Ornstein-Uhlenbeck type processes. In preparation.

Barndorff-Nielsen, O. E. and N. Shephard (1998). Aggregation and model construction for volatility models. Unpublished discussion paper: Nuffield College, Oxford.

Black, F. (1976). Studies of stock price volatility changes. *Proceedings of the Business and Economic Statistics Section*, 177–181.

Campbell, J. Y. and A. S. Kyle (1993). Smart money, noise trading and stock price behaviour. *Rev. Economic Studies 60*, 1–34.

Drost, F. C. and T. E. Nijman (1993). Temporal aggregation of GARCH processes. *Econometrica 61*, 909–927.

Drost, F. C. and B. Werker (1996). Closing the GARCH gap: continuous time GARCH modelling. *J. Econometrics 74*, 31–57.

Ghysels, E., A. C. Harvey, and E. Renault (1996). Stochastic volatility. In C. R. Rao and G. S. Maddala (Eds.), *Statistical Methods in Finance*, pp. 119–191. Amsterdam: North-Holland.

Glosten, L. R., R. Jagannathan, and D. Runkle (1993). Relationship between the expected value and the volatility of the nominal excess return on stocks. *J. Finance 48*, 1779–1802.

Nelson, D. B. (1991). Conditional heteroskedasticity in asset pricing: a new approach. *Econometrica 59*, 347–370.

Sentana, E. (1991). Quadratic ARCH models: a potential re-intepretation of ARCH models. *Rev. Economic Studies 62*, 639–61.

An extension of P. Lévy's distributional properties to the case of a Brownian motion with drift.*

S.E. Graversen
Department of Theoretical Statistics and MaPhySto†,
University of Aarhus, DK-8000 Aarhus C, Denmark.

A.N. Shiryaev
Steklov Mathematical Institute,
Gublina str. 8, 117966 Moscow,
and MaPhySto†

Abstract

We extend the well known P. Lévy theorem on the distributional identity $(M_t - B_t, M_t) \simeq (|B_t|, L(B)_t)$, where (B_t) is a standard Brownian Motion and $(M_t) = (\sup_{0\leq s\leq t} B_s)$ to the case of Brownian Motion with drift λ. Processes of the type

$$dX_t^\lambda = -\lambda sgn(X_t^\lambda)\, dt + dB_t$$

appears naturally in the generalisation.

Key Words: Brownian Motion, Local Time, Markov Processes.

1 Introduction

A classical result of Paul Lévy states that if $B = (B_t)_{0\leq t\leq 1}$ is a standard *Brownian motion* $(B_0 = 0, EB_t = 0, EB_t^2 = t)$ then

$$(M - B, M) \stackrel{law}{=} (|B|, L(B)) \tag{1}$$

*This note is an excerpt from the MaPhySto Research Report No. 1998-22 "An extension of P. Lévy's distributional properties to the case of a Brownian motion with drift" (to appear in Bernoulli). For the proof of the main theorem and derived results, we refer to the full paper.

†MaPhySto – Centre for Mathematical Physics and Stochastics, funded by a grant from The Danish National Research Foundation.

i.e. $((M_t - B_t, M_t); 0 \le t \le 1) \stackrel{law}{=} (|B_t|, L(B)_t; 0 \le t \le 1)$ where $M = (M_t)_{0 \le t \le 1}$, $M_t = \max_{0 \le s \le t} B_s$, and $L(B) = (L(B)_t)_{0 \le t \le 1}$ is the local time of B at zero:

$$L(B)_t = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{(|B_s| \le \epsilon)}\, ds. \tag{2}$$

(See, for example, [Revuz−Yor (1994);Ch.VI].)

The main aim of this note is to give an extension of the distributional property (1) to the case of a Brownian motion *with drift* B^λ where $B^\lambda = (B_t^\lambda)_{0 \le t \le 1}$, $B_t^\lambda = B_t + \lambda t$. Let's denote $M^\lambda = (M_t^\lambda)_{0 \le t \le 1}$, $M_t^\lambda = \max_{0 \le s \le t} B_s^\lambda$.

For our presentation the following process $X^\lambda = (X_t^\lambda)_{0 \le t \le 1}$ defined as the unique strong solution of the stochastic differential equation

$$dX_t^\lambda = -\lambda\, sgn X_t^\lambda\, dt + dB_t, \quad X_0^\lambda = 0, \tag{3}$$

plays a key role. (Here $sgn\, x$ is defined to be 1 on $\Re_+$, -1 on $\Re_-$ and 0 at 0.) In particular we shall see that the process $|X^\lambda| = (|X_t^\lambda|)_{0 \le t \le 1}$ realizes an explicit construction of the process RBM$(-\lambda)$ i.e. a *reflecting Brownian motion with drift* $(-\lambda t)$.

2 Main result

Theorem 1 *For any* $\lambda \in \Re$

$$(M^\lambda - B^\lambda, M^\lambda) \stackrel{law}{=} (|X^\lambda|, L(X^\lambda)) \tag{4}$$

i.e. $(M_t^\lambda - B_t^\lambda, M_t^\lambda); 0 \le t \le 1) \stackrel{law}{=} (|X_t^\lambda|, L(X^\lambda)_t); 0 \le t \le 1)$ *where*

$$L(X^\lambda)_t = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{(|X_s^\lambda| \le \epsilon)}\, ds.$$

3 Some remarks

The theorem of P. Lévy (1) and its extension (4) given above have both "two-dimensional" character in the sense that they are statements for pairs of processes $((M^\lambda - B^\lambda), M^\lambda)$ and $(|X^\lambda|, L(X^\lambda))$.
M. Yor has pointed out the connection between Theorem 1 and 2 above and the results in the papers [Kinkladze (1982)] and [Fitzsimmons (1987)]. From paper [Kinkladze (1982)] one may obtain easily the corresponding following "one-dimensional" result saying that $M^\lambda - B^\lambda \stackrel{law}{=}$ RBM$(-\lambda)$. (For the notion of RBM$(-\lambda)$ see Definition 1 in [Kinkladze (1982)]). Indeed by Theorem 1 and 2 in [Kinkladze (1982)] the process $Y^\lambda \equiv$RBM$(-\lambda)$ can be realized with some Brownian motion B in the form

$$Y_t^\lambda = \sup_{0 \le s \le t} (-\lambda(t-s) - (B_t - B_s)), \quad t \ge 0.$$

So, $Y_t^\lambda = \sup_{0\le s\le t}((\lambda s + B_s) - (\lambda t + B_t)$ and as a corollary $Y^\lambda \equiv M^\lambda - B^\lambda$ with $B_t^\lambda = \lambda t + B_t$. Together with formula (21) of Theorem 2 we obtain that $M^\lambda - B^\lambda \stackrel{law}{=} |X^\lambda|$. In connection with this formula it is usefull to remark that the process X^λ has appeared in many different problems however the very natural property $\mathrm{RBM}(-\lambda) \stackrel{law}{=} |X^\lambda|$, apparently, was not noted before.

It is very reasonable to ask about possible extensions of the result $M^\lambda - B^\lambda \stackrel{law}{=} |X^\lambda|$ for more general class of processes $Z = (Z_t)_{t\ge 0}$ besides the processes $B^\lambda = (B_t^\lambda)_{t\ge 0},\ \lambda \in \Re$. According to [Fitzsimmons (1987)] if $Z = (Z_t)_{t\ge 0}$ is a conservative real valued diffusion process and the process $\max Z - Z$ is a time homogenuous strong Markov process then necessarily $Z = B^{\lambda,\sigma}$, where $B_t^{\lambda,\sigma} = \lambda t + \sigma B_t$ with $\lambda \in \Re$, $\sigma > 0$. So, this result shows that in some sense a direct extension of the P. Lévy's result is possible only for Brownian motion with drift. That is, exactly the framework of Theorem 1 above.

References

[1] Revuz D., Yor M. *Continuous Martingales and Brownian Motion*, Second ed.,1994. Springer-Verlag.

[2] Ikeda N., Watanabe Sh. *Stochastic Differential Equations and Diffusion Processes*, 1981. North Holland.

[3] Liptser R., Shiryaev A.N. *Statistics of Random Processes*, Vol. 1, 1977. Springer-Verlag.

[4] Karatzas I., Shreve S. *Brownian Motion and Stochastic Calculus*, 1988. Springer-Verlag.

[5] Kinkladze G. N., A Note on the Structure of Processes the Measure of which is Absolutely Contonuous with Respect to the Wiener Process Modulus Measure, *Stochastics*, 1982, Vol. **8**, 39-44.

[6] Fitzsimmons P. J., A Converse to a Theorem of P. Lévy, *The Annals of Probability*, 1987, Vol. **15**, No. 4, 1515-1523.

Multiparameter Levy Stable Noise and a Selection Procedure

A. R. Soltani*

Let Z be a random measure on bounded Borel sets in R^N with values in a Banach space of jointly symmetric α-stable random variables for which $E[\exp(i\gamma Z(B))] = \exp(-c\lambda(B)|\gamma|^\alpha),\ \gamma \in R,\ \alpha \in (0,2];\ \lambda(B)$ is the Lebesgue measure of B. Define $Z(t)$ to be restriction of the random measure Z to the rectangle $(0,t] = \prod(0,t_j],\ i.e.,\ Z(t) = Z((0,t]),\ t \in R^{N+}$, then $Z(t),\ t \in R^{N+}$, defines a multiparameter Levy stable noise which is a random field with independent increments. Processes of this type have been the center of many research activities, [1], [2], [3], [4]. From the work in [1] concerning the existence of a jointly continuous version of local time for d-dimensional Levy stable fields we were motivated to prove the following:

let $Q = (0,v]$ be a rectangle in R^{N+}, then for each positive integer m and each $(t^1,\ldots,t^{2m}) \in Q^{2m}, \alpha \in (1,2]$,

$$\int_{R^{2m}} |E[\exp(i\sum_{j=1}^{2m}\gamma_j Z(t^j))]| \prod_{j=1}^{2m}|\gamma_j|^\eta d\gamma_j \le c(m)\prod_{k=1}^{N}\prod_{j=1}^{2m}(t_k^{(j)} - t_k^{(j-1)})^{-(1+\eta)/\alpha,} \tag{1.1}$$

where $t_k^{(j)}$ is the jth order statistic of $t_k^1,\ldots,t_k^{2m},\ j = 1,\ldots,2m,\ k = 1,\ldots,N$, $c(m)$ is a conctant, and $\eta > 0$ subject to $(1+\eta)/\alpha < 1$.

Note that as $\prod_{j=1}^{2m}|\gamma_j|^\eta$ is not integrable, one can not neglect the characteristic

*Department of Statistics College of Sciences, Shiraz University, Shiraz, Iran

function term of the integrand in (1.1), and has to estimate it properly.

In this work we present a proof for (1.1). It happens to be an interesting problem. No advanced theorem is used. To make readers curious to go through the detail of the proof, for $m = 2$ and for different choices of points $t \in Q$ the integral splites into sum of 104 elementary integrals that each has to be treated in a certain way; for m=3 the number of elementary integrals exceeds 1000. A selection procedure is performed to take care of the integration procedure, it is a two dimensional selection procedure which is new. At this stage we only sketch the method and the procedure for $N = 2$.

Step1: Small rectangles. we may assume that $t_1^1, t_1^2, \ldots, t_1^{2m}$ are ordered, i.e. $0 < t_1^1 < t_1^2 < \ldots < t_1^{2m} \leq v_1$. Let each π be a permutation on $\{1, \ldots, 2m\}$. we use π to identify the positions of the 2th components of $t^j, j = 1, \ldots, 2m,$ i.e. t_2^j is the $\pi(j)$th order statistic of $t_2^1, \ldots, t_2^{2m}$. The lines which are parallel to the coordinate axes and pass through each $t^j, j = 1, \ldots, 2m$ partition each rectangle $[0, t^j]$ into $j \times \pi(j)$ disjoint rectangles. Let us call these disjoint rectangles *small rectangles*. If a small rectangle D is contained in $[0, t^j]$, the number j is assigned to it, $j = 1, \ldots, 2m$. The *index* set of a small rectangle D is defined to be the set of all numbers in $\{1, 2, \ldots, 2m\}$ which are assigned to D and is denoted by $I(D)$.

Step2: Properties of Z. Since $Z(.)$ has independent increments we may write

$$Z(t^j) = \sum_{\{D: j \in I(D)\}} Z(D),$$

where the terms are independent of each other. This implies that

$$\sum_{j=1}^{2m} \gamma_j Z(t^j) = \sum_D (\sum_{j \in I(D)} \gamma_j) Z(D),$$

which in turn implies that

$$|E[\exp\left(i \sum_{j=1}^{2m} \gamma_j Z(t^j)\right)]| = \prod_D \exp(-\left|\sum_{j \in I(D)} \gamma_j\right|^\alpha \lambda(D)). \tag{1.2}$$

Step 3: Change of variable. let $I_j = \{k : \ k \geq j : \pi(k) \geq \pi(j)\}$ and $I_j^* = I_j - \{j\}$ and let

$$\beta_j = \sum_{k \in I_j} \gamma_k, \qquad \delta_j = \sum_{k \in I_j^*} \gamma_k. \tag{1.3}$$

Now $\gamma_j = \beta_j - \delta_j$, therefore $\prod_{j=1}^{2m} |\gamma_j|^\eta \leq 2^\eta \sum' \prod_{j=1}^{2m} c_j^\eta$, where c_j is either $|\beta_j|$ or $|\delta_j|$, $j = 1, \ldots, 2m-1$, $c_{2m} = |\beta_{2m}|$. If $I_j^* = \phi$, then $c_j = |\beta_j|$. Thus the integral in (1.1) is dominated by

$$2^\eta \sum\nolimits' \int_{R^{2m}} \prod_D \exp\left(-\left|\sum_{j \in I(D)} \gamma_j\right|^\alpha \lambda(D)\right) \prod_{j=1}^{2m} c_j^\eta d\beta_j. \tag{1.4}$$

Step 4: Integration and estimation. (I). If $c_j = |\delta_j|$, then we choose a small rectangle D in the column $[j-1, j] \times [0, v_2]$ for which $I(D) = I_j^*$ and use the inequality that for $a > 0$, $\eta > 0$, $e^{-ax^\alpha} x^\eta \leq (\eta/\alpha)^{\eta/\alpha} a^{-\eta/\alpha}$ for all $x \geq 0$ to estimate

$$\exp\left(-\left|\sum_{j \in I(D)} \gamma_j\right|^\alpha \lambda(D)\right) \left|\sum_{k \in I_{j^*}} \gamma_k\right|^\eta \leq (\eta/\alpha)^{\eta/\alpha} [\lambda(D)]^{-\eta/\alpha}. \tag{1.5}$$

In this case we also use the small rectangle with upper right vertex t_j, D_j, to perform the following integration

$$\int_R \exp(-|\beta_j|^\alpha \lambda(D_j)) d\beta_j \leq k_0 [\lambda(D_j)]^{-1/\alpha}, \tag{1.6}$$

where $k_0 = \int_R \exp(-|y|^\alpha)$dy, note that $I(D_j) = I_j$.

(II). If $c_j = |\beta_j|$, then we must *decide* on either using the small rectangle D_j to obtain

$$\int_R \exp(-|\beta_j|^\alpha \lambda(D_j)) |\beta_j|^\eta d\beta_j \leq k_1 [\lambda(D_j)]^{-(1+\eta)/\alpha}, \tag{1.7}$$

where $k_1 = \int_R \exp(-|y|^\alpha) |y|^\eta dy$, or using D_j as in (1.6) and look for a small rectangle D in $[j, j-1] \times [0, v_2] - D_j$ to obtain that

$$\exp(-|\beta_j|^\alpha \lambda(D)) |\beta_j|^\eta \leq (\eta/\alpha)^{\eta/\alpha} [\lambda(D)]^{-\eta/\alpha} \tag{1.8}$$

For performing this procedure for $j = 1, \ldots, 2m$ a special caution is needed for choosing each small rectangle. Namely if at stage j, D_j is used in (1.7), then no other small rectangle which has an edge length as D_j should be used in cases I or II in other stages. Also if at stage j a small rectangle D is used for (1.5) or (1.8) then any small rectangle which has an edge length as D is only allowed to be used for (1.6) in other stages. Finally, if at stage j D_j is used for (1.6), then any small rectangle which has an edge length as D_j is only allowed to be used for (1.5) or (1.8) in other stages.

The product in (1.4) is over all small rectangles and at most $2 \times 2m$ small rectangles are used in the procedure given above. Each small rectangle D_j, $j = 1, \ldots, 2m$ is used in the procedure. For those small rectangles which are of no use in our procedure; we use the estimate that $\exp(-|\sum_{j \in I(D)} \gamma_j|^\alpha \lambda(D)) \leq 1$. Now, based on the above procedure, we need a direction for choosing our desired small rectangles and to prove that always our desired rectangles are available. Our two dimensional selection procedure provides the right direction.

References

[1] Ehm, W. (1981). Sample function properties of multiparameter stable processes. Zeit. Wahrscheinlichkeitstheor. Verw. Geb. 56 195-228.

[2] Dozzi, M and Soltani, A. R. (1997). Local time for stable moving average processes: Holder conditions. Stochastic Processes and their Appl. 68 no. 2 195-207.

[3] Nolan, J. P. (1989). Local nondeterminism and local times for stable processes. Probab. Th. Rel. Fields. 82 387-410.

[4] Soltani, A. R. (1992). Local nondeterminism for moving average stable processes. Bull. Iranian Math. Soc. vol. 18 7-16 .

Optimal Consumption and Portfolio in a Jump Diffusion Market

Nils Christian Framstad[1], Bernt Øksendal[1,2], Agnès Sulem[3]

Extended Abstract - 21/12/98

1 The Model

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a given filtration $(\mathcal{F}_t)_{t\geq 0}$. We consider a market consisting of two securities or investment possibilities of the form:

$$dX(t) = (rX(t) - c(t))dt, \quad X(0) = x, \tag{1}$$

$$dY(t) = \alpha Y(t)dt + \sigma Y(t)dW(t) + Y(t^-)\int_{\mathbb{R}} \beta(\eta)\tilde{N}(dt, d\eta), \quad Y(0) = y, \tag{2}$$

where

$$\tilde{N}([0,t]\times U) = N([0,t]\times U) - tm(U)$$

is the compensator of an (homogeneous) Poisson random measure $N([0,t]\times U)$ on $\mathbb{R}^+ \times \mathbb{R}$ with intensity measure $m([0,t]\times U) = E[N([0,t]\times U)] = t\nu(U)$, where $d\nu(\eta)$ is a σ-finite measure on the Borel set U of $\mathbb{R}$. We also assume that $\beta(\eta) \geq 0$ everywhere (see [3]). Here $c(t) = c(t,\omega) \geq 0$ is our consumption rate process, which can be chosen optimally at any instant of time t.

[1]Dept. of Mathematics, University of Oslo, P. O. Box 1053 Blindern, N–0316 Oslo, Norway, email: oksendal@math.uio.no

[2]Norwegian School of Economics and Business Administration, Helleveien 30, N–5035 Bergen-Sandviken, Norway

[3]INRIA, Domaine de Voluceau-Rocquencourt BP 74105, F–78153 Le Chesnay Cedex, France, email: Agnes.Sulem@inria.fr

2 No transaction costs.

In this case, we assume that at any time t, the investor is free to transfer money from one investment to another without transaction fees. Such a portfolio is conveniently characterised by the fraction

$$u(t) = \frac{Y(t)}{X(t) + Y(t)} \tag{3}$$

of the total wealth

$$Z(t) = X(t) + Y(t) \tag{4}$$

invested in the risky asset at time t. With such a portfolio choice $u = u(t, \omega)$ and consumption rate $c = c(t, \omega)$, the wealth process $Z(t) = Z^{c,u}(t)$ is given by the stochastic differential equation:

$$dZ(t) = (r(1-u)+\alpha u)Z(t)dt - c(t)dt + \sigma u Z(t)dW(t) + uZ(t^-)\int_{\mathbb{R}} \beta(\eta)\tilde{N}(dt, d\eta), \quad Z(0) = z = x+y.$$

Define the performance criterion by

$$J^{c,u}(s, z) = E^z\left(\int_0^\tau e^{-\delta(s+t)} \frac{c^\gamma(t)}{\gamma} dt\right) \tag{5}$$

where $0 < \gamma < 1$ is a constant and

$$\tau = \tau(\omega) = \inf\{t > 0; Z(t) \leq 0\} \tag{6}$$

is the time of bankruptcy. We want to find $\phi(s, z)$ and c^*, u^* such that

$$\phi(s, z) = \sup_{c,u} J^{c,u}(s, z) = J^{c^*,u^*}(s, z). \tag{7}$$

In the no jump case ($\beta(\eta) \equiv 0$), this is the well-known Merton problem which was solved by Merton [4]. He proved that if

$$\delta > \gamma\left(r + \frac{(\alpha - r)^2}{2\sigma^2(1-\gamma)}\right) \tag{8}$$

then the optimal strategy is :

$$c_0^*(s,z) = (\gamma K_0)^{\frac{1}{\gamma-1}} z, \quad u_0^*(s,z) = \frac{\alpha - r}{\sigma^2(1-\gamma)} \tag{9}$$

and this leads to the value function

$$\phi_0(s,z) = e^{-\delta s} K_0 z^\gamma \tag{10}$$

with

$$K_0 = \frac{1}{\gamma}\left(\frac{1}{1-\gamma}(\delta - \gamma r - \frac{\gamma(\alpha-r)^2}{2\sigma^2(1-\gamma)})\right)^{\gamma-1} \tag{11}$$

(see also Davis and Norman [2, section 2]. We prove that in the jump diffusion model, the problem has a similar solution. More precisely, we have:

Theorem 2.1. *Let $V = u_\beta{}^*$ be the solution of the equation*

$$\alpha - r + V\sigma^2(\gamma-1) + \int_{\mathbb{R}} \left((1+V\beta(\eta))^{\gamma-1} - 1\right)\beta(\eta)d\nu(\eta) = 0 \tag{12}$$

and suppose that

$$\rho := \delta - (r + (\alpha - r)u_\beta{}^*)\gamma + \frac{1}{2}\sigma^2(u_\beta{}^*)^2\gamma(\gamma-1) + \int_{\mathbb{R}} \left((1+u_\beta{}^*\beta(\eta))^{\gamma-1} - 1 - u_\beta{}^*\gamma\beta(\eta)\right)d\nu(\eta) > 0.$$

Then, the value function $\phi = \phi_\beta$ of problem (7) is given by

$$\Phi_\beta(s,z) = e^{-\delta s} K z^\gamma \tag{13}$$

where K is the solution of the equation

$$\frac{1-\gamma}{\gamma}(K\gamma)^{\frac{\gamma}{\gamma-1}} - K\rho = 0. \tag{14}$$

The corresponding optimal portfolio is the constant fraction u_β^* and the optimal consumtion rate is given by

$$c_\beta^* = (\gamma K)^{\frac{1}{\gamma-1}} z. \tag{15}$$

3 Proportional Transaction Costs

We consider now the case when the investor pays fractions λ and μ of the amount transacted on stock, on purchase and sale respectively. This leads to the equations:

$$dX(t) = (rX(t) - c(t))dt - (1+\lambda)d\mathcal{L}_t + (1-\mu)d\mathcal{M}_t, \quad X(0) = x, \tag{16}$$

$$dY(t) = \alpha Y(t)dt + \sigma Y(t)dW(t) + Y(t^-)\int_{\mathbb{R}} \beta(\eta)\tilde{N}(dt, d\eta) + d\mathcal{L}_t - d\mathcal{M}_t, \quad Y(0) = y, \tag{17}$$

where $\mathcal{L}_t$ and $\mathcal{M}_t$ are cadlag non decreasing processes representing the cumulative purchase and sale amounts of stock up to time t respectively. In this case, we define the bankruptcy time T by

$$T = \inf\{t > 0; (X(t), Y(t)) \notin \mathcal{S}\}$$

where $\mathcal{S}$ is the solvency region

$$\mathcal{S} = \{(x,y) \in \mathbb{R}^2; x + (1-\mu)y \geq 0 \text{ and } x + (1+\lambda)y \geq 0\}.$$

The performance criterion for a consumption c and a portfolio $(\mathcal{L}, \mathcal{M})$ is given by

$$J^{c,\mathcal{L},\mathcal{M}}(x,y) = E^{x,y}(\int_0^T e^{-\delta t}\frac{c^\gamma(t)}{\gamma}dt).$$

In the no-jump case ($\beta(\eta) \equiv 0$), this problem was solved in [2] and [1]. Let $v(x,y)$ be the value function that is

$$\Phi(x,y) = \sup_{c,\mathcal{L},\mathcal{M}} J^{c,\mathcal{L},\mathcal{M}}(x,y) = J^{c^*,\mathcal{L}^*,\mathcal{M}^*}(x,y). \tag{18}$$

The variational inequality associated to this problem is

$$\max\left\{A\Phi + \max_{c\geq 0}(-c\frac{\partial\Phi}{\partial x} + \frac{c^\gamma}{\gamma}) + \int_{\mathbb{R}}(\Phi(x, y+\beta(\eta)y) - \Phi(x,y) - \beta(\eta)y\frac{\partial\Phi}{\partial y})d\nu(\eta), L\Phi, M\Phi\right\} = 0 \qquad \text{in } \mathcal{S}$$

$$\Phi = 0 \qquad \text{on } \partial$$

where

$$A\Phi = \frac{1}{2}\sigma^2 y^2\frac{\partial^2\Phi}{\partial y^2} + \alpha y\frac{\partial\Phi}{\partial y} + rx\frac{\partial\Phi}{\partial x} - \delta\Phi, \tag{19}$$

$$L\Phi = -(1+\lambda)\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y}, \tag{20}$$

$$M\Phi = (1-\mu)\frac{\partial \Phi}{\partial x} - \frac{\partial \Phi}{\partial y}. \tag{21}$$

The value function Φ has, like in the no jump case, the homothetic property

$$\Phi(\rho x, \rho y) = \rho^{\gamma}\Phi(x,y) \quad \forall \rho > 0.$$

This enables to reduce the dimension of the problem.

References

[1] M. Akian, J. L. Menaldi, and A. Sulem, *On an investment-consumption model with transaction costs*, SIAM J. Control and Optim. **34** (1996), no. 1, 329–364.

[2] M. Davis and A. Norman, *Portfolio selection with transaction costs*, Math. Oper. Res. **15** (1990), 676–713.

[3] J. Jacod and A. Shiryaev, *Limit theorems for stochastic processes*, Springer Verlag, 1987.

[4] R. C. Merton, *Optimum consumption and portfolio rules in a continuous time model*, J. Economic Theory **3** (1971), 373–413.

ON STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY CAUCHY PROCESS AND THE OTHER α–STABLE MOTIONS

P.A. Zanzotto

Dipartimento di Matematica – Università di Pisa
Via F. Buonarroti, 2 – 56127 Pisa, Italy

First recall the Engelbert and Schmidt characterization ([1]) of weak existence and uniqueness for the one–dimensional SDE

$$dX_t = b(X_t)dW_t \quad , \quad t \geq 0 \tag{0}$$

where b is Borel measurable and W denotes Wiener process (cf. also [3], Theorems 5.4, 5.7). It is given in terms of the singularity sets

$$I = \{x \in \mathbb{R} : \int_{x^-}^{x^+} b^{-2}(y)dy = \infty\} \ , \ N = \{x \in \mathbb{R} : b(x) = 0\}.$$

Theorem (Engelbert and Schmidt). *Weak existence holds for equation* (0) *with arbitrary initial distribution if and only if* $I \subset N$. *In that case uniqueness in law holds for every initial distribution if and only if* $I = N$.

In this talk we consider the following generalization of the above SDE

$$dX_t = b(X_{t^-})dZ_t \quad , \quad t \geq 0 \tag{1}$$

where b is a Borel measurable real function and Z denotes a strictly α–stable Lévy process, $0 < \alpha \leq 2$, a symmetric Cauchy process when $\alpha = 1$, starting from 0.

In the following we will omit the adjective "strictly". *Moreover, when we consider such equation driven by* Z *with parameter* $\alpha \neq 1, 2$, *we always make the assumption that either* Z *is symmetric or* b *is non–negative;* however, for the sake of shortness *we will omit it in the statements. In the same way, we will simply write "Cauchy process" to mean symmetric Cauchy process.*

For the above class of equations with $\alpha \neq 1$ and Z symmetric, weak solutions were already investigated in [4]. Here we pursue that study and improve the results obtained there from various point of view (see [6]).

First of all, we extend the above recalled weak existence and uniqueness exact criteria of Engelbert and Schmidt (corresponding to the case $\alpha = 2$), to the class of the above equations driven by Z with parameter α, $1 < \alpha \leq 2$.

More precisely, for a fixed α, $1 < \alpha \leq 2$, define the singularity set

$$I(\alpha) = \{x \in \mathbb{R} : \int_{x^-}^{x^+} |b(y)|^{-\alpha}dy = \infty\} \tag{2}$$

(while N still denotes the set of zeros of b). Then we have the following result.

Theorem 1. ([6], (3.5), (3.21)). *The above assumptions being in force, for a fixed* α, $1 < \alpha \leq 2$ *consider equation* (1) *driven by* Z *with parameter* α*: then weak existence holds for every initial distribution if and only if* $I(\alpha) \subset N$. *In that case uniqueness in law holds for every initial distribution if and only if* $I(\alpha) = N$.

The existence part of the above Theorem is to be compared with the following result that deals with nontrivial solutions, a solution being termed trivial if a.s., $X_t = X_0$ for all t; it was obtained in [4].

Theorem ([4], Th. (2.32)). *Consider equation* (1) *driven by a symmetric* Z *with* $\alpha, 1 < \alpha \leq 2$. *Then the following properties are equivalent:*
a) *For every* $x \in \mathbb{R}$ *there exists a nontrivial solution starting from* x.
(b) *Function* $|b|^{-\alpha}$ *is locally integrable.*

Such existence criterion is clearly restrictive for the coefficient b, since in general $I(\alpha)$ is not empty.

Sufficient existence conditions with a fixed starting point are based in [4] on the condition expressed in the following

Definition. Let $x \in \mathbb{R}$.. The coefficient b in (1) is said to *satisfy condition (H) with respect to* x (for short $(H)(x)$) if

$$\int_0^t ds \left(\int_{|y|<L} \frac{1}{|b(x+y)|^\alpha} f(s,y)dy \right) < \infty \quad \text{for all } \ t > 0, \ L > 0,$$

f denoting the α–stable transition density ($0 < \alpha \leq 2$). (We set $|b(x)|^{-\alpha} = +\infty$ if $b(x) = 0$).

It turns out that condition $H(x)$ can be used also in the case of a driving Cauchy process. So we have the following theorem which improves Th. (2.5) in [4] including the case $\alpha = 1$ (and removing the symmetry assumption when $\alpha \neq 1, 0 < \alpha < 2$).

Theorem 2 ([6], (4.3)). *Consider equation* (1). *Let* $x \in \mathbb{R}$ *and assume that* b *satisfies condition* (H) *with respect to* x.

(a) *Let* $1 \leq \alpha \leq 2$. *Under the stated assumptions there exists a nontrivial solution of* (1) *starting from* x.

(b) *Let* $0 < \alpha < 1$. *In addition to the above assumptions, we suppose that there exists* $U > 0$ *such that the set* $\{x \in \mathbb{R} : |b(x)| > U\}$ *has finite Lebesgue measure. Then there exists a nontrivial solution of* (1) *starting from* x.

Condition (H) is related with integrability conditions of b alone. So in particular from Th. 2 we deduce the following

Theorem 3 ([6], (4.14)). *Consider equation* (1) *driven by a Cauchy process. Assume that there exists a real number* $\delta > 1$ *such that* $|b|^{-\delta}$ *is locally integrable. Then for any law* μ *on* $\mathbb{R}$*, there exists a nontrivial solution with initial distribution* μ*.*

A process is termed a *solution* of (1) *on an interval* if it solves the equation up to the first exit time of the interval ("local" solution). For the study of "local" solutions, we employ the following "local" version of (H):

Definition. Let x be a real number. We say that the coefficient b in (1) satisfies *condition (LH) with respect to* x (for short $(LH)(x)$) if there exists a real number $\epsilon > 0$ such that

$$\int_0^\epsilon ds \left(\int_{|y| \le \epsilon} \frac{1}{|b(x+y)|^\alpha} f(s,y) dy \right) < \infty \ .$$

As a sufficient existence condition for "local" nontrivial solutions starting from a fixed point x, condition $(LH)(x)$ completely unifies the cases of equation (1) driven by processes Z's with different $\alpha,\ 0 < \alpha \le 2$:

Theorem 4 ([6], (4.17)). *Consider equation* (1) *driven by an* α*–stable Lévy process* Z*,* $0 < \alpha \le 2$ *(a Cauchy process when* $\alpha = 1$*). Assume that the coefficient* b *satisfies condition (LH) with respect to a real number* x*. Then there exists a nontrivial solution of* (1) *on the interval* $[x-\epsilon, x+\epsilon]$ *starting from* x*,* ϵ *denoting any number such that the relation defining (LH)(x) be satisfied.*

When $1 < \alpha \le 2$, condition $(LH)(x)$ is also necessary for the existence of a "global" (i.e. defined for all t) or of a "local" nontrivial solution starting from x.

Proposition. *Consider* (1) *driven by* Z*,* $1 < \alpha \le 2$*. Then condition* $(LH)(x)$ *is necessary for the existence of a nontrivial "local" solution (and so also for the existence of a nontrivial solution) starting from* x*.*

When $0 < \alpha \le 1$, we do not know if a similar necessity property holds for condition $(LH)(x)$: however, from examples it seems that condition $(LH)(x)$ is not far from a necessary one, also in this case.

Condition $(H)(x)$ is, in a sense, a basic one because, also in the "local" existence case, sufficiency of $(LH)(x)$ goes back to results involving $(H)(x)$. However $H(x)$ is not necessary in general, as one sees from examples.

The following result illustrates the role of condition $H(x)$ in the case $1 < \alpha \le 2$: it improves the above recalled Th. (2.32) as well as Th. (3.17) of [4].

Theorem 5 ([6], (4.22)). *Consider equation* (1) *driven by an* α*–stable Lévy process* $Z, 1 < \alpha \le 2$*. Then the following propositions are equivalent.*

(a) *There exists* $x_0 \in \mathbb{R}$ *such that* b *satisfies condition (H) with respect to* x_0*;*
(b) *Coefficient* b *satisfies condition (H) with respect to any* x*;*
(c) *Coefficient* b *satisfies condition (LH) with respect to any* x*;*
(d) *The set* $I(\alpha)$ *is empty;*

(e) *Function* $|b|^{-\alpha}$ *is locally integrable*;
(f) *For every* x *there exists a nontrivial solution starting from* x*;*
(g) *For every* x *there exists a compact non degenerate or an open interval* A_x *containing* x *in the interior and a nontrivial solution on* A_x *starting from* x*.*
(h) *For every law* η *on* $\mathbb{R}$*, there exists a nontrivial solution with initial law* η*.*

As to the tools, we mention here the criteria of integrability with respect to α–stable Lévy processes in [2] as well as some representation properties in [5]. There we characterize the class of quadratic pure jump semimartingales (i.e. SM with vanishing continuous martingale part) which can be represented as a stochastic integral with respect to stable Lévy processes.

Consider a semimartingale X and suppose that there exists a predictable process H and an α–stable Lévy process Z $(0 < \alpha < 2)$ such that

$$X_t = \int_0^t H_s dZ_s \ , \ t \geq 0 : \tag{3}$$

then the jump–measure of X has compensator

$$\pi(ds, dx) = |H_s|^\alpha q_\alpha(ds, dx)$$

where q_α is the compensator of the jump–measure of Z i.e. $q_\alpha(ds, dx) = ds \otimes \nu(dx), \nu$ denoting Lévy measure.

It turns out that the above condition is also sufficient to have a representation formula of the form (3). Indeed, still denoting by π the compensator of the jump–measure μ of a generic semimartingale, let us introduce the following condition

(4). *There is a predictable process* $H(s, \omega)$ *such that*

$$\pi(\omega, ds, dx) = |H(s, \omega)|^\alpha q_\alpha(ds, dx)$$

where $q_\alpha(ds, dx) = ds \otimes \nu(dx), \nu$ *denoting the Lévy measure of an* α*–stable Lévy process.*

Then we have the following

Theorem 6. ([5], Th. 1). a) *Let* X *be a quadratic pure jump semimartingale* X *satisfying condition* (4) *with* $\alpha = 1$ *and such that, for all* t

$$X_t = \int_{]0,t]} \int_{|x| \leq 1} x(\mu - \pi)(ds, dx) + \int_{]0,t]} \int_{|x|>1} x\mu(ds, dx) + aA_t$$

where $a \in \mathbb{R}$ *and* $A_t(\omega) = \int_0^t |H_s(\omega)| ds$*.*

Then there is an 1*–stable Lévy process* Z *with Lévy measure* ν *and drift* at*, possibly defined on some extension of the original probability space, such that*

$$X_t = \int_0^t |H_s| dZ_s \ , \ t \geq 0.$$

b) *The assumptions are the same as above, except for the process A that now is such that* $A_t(\omega) = \int_0^t H_s(\omega)ds$, $t \geq 0$.

Then the following representation formula holds

$$X_t = \int_0^t H_s dZ_s^* \ , \ t \geq 0,$$

where Z^* *denotes a process with the same law as* Z *in* a)*, possibly defined on some extension of the original probability space.*

Theorem 7 ([5], Th. 2). a) *Let* X *be a purely discontinuous local martingale (resp. be of pure jump type with locally finite variation paths). Assume that it verifies condition* (4) *with* $\alpha, 1 < \alpha < 2$ *(resp. with* $\alpha, 0 < \alpha < 1$*).*

Then there is an α*–stable Lévy process* Z *with Lévy measure* ν*, possibly defined on some extension of the original probability space, such that*

$$X_t = \int_0^t |H_s| dZ_s \ , \ t \geq 0.$$

b) *All the assumptions in* a) *hold. In addition we assume that* ν *in* (4) *is symmetric. Then we have also the representation formula*

$$X_t = \int_0^t H_s dZ_s^*, \ t \geq 0,$$

where Z^* *is like* Z *in* a).

REFERENCES

[1] Engelbert H.J. and Schmidt W., On one–dimensional stochastic differential equations with generalized drift, Lecture Notes in Control and Information Sciences, **69**, 143–155, Springer–Verlag, Berlin (1985).

[2] Kallenberg O., Some time change representations of stable integrals, via predictable transformations of local martingales, *Stoch. Process. Appl.* **40** (1992), 199–223.

[3] Karatzas I., Shreve S.E., *Brownian Motion and Stochastic Calculus*, 2nd ed., Springer, New York, (1994).

[4] Zanzotto P.A., On solutions of one–dimensional stochastic differential equations driven by stable Lévy motion, *Stoch. Process. Appl.* **68** (1997), 209–228.

[5] Zanzotto P.A., Representation of a class of semimartingales as stable integrals, *Theory Probab. Appl.*, **43**, 1998, 808–818.

[6] Zanzotto P.A., On stochastic differential equations driven by Cauchy process and the other stable Lévy motions, Technical Report 2.312.1127, Dip. di Matematica, Univ. di Pisa, 1998.

WHITE NOISE GENERALIZATIONS OF THE CLARK-OCONE THEOREM WITH APPLICATION TO MATHEMATICAL FINANCE

KNUT AASE[1),2)], BERNT ØKSENDAL[2),1)] AND JAN UBØE[3)]

[1)]Norwegian School of Economics and Business Administration

Helleveien 30, N-5035 Bergen - Sandviken, Norway

[2)]Department of Mathematics, University of Oslo, Box 1053 Blindern, N-0316 Oslo, Norway

[3)]Stord/Haugesund College, Skåregaten 103, N-5 500, Haugesund, Norway

1. EXTENDED ABSTRACT

Let $B_t(\omega) = B(t,\omega); t \geq 0, \omega \in \Omega$ be a 1-dimensional Wiener process (Brownian motion) on a probability space $(\Omega, \mathcal{F}, P)$ such that $B(0,\omega) = 0$ a.s. P. For $t \geq 0$ let $\mathcal{F}_t$ be the σ-algebra generated by $\{B(s,\cdot); s \leq t\}$. Fix $T > 0$. The *Clark-Ocone theorem* states that if $F = F(\omega) \in L^2(P)$ is $\mathcal{F}_T$-measurable and $F \in \mathbf{D}_{1,2}$ (see definitions below), then

$$F(\omega) = E[F] + \int_0^T E[D_t F|\mathcal{F}_t](\omega) dB_t(\omega) \tag{1.1}$$

where $D_t F = \frac{dF}{d\omega}(t)$ denotes the Malliavin derivative of F at t. This result and its generalizations have important applications in economics, where (basically) $E[D_t F|\mathcal{F}_t]$ represents the replicating portfolio of a given T-claim F. (See, e.g., [KO], [Ø])

Usually this result is presented and proved in the context of analysis on the Wiener space $\Omega = C_0([0,T])$, the space of all real continuous functions on $[0,T]$ starting at 0. Then one can identify each Wiener process path $B(\cdot,\omega)$ with one element $\omega(\cdot) \in \Omega$, which is a computational advantage. It is in this setting that the Malliavin derivative and its properties are usually studied. See, e.g., [N], [U]. However, the drawback with this setting is that the Malliavin derivative only exists for $F \in \mathbf{D}_{1,2}$. This excludes many interesting applications. For example, in mathematical finance one is interested in computing the replicating portfolios of a given T-claim F. If, say, the claim is a digital option of the form

$$F(\omega) = \mathcal{X}_{[K,\infty)}(B_T(\omega)) = \begin{cases} 1 & \text{if } B_T(\omega) \geq K \\ 0 & \text{if } B_T(\omega) < K \end{cases} \tag{1.2}$$

then $D_t F$ does not exist and formula (1.1) cannot be applied. The purpose of this paper is to present a new proof of the Clark-Ocone formula in the setting of white noise analysis. One of the advantages with this approach is that it allows a generalization of the Clark-Ocone formula which is valid for all $\mathcal{F}_T$-measurable $F \in \mathcal{G}^*$, a space of stochastic distributions which contains $L^2(\mu)$, where μ is the white noise probability measure (μ corresponds to P in the Wiener space setting). The generalization has the form (See Theorem 3.15)

$$F(\omega) = E[F] + \int_0^T E[D_t F|\mathcal{F}_t] \diamond W_t dt \tag{1.3}$$

where $\diamond$ denotes the Wick product and $W_t \in (\mathcal{S})^*$ is white noise. $E[F]$ is the generalized expectation of $F \in \mathcal{G}^*$, $E[D_t F|\mathcal{F}_t]$ is the generalized expectation and the integral on the right hand side is an $(\mathcal{S})^*$-valued (Bochner) integral. In view of the identity

$$\int_0^T Y(t,\omega)\delta B(t) = \int_0^T Y(t,\omega) \diamond W_t dt \tag{1.4}$$

valid for all Skorohod integrable processes $Y(t,\omega)$, we see that (1.3) is indeed a generalization of (1.1). In fact, if $F \in L^2(\mu)$ then (1.3) simplifies to

$$F(\omega) = E[F] + \int_0^T E[D_t F|\mathcal{F}_t] dB(t) \tag{1.5}$$

where $D_t F \in \mathcal{G}^*, E[D_t F|\mathcal{F}_t] \in L^2(\mu)$ for a.a. t and

$$E\Big[\int_0^T E[D_t F|\mathcal{F}_t]^2 dt\Big] < \infty \qquad \text{(Theorem 3.11)} \tag{1.6}$$

We emphasize that in the Wiener space setting another generalization of (1.1) has been obtained by S. Ustunel [U, Theorem 1 p.44]. His generalization is valid for all $F \in \mathbf{D}_{-\infty}$, the *Meyer-Watanabe distributions*. Since $\mathbf{D}_{-\infty} \subsetneq \mathcal{G}^*$, our result is more general. Moreover, our approach is entirely different. Recently other approaches to the Malliavin calculus and the Clark-Ocone theorem have been given by F. E. Benth [B], M. de Faria, M. J. Oliveira & L. Streit [dOS], and G. Våge [V].

Our white noise setup can be easily modified to cover more general situations. This is demonstrated in Sections 4-6. In Section 4 we prove the following multidimensional version of the generalized Clark-Ocone theorem:

Let $B(t,\omega) := (B_1(t,\omega_1), \ldots, B_m(t,\omega_m)); \omega = (\omega_1, \ldots, \omega_m) \in \Omega$ be m-dimensional Brownian motion with filtration $\mathcal{F}_T^{(m)}$. Then if $F \in (\mathcal{G}^*)^m$ is $\mathcal{F}_T^{(m)}$-measurable, we have

$$F(\omega) = E[F] + \int_0^T \left(\sum_{j=1}^m E\left[\frac{\partial F}{\partial \omega_j}(t,\omega) \Big| \mathcal{F}_T^{(m)} \right] \diamond W_j(t,\omega_j) \right) dt \tag{1.7}$$

where we have used the notation $\left(\frac{\partial F}{\partial \omega_1}, \ldots, \frac{\partial F}{\partial \omega_m}\right)$ for the Malliavin gradient of F at t (Theorem 4.2).

If we replace the *Gaussian* white noise probability measure μ by the *Poissonian* white noise probability measure ν (on the same underlying space $\mathcal{S}'(\mathbf{R})$), then we obtain a similar theory where Gaussian white noise $W(t)$ is replaced by Poissonian white noise $V(t)$ and Brownian motion $B(t)$ is replaced by compensated Poisson process $Q(t)$. The spaces $\mathcal{G}^* = \mathcal{G}^*(\nu)$ can be defined in a similar way as for the Gaussian case and the Malliavin gradient too. Thus we obtain the following generalized Clark-Ocone theorem for the compensated Poisson process:

If $F \in \mathcal{G}^*(\nu_m)$ is measurable with respect to the filtration $\mathcal{H}_t^{(m)}$ of the m-dimensional compensated Poisson process $Q(t)$, then

$$F(\omega) = E[F] + \int_0^T \left(\sum_{j=1}^m E\left[\frac{\partial F}{\partial \omega_j}(t,\omega) \Big| \mathcal{H}_T^{(m)} \right] \diamond V_j(t,\omega_j) \right) dt$$

This result is proved in Section 5 (See Theorem 5.2).

Then, in Section 6 we point out how the above theory can be modified to cover the case with combinations of Gaussian and Poissonian noises.

Finally, in Section 7 we apply our results to compute the replicating portfolios for the European call option in a Poisson Black and Scholes type market:
Consider a market $X(t) = (A(t), S(t))$ consisting of two investment possibilities:

(i) *a bank account*, where the price $A(t)$ at time t is given by

$$dA(t) = \rho(t)A(t)dt \ ; \ A(0) = 1 \tag{7.1}$$

(ii) *a stock*, where the price $S(t)$ at time t is given by

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dQ(t) \ ; \ S(0) = x > 0 \tag{7.2}$$

where $\rho(t), \mu(t)$, and $\sigma(t)$ are *deterministic* functions in $L^2[0,T]$ ($T > 0$ constant), $\sigma(t) \geq \epsilon$ for some $\epsilon > 0$. As before $Q(t) = P(t) - t$ is the compensated Poisson process. It is well known (see, e.g., [HØ, Example 2.2]) that the solution of (7.2) is given by

$$S(t) = x \exp\left[\int_0^t \ln[1+\sigma(s)]dQ(s) + \int_0^t (\mu(s) - \sigma(s) + \ln[1+\sigma(s)])ds \right] \tag{7.3}$$

Let $(\xi(t,\omega), \eta(t,\omega))$ be a *portfolio*, i.e., $\xi(t), \eta(t)$ gives the number of units of investments #1, #2, respectively, held by an agent at time t. The total value $V(t)$ at time t of such a portfolio is then given by

(7.4) $$V(t) = \xi(t)A(t) + \eta(t)S(t)$$

Assume that the portfolio is *self-financing*, in the sense that

(7.5) $$dV(t) = \xi(t)dA(t) + \eta(t)dS(t)$$

¿From (7.4) we get

(7.6) $$\xi(t) = \frac{V(t) - \eta(t)S(t)}{A(t)}$$

which substituted in (7.5) gives

(7.7) $$dV(t) = \rho(t)V(t)dt + \sigma(t)\eta(t)S(t)\left(\frac{\mu(t) - \rho(t)}{\sigma(t)}dt + dQ(t)\right)$$

Define

(7.8) $$u(t) = \frac{\mu(t) - \rho(t)}{\sigma(t)}$$

Suppose

(7.9) $$u(t) \leq 1 - \epsilon_1 \qquad \text{for some } \epsilon_1 > 0$$

Our main result in this section is the following:

THEOREM 7.1

The replicating portfolio $\xi(t), \eta(t)$ for a European call option with payoff

$$F(\omega) = (S(T) - K)^+$$

in the Poissonian market defined by (7.1), (7.2) and satisfying (7.9), is given by (7.6) and

(7.25) $$\eta(t) = e^{-\int_t^T \rho(s)ds}\sigma(t)^{-1}S(t)^{-1}\ln[1 + \sigma(t)]E_\nu[\mathcal{X}_{[K,\infty)}(Y^y(T-t))Y^y(T-t)]_{y=S(T)}$$

with $Y^y(t)$ given by

$$Y^y(t) = y\exp\left[\int_0^t \ln[1 + \sigma(s)]dQ(s) + \int_0^t (\mu(s) - \sigma(s) + \ln[1 + \sigma(s)](1 - u(s)))ds\right]$$

REMARK

The hedging formula (7.25) appears to be new. Note that the alternative approach often used to compute hedging strategies (the PDE approach) seems difficult to apply here because it involves the calculation of

$$\frac{\partial f}{\partial x}(T-t,x)$$

where $f(T-t,x)$ is the price at time $T-t$ if $S(t)=x$. One can express f in terms of an expectation with respect to ν and this leads to a series expansion for f. This series, however, cannot be differentiated term by term. Pricing in models described by diffusions plus jumps is treated in [CG]. However, that paper does not study the question of finding replicating portfolios.

ACKNOWLEDGEMENTS

This work is supported by VISTA, a research cooperation between The Norwegian Academy of Science and Letters and Den Norske Stats Oljeselskap (Statoil). The authors also wish to thank Ole Barndorff-Nielsen, Fred Espen Benth, Peter Leukert, and Gjermund Våge for several useful comments.

REFERENCES

[AØU] K. Aase, B. Øksendal, and J. Ubøe: Using the Donsker delta function to compute hedging strategies. Preprint University of Oslo, No 6, 1998.

[B] F. E. Benth: The Malliavin derivative of generalized random variables. MaPhySto Research Report, No 11, 1998.

[BC] R. F. Bass and M. Cranston: The Malliavin calculus for pure jump processes and applications to local time. *Annals of Probability*,14, 490–532, 1986.

[BG] F. E. Benth and J. Gjerde: A remark on the equivalence between Poisson and Gaussian stochastic partial differential equations, *Potential Analysis*, 8,179–193, 1998.

[BGJ] K. Bichteler, J.-B. Graveraux, and J. Jacod: *Malliavin Calculus for Processes with Jumps*. Gordon & Breach 1987.

[BP] F. E. Benth and J. Potthoff: On the martingale property for generalized stochastic processes. *Stochastics*, 58, 349–367, 1996.

[BVW] R. Boiel, P. Varaiya, and E. Wong: Martingales on jump processes; Part I Representation Results; Part II Applications, *SIAM J. Control*, 13, 999–1061, 1975.

[CG] J. D. Cummings and H. Geman: Pricing catastrophe insurance futures and call spreads: an arbitrage approach. *Journal of Fixed Income*, March, 1–24, 1995.

[dOS] M. de Faria, M. J. Oliveira and L. Streit: A generalized Clark-Ocone formula. Preprint IFM 2/98, Universidade de Lisboa, 1998.

[DPV] T. Deck, J. Potthoff, and G. Våge: A review of white noise analysis from a probabilistic standpoint. *Acta Applicandae Mathematicae*, 48, 91–112, 1997.

[ET] R. J. Elliott and A. H. Tsoi: Integration by parts for Poisson processes. *Journal of Multivariate Analysis*, 44, 179–190, 1993.

[GKS] M. Grothaus, Y. G. Kondratiev, and L. Streit: Regular generalized functions in Gaussian analysis. Manuscript 1998.

[HKPS] T. Hida, H.-H. Kuo, J. Potthoff, and L. Streit: *White Noise Analysis*. Kluwer 1993.

[HØ] H. Holden and B. Øksendal: A White Noise Approach to Stochastic Differential Equations Driven by Wiener and Poisson Processes, to appear in M. Oberguggenberger (editor): *Proceedings on Nonlinear Theory of Generalized Functions ESI 97*, Vienna, 1998.

[HØUZ] H. Holden, B. Øksendal, J. Ubøe, and T.-S. Zhang: *Stochastic Partial Differential Equations - A Modeling, White Noise Functional Approach*, Birkhäuser, Boston 1996.

[I] K. Itô: Multiple Wiener integral. *J. Math. Soc. Japan*, 3, 157–169, 1951.

[J] J. Jacod: Multivariate Point Processes; predictable projection, Radon-Nikodym derivative, representation of martingales. *Probability Theory and Related Fields*, 31, 235–253, 1975.

[KO] I. Karatzas and D. Ocone: A generalized Clark representation formula, with application to optimal portfolios. *Stochastics and Stochastic Reports*, 34, 187–220, 1991.

[N] D. Nualart: *The Malliavin Calculus and Related Topics*. Springer-Verlag, 1995.

[Ø] B. Øksendal: An introduction to Malliavin calculus with applications to economics. Working paper, No 3/96, Norwegian School of Economics and Business Administration, 1996.

[PT] J. Potthoff and M. Timpel: On a dual pair of smooth and generalized random variables. *Potential Analysis*, 4, 637–654, 1995.

[U] A. S. Üstünel: *An Introduction to Analysis on Wiener Space*. Springer Lecture Notes in Mathematics, No 1610, Springer-Verlag, 1995.

[V] G. Våge: A note on the Clark-Ocone theorem. Manuscript, 1998.

Additional references to works on Lévy Processes

Applebaum, D. (1998). Operator-valued stochastic differential equations arising from unitary group representations, *Technical report*.

Applebaum, D. & Brooks, M. (1996). Infinite series of quantum spectral stochastic integrals, *J. Operator Theory* **36**(2): 295–316.

Applebaum, D. & Cohen, S. (1997). Stochastic parallel transport along Lévy flows of diffeomorphisms, *J. Math. Anal. Appl.* **207**(2): 496–505.

Applebaum, D. & Kunita, H. (1997). Invariant measures for Lévy flows of diffeomorphisms, *Technical report*.

Applebaum, D. & Wu, J.-L. (1997). Stochastic partial differential equations driven by Lévy space-time white noise, *Technical report*.

Barchielli, A. & Paganoni, A. (1996). A note on a formula of the Lévy-Khinchin type in quantum probability, *Nagoya Math. J.* **141**: 29–43.

Basrak, B., Davis, R. & Mikosch, T. (1998). The sample acf of a simple bilinear process with dependent heavy-tailed steps, *Technical report*, Institute of Mathematics and Computing Science of the University of Groningen.

Baxter, G. & Donsker, M. D. (1957). On the distribution of the supremum functional for processes with stationary independent increments, *Trans. Amer. Math. Soc.* **85**: 73–87.

Bertoin, J. (1992). An extension of Pitman's theorem for spectrally positive Lévy processes, *Ann. Probab.* **20**(3): 1464–1483.

Bertoin, J. (1996). *Lévy processes*, Cambridge University Press, Cambridge.

Bertoin, J. (1997). Exponential decay and ergodicity of completely asymmetric Lévy processes in a finite interval, *Ann. Appl. Probab.* **7**(1): 156–169.

Bingham, N. H. (1976). Continuous branching processes and spectral positivity, *Stochastic Processes Appl.* **4**(3): 217–242.

Braverman, M. (1997). Suprema and sojourn times of Lévy processes with exponential tails, *Stochastic Process. Appl.* **68**(2): 265–283.

Braverman, M. & Samorodnitsky, G. (1998). Symmetric infinitely divisible processes with sample paths in Orlicz spaces and absolute continuity of infinitely divisible processes, *Stochastic Process. Appl.* **78**(1): 1–26.

Chaumont, L. (1996). Conditionings and path decompositions for Lévy processes, *Stochastic Process. Appl.* **64**(1): 39–54.

Chiang, T., Chow, Y. & Lee, Y. (1994). Exact formulas of certain functional integrals on Wiener spaces, *Stochastics Stochastics Rep.* **50**: 211–223.

Choi, G. S. & Sato, K.-i. (1995). Recurrence and transience of operator semi-stable processes, *Proc. Japan Acad. Ser. A Math. Sci.* **71**(5): 87–89.

Dalang, R. C. & Hou, Q. (1997). On Markov properties of Lévy waves in two dimensions, *Stochastic Process. Appl.* **72**(2): 265–287.

Damien, P., Laud, P. W. & Smith, A. F. (1995). Approximate random variate generation from infinitely divisible distributions with applications to Bayesian inference, *J. R. Statist. Soc. B* **57**: 547–563.

Dassios, A. (1996). Sample quantiles of stochastic processes with stationary and independent increments, *Ann. Appl. Probab.* **6**(3): 1041–1043.

Dozzi, M. & Soltani, A. R. (1997). Local time for stable moving average processes: Hölder conditions, *Stoch. Proc. Appl.* **68**(2): 195–207.

Dudley, R. M. (1966). Lorentz-invariant Markov processes in relativistic phase space, *Ark. Mat.* **6**: 241–268.

Dudley, R. M. (1967). A note on Lorentz-invariant Markov processes, *Ark. Mat.* **6**: 575–581 (1967).

Dudley, R. M. (1973). Asymptotics of some relativistic Markov processes, *Proc. Nat. Acad. Sci. U.S.A.* **70**: 3551–3555.

Dudley, R. M. (1974). Recession of some relativistic Markov processes, *Rocky Mountain J. Math.* **4**: 401–406.

Ferguson, T. S. & Klass, M. J. (1972). A representation of independent increment processes without Gaussian components, *Ann. Math. Statist.* **43**: 1634–1643.

Franz, U. (1998). Classical Markov processes from quantum Lévy processes, *Technical report*, Université Louis Pasteur.

Franz, U. (1999). Classical Markov processes from quantum Lévy processes, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **2**(1): 105–129.

Gangolli, R. (1964). Isotropic infinitely divisible measures on symmetric spaces, *Acta Math.* **111**: 213–246.

Gangolli, R. (1965). Sample functions of certain differential processes on symmetric spaces, *Pacific J. Math.* **15**: 477–496.

Getoor, R. K. & Sharpe, M. J. (1994). Local times on rays for a class of planar Lévy processes, *J. Theoret. Probab.* **7**(4): 799–811.

Gīhman, Ĭ. Ī. & Skorohod, A. V. (1980). *The theory of stochastic processes. I, Chapter IV, Processes with Independent Increments*, english edn, Springer-Verlag, Berlin.

Hansen, B. G. (1996). Stability and self-decomposability of semi-group valued random variables, *Statist. Neerlandica* **50**(2): 295–305.

Itô, K. (1956). Spectral type of the shift transformation of differential processes with stationary increments, *Trans. Amer. Math. Soc.* **81**: 253–263.

Küchler, U. & Sørensen, M. (1994). Exponential families of stochastic processes and Lévy processes, *J. Statist. Plann. Inference* **39**(2): 211–237.

Le Gall, J.-F. & Le Jan, Y. (1998a). Branching processes in Lévy processes: Laplace functionals of snakes and superprocesses, *Ann. Probab.* **26**(4): 1407–1432.

Le Gall, J.-F. & Le Jan, Y. (1998b). Branching processes in Lévy processes: the exploration process, *Ann. Probab.* **26**(1): 213–252.

Leblanc, B. & Yor, M. (1998). Lévy processes in finance: A remedy to the non-stationarity of continuous martingales, *Finance Stoch.* **2**: 399–408.

Lee, M.-L. T. & Whitmore, G. A. (1993). Stochastic processes directed by randomized time, *J. Appl. Prob.* **30**: 302–314.

Lévy, P. (1934). Sur les intégrales dont les éléments sont des variables aléatoires indépendantes, *Ann. Scuola Norm. Sup. Pisa (2)* **3**: 217–218.

Lévy, P. (1939). Sur certains processus stochastiques homogènes, *Compositio Math.* **7**: 283–339.

Lévy, P. (1954). *Théorie de l'addition des variables aléatoires*, Gauthier-Villars & Cie, Paris. Suivi d'une note de M. Loève. Deuxième édition revue et augmentée.

Lévy, P. (1965). *Processus stochastiques et mouvement brownien*, Gauthier-Villars & Cie, Paris. Suivi d'une note de M. Loève. Deuxième édition revue et augmentée.

Lyons, R. (1996). Probabilistic aspects of infinite trees and some applications, *Trees (Versailles, 1995)*, Birkhäuser, Basel, pp. 81–94.

Maejima, M. (1996). Limit theorems related to a class of operator-self-similar processes, *Nagoya Math. J.* **142**: 161–181.

Pakes, A. G. (1996). A hitting time for Lévy processes, with applications to dams and branching processes, *Ann. Fac. Sci. Toulouse Math. (6)* **5**(3): 521–544.

Protter, P. & Talay, D. (1997). The Euler scheme for Lévy driven stochastic differential equations, *Ann. Probab.* **25**(1): 393–423.

Rentzsch, C. (1999). Lévy-Khintchine representation on local Sturm-Liouville hypergroups, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **2**(1): 79–104.

Rogers, L. C. G. (1984). A new identity for real Lévy processes, *Ann. Inst. Henri Poincaré* **20**: 21–34.

Rosiński, J. (1991). On a class of infinitely divisible processes represented as mixtures of Gaussian processes, *Stable processes and related topics (Ithaca, NY, 1990)*, Birkhäuser Boston, Boston, MA, pp. 27–41.

Sato, K.-i. (1990). Subordination depending on a parameter, *Probability theory and mathematical statistics, Vol. II (Vilnius, 1989)*, "Mokslas", Vilnius, pp. 372–382.

Sato, K.-i. (1998). Multivariate distributions with selfdecomposable projections, *J. Korean Math. Soc.* **35**(3): 783–791.

Sato, K.-i. & Steutel, F. W. (1998). Note on the continuity of infinitely divisible distributions and canonical measures, *Statistics* .

Sato, K.-i., Watanabe, T., Yamamuro, K. & Yamazato, M. (1996). Multidimensional process of Ornstein-Uhlenbeck type with nondiagonalizable matrix in linear drift terms, *Nagoya Math. J.* **141**: 45–78.

Sato, K.-i., Watanabe, T. & Yamazato, M. (1994). Recurrence conditions for multidimensional processes of Ornstein-Uhlenbeck type, *J. Math. Soc. Japan* **46**(2): 245–265.

Sato, K.-i. & Yamamuro, K. (1998). On selfsimilar and semi-selfsimilar processes with independent increments, *J. Korean Math. Soc.* **35**: 207–224.

Sato, K.-i. & Yamazoto, M. (1984). Operator-selfdecomposable distributions as limit distributions of processes of Ornstein-Uhlenbeck type, *Stochastic Process. Appl.* **17**: 73–100.

Schoutens, W. & Teugels, J. L. (1998). Lévy processes, polynomials and martingales, *Comm. Statist. Stochastic Models* **14**(1-2): 335–349.

Shieh, N.-R. (1998). Multiple points of dilation-stable Lévy processes, *Ann. Probab.* **26**(3): 1341–1355.

Skorohod, A. V. (1991). *Random processes with independent increments*, Kluwer Academic Publishers Group, Dordrecht.

Soltani, A. R. & Moeanaddin, R. (1994). On dispersion of stable random vectors and its application in the prediction of multivariate stable processes, *J. Appl. Probab.* **31**(3): 691–699.

Szulga, J. (1991). Multiple stochastic integrals with respect to symmetric infinitely divisible random measures, *Ann. Probab.* **19**(3): 1145–1156.

Urbanik, K. (1996). Autoregressive Laplace functionals on stochastic processes, *Probab. Math. Statist.* **16**(2): 243–260.

Us, G. F. (1997). Towards a coloured noise analysis, *Methods Funct. Anal. Topology* **3**(2): 83–99.

Walker, S. G., Damien, P., Laud, P. W. & Smith, A. F. (1999). Bayesian nonparametric inference for random distributions and related functions, *J.R. Statist. Soc. B* **61**: 1–26.

Wolpert, R. L. & Ickstadt, K. (1998). Poisson/gamma random field models for spatial statistics, *Biometrika* **85**(2): 251–267.

Zanzotto, P. A. (1997). On solutions of one-dimensional stochastic differential equations driven by stable Lévy motion, *Stochastic Process. Appl.* **68**(2): 209–228.