HIGHER SKEIN MODULES

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Abstract. We introduce higher skein modules of links generalizing the Conway skein module. We show that these modules are closely connected to the HOMFLY polynomial.

1. Introduction

The notion of a skein module of links arises naturally from the study of link polynomials. For instance, the one-variable Conway polynomial ∇ of links in the 3-sphere S^3 satisfies the fundamental skein relation $\nabla(X_+) = \nabla(X_-) + h\nabla(X_0)$ where h is the variable and X_+, X_-, X_0 are any three oriented links coinciding outside a 3-ball and looking as in Figure 1 inside this ball. This suggests to consider the $\mathbb{Z}[h]$-module generated by the isotopy classes of oriented links in S^3 modulo the relations $X_+ - X_- - hX_0 = 0$ corresponding to all triples (X_+, X_-, X_0) as above. This is the Conway skein module of S^3. Applying similar definitions to links in an oriented 3-manifold M, we obtain the skein module of M, cf. [4] and [5].

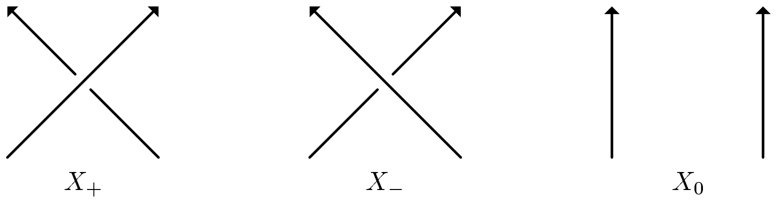

Figure 1. X_+, X_-, and X_0.

In this paper we introduce "higher" versions of the Conway skein module. Our approach is inspired by the theory of Vassiliev link invariants. In that theory one considers singular links, i.e., links with double points as in Figure 2. Each double point $X_\bullet$ is resolved into the formal difference $X_+ - X_-$ of the positive and negative crossings. In this way each singular link with n double points is resolved into a formal linear combination of 2^n (non-singular) links. This yields the Vassiliev filtration $V = V_0 \supset V_1 \supset V_2 \supset \dots$ where V is the abelian group freely generated by the isotopy classes of oriented links and V_n is its subgroup generated by the resolutions of singular links with n double points. The consecutive quotients V_n/V_{n+1} are among the main objects of the theory of Vassiliev invariants.

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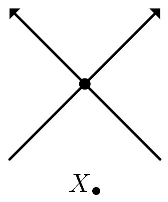

FIGURE 2. A double point $X_\bullet$.

We consider here a deformation of the Vassiliev filtration. The idea is to resolve the double points via the formula in Figure 3. More precisely, for an oriented 3-manifold M denote by $A(M)$ the free $\mathbb{Z}[h]$-module generated by the isotopy classes of oriented (non-empty) links in M. Here $\mathbb{Z}[h]$ is the ring of polynomials in h with integer coefficients. By a singular link in M, we mean an immersion of a finite system of oriented circles in M with only double transversal intersections. Using the formula

$$r(X_\bullet) = X_+ - X_- - hX_0,$$

we resolve each singular link L with n double points into a formal sum $r(L) \in A(M)$ of 3^n terms. Denote by A_n the $\mathbb{Z}[h]$-submodule of $A(M)$ generated by $r(L)$ where L runs over all singular links with n double points. Clearly, $A(M) = A_0 \supset A_1 \supset A_2 \supset \dots$. The quotient A_0/A_1 is the Conway skein module of M. We call the $\mathbb{Z}[h]$-modules A_n/A_{n+1} with $n = 1, 2, \dots$ the higher Conway skein modules of M. Our aim is to compute them (at least partially) in the case $M = S^3$. In the sequel, we restrict ourselves to links in S^3 unless explicitly stated to the contrary. Set $A = A(S^3)$.

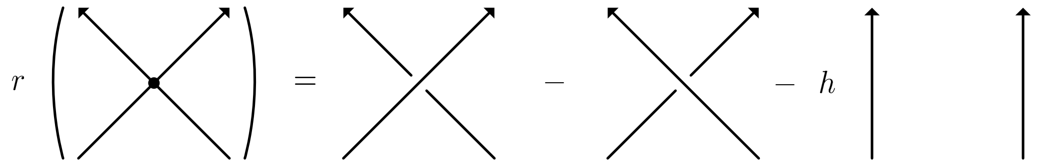

FIGURE 3. The resolution r of a double point.

Recall first the structure of $A_0/A_1 = A/A_1$. The Conway polynomial ∇ (normalised so that its value on an unknot is 1) defines a $\mathbb{Z}[h]$-linear epimorphism $A/A_1 \to \mathbb{Z}[h]$. Its kernel is a free abelian group freely generated by the classes in A/A_1 of the trivial links with ≥ 2 components. A standard argument (reproduced below) shows that multiplication by h annihilates these classes. Therefore $A/A_1 = h\text{-torsion} \oplus \mathbb{Z}[h]$. (By the h-torsion of a $\mathbb{Z}[h]$-module M we mean the set $\{a \in M \mid ha = 0\}$).

It turns out that similar results hold for the higher skein modules. To state our theorems we introduce a two-parameter family of singular links G_n^l with n double points where $n \geq 0$ and $l = 0, 1, \dots, n$, see Figure 4. The singular link G_n^l is formed by $n - l$ embedded circles crossing one immersed circle with l curls. According to our definitions, G_n^l represents an element $r(G_n^l) \in A_n/A_{n+1}$.

Our main theorem on the structure of A_n/A_{n+1} says the following.

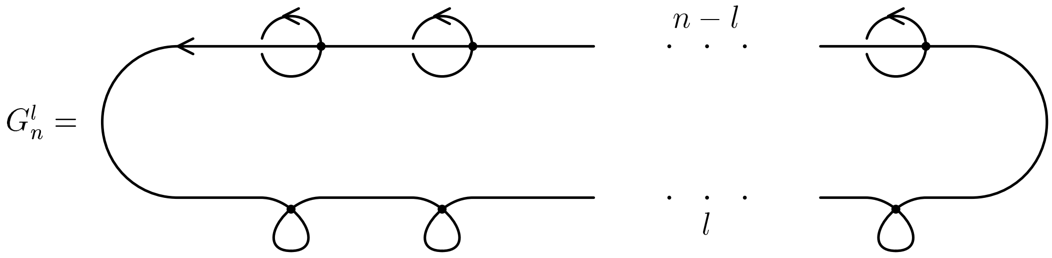

FIGURE 4. The singular link G_n^l.

Theorem 1.1. *For each $n \geq 0$,*

$$A_n/A_{n+1} = h\text{-}torsion \oplus \bigoplus_{l=0}^{n} \mathbb{Z}[h]\, r(G_n^l).$$

This theorem may be reformulated by saying that the quotient of the module A_n/A_{n+1} by its h-torsion is a free $\mathbb{Z}[h]$-module of rank $n+1$ with free generators $r(G_n^l) \mod A_{n+1}$ where $l = 0, 1, \dots, n$. The h-torsion of A_n/A_{n+1} is in general quite big. Indeed setting $h = 0$ we obtain a projection $A \to V$ mapping the filtration $A = A_0 \supset A_1 \supset \dots$ onto the Vassiliev filtration $V = V_0 \supset V_1 \supset \dots$. This induces a $\mathbb{Q}$-linear epimorphism from $\mathbb{Q} \otimes (A_n/A_{n+1})$ onto the vector space $\mathbb{Q} \otimes (V_n/V_{n+1})$ isomorphic to the vector space of chord diagrams with n chords modulo the 4T- and 1T-relations. It follows from Theorem 1.1 that $\mathbb{Q} \otimes (h - \text{torsion of}\, A_n/A_{n+1})$ lies in the latter vector space of chord diagrams as a subspace of codimension $n+1$.

In this paper we shall be mainly interested in the $\mathbb{Z}[h]$-free part of A_n/A_{n+1}. Tensoring with the ring of Laurent polynomials $\Lambda = \mathbb{Z}[h, h^{-1}]$ we obtain the following.

Corollary 1.2. *For each $n \geq 0$,*

$$\Lambda \otimes_{\mathbb{Z}[h]} (A_n/A_{n+1}) = \bigoplus_{l=0}^{n} \Lambda\, r(G_n^l) \quad \textit{and} \quad \Lambda \otimes_{\mathbb{Z}[h]} (A/A_n) = \bigoplus_{\substack{l,m \geq 0 \\ l+m<n}} \Lambda\, r(G_{l+m}^l).$$

Corollary 1.3. *There are unique $\mathbb{Z}[h]$-linear homomorphisms $\nabla_{l,m} : A \to \Lambda$ numerated by pairs of non-negative integers (l, m) such that for any $a \in A$,*

$$a = \sum_{l,m} \nabla_{l,m}(a)\, r(G_{l+m}^l) \in \varprojlim{}_n \Lambda \otimes_{\mathbb{Z}[h]} (A/A_n).$$

Applying this to any oriented link L we obtain an expansion

$$L = \sum_{l,m} \nabla_{l,m}(L)\, r(G_{l+m}^l) \in \varprojlim{}_n \Lambda \otimes_{\mathbb{Z}[h]} (A/A_n).$$

By definition of $\nabla_{l,m}$, we have

$$\nabla_{l,m}(A_{l+m+1}) = 0 \quad \text{and} \quad \nabla_{l,m}(r(G_{m'}^{l'})) = \delta_l^{l'} \delta_{l+m}^{m'}$$

where δ is the Kronecker delta. It is easy to check that $h^{l+m}\nabla_{l,m}(A) \subset \mathbb{Z}[h]$, for all l, m. In particular, $\nabla_{0,0}$ annihilates A_1 and maps the trivial knot G_0^0 into 1. Hence $\nabla_{0,0} = \nabla$ is the Conway polynomial. The following theorem computes $\nabla_{l,m}$ from $\nabla_{l,0}$.

Theorem 1.4. *For any $l \geq 0, m \geq 1$ and any oriented link L,*

$$\nabla_{l,m}(L) = \frac{h^{-l}}{m!}(h^l \nabla_{l,0}(L))^{(m)}$$

where $f^{(m)}$ is the m-th derivative of a Laurent polynomial $f \in \Lambda$. In particular,

$$\nabla_{0,m}(L) = (m!)^{-1}(\nabla(L))^{(m)}.$$

Theorem 1.4 and the inclusion $h^l \nabla_{l,0}(L) \in \mathbb{Z}[h]$ imply the following.

Corollary 1.5. *For all links L and all l, m we have that $h^l \nabla_{l,m}(L) \in \mathbb{Z}[h]$.*

It turns out that the sequence of link polynomials $\nabla_{l,0}$ with $l = 0, 1, \ldots$ is equivalent to the Homfly polynomial. By the Homfly polynomial, $\tilde{\nabla}$, we shall mean the (unique) mapping from the set of isotopy classes of oriented links in S^3 into the ring of Laurent polynomials $\mathbb{Z}[x, x^{-1}, h, h^{-1}]$ which is uniquely characterised by the following two properties:

(i) the value of $\tilde{\nabla}$ on an unknot is equal to 1;

(ii) for any three oriented links X_+, X_-, X_0 coinciding outside a 3-ball and looking as in Figure 1 inside this ball, we have that

$$x\tilde{\nabla}(X_+) - x^{-1}\tilde{\nabla}(X_-) = h\tilde{\nabla}(X_0). \tag{1.1}$$

Clearly, the Conway polynomial ∇ is obtained from $\tilde{\nabla}$ by the substitution $x = 1$.

Theorem 1.6. *For any oriented link L, the formal power series*

$$P(L)(h, u) = \sum_{l \geq 0} (-h)^l \nabla_{l,0}(L)(h)\, u^l$$

is a reparametrisation of $\tilde{\nabla}(L)$.

The precise form of the reparametrisation in this theorem is a little technical (of course it does not depend on the choice of L). We shall give a detailed statement in Section 6. Theorems 1.4 and 1.6 imply that the polynomials $\{\nabla_{l,m}\}_{l,m}$ determine and are determined by the Homfly polynomial.

Remarks 1.7. 1. Applying the definition of the resolution r inductively to all l curls of G^l_{l+m}, we obtain

$$r(G^l_{l+m}) = -hu\, r(G^{l-1}_{l+m-1}) = h^2u^2 r(G^{l-2}_{l+m-2}) = \ldots = (-h)^l u^l r(G^0_m)$$

where the variable u acts on A as the disjoint union with an unknot. (Note that multiplication by hu maps each A_n into A_{n+1}). This gives for each oriented link L, an expansion

$$L = \sum_{l,m} \nabla^0_{l,m}(L)\, u^l r(G^0_m) \in \varprojlim{}_n \Lambda \otimes_{\mathbb{Z}[h]} (A/A_n)$$

where

$$\nabla^0_{l,m}(L) = (-h)^l \nabla_{l,m}(L) \in \mathbb{Z}[h].$$

2. It is instructive to set $h = \pm 1$ in our constructions. To this end, consider the abelian group V freely generated by the isotopy classes of oriented links. For $\varepsilon = \pm 1$, consider the homomorphism $A \to V$ mapping any element $\sum_i p_i(h) L_i \in A$ into $\sum_i p_i(\varepsilon) L_i \in V$ where $p_i(h) \in \mathbb{Z}[h]$ and $\{L_i\}_i$ are oriented links. For $n \geq 0$, let V^ε_n be the image of A_n under this homomorphism. Clearly, V^ε_n is the subgroup of V generated by the resolutions of singular links with n double points where we use

the resolution, r^{ε}, obtained from the one in Figure 3 by setting $h = \varepsilon$. Corollary 1.2 allows us compute the quotients associated with the filtration $V = V_0^{\varepsilon} \supset V_1^{\varepsilon} \supset V_2^{\varepsilon} \supset \dots$. Namely, $V^{\varepsilon}/V_n^{\varepsilon}$ is a free abelian group of rank $n(n+1)/2$ freely generated by the elements $r^{\varepsilon}(G^l_{l+m}) \mod V_n^{\varepsilon}$ with $l+m < n$. As above, for any oriented link L, we have an expansion

$$L = \sum_{l,m\geq 0} p^{\varepsilon}_{l,m}(L)\, r^{\varepsilon}(G^l_{l+m}) \in \varprojlim{}_n V/V_n^{\varepsilon}$$

with $p^{\varepsilon}_{l,m}(L) \in \mathbb{Z}$. Theorem 1.4 implies that

$$p^{\varepsilon}_{l,m}(L) = \nabla_{l,m}(\varepsilon) = \varepsilon^{-l}(m!)^{-1}(h^l \nabla_{l,0})^{(m)}(\varepsilon)$$

where $\nabla_{l,m} = \nabla_{l,m}(L) \in \Lambda$. Thus, the numbers $p^{\varepsilon}_{l,m}(L)$ are the coefficients in the expansion of $h^l\nabla_{l,0}$ as a formal power series in $h - \varepsilon$:

$$\nabla_{l,0}(L) = \varepsilon^l h^{-l} \sum_{m\geq 0} p^{\varepsilon}_{l,m}(L)(h-\varepsilon)^m.$$

By Theorem 1.6, the numbers $\{p^{\varepsilon}_{l,m}(L)\}$ determine the Homfly polynomial and are determined by it.

3. In the sequel to this paper the authors will consider higher Homfly skein modules and higher Kauffman skein modules of 3-manifolds.

The paper is organised as follows. In Section 2 we prove that A_n/A_{n+1} is generated by the h-torsion and the generators specified in Theorem 1.1. In Section 3 we introduce a certain quotient of the vector space generated by chord diagrams modulo the 4T-relation. This quotient is used in Section 4 where we complete the proof of Theorem 1.1. In Section 5 we prove Theorem 1.4. In Section 6 we prove Theorem 1.6.

4. Assume we have a pair of oriented links L_1 and L_2 with the same Homfly polynomial. Then by Theorem 1.6 and Theorem 1.4 we see that $\nabla_{l,m}(L_1) = \nabla_{l,m}(L_2)$ for all l, m. So for the descending filtration $\Lambda\otimes_{\mathbb{Z}[h]}A = \Lambda\otimes_{\mathbb{Z}[h]}A_0 \supset \Lambda\otimes_{\mathbb{Z}[h]}A_1 \supset \dots$ we have by Corollary 1.2 that

$$L_1 - L_2 \in \bigcap_n \left(\Lambda \otimes_{\mathbb{Z}[h]} A_n\right).$$

It is well know that if an oriented link L_2 is the mutant of some other oriented link L_1, then $\tilde{\nabla}(L_1) = \tilde{\nabla}(L_2)$ (see [3]). Hence it is certainly clear from this that

$$\bigcap_n \left(\Lambda \otimes_{\mathbb{Z}[h]} A_n\right) \neq \{0\}.$$

2. The 8T-relation and generators of A_n/A_{n+1}

We begin with a fundamental relation for singular links, which we call the 8T-relation.

Proposition 2.1. *We have the identity in Figure 5, once all double points are resolved as in Figure 3.*

It is understood that all eight local pictures in Figure 5 are completed by one and the same singular tangle to form eight singular links in the 3-sphere. Alternatively, one may view the identity in Figure 5 as a formal relation between singular tangles which lies in the kernel of the resolution map r.

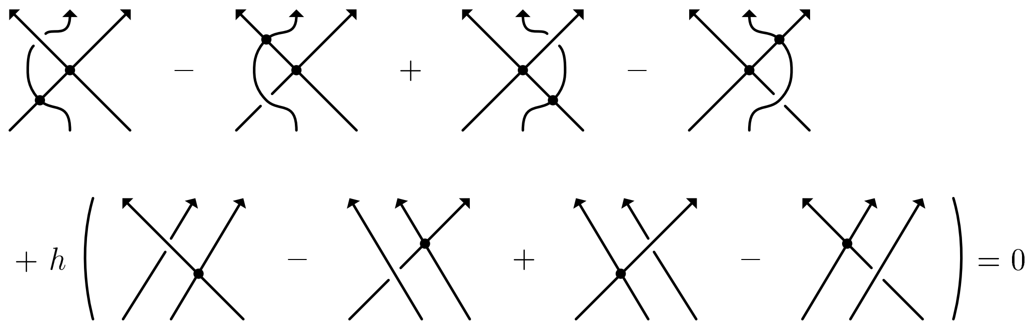

FIGURE 5. The 8T-relation.

Proof. Consider the strand leading from the second input to the second output in the first four pictures. This strand contains one double point and one over/under-crossing. Resolve this double point in each of these four pictures. This yields an algebraic sum of eight terms with coefficient h^0 and four terms with coefficient h. The sum of eight terms vanishes while the sum of four terms is exactly the opposite of the sum in the second row in Figure 5.

□

We shall need the following lemma

Lemma 2.2. *Let $a \in A_l$. Then disjoint union with ha maps A_n to A_{n+l+1}.*

Proof. It suffices to prove that for any singular link L with l double points and any singular link L' with n double points, $hr(L \amalg L') \in A_{n+l+1}$. Consider the singular link N with $n+l+1$ double points obtained from L and L' as in Figure 6. The result of resolving the double point in the center is $-hL \amalg L'$, hence $hr(L \amalg L') = -r(N) \in A_{n+l+1}$.

□

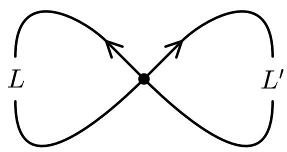

FIGURE 6. The singular link N.

Proposition 2.3. *For any $n \geq 0$,*

$$A_n/A_{n+1} = h\text{-}torsion \oplus span_{\mathbb{Z}[h]}\{r(G_n^0), \ldots, r(G_n^n)\}.$$

Proof. We first derive some consequences of the 8T-relation. For any $(3,3)$-tangle with $n-1$ double points we complete the eight local pictures in Figure 5 with that tangle so as to obtain eight singular links. The first four pictures in Figure 5 yield after resolution of double points elements of A_{n+1}, which shall be ignored in the following calculations proceeding in A_n/A_{n+1}. Thus we can complete the second row in Figure 5 by any $(3,3)$-tangle with $n-1$ double points and obtain a 4-term relation in A_n/A_{n+1}.

Let us now connect the middle top strand to the bottom left strand and add a negative crossing at the bottom in the four pictures in the second row of Figure 5. By the argument above, we obtain a valid identity in A_n/A_{n+1}, see Figure 7.

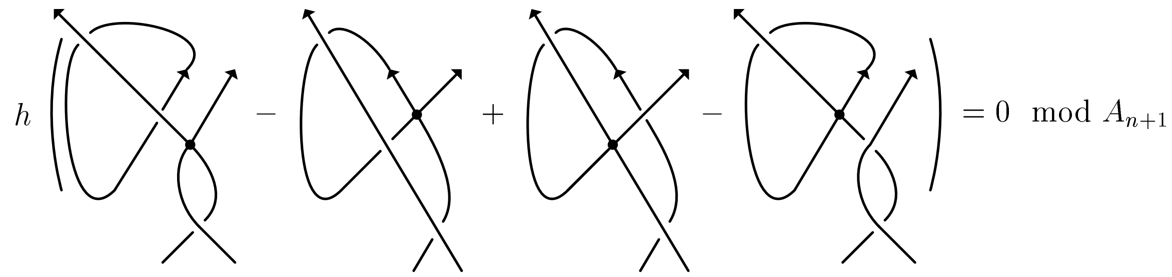

FIGURE 7.

Observe that the first term in the equation in Figure 7 is in A_{n+1} by Lemma 2.2. Hence we obtain the *basic relation* in A_n/A_{n+1}, see Figure 8.

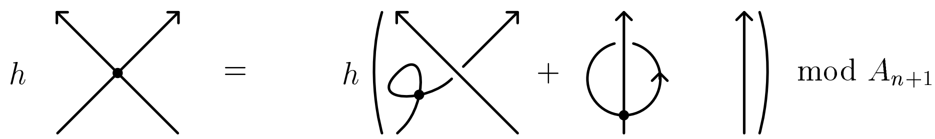

FIGURE 8. The basic relation in A_n/A_{n+1}.

To prove the proposition it is enough to show that for any singular link L with n double points,

$$hr(L) \in h\,\mathrm{span}_{\mathbb{Z}[h]}\{r(G_n^0), \ldots, r(G_n^n)\} \mod A_{n+1}.$$

The basic relation implies that $hr(L) = \sum_{i=1}^m hr(L_i) \mod A_{n+1}$ where $m = 2^n$ and $L_1, \ldots, L_m$ are singular links with n double points such that all their double points are as on the right-hand side of the equality in Figure 9. In other words, each L_i is obtained from a non-singular link by inserting a certain number, say l_i, curls and attaching $n - l_i$ small unknotted circles meeting L_i in one point as in Figure 9.

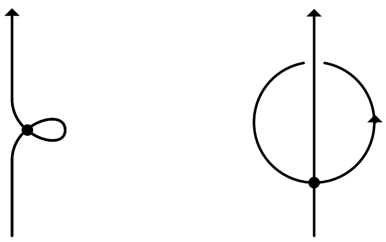

FIGURE 9. Singular tangles T_1 and T_2.

To compute $hr(L_i) \mod A_{n+1}$ we can use the same method as in the usual recursive computation of the Conway polynomial of a link. The role of the skein relation is played here by the fact that we are computing modulo A_{n+1}. This shows that each $hr(L_i) \mod A_{n+1}$ expands as a linear combination over $\mathbb{Z}[h]$ of certain $hr(L_i^j)$ where each L_i^j is a disjoint union of singular links of the form G_t^s. (We note that the order of any two double points along the strand in Figure 4 can be changed in an arbitrary way without changing the resolution r. Indeed, since any curl resolves to a disjoint union of an unknot times $-h$ and the rest, we can move this unknot anywhere and reattach it). By Lemma 2.2, if L_i^j is disconnected then $hr(L_i^j) = 0 \mod A_{n+1}$ so that we need to consider only connected L_i^j. Then by the remarks above, $L_i^j = G_n^{l_i}$. Hence, $hr(L_i) = hp_i(h)r(G_n^{l_i}) \mod A_{n+1}$ where $p_i(h) \in \mathbb{Z}[h]$. This completes the proof of the proposition.

□

Remark 2.4. The arguments given in the proof of Proposition 2.3 allow us to compute the coefficients in the expansion

$$hr(L) = h \sum_{l=0}^{n} q_l \, r(G_n^l) \mod A_{n+1}$$

where L is a singular link with n double points and $q_l \in \mathbb{Z}[h]$. Denote by $\operatorname{sing}(L)$ the set of double points of L. For each subset $X \subset \operatorname{sing}(L)$ denote by L_X the non-singular link obtained from L as follows: all double points of L belonging to X are replaced with negative crossings and all other double points of L are smoothed. Then

$$q_l = \sum_{\substack{X \subset \operatorname{sing}(L) \\ \operatorname{card}(X) = l}} \nabla(L_X)$$

where ∇ is the Conway polynomial of links.

3. The 4TS-relation for chord diagrams

3.1. **Chord diagrams.** By a chord diagram we mean a finite family of oriented circles with finitely many disjoint chords attached to them. Here a chord connects either two distinct points of the same circle or two points belonging to different circles. In our pictures, chords are represented by fat dots, while the ordinary intersections of strands should be ignored.

For $n \geq 0$, let ch_n be the vector space over $\mathbb{Q}$ generated by chord diagrams with n chords. We shall consider the following two quotients, C_n and D_n, of ch_n:

$$C_n = ch_n/4TS \quad \text{and} \quad D_n = C_n/4T = ch_n/4T, 4TS$$

where 4T is the standard 4-term relation for chord diagrams shown in Figure 10 and 4TS is the 4-term relation for chord diagrams shown in Figure 11. The abbreviation 4TS should become more clear in Section 3.3.

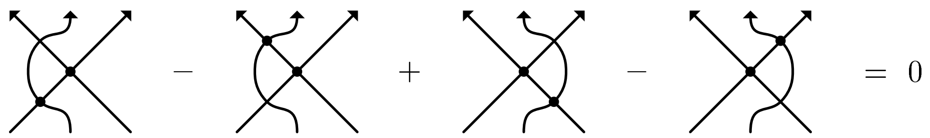

FIGURE 10. The 4T-relation.

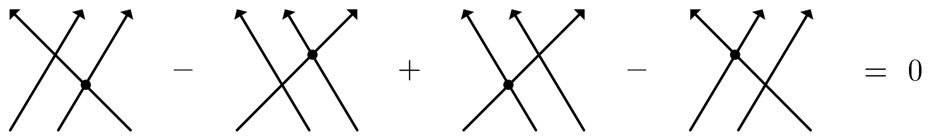

FIGURE 11. The 4TS-relation.

The vector spaces ch_n, C_n, and D_n have the structure of a module over the polynomial ring $\mathbb{Q}[u]$ on one variable u. The variable u acts on a chord diagram d by adding an oriented circle g_0 without chords: $ud = d \coprod g_0$.

Every $\mathbb{Q}[u]$-module M has a completion $\hat{M}$ defined as the limit of the projective system $M \leftarrow M/uM \leftarrow M/u^2M \leftarrow ....$ Note that $\hat{M} = \varprojlim_e M/u^eM$ is a module over the ring of formal power series $\mathbb{Q}[[u]]$. The next proposition computes $\hat{C}_n$ and $\hat{D}_n$.

Theorem 3.1. *For any $n \geq 0$, the projection $C_n \to D_n$ induces an ismorphism $\hat{C}_n \to \hat{D}_n$. The completion $\hat{C}_n = \hat{D}_n$ is a free $\mathbb{Q}[[u]]$-module of rank $n+1$ freely generated by the classes of the chord diagrams g_n^l, where g_n^l is the underlying chord diagram of the singular link G_n^l shown in Figure 4, $l = 0, 1, \ldots, n$.*

The proof of this theorem given at the end of this section is based on a study of three operators acting on chord diagrams, specifically the operator adding an isolated chord and the smoothing and forgetting operators.

3.2. **Adding an isolated chord.** Let T denote the operation on chord diagrams, which adds an isolated chord. This operation is of course not well-defined on the chord diagrams themselves, but is well-defined provided we consider the diagrams modulo 4TS.

Lemma 3.2. *The operation of adding an isolated chord induces a well-defined $\mathbb{Q}[u]$-linear homomorphism $T : C_n \to C_{n+1}$.*

Proof. Consider the corollary of the 4TS-relation in Figure 11, obtained by connecting the left top end to the left bottom end on all four pictures. The first and second term cancel and one obtains the equality of the middle two terms. This exactly shows that an isolated chord can be moved from one place to any other modulo 4TS.

□

3.3. **Smoothing and forgetting operators.** For each $n = 1, 2, ...$, we define two homomorphisms $D_n \to D_{n-1}$, called the smoothing and forgetting operators. The smoothing operator, S, maps a chord diagram $d \in D_n$ into the sum $S(d) = \sum_c S(d; c) \in D_{n-1}$ where c runs over all chords of d and $S(d; c)$ is d with chord c smoothed as shown in Figure 12. This operator maps 4T and 4TS into 4TS and 0, respectively, and defines therefore a $\mathbb{Q}[u]$-linear homomorphism $D_n \to D_{n-1}$. Note that the 4TS-relation is nothing else than the "4T-relation smoothed", hence the name 4TS.

The forgetting operator, F, maps a chord diagram $d \in D_n$ into the sum $F(d) = \sum_c F(d; c) \in D_{n-1}$ where c runs over all chords of d and $F(d; c)$ is d with chord c forgotten. It is easy to check that F maps both 4T and 4TS into 0 and defines a $\mathbb{Q}[u]$-linear homomorphism $D_n \to D_{n-1}$.

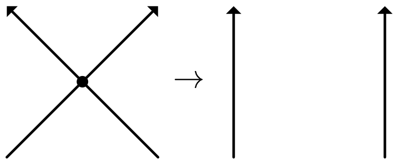

FIGURE 12. The smoothing of a chord.

The operator $T : C_n \to C_{n+1}$ constructed above induces an operator $D_n \to D_{n+1}$ denoted by the same symbol T. We have the following commutation relations:

$$SF - FS = 0, \quad ST - TS = u, \quad FT - TF = \mathrm{id}.$$

This implies the useful commutation relation $(S - uF)T = T(S - uF)$.

3.4. **Proof of Theorem 3.1.** Both relations 4T and 4TS are void for chord diagrams without chords. This implies that $C_0 = D_0$ is the free $\mathbb{Q}[u]$-module of rank 1 generated by g_0^0. Clearly, $\hat{C}_0 = \hat{D}_0 = \mathbb{Q}[[u]]g_0^0$.

Assume now that the claim of the theorem holds for $n-1$ and prove it for n. The proof consists of two parts. First we show that the classes of the chord diagrams $g_n^0, \dots, g_n^n$ generate $\hat{C}_n$. Then we show that they are linearly independent in $\hat{D}_n$. These facts and the surjectivity of the projection $\hat{C}_n \to \hat{D}_n$ would give the inductive step.

Let us prove that $g_n^0, \dots, g_n^n$ generate $\hat{C}_n$. By the inductive assumption, $\hat{C}_{n-1}$ is a free $\mathbb{Q}[[u]]$-module freely generated by $g_{n-1}^0, \dots, g_{n-1}^{n-1}$. Clearly, $T(g_{n-1}^l) = g_n^{l+1}$. Therefore the image of the operator $\hat{T} : \hat{C}_{n-1} \to \hat{C}_n$ induced by $T : C_{n-1} \to C_n$ is generated by $g_n^1, \dots, g_n^n$. It remains to prove that g_n^0 generates the $\mathbb{Q}[[u]]$-module $\hat{C}_n/\hat{T}(\hat{C}_{n-1})$.

We shall show that any chord diagram d with n chords can be expanded modulo 4TS and modulo $T(C_{n-1})$ in the form $d = \nu g_n^0 + ud'$ where $\nu \in \mathbb{Z}$ and $d' \in C_n$. Iterating this expansion we obtain that g_n^0 generates the $(\mathbb{Q}[u]/u^e)$-module

$$C_n/(u^e C_n + T(C_{n-1})) = \hat{C}_n/(u^e \hat{C}_n + \hat{T}(\hat{C}_{n-1}))$$

for any $e \geq 0$. This implies that g_n^0 generates $\hat{C}_n/\hat{T}(\hat{C}_{n-1})$ over $\mathbb{Q}[[u]]$.

The 4TS-relation implies the relation in Figure 7 where we remove h, ignore the over/under-crossings and interpret the four terms as local pictures of chord diagrams. Therefore the basic relation in Figure 8 with h removed (again ignoring the over/under-crossings) holds in C_n/uC_n. Quotienting by $T(C_{n-1})$ we obtain the equalities in Figure 13. Applying these equalities we can expand d as a finite sum $\nu g_n^0 + \sum_j d_j$ where $\nu \in \mathbb{Z}$ and each $d_j \in C_n$ is a disjoint union of several chord diagrams of type g_m^0 with $m < n$. Each such d_j belongs to $uC_n + T(C_{n-1})$. Indeed, the relation in Figure 13 implies that

$$g_m^0 \amalg g_{m'}^0 = g_{m-1}^0 \amalg g_{m'+1}^0 = \dots = g_0^0 \amalg g_{m+m'}^0 = ug_{m+m'}^0 \mod T(C_{n-1}), 4TS$$

for all $m, m' \geq 0$. Hence, $d = \nu g_n^0 + ud' \mod T(C_{n-1})$ with $d' \in C_n$.

It remains to show that the classes of the chord diagrams $g_n^0, \dots, g_n^n$ are linearly independent in $\hat{D}_n$. It follows from definitions that

$$F(g_n^l) = l g_{n-1}^{l-1} + u(n-l) g_{n-1}^l \tag{3.1}$$

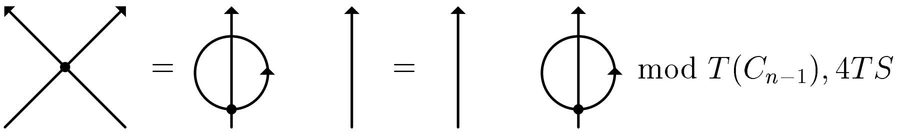

FIGURE 13

for $l \geq 1$, and $F(g_n^0) = ung_{n-1}^0$. Similarly,

$$S(g_n^l) = ulg_{n-1}^{l-1} + (n-l)g_{n-1}^l$$

for $l \geq 1$, and $S(g_n^0) = ng_{n-1}^0$. Now, the $\mathbb{Q}[u]$-linear homomorphisms $S, F : D_n \to D_{n-1}$ induce $\mathbb{Q}[[u]]$-linear homomorphisms $\hat{S}, \hat{F} : \hat{D}_n \to \hat{D}_{n-1}$. If there is a linear relation $\sum_{l=0}^n k_l g_n^l = 0$ in $\hat{D}_n$ (where $k_l \in \mathbb{Q}[[u]]$) then applying $\hat{F}$ and $\hat{S}$ we obtain two linear relations between the classes of $g_{n-1}^0, \ldots, g_{n-1}^{n-1}$ in $\hat{D}_{n-1}$. By the inductive assumption, these classes are linearly independent. This gives two systems of linear equations on $\{k_l\}$: first, $k_{l+1}(l+1) + k_l u(n-l) = 0$ and second, $k_{l+1}u(l+1) + k_l(n-l) = 0$ for $l = 0, 1, \ldots, n-1$. The only solution is $k_l = 0$ for all l. Thus, the classes of $g_n^0, \ldots, g_n^n$ are linearly independent in $\hat{D}_n$ which completes the inductive step and the proof of the theorem.

□

4. Proof of Theorem 1.1

4.1. **Algebraic preliminaries.** Let K be a commutative algebra over the field of rational numbers $\mathbb{Q}$. (In the sequel, K will be the polynomial ring $\mathbb{Q}[u]$). For any K-module M, denote by $M[[v]]$ the set of formal power series on the variable v with coefficients in M. We provide $M[[v]]$ with the structure of a module over the ring of formal power series $K[[v]]$ in the obvious way. We have $M \subset M[[v]]$: an element $a \in M$ is identified with the formal power series $a + 0 \cdot v + 0 \cdot v^2 + \ldots$.

For K-modules $M_0, M_1, \ldots$, the product $\prod_{k \geq 0} M_k$ is the K-module consisting of the series $a_0 + a_1 + \ldots$ with $a_k \in M_k$, $k \geq 0$. The addition and multiplication by elements of K are defined coordinate-wise. Applying this construction to the $K[[v]]$-modules $M_0[[v]], M_1[[v]], \ldots$ we obtain a $K[[v]]$-module $\prod_{k \geq 0} M_k[[v]]$. It is easy to observe that $\prod_{k \geq 0} M_k[[v]] = (\prod_{k \geq 0} M_k)[[v]]$.

Let $M_0, M_1, \ldots$ be K-modules provided with K-linear homomorphisms $\alpha : M_k \to M_{k-1}$ for all $k \geq 1$. These morphisms extend by linearity to $K[[v]]$-linear homomorphisms $M_k[[v]] \to M_{k-1}[[v]]$ also denoted α. We define a $K[[v]]$-linear endomorphism, α, of $\prod_{k \geq 0} M_k[[v]]$ by

$$\alpha(\sum_{k \geq 0} a_k) = \sum_{k \geq 1} \alpha(a_k)$$

where $a_k \in M_k[[v]]$ and $\alpha(a_k) \in M_{k-1}[[v]]$. Finally, we define a $K[[v]]$-linear endomorphism $e^{v\alpha}$ of $\prod_{k \geq 0} M_k[[v]]$ by

$$e^{v\alpha}(\sum_{k \geq 0} a_k) = \sum_{m \geq 0} \frac{v^m}{m!} \alpha^m(\sum_{k \geq 0} a_k) = \sum_{k \geq 0} \left(\sum_{m \geq 0} \frac{v^m}{m!} \alpha^m(a_{k+m}) \right).$$

It is easy to check that $e^{v\alpha}$ is well defined and invertible with inverse $e^{-v\alpha}$.

4.2. **Framed singular links.** Recall that a framed link is a link provided with a homotopy class of non-zero normal vector fields. Let $\mathcal{L}^f$ denote the set of the isotopy classes of framed oriented links in the 3-sphere. Denote by A^f the $\mathbb{Z}[h]$-module freely generated by the set $\mathcal{L}^f$. We define a filtration in A^f using framed singular links as follows. A framed singular link is a singular link (as defined in the introduction) provided with a homotopy class of non-zero normal vector fields. In a neighbourhood of a double point as in Figure 2 the vector field should be orthogonal to the plane of the picture and directed towards the reader. Note that we can keep the framing when resolving a double point of a framed singular link as in Figure 3. (Here and below we use the standard convention for the framings of links presented by link diagrams: the framings are orthogonal to the plane of the pictures and are directed towards the reader). Using the formula in Figure 3, we resolve each framed singular link L with n double points into a formal sum $r(L) \in A^f$ of 3^n terms. Denote by A^f_n the $\mathbb{Z}[h]$-submodule of A^f generated by $r(L)$ where L runs over all framed singular links with n double points. Clearly, $A^f = A^f_0 \supset A^f_1 \supset A^f_2 \supset \dots$. Forgetting the framing, we obtain a projection $A^f \to A$ mapping each A^f_n onto A_n.

4.3. **The smoothed Kontsevich invariant.** The Kontsevich invariant of framed links is a mapping $\mathcal{L}^f \to \prod_{k\geq 0} ch_k/4T$ where $ch_k/4T$ is the vector space over $\mathbb{Q}$ generated by chord diagrams with k chords modulo 4T (see e.g. [2] and [1]). Quotienting further by 4TS we obtain a mapping $\mathcal{L}^f \to \prod_{k\geq 0} D_k$ where $D_k = ch_k/(4T, 4TS)$ is the $\mathbb{Q}[u]$-module considered in Section 3. We extend the latter mapping to an additive homomorphism $z : A^f \to \prod_{k\geq 0} D_k[[v]]$ such that for any $a \in A^f$ and any $g(h) \in \mathbb{Z}[h]$ we have

$$z(g(h)a) = g(e^{v/2} - e^{-v/2})\, z(a) \tag{4.1}$$

(i.e. each entry of h is traded for the formal power series $e^{v/2} - e^{-v/2}$).

In the next lemmas we shall combine z with the $\mathbb{Q}[u]$-linear smoothing and forgetting operators $S, F : D_k \to D_{k-1}$ defined in Section 3. It is convenient to use the following formulas for the endomorphisms e^{vS} and e^{vF} of $\prod_{k\geq 0} D_k[[v]]$ induced by S and F: for a chord diagram d,

$$e^{vS}(d) = \sum_C v^{\mathrm{card}(C)} S(d; C) \quad \text{and} \quad e^{vF}(d) = \sum_C v^{\mathrm{card}(C)} F(d; C) \tag{4.2}$$

where C runs over all subsets of the set of chords of d and $S(d; C)$ (resp. $F(d; C)$) is d with all the chords $c \in C$ smoothed (resp. forgotten).

Lemma 4.1. *For every $n \geq 0$,*

$$(e^{vS} z)(A^f_n) \subset \prod_{k\geq n} D_k[[v]].$$

Moreover, if L is a framed singular link in S^3 with n double points then

$$p_n e^{vS} z(r(L)) = d(L) \mod v$$

where p_n is the projection $\prod_{k\geq n} D_k[[v]] \to D_n[[v]]$, $r(L)$ is the element of A^f_n represented by L and $d(L) \in D_n$ is represented by the underlying chord diagram of L.

Proof. We begin with the second claim of the lemma. It is clear that computing modulo v we obtain $p_n e^{vS} z(r(L)) = p_n z(r(L)) \mod v$. Since the mapping z maps each coefficient h in the resolution $r(L)$ into $e^{v/2} - e^{-v/2} = v + v^3/24 + \dots$, the terms of this resolution with non-trivial powers of h contribute 0 to $z(r(L)) \mod v$. Hence, $z(r(L)) \mod v$ is just the Kontsevich invariant of the standard Vassiliev resolution of L. The n-th term $p_n z(r(L)) \mod v$ of this invariant is well known to be $d(L)$.

To prove the first claim of the lemma, recall that the Kontsevich invariant z can be applied to a framed tangle, the chords being attached to the underlying 1-manifold of the tangle. Here the orientation of this 1-manifold and the order of its endpoints is remembered while its embedding into the 3-space is forgotten. The smoothing operator S and the exponential e^{vS} extend to chord diagrams based on tangles and their formal linear combinations over $\mathbb{Q}[[v]]$ in the obvious way.

Let X_+, X_-, and X_0 be the framed oriented tangles drawn in Figure 1. Set

$$Z_\bullet = z(X_+) - z(X_-) - (e^{v/2} - e^{-v/2})\, z(X_0).$$

We claim that

$$p_0 e^{vS}(Z_\bullet) = 0$$

where p_0 is the projection to the module of chord diagrams with 0 chords. (The projection p_0 annihilates all chord diagrams with at least one chord).

By definition,

$$z(X_+) = \sum_{m\geq 0} \frac{t^m(X)}{2^m m!} \quad \text{and} \quad z(X_-) = \sum_{m\geq 0} \frac{(-1)^m t^m(X)}{2^m m!}$$

where X is the underlying 1-manifold of X_+ and X_- and t^m attaches to X exactly m parallel chords connecting two components of X. By definition, $I_0 = z(X_0)$ is the chord diagram consisting of two vertical arcs and no chords. Then

$$\begin{aligned}
p_0 e^{vS}(Z_\bullet) &= 2p_0 e^{vS}\left(\sum_{\substack{m\geq 0\\ m \text{ odd}}} \frac{t^m(X)}{2^m m!}\right) - (e^{v/2} - e^{-v/2}) p_0 e^{vS}(I_0) \\
&= 2\sum_{\substack{m\geq 0\\ m \text{ odd}}} \frac{(vS)^m}{m!}\left(\frac{t^m(X)}{2^m m!}\right) - (e^{v/2} - e^{-v/2}) I_0 \\
&= 2\sum_{\substack{m\geq 0\\ m \text{ odd}}} \frac{v^m}{2^m m!}\, I_0 - (e^{v/2} - e^{-v/2}) I_0 = 0.
\end{aligned}$$

Here we used the obvious equality $S^m(t^m(X)) = m! I_0$ for any odd m.

Let us prove the first claim of the lemma. It suffices to prove that for any framed singular link L in S^3 with n double points, $e^{vS} z(r(L))$ expands as a sum of chord diagrams with $\geq n$ chords. Let $B_1, \dots, B_n$ be small 3-balls surrounding the double points of L and let B be their complement in the 3-sphere. Resolving L we obtain an algebraic sum of 3^n framed links which coincide in B and represent a framed tangle $\tau \subset B$. The Kontsevich invariant z of these links can be computed in two steps: first compute z for τ and for the tangles sitting in B_i, $i = 1, \dots, n$, then glue the resulting chord diagrams based on tangles along their common endpoints on ∂B. Thus $z(r(L))$ may be obtained by gluing $z(\tau)$ and n copies of $Z_\bullet$. Therefore by Formula 4.2, $e^{vS} z(r(L))$ may be obtained by gluing $e^{vS} z(\tau)$ and n copies of

$e^{vS}(Z_\bullet)$. By the result above, each of these n copies expands as a formal sum of chord diagrams with ≥ 1 chords. Therefore $e^{vS}z(r(L))$ expands as a sum of chord diagrams with $\geq n$ chords.

$\square$

Lemma 4.2. *For every $n \geq 0$,*

$$e^{v(S-uF)}z(A_n^f) \subset \prod_{k=0}^{n}(uv)^{n-k}D_k[[v]] \times \prod_{k\geq n+1} D_k[[v]].$$

Moreover, if L is a framed singular link in S^3 with n double points then

$$e^{v(S-uF)}z(r(L)) = e^{-uvF}d(L) \mod v\prod_{k=0}^{n}(uv)^{n-k}D_k[[v]] \times \prod_{k\geq n+1} D_k[[v]].$$

Proof. Since the operators S and F commute, $e^{v(S-uF)} = e^{vS}e^{-uvF} = e^{-uvF}e^{vS}$. By Lemma 4.1, if $a \in A_n^f$ then $e^{vS}z(s) = \sum_{i\geq n} a_i$ with $a_i \in D_i[[v]]$. Moreover, for $a = r(L)$, we have $a_n \in d(L) + vD_n[[v]]$. It remains to observe that e^{-uvF} maps $D_i[[v]]$ into $\prod_{k=0}^{i}(uv)^{i-k}D_k[[v]]$.

$\square$

4.4. **Proof of Theorem 1.1.** Let $E_k = D_k/T(C_{k-1})$ be the quotient of D_k by the subspace generated by chord diagrams with isolated chords. It follows from Theorem 3.1 that $\hat{E}_k$ is the free $\mathbb{Q}[[u]]$-module generated by $g_k = g_k^0$. This implies that for all $m \geq 0$,

$$u^m E_k/u^{m+1}E_k = \mathbb{Q}u^m g_k. \tag{4.3}$$

Denote by J the composition of the mapping $e^{v(S-uF)}z : A^f \to \prod_{k\geq 0} D_k[[v]]$ and the projection

$$proj : \prod_{k\geq 0} D_k[[v]] \to \prod_{k\geq 0} E_k[[v]].$$

Note that if a framed link L' is obtained from a framed link L by inserting a $+1$ framing twist then $z(L') = e^{T/2}z(L)$ and therefore

$$J(L') = proj(e^{v(S-uF)}e^{T/2}z(L)) = proj(e^{T/2}e^{v(S-uF)}z(L)) = J(L).$$

Here the second equality follows from the fact that $S - uF$ commutes with T, so that $e^{v(S-uF)}$ commutes with $e^{T/2}$. Thus, J is framing-independent and induces an additive homomorphism from A to $\prod_{k\geq 0} E_k[[v]]$. Denote this homomorphism by j.

It follows from Lemma 4.2 that for every $n \geq 0$,

$$j(A_n) \subset \prod_{k=0}^{n}(uv)^{n-k}E_k[[v]] \times \prod_{k\geq n+1} E_k[[v]].$$

Therefore j induces an additive homomorphism

$$j_n : A_n/A_{n+1} \to \prod_{k=0}^{n}(uv)^{n-k}E_k[[v]]/u(uv)^{n-k}E_k[[v]]$$

$$= \bigoplus_{k=0}^{n}(uv)^{n-k}E_k[[v]]/u(uv)^{n-k}E_k[[v]].$$

Formula 4.3 implies that for each $a \in A_n/A_{n+1}$,

$$j_n(a) = \sum_{k=0}^{n} (uv)^{n-k} j_n^k(a) g_k, \tag{4.4}$$

where $j_n^k(a)$ is a uniquely defined element of $\mathbb{Q}[[v]]$. This gives $n+1$ additive homomorphisms

$$j_n^0, j_n^1, \dots, j_n^n : A_n/A_{n+1} \to \mathbb{Q}[[v]]$$

satisfying (4.1) for any $a \in A_n/A_{n+1}$ and any $g(h) \in \mathbb{Z}[h]$.

In light of Proposition 2.3, to finish the proof of Theorem 1.1, we just need to show that the elements $a^l = r(G_n^l)$ of A_n/A_{n+1} represented by the singular links G_n^l with $l = 0, \dots, n$ are linearly independent over $\mathbb{Z}[h]$. It suffices to show that the $(n+1) \times (n+1)$-matrix $(j_n^k(a^l))_{k,l}$ over $\mathbb{Q}[[v]]$ is non-degenerate. To this end it suffices to compute this matrix modulo v and to show that the resulting matrix over $\mathbb{Q}$ is non-degenerate.

By Lemma 4.2,

$$(uv)^{n-k} j_n^k(a^l)\, g_k = \frac{(-uvF)^{n-k}(g_n^l)}{(n-k)!} \mod (u,v)(uv)^{n-k} E_k[[v]]$$

where (u, v) is the ideal of $\mathbb{Q}[u][[v]]$ generated by u and v and g_n^l is the underlying chord diagram of G_n^l. Therefore

$$j_n^k(a^l)\, g_k = (-1)^{n-k}((n-k)!)^{-1} F^{n-k}(g_n^l) \mod (u,v).$$

It follows from Formula 3.1 that $F^{n-k}(g_n^l) = 0 \mod u$ if $l < n-k$ and $F^{n-k}(g_n^l) = l!\, g_k^0 = l!\, g_k \mod u$ if $l = n-k$. This gives $j_n^k(a^l) = 0 \mod v$ if $l < n-k$ and $j_n^k(a^l) = (-1)^l \mod v$ if $l = n-k$. Therefore the matrix $(j_n^k(a^l))_{k,l}$ is non-degenerate which completes the proof of the theorem.

Remark 4.3. An easy calculation shows that

$$e^{vS} e^{T/2} = e^{vu/2} e^{T/2} e^{vS}.$$

From this we observe that $p_0 e^{vS} z(L) \mod uD_0[[v]]$ is independent of the framing of the link L. Since $D_0 \cong \mathbb{Q}[u]$, we have that $D_0[[v]]/uD_0[[v]] \cong \mathbb{Q}[[v]]$. So for an oriented link L we can now define $\sigma(L) \in \mathbb{Q}[[v]]$ by

$$\sigma(L) = p_0 e^{vS} z(L) \mod uD_0[[v]].$$

By Lemma 4.1 we see that

$$\sigma(X_+) - \sigma(X_-) = (e^{v/2} - e^{-v/2})\sigma(X_0),$$

for any triple (X_+, X_-, X_0) as in Figure 1. From this we see that

$$\nabla(L)(e^{v/2} - e^{-v/2}) = \sigma(L)/\sigma(G_0^0)$$

for any oriented link L, where G_0^0 is an oriented unknot. Let us now compute $\sigma(G_0^0)$. Consider the singular link G_1^1. By Lemma 4.1 we have that $p_0 e^{vS} z(r(G_1^1)) = 0$. Now

$$z(r(G_1^1)) = e^{T/2} z(G_0^0) - e^{-T/2} z(G_0^0) - (e^{v/2} - e^{-v/2}) z(G_0^0)^2$$

so

$$e^{vu/2} p_0 e^{vS} z(G_0^0) - e^{-vu/2} p_0 e^{vS} z(G_0^0) = (e^{v/2} - e^{-v/2})(p_0 e^{vS} z(G_0^0))^2$$

in $D_0[[v]] \cong \mathbb{Q}[u][[v]]$. Hence

$$p_0 e^{vS} z(G_0^0) = \frac{e^{vu/2} - e^{-vu/2}}{e^{v/2} - e^{-v/2}}$$

and therefore

$$\sigma(G_0^0) = \frac{v}{e^{v/2} - e^{-v/2}}.$$

If we write

$$\sigma = \sum_{k=1}^{\infty} \sigma_k v^k$$

then we can easily describe the weight system $w_k : ch_k \to \mathbb{Q}$ which composed with z gives σ_k. It is given by

$$w_k = p_0 \frac{S^k}{k!} \mod uD_0.$$

It is clear that the operator $p_0 e^S$ on a given chord diagram simply just smoothes all chords in the diagram. The result is a power of u equal to the number of resulting components. Hence w_k on a chord diagram with k chords is 1 if and only if smoothing all chords results in a connected diagram. Thus we see that w_k vanishes on $4T, 4TS, 1T, uch_k$ and takes the value 1 on the diagram $g_k^k \in ch_k$ for all k. By Theorem 3.1 any weight system on ch_k with these properties equals w_k.

5. Proof of Theorem 1.4

In this section we study differentiation of link invariants and prove Theorem 1.4. A part of our results apply to links in arbitrary 3-manifolds and to arbitrary resolutions of double points.

5.1. **Differentiation of link invariants.** Let K be a commutative ring endowed with a differential, i.e., with an additive homomorphism $x \mapsto x' : K \to K$ such that $(xy)' = x'y + xy'$ for any $x, y \in K$. For an oriented 3-manifold M, denote by $\mathcal{A} = \mathcal{A}(M, K)$ the free K-module generated by the isotopy classes of oriented links in M. There is a unique additive homomorphism $d : \mathcal{A} \to \mathcal{A}$ satisfying the following two conditions:

(i) d maps the generators of $\mathcal{A}$ represented by oriented links into 0;

(ii) for any $k \in K$, $a \in \mathcal{A}$, we have $d(ka) = k'a + kd(a)$.

We can compute d explicitly as follows: if $a = \sum_i k_i L_i \in \mathcal{A}$ where $k_i \in K$ and $\{L_i\}_i$ are oriented links in M then

$$d(a) = \sum_i k_i' L_i \in \mathcal{A}. \tag{5.1}$$

Note that $d(ua) = ud(a)$ where u acts on $\mathcal{A}$ as the disjoint union with an unknot.

The dual K-module $\mathcal{A}^* = \mathrm{Hom}_K(\mathcal{A}, K)$ can be identified with the module of K-valued isotopy invariants of oriented links in M. For any $P \in \mathcal{A}^*$, consider the K-linear homomorphism $d^*(P) : \mathcal{A} \to K$ sending the generator of $\mathcal{A}$ represented by an oriented link L into $(P(L))'$. We can explicitly compute $d^*(P)$ as follows: if $a = \sum_i k_i L_i \in \mathcal{A}$ as above then

$$d^*(P)(a) = \sum_i k_i \,(P(L_i))'.$$

It is clear that $d^* : \mathcal{A}^* \to \mathcal{A}^*$ is an additive homomorphism such that $d^*(kP) = k'P + kd^*(P)$ for any $k \in K, P \in \mathcal{A}^*$. The following lemma yields the fundamental relation between d^* and d.

Lemma 5.1. *For any $a \in \mathcal{A}$ and $P \in \mathcal{A}^*$,*

$$(P(a))' = d^*(P)(a) + P(d(a)).$$

Proof. If $a = \sum_i k_i L_i$ as above, then

$$\begin{aligned}(P(a))' = (P(\sum_i k_i L_i))' &= \sum_i (k_i P(L_i))' \\ &= \sum_i k_i (P(L_i))' + \sum_i k_i' P(L_i) \\ = \sum_i k_i (P(L_i))' + P(\sum_i k_i' L_i) &= d^*(P)(a) + P(d(a)).\end{aligned}$$

□

We describe now the behaviour of any Vassiliev-type filtration in $\mathcal{A} = \mathcal{A}(M, K)$ under the differential d. Fix a finite formal linear combination $\sum_j k_j T_j$ where $k_j \in K$ and each T_j is a tangle in the 3-ball with two inputs and two outputs. Consider a resolution R of a double point (Figure 2) defined by $R(X_\bullet) = \sum_j k_j T_j$. In this way we resolve each singular link $L \subset M$ into a formal sum $R(L) \in \mathcal{A}$. Denote by $\mathcal{A}_n$ the K-submodule of $\mathcal{A}$ generated by $R(L)$ where L runs over all singular links with n double points in M. Clearly, $\mathcal{A} = \mathcal{A}_0 \supset \mathcal{A}_1 \supset \dots$.

Lemma 5.2. *For each $n \geq 0$, $d(\mathcal{A}_{n+1}) \subset \mathcal{A}_n$.*

Proof. Observe first that the definition of the differential d extends word for word to linear combinations of oriented tangles in oriented 3-manifolds with coefficients in K (use Formula 5.1). Note that the usual gluing of tangles extends by linearity to their linear combinations. It is clear that if EF is the result of gluing of two tangles (or linear combinations there of) E, F then $d(EF) = d(E)F + Ed(F)$.

To prove the lemma, it is enough to prove that for any singular link L with $n+1$ double points, $d(R(L))$ is a linear combination of the resolutions of singular links with n double points. We shall prove a more general claim: for any singular tangle L with $n+1$ double points, $d(R(L))$ is a linear combination of the resolutions of singular tangles with n double points. The proof goes by induction. For $n = 0$ the claim is obvious. Assume that the claim holds for $n < N$ and prove it for $n = N$. Consider a singular tangle L with $N+1$ double points. Choose a double point x of L and split L into two pieces: the singular tangle $X_\bullet$ in a 3-ball neighborhood of x and the complementary singular tangle τ in the complement of this 3-ball. It follows from definitions that $R(L) = \sum_j k_j T_j R(\tau)$. Note that $d(T_j) = 0$. Hence

$$\begin{aligned}d(R(L)) &= \sum_j k_j' T_j R(\tau) + \sum_j k_j T_j d(R(\tau)) \\ &= \sum_j k_j' R(T_j \tau) + R(X_\bullet) d(R(\tau)).\end{aligned}$$

Since τ is a singular tangle with N double points, the inductive assumption implies that $d(R(\tau))$ is a linear combination of the resolutions of singular tangles with $N-1$ double points. Hence, $R(X_\bullet)d(R(\tau))$ is a linear combination of the resolutions of singular tangles with N double points. This proves the inductive step.

□

Corollary 5.3. *If $P \in \mathcal{A}^*$ annihilates $\mathcal{A}_n$ with $n \geq 0$ then $d^*(P)$ annihilates $\mathcal{A}_{n+1}$.*

Proof. For any $a \in \mathcal{A}_{n+1} \subset \mathcal{A}_n$, we have $P(a) = 0$. By the previous lemma, $P(d(a)) = 0$. By Lemma 5.1, $d^*(P)(a) = (P(a))' - P(d(a)) = 0$.

Corollary 5.4. *For any $n \geq 1$, the differential $d : \mathcal{A} \to \mathcal{A}$ induces an additive homomorphism $\mathcal{A}/\mathcal{A}_{n+1} \to \mathcal{A}/\mathcal{A}_n$. Its restriction $\mathcal{A}_n/\mathcal{A}_{n+1} \to \mathcal{A}_{n-1}/\mathcal{A}_n$ is K-linear.*

5.2. **Proof of Theorem 1.4.** We apply the differentials d and d^* introduced above in the case $M = S^3$ and $K = \Lambda = \mathbb{Z}[h, h^{-1}]$ with usual differentiation of Laurent-polynomials in h. Note that in this case $\mathcal{A} = \Lambda \otimes_{\mathbb{Z}[h]} A$ and $\mathcal{A}/\mathcal{A}_n = \Lambda \otimes_{\mathbb{Z}[h]} (A/A_n)$ for all $n \geq 0$.

Recall that

$$\nabla^0_{l,m} = (-h)^l \nabla_{l,m} \in \mathrm{Hom}_{\mathbb{Z}[h]}(A, \Lambda) = \mathrm{Hom}_\Lambda(\mathcal{A}, \Lambda) = \mathcal{A}^*.$$

We should prove that

$$\nabla^0_{l,m} = (m!)^{-1}(d^*)^m(\nabla^0_{l,0}).$$

It is enough to prove that

$$d^*(\nabla^0_{l,m-1}) = m\nabla^0_{l,m}$$

for all $l \geq 0, m \geq 1$. By Corollary 5.3, both sides annihilate $\mathcal{A}_{l+m+1}$ and determine Λ-linear homomorphisms $\mathcal{A}/\mathcal{A}_{l+m+1} \to \Lambda$. It suffices to verify that these homomorphisms coincide on the generators $u^s r(G^0_t)$ of $\mathcal{A}/\mathcal{A}_{l+m+1}$ where $s + t \leq l + m$. Set $a_t = r(G^0_t) \in \mathcal{A}$. By definition (cf. Remark 1.7 1.), $\nabla^0_{l,m}(u^s a_t) = \delta^s_l \delta^t_m$. By Lemma 5.1,

$$d^*(\nabla^0_{l,m-1})(u^s a_t) = (\nabla^0_{l,m-1}(u^s a_t))' - \nabla^0_{l,m-1}(u^s d(a_t)) = -\nabla^0_{l,m-1}(u^s d(a_t)).$$

If $t = 0$, then $d(a_t) = d(r(G_0)) = d(G_0) = 0$ and

$$-\nabla^0_{l,m-1}(u^s d(a_t)) = 0 = m\delta^s_l \delta^t_m.$$

Assume that $t \geq 1$. It is clear that G_t is the closure of the t-th power of the singular tangle T_2 drawn in Figure 9. Therefore $r(G_t)$ is the closure of $r(T^t_2)$. It follows from definitions that $d(r(T_2)) = -I$ where I is the unknotted vertical strand oriented upwards. Therefore,

$$d(r(T^t_2)) = d((r(T_2))^t) = -t(r(T_2))^{t-1} = -tr(T^{t-1}_2).$$

Taking the closures, we obtain that $d(r(G_t)) = -t\, r(G_{t-1})$. Thus,

$$-\nabla^0_{l,m-1}(u^s d(a_t)) = t\nabla^0_{l,m-1}(u^s a_{t-1}) = t\delta^s_l \delta^{t-1}_{m-1} = m\delta^s_l \delta^t_m.$$

Remark 5.5. Weight systems can be obtained for the derivatives of the Conway polynomial using the weight systems for the Conway polynomial described in Remark 4.3. This will be treated elsewhere.

6. Proof of Theorem 1.6

For any oriented link L, we define a two variable formal power series $P(L) \in \mathbb{Z}[h][[u]]$ by

$$P(L)(h, u) = \sum_{l=0}^{\infty} (-h)^l \nabla_{l,0}(L)(h) u^l.$$

This definition extends to singular links by $P(L) = P(r(L))$.

The function $L \mapsto P(L)$ is u-linear, i.e., $P(uL) = uP(L)$ for any oriented link or singular link L. Furthermore, $u^n | P(A_n)$, since $\nabla_{l,0}(A_n) = 0$ for $l < n$. To prove Theorem 1.6 we establish the following theorem.

Theorem 6.1. *There is a pair of formal power series $\alpha, \beta \in \mathbb{Z}[[u]]$ such that P satisfies the skein relation*

$$P(X_+) = (1 + h\alpha)P(X_-) + (h + h\beta)P(X_0)$$

for any three oriented links X_+, X_-, X_0 coinciding outside a 3-ball and looking as in Figure 1 inside this ball. The power series α and β are described in Proposition 6.2.

It follows from this theorem that P is a reparametrised version of the Homfly link polynomial.

Proposition 6.2. *There is a unique pair of formal power series $\alpha, \beta \in \mathbb{Z}[[u]]$ which satisfy the equations*

$$\alpha = -u + \beta(\alpha + \beta u) \tag{6.1}$$

$$\beta = \alpha(\alpha + \beta u) \tag{6.2}$$

and such that $\alpha = -u \mod u^2$ and $\beta = 0 \mod u^2$.

Proof. Let us assume first that α and β exist and show their uniqueness. Since $\alpha = -u \mod u^2$, the formal power series $\alpha + u\beta$ is divisible by u so there exists $\gamma \in \mathbb{Z}[[u]]$ such that $\alpha + u\beta = u\gamma$. Clearly, the free term of γ is -1, so that γ is invertible in $\mathbb{Z}[[u]]$. Multiplying Formula 6.2 by u and adding it to Formula 6.1, we obtain $u\gamma = -u + u\gamma(\beta + u\alpha)$ which is equivalent to $\gamma = -1 + \gamma(\beta + u\alpha)$. This implies that $\beta + u\alpha = 1 + \gamma^{-1}$ so that $\beta = 1 + \gamma^{-1} - u\alpha$. Substituting this expression for β in the formula $\alpha + u\beta = u\gamma$ we obtain a linear equation on α which yields

$$\alpha = u(1 - u^2)^{-1}(\gamma - \gamma^{-1} - 1). \tag{6.3}$$

We can also determine β from the equality $\alpha + u\beta = u\gamma$. This gives

$$\beta = (1 - u^2)^{-1}(1 - u^2\gamma + \gamma^{-1}). \tag{6.4}$$

Substituting these expressions into 6.1 and 6.2, we easily observe that these two equations are equivalent to the following equation on γ:

$$u^2\gamma^3 - (u^2 + 1)\gamma - 1 = 0. \tag{6.5}$$

Now, expanding $\gamma = -1 + \sum_{k \geq 1} a_k u^k$ with $a_k \in \mathbb{Z}$ we inductively compute the coefficients of γ from 6.5. Hence there is only one formal power series γ satisfying this equation. (In fact, γ is a formal power series in u^2). This proves uniqueness of α and β. Conversely, definining α, β, γ by Formulas 6.3 - 6.5 we obtain α, β satisfying the conditions of the proposition.

□

The key ingredient in the proof of Theorem 6.1 is the following local relation.

Lemma 6.3. *Let $\alpha, \beta \in \mathbb{Z}[[u]]$ be as in Proposition 6.2. Then*

$$P(X_\bullet) = h\alpha P(X_-) + h\beta P(X_0) \tag{6.6}$$

where $X_\bullet$ is the double point as in Figure 2.

We note that Theorem 6.1 follows directly from this lemma, since by definition

$$P(X_\bullet) = P(r(X_\bullet)) = P(X_+) - P(X_-) - hP(X_0).$$

Proof. By connecting the middle top strand to the bottom left strand and by adding a negative crossing at the bottom of the 8 terms in the 8T-relation (as we did in Section 2), we obtain, after canceling the first and the fifth term, the "6T-relation" in Figure 14, which holds in A once all double points have been resolved.

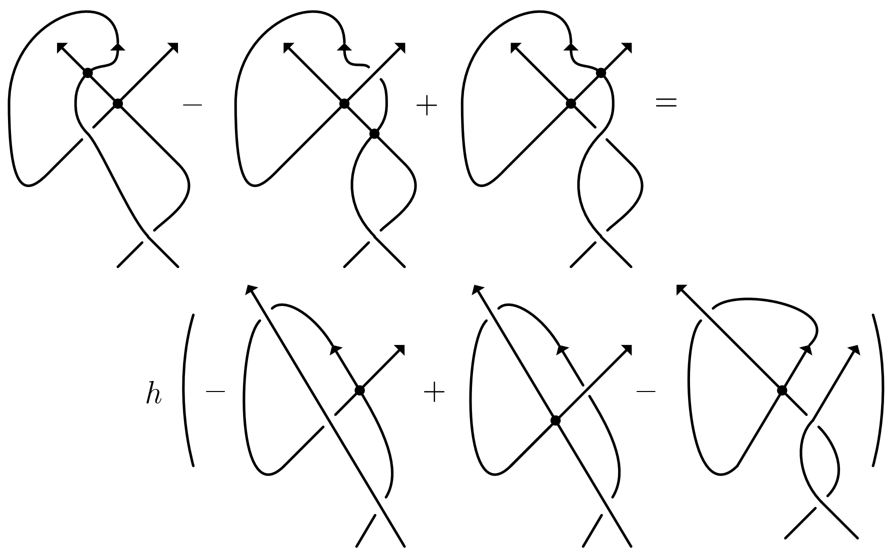

FIGURE 14. The 6T-relation.

We now claim that P vanishes on any singular link obtained by completing the singular $(1,1)$-tangle T_2 in Figure 9 by any singular (1,1)-tangle. Let L be such a singular link with n double points. We apply the argument given in the proof of Proposition 2.3 to all double points of L except to the one of $T_2 \subset L$.. This and the proof of Lemma 2.2 show that $hr(L) \in A$ may be expanded as a finite sum $\sum_{l=0}^{n-1} hq_l r(G_n^l) + \sum_j ha_j r(L_j)$ where $q_l, a_j \in \mathbb{Z}[h]$ and each L_j is a singular link with $\geq n+1$ double points obtained as a completion of T_2 by a singular (1,1)-tangle with $\geq n$ double points. Repeating this argument inductively, we see that for any $N \geq n$ there is a finite expansion

$$hr(L) = \sum_{m=n}^{N} \sum_{l=0}^{m-1} hq_{l,m}\, r(G_m^l) + \sum_j ha_{j,N}\, r(L_{j,N}) \in A$$

where $q_{l,m}, a_{j,N} \in \mathbb{Z}[h]$ and $L_{j,N}$ are singular links with $\geq N+1$ double points obtained by a completion of T_2. Note that $P(G_m^l) = P(r(G_m^l)) = 0$ for $l \leq m-1$ and $P(L_{j,N})$ is divisible by u^{N+1}. Therefore $P(L)$ is divisible by any power of u, hence it vanishes.

From the above we conclude that P satisfies the relation in Figure 15. From this formula we see that $P(X_\bullet) = -huP(X_-) \mod u^2$, since the last three terms are divisible by u^2. This agrees with (6.6) modulo u^2. We shall now prove Formula (6.6)

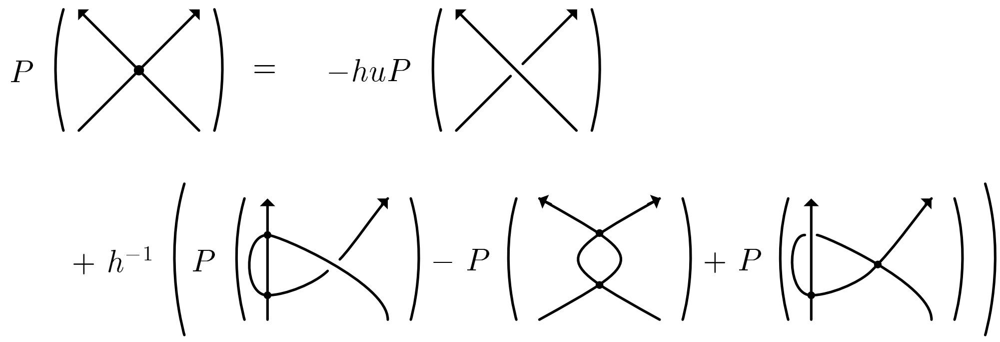

FIGURE 15.

by proving it mod u^k by induction on k. Let us assume that (6.6) holds modulo u^k. Since both α and β are divisible by u, we can apply our mod u^k formula for $P(X_\bullet)$ to each of the double points in the last three terms in Figure 15, and obtain a formula for $P(X_\bullet)$ mod u^{k+1}. When we do that, we obtain

$$P(X_\bullet) = h(-u + \beta(\alpha + \beta u))P(X_-) + h\alpha(\alpha + \beta u)P(X_0) \mod u^{k+1}.$$

By Proposition 6.2, this is exactly Formula (6.6) modulo u^{k+1}.

□

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