Phillips-Sarnak's Conjecture for Hecke Groups with Primitive Character

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Abstract

We prove a conjecture of Phillips and Sarnak about the disappearance of embedded eigenvalues for the Laplacian $A(\Gamma_0(N), \chi)$, where χ is a primitive character mod N, under an analytic family of character perturbations $\chi^{(\alpha)}$ with $\chi^{(0)} = \chi$. Eigenvalues with odd eigenfunctions turn into resonances and odd eigenfunctions into residues of Eisenstein series. This indicates that the Weyl law is violated by the operators $A(\Gamma_0(N), \chi^{(\alpha)})$ for $0 < |\alpha| < \varepsilon$ and some $\varepsilon > 0$.

Introduction

It was proved by [Se1] that the Laplacian $A(\Gamma)$ for congruence subgroups Γ of the modular group $\Gamma_{\mathbb{Z}}$ has an infinite sequence of embedded eigenvalues $\{\lambda_i\}$ satisfying a Weyl law $\#\{\lambda_i \leq \lambda\} \sim \frac{|F|}{4\pi}\lambda$ for $\lambda \to \infty$. Here $|F|$ is the area of the fundamental domain F of the group Γ, and the eigenvalues λ_i are counted according to multiplicity. The same holds true for the Laplacian $A(\Gamma; \chi)$, where χ is a character on Γ and $A(\Gamma; \chi)$ is associated with a congruence subgroup Γ_1 of Γ. It is an important question whether this is a characteristic of congruence groups or it may hold also for some non-congruence subgroups of $\Gamma_{\mathbb{Z}}$. To investigate this problem Phillips and Sarnak studied perturbation theory for Laplacians $A(\Gamma)$ with regular perturbations derived from modular forms of weight 4 [P-Sa1] and singular perturbations by characters derived from modular forms of weight 2 [P-Sa2]. Their work on singular perturbations was inspired by work of Wolpert [W1], [W2]. See

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also [DIPS] for a short version of these ideas and related conjectures. Central to their approach was the application of perturbation theory and in that connection the evaluation of the integral of the product of the Eisenstein series $E_k(s_j)$ at an eigenvalue $\lambda_j = s_j(1-s_j)$ with the first order perturbation M applied to the eigenfunction v_j. If this integral $I_k(s_j)$, which we call the Phillips-Sarnak integral, is non-zero for at least one of the Eisenstein series $E_k(s_j)$, then the eigenvalue λ_j disappears under the perturbation $aM+\alpha^2 N$ for small $\alpha \neq 0$ and becomes a resonance. This follows from the fact that Im $\lambda_j''(0)$ is proportional to the sum over k of $|I_k(s_j)|^2$, a fact known as Fermi's Golden Rule. The strategy of Phillips and Sarnak is on the one hand to prove this rule for Laplacians $A(\Gamma)$ and on the other hand to prove that $I(s_j) \neq 0$ under certain conditions. For congruence groups with singular character perturbation closing two or more cusps a fundamental difficulty presents itself due to the appearance of new resonances of $A(\Gamma, \alpha)$ for $a \neq 0$, which condense at every point of the continuous spectrum of A as $\alpha \to 0$. These resonances (poles of the S-matrix) were discovered by [Se2] for the group $\Gamma(2)$ with singular character perturbation closing 2 cusps, so we call them the Selberg resonances. Any method of proving that eigenvalues become resonances or remain eigenvalues has to deal with these resonances, which arise from the continuous spectrum of the cusps, which are closed by the perturbation. This makes the problem very difficult in that case. In the case of regular perturbations derived from cusp forms it is not too difficult to prove Fermi's Golden Rule, but it is very hard to prove that the integral is not zero.

We consider instead as our basic operator $A(\Gamma, \chi)$, where $\Gamma = \Gamma_0(N)$ is the Hecke group of level N and χ is the primitive character $\Gamma_0(N)$ defined by a real primitive Dirichlet character mod N. These characters are fundamental in number theory, since they are related to quadratic fields. For $N \equiv 1$ mod 4 this character keeps all cusps open. For $N = 4n$, where n is a square-free integer and $n \equiv 3$ mod 4 or $n \equiv 2$ mod 4, the primitive character χ on $\Gamma_0(N)$ closes one third of the cusps in the first case and one half of the cusps in the second case. This is described in Section 1 and the result formulated as Theorem 1.1. The special case of $\Gamma_0(8)$ was treated in [B-V], where in addition the ω-

form was explicitly constructed such that the point $\alpha = 0$ corresponds to $\Gamma_0(8)$ with trivial character. Also, the S-matrix of $A(\Gamma_0(8), \chi^{(1/2)})$ was calculated and turned out to be off-diagonal.

In Section 2 we discuss the Eisenstein series, and in Section 3 we prove the Weyl law for eigenvalues of $A(\Gamma_0(N), \chi)$ (Theorem 3.6), using the factorization formula for the Selberg zeta function [V] and Huxley's explicit formula for the scattering matrix of $\Gamma_1(N)$ [Hu].

In Section 4 we introduce the perturbation of $A(\Gamma, \chi)$ by a group of characters $\chi(\alpha)$ or, equivalently, by a family of operators $aM + \alpha^2 N$, where M and N are Γ-invariant first order differential operators and multiplication operators, respectively, with coefficients derived from a modular form ω of weight 2.

In Section 5 we construct explicitly the form ω from the holomorphic Eisenstein series of weight 2 for the modular group in such a way that the corresponding perturbation is regular relative to $A(\Gamma, \chi)$, leaving the same cusps closed and open as $A(\Gamma, \chi)$ (Theorem 5.1). This is based on the existence of a form ω, which contains the original Eisenstein series with a non-zero coefficient (Appendix 1).

In Section 6 we introduce the Hecke theory for the operators $A(\Gamma_0(N), \chi)$. This is important for proving that the Dirichlet L-series associated with the eigenfunctions of $A(\Gamma_0(N), \chi)$, which are also eigenfunctions of the Hecke operators, have no zeros on the lines $\operatorname{Re} s = 1$ and $\operatorname{Re} s = 0$ (Theorem 6.1). Here we make use of the Rankin-Selberg convolution (Appendix 2) and the functional equation for the Dirichlet L-series (Appendix 3).

In section 7 we prove, using the non-vanishing of the Dirichlet L-series for eigenfunctions, that the Phillips-Sarnak integral is different from zero for all odd eigenfunctions with eigenvalue $\lambda > \frac{1}{4}$ (Theorem 7.1).

Section 8 contains the general perturbation theory, which allows to conclude from the non-vanishing of the Phillips-Sarnak integral, that all odd eigenfunctions turn into resonance functions (Theorem 8.5). The operator M, which is derived from the real part of the form ω, maps odd functions into even functions and even into odd. Therefore, the

Phillips-Sarnak integral is always zero for even eigenfunctions, and it remains an open question whether they stay or leave under this perturbation.

There is another perturbation $\alpha\tilde{M} + \alpha^2 N$, where $\tilde{M}$ is derived from $\mathrm{Im}\,\omega$ and $\tilde{M}$ preserves parity. This perturbation, however, is completely different and in some sense trivial. Although the Phillips-Sarnak integral is non-zero for even eigenfunctions, it does not follow that the eigenvalues disappear. Quite the contrary happens. All eigenvalues and resonances remain constant, because the Laplacians $L(\alpha)$ are unitarily equivalent to L via multiplication by an automorphic phase function (Remark 8.6).

The proof that the Phillips-Sarnak integral is not zero utilizes strong arithmetical properties based on Hecke theory and is specific for the operators $A(\Gamma_0(N), \chi)$. The general perturbation theory makes it possible, however, to draw some conclusions about the eigenvalues more globally (Remark 8.7). Thus, eigenvalues of $A(\Gamma_0(N), \chi)$ with odd eigenfunctions after leaving the spectrum for $\alpha \neq 0$ can then only become eigenvalues for isolated values of $\alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right)$.

1 The group $\bar{\Gamma}_0(N)$ with real primitive character

We consider the Hecke congruence group $\bar{\Gamma}_0(N)$ together with its one-dimensional unitary representation $\hat{\chi}$, also called a character of the group. We are interested here only in arithmetically important characters, coming from real primitive Dirichlet characters χ mod N. We have, following Hecke,

$$\chi(\gamma) = \chi_N(n),\ \gamma = \begin{pmatrix} a & b \\ Nc & n \end{pmatrix} \in \bar{\Gamma}_0(N).\ an - bcN = 1.$$

It is well-known (see [D]) that the real primitive characters mod $N = |d|$ are identical with the symbols

$$\left(\frac{d}{n}\right)$$

where d is a product of relatively prime factors of the form

$$-4, 8, -8, (-1)^{(p-1)/2}p,\ p > 2. \tag{1.1}$$

We have

$$\left(\frac{d_1 d_2}{n}\right) = \left(\frac{d_1}{n}\right)\left(\frac{d_2}{n}\right)$$

provided $(d_1, d_2) = 1$.

By definition

$$\left(\frac{-4}{n}\right) = \chi_4(n) = \begin{cases} 1, & n \equiv 1 \mod 4 \\ -1, & n \equiv 3 \mod 4 \\ 0, & \text{otherwise} \end{cases}$$

$$\left(\frac{8}{n}\right) = \chi_8(n) = \begin{cases} 1, & n \equiv \pm 1 \mod 8 \\ -1, & n \equiv \pm 3 \mod 8 \\ 0, & \text{other}\omega\text{ise} \end{cases}$$

$$\left(\frac{-8}{n}\right) = \chi_4(n)\chi_8(n).$$

We have also, by the law of quadratic reciprocity for the Legendre symbol

$$\left(\frac{n}{p}\right) = \left(\frac{p'}{n}\right) \text{ where } p' = (-1)^{(p-1)/2} p \tag{1.2}$$

provided n is an odd square-free integer. By Kronecker's extension of Legendre's symbol we have

$$\left(\frac{p'}{2}\right) = \left(\frac{2}{p}\right)$$

or, more generally,

$$\left(\frac{p'}{2^k m}\right) = \left(\frac{2^k m}{p}\right) \; k \in \mathbb{Z},\; k \geq 1.$$

Thus, relation (1.2) holds whether n is odd or even. It holds also in the more general form

$$\left(\frac{n}{l}\right) = \left(\frac{l'}{n}\right)$$

where

$$l^{'} = (-1)^{1/2(l-1)}l$$

and $l = p_1 p_2, ...,$ that is if l is any square-free odd positive integer, and $l^{'} = p_1' p_2' ...$ (see (1.2)).

Finally, we recall that

$$\left(\frac{n}{p}\right) = \begin{cases} +1 & \text{if} \quad nRp \\ -1 & \text{if} \quad nNp \end{cases}$$

for p an odd prime and $(p, n) = 1$. We also define

$$\left(\frac{n}{p}\right) = 0 \quad p\,|n\ .$$

Here by definition. nRp just means that there exists an integer x, such that $x^2 \equiv n$ mod p, in the case of nNp such integer does not exist.

For odd n we also have

$$\left(\frac{2}{n}\right) = \left(\frac{8}{n}\right), \left(\frac{-2}{n}\right) = \left(\frac{-8}{n}\right).$$

This explicit definition of the symbol $\left(\frac{d}{n}\right)$ is important in order to calculate the values of the character χ on the parabolic generators of the Hecke group.

The numbers

$$(-1)^{1/2(p-1)}p,\ p > 2,$$

are all congruent to 1 mod 4, and the products of relatively prime factors, i.e. distinct factors, each of this form, comprise all square-free integers, positive and negative, that are congruent to 1 mod 4. In addition, we get all such numbers, multiplied by -4, that is, all numbers $4N$, where N is square free and congruent to 3 mod 4. Finally, we get all such numbers, multiplied by ± 8, which is equivalent to saying all numbers $4N$ where N is congruent to 2 mod 4 (see [D]).

By this we have obtained all real primitive Dirichlet characters. But we need only even characters here, since we consider the projective Hecke group $\bar{\Gamma}_0(N) \subset PSL(2,\mathbb{R})$, that means we identify two matrices

$$\begin{pmatrix} a & b \\ N_c & d \end{pmatrix} \sim \begin{pmatrix} -a & -b \\ -N_c & -d \end{pmatrix}$$

and $\chi(d) = \chi(-d)$. According to this classification of primitive even real characters, we will consider three different choices of N in $\bar{\Gamma}_0(N)$.

1) $\bar{\Gamma}_0(N_1)$, $N_1 = \Pi_{p>2}(-1)^{1/2(p-1)}p$, $N_1 > 0$.

That means N_1 is any positive square-free integer and $N_1 \equiv 1 \bmod 4$.

2) We take $M_2' = \Pi_{p>2}(-1)^{1/2(p-1)}p$, $M_2' < 0$, and we consider $\bar{\Gamma}_0(4N_2)$, where $N_2 = -M_2'$. That means, N_2 is any square-free positive integer $N_2 \equiv 3 \bmod 4$.

3) We take $M_2' = \Pi_{p>2}(-1)^{1/2(p-1)}p$, and we define $N_3 = 2\left|M_3'\right|$.

We have, N_3 is any square-free positive integer, and $N_3 \equiv 2 \bmod 4$.

Then we consider $\bar{\Gamma}_0(4N_3)$.

We now recall the basic properties of the group $\bar{\Gamma}_0(N)$, having in mind our choices 1), 2), 3) for N. It is well-known that for any N

$$\left[\bar{\Gamma}_0(1) : \bar{\Gamma}_0(N)\right] = N\prod_{p|N}(1+1/p) = m$$

$$n_2 = \begin{cases} 0 & , \ 4\,|N \\ \prod_{p|N}\left(1+\left(\frac{-1}{p}\right)\right) & , \text{otherwise} \end{cases} \qquad \left(\frac{-1}{p}\right) = \begin{cases} 0, & p=2 \\ 1, & p\equiv 1 \mod 4 \\ -1, & p \equiv 3 \mod 4 \end{cases}$$

$$n_3 = \begin{cases} 0 & , \ 9\,|N \\ \prod_{p|N}\left(1+\left(\frac{-3}{P}\right)\right) & , \text{otherwise} \end{cases} \qquad \left(\frac{-3}{p}\right) = \begin{cases} 0, & p=3 \\ 1, & p\equiv 1 \mod 3 \\ -1, & p \equiv 2 \mod 3 \end{cases}$$

$$h = \sum_{\substack{d|N \\ d>0}} \varphi((d, N/d))$$

$$g = 1 + \frac{m}{12} - \frac{n_2}{4} - \frac{n_3}{3} - \frac{h}{2}.$$

Here m is the index of $\bar{\Gamma}_0(N)$ in the modular group, n_2 is the number of $\bar{\Gamma}_0(N)$ inequivalent elliptic points of order 2 (n_3, of order 3), h is the number of $\bar{\Gamma}_0(N)$ inequivalent cusps, g is the genus, $\varphi(n)$ is the Euler function, $\varphi(1) = 1$, $\varphi(n) = n(1-1/p_1)(1-1/p_2)...(1-1/p_k)$, $n = p_1^{l_1} p_2^{l_2} ... p_k^{l_k}$.

For our purposes it is important to see the parabolic generators of our groups and corresponding cusps of the canonical fundamental domains.

Case 1). For $\bar{\Gamma}_0(N_1)$ we have $N_1 = p_1 p_2 ... p_k$, a product of odd different primes. Then $h_1 = \sum\limits_{\substack{d|N_1 \\ d>0}} \varphi((d, N_1/d)) = d(N_1)$, the number of positive divisors of the positive integer N_1. Let $\sigma_d \in \Gamma_0(1)$, $d\,|N$, $d > 0$

$$\sigma_d = \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix}. \tag{1.3}$$

We can take a complete set of inequivalent cusps for $\bar{\Gamma}_0(N_1)$ the set of points $z_d = 1/d \in \mathbb{R}$, $d\,|N$, $d > 0$. We define them as $\bar{\Gamma}_d = \left\{ \gamma \in \bar{\Gamma}_0(N_1)\, | \gamma z_d = z_d \right\}$. Let S_d be the generator of $\bar{\Gamma}_d$. We can find S_d from the condition $S_d' \in \bar{\Gamma}_0(N_1)$

$$S_d' = \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix} \begin{pmatrix} 1 & m_d' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -d & 1 \end{pmatrix} = \begin{pmatrix} 1 - dm_d' & m_d' \\ -d^2 m_d' & 1 + dm_d' \end{pmatrix} \tag{1.4}$$

where we have to take the minimum $\left| m_d' \right|$ (width). That gives $m_d = N_1/d$, and we obtain

$$S_d = \begin{pmatrix} 1 - N_1 & N_1/d \\ -dN_1 & 1 + N_1 \end{pmatrix} \; d > 0, \; d\,|N_1 \, . \tag{1.5}$$

Since our character $\chi = \chi_{N_1}$ is mod N_1, we obtain

$$\chi_{N_1}(S_d) = \chi_{N_1}(1 + N_1) = \chi_{N_1}(1) = 1 \text{ for any } d\,|N_1 \, , \; d > 0. \tag{1.6}$$

Case 2) $\bar{\Gamma}_0(4N_2)$, N_2 is the product of different odd primes. Then we have

$$h_2 = \sum_{\substack{d|4N_2 \\ d>0}} \varphi((d, 4N_2/d)).$$

Since $\varphi(2) = 1$ we obtain $h_2 = d(4N_2)$, the divisor function of $4N_2$. For any $d\,|4N_2$ we introduce the matrix $\sigma_d \in \Gamma_0(1)$ (1.3). Again we take as a complete set of inequivalent cusps for $\bar{\Gamma}_0(4N_2)$ the set of points $z_d = 1/d \in \mathbb{R}$, $d\,|4N_2\,$, $d > 0$. We define in analogy with the first case

$$\bar{\Gamma}_d = \left\{\gamma \in \bar{\Gamma}_0(4N_2)\,|\gamma z_d = z_d\right\} \tag{1.7}$$

and for the generator $S_d = S_d^{(2)}$ we have

$$S_d = \begin{pmatrix} 1 - dm_d & m_d \\ -d^2 m_d & 1 + dm_d \end{pmatrix} \tag{1.8}$$

where for m_d we have the minimum $\left|m_d^{'}\right|$, when $S_d^{'} \in \Gamma_0(4N_2)$ (see (1.4)). We have three possibilities now. In case (i) $d\,|N_2\,$, then we have $4N_2\,|d^2 m_d$ and $m_d = 4N_2/d$. We obtain

$$S_d = \begin{pmatrix} 1 - 4N_2 & 4N_2/d \\ -4N_2 d & 1 + 4N_2 \end{pmatrix},\ d\,|N_2\ ,\ d > 0. \tag{1.9}$$

In case (ii) $d = 2d_1$, $d_1\,|N_2\,$, then we have $4N_2\,|d^2 m_d$ and $m_d = N_2/d_1$. We get

$$S_d = \begin{pmatrix} 1 - 2N_2 & N_2/d_1 \\ -4d_1 N_2 & 1 + 2N_2 \end{pmatrix},\ d_1\,|N_2\ ,\ d_1 > 0,\ d = 2d_1. \tag{1.10}$$

Finally in case (iii) we have $d = 4d_2$, $d_2\,|N_2\,$. Then $4N_2\,|16d_2^2 m_d\,$, and we get $m_d = N_2/d_2$. We obtain

$$S_d = \begin{pmatrix} 1 - 4N_2 & N_2/d_2 \\ -16d_2 N_2 & 1 + 4N_2 \end{pmatrix},\ d_2\,|N_2\ ,\ d_2 > 0,\ d = 4d_2. \tag{1.11}$$

Case 3) For $\bar{\Gamma}_0(4N_3)$ we have $N_3 = 2n$, $n = p_1, p_2 ... p_k$ is the product of different odd primes. We get

$$h_3 = \sum_{\substack{d|4N_3 \\ d>0}} \varphi((d, 4N_3/d)) = d(4N_3). \tag{1.12}$$

We take as the set of all $\bar{\Gamma}_0(4N_3)$ inequivalent cusps the set of points

$$z_d = 1/d,\ d\,|4N_3\ ,\ d > 0, \tag{1.13}$$

then we define in analogy to (1.7)

$$\bar{\Gamma}_d = \left\{ \gamma \in \bar{\Gamma}_0(4N_3 \,|\gamma z_d = z_d \right\}$$

and its generator S_d, given by (1.8) with $d\,|4N_3\,,\ d > 0$. Similar to (1.4) for m_d we take the minimum $\left|m_d'\right|$, when $S_d' \in \bar{\Gamma}_0(4N_3)$. We have 4 possibilities now,

$$\text{(i) } d\,|n \;\; \text{(ii) } d = 2d_1, d_1\,|n \;\; \text{(iii) } d = 4d_2, d_2\,|n \;\; \text{(iv) } d = 8d_3, d_3\,|n\ .$$

Analogous to (1.8), (1.10) we obtain

$$S_d = \begin{cases} \begin{pmatrix} 1-4N_3 & 4N_3/d \\ -8dn & 1+4N_3 \end{pmatrix} & d\,|n\ ,\ d>0 \\ \begin{pmatrix} 1-2N_3 & N_3/d_1 \\ -4d_1N_3 & 1+2N_3 \end{pmatrix} & d = 2d_1,\ d_1\,|n\ ,\ d_1 > 0 \\ \begin{pmatrix} 1-2N_3 & n/d_2 \\ -16d_2n & 1+2N_3 \end{pmatrix} & d = 4d_2,\ d_2\,|n\ ,\ d_2 > 0 \\ \begin{pmatrix} 1-4N_3 & n/d_3 \\ -64d_2n & 1+4N_3 \end{pmatrix} & d = 8d_3,\ d_3\,|n\ ,\ d_3 > 0. \end{cases} \tag{1.14}$$

In (1.6) we calculated the values of $\chi_{N_1}(S_d)$, $d\,|N_1\,,\ d > 0$. Now we do that for all other cases. We have in Case 2) of $4N_2$ with either (i) $d\,|N_2$ or (iii) $d = 4d_2,\ d_2\,|N_2$

$$\chi_{4N_2}(S_d) = \chi_{4N_2}(1+4N_2) = \chi_{4N_2}(1) = 1,\ d\,|N_2,\ d>0 \tag{1.15}$$

$$\chi_{4N_2}(S_d) = 1,\ d = 4d_2,\ d_2\,|N_2\ ,\ d_2 > 0 \text{ (see (1.11))} \tag{1.16}$$

For the case (ii) $d = 2d_1,\ d_1\,|N_2\,,\ d_1 > 0$ we have to calculate

$$\chi_{4N_2}(S_d) = \chi_{4N_2}(1+2N_2) \text{ (see (1.10))}. \tag{1.17}$$

We obtain

$$\begin{aligned} \chi_{4N_2}(1+2N_2) = \left(\frac{4N_2}{1+2N_2}\right) &= \left(\frac{-4}{1+2N_2}\right)\left(\frac{-N_2}{1+2N_2}\right) \\ &= \chi_4(1+2N_2)\left(\frac{N_2'}{1+2N_2}\right) \\ &= \chi_4(1+2N_2)\left(\frac{1+2N_2}{2N_2}\right) = \chi_4(1+2N_2). \end{aligned} \tag{1.18}$$

Since $N_2 \equiv 3 \bmod 4$ we get $\chi_4(1+2N_2) = -1$ and then

$$\chi_{4N_2}(S_d) = \chi_{4N_2}(1+2N_2) = -1, \quad d = 2d_1,\ d_1\,|N_2\ ,\ d_1 > 0. \tag{1.19}$$

In Case 3) $\bar{\Gamma}_0(4N_3)$ we have (see (1.14))

$$\underset{d|n}{\chi_{4N_3}}(S_d) = \underset{d_3|n}{\chi_{4N_3}}(S_{8d_3}) = 1 \tag{1.20}$$

$$\underset{d|n}{\chi_{4N_3}}(S_{2d_1}) = \underset{d_2|n}{\chi_{4N_3}}(S_{4d_2}) = \chi_{4N_3}(1+2N_3) = -1. \tag{1.21}$$

From the basic properties of the symbol $\left(\frac{d}{n}\right)$ (see the beginning of this section) follows

$$\chi_{4N_3}(1+2N_3) = \left(\frac{4N_3}{1+2N_3}\right) = \begin{cases} \chi_8(1+2N_3)\left(\frac{1+2N_3}{M_3'}\right), & M_3' > 0 \\ \chi_8(1+2N_3)\chi_4(1+2N_3)\left(\frac{1+2N_3}{M_3'}\right), & M_3' < 0 \end{cases}$$

where $M_3' = \prod\limits_{p>2}(-1)^{(p-1)/2}p$. Then we have

$$\left(\frac{1+2N_3}{M_3'}\right) = \left(\frac{M_3}{1+2N_3}\right) = 1$$

where $M_3 = \prod\limits_{p>2} p$, corresponding to the product M_3'. Next $2N_3 = 4\left|M_3'\right|$ and we have

$$\chi_4(1+2N_3) = 1. \tag{1.22}$$

Since $N_3 \equiv 2 \bmod 4$ we have finally

$$\chi_8(1+2N_3) = -1. \tag{1.23}$$

We have proved the following theorem.

Theorem 1.1 *1) For the group $\bar{\Gamma}_0(N_1)$, N_1 a square-free positive integer, $N_1 \equiv 1 \bmod 4$, and its arithmetical character $\hat{\chi}_{N_1} = \left(\frac{N_1}{\cdot}\right)$ we have a complete system of $\bar{\Gamma}_0(N_1)$ inequivalent cusps z_d given by $z_d = \frac{1}{d}$, $d\,|N_1\,,\ d > 0$. The system of all parabolic generators S_d is given by (1.5). Then all the above-mentioned cusps are open relative to this character. That precisely means that $\chi_{N_1}(S_d) = 1$. We are also saying in this case that the character χ is regular for the group $\bar{\Gamma}_0(N_1)$ (see Section 2).*

2) For the group $\bar{\Gamma}_0(4N_2)$*,* N_2 *a square-free positive integer,* $N_2 \equiv 3$ *mod 4, and its arithmetical character* $\chi_{4N_2} = \left(\frac{4N_2}{\cdot}\right)$ *we have the complete system of* $\bar{\Gamma}_0(4N_2)$ *inequivalent cusps* $z_d = 1/d$*,* $d\,|4N_2$*,* $d > 0$*. The system of all parabolic generators* S_d *is given by (1.9), (1.10), (1.11). The character* χ_{4N_2} *is singular for the group* $\bar{\Gamma}_0(4N_2)$*, two third of the cusps* z_d *are open and one third is closed by the character* χ*. That precisely means, that for open cusps* z_d*,* $d\,|N_2$*,* $d > 0$*, or* $d = 4d_2$*,* $d_2 > 0$*,* $d_2\,|N_2$*, we have* $\chi(Sd) = 1$ *(see (1.15), (1.16)). For closed cusps* z_d*,* $d = 2d_1$*,* $d_1 > 0$*,* $d_1\,|N_2$*,* $\chi(S_d) = -1$ *(see (1.19)).*

3) For the group $\bar{\Gamma}_0(4N_3)$*,* N_3 *a square-free positive integer,* $N_3 \equiv 2$ *mod 4, and its arithmetical character* $\chi_{4N_3} = \left(\frac{4N_3}{\cdot}\right)$ *we have the complete system of* $\bar{\Gamma}_0(4N_3)$ *inequivalent cusps* $z_d = 1/d$*,* $d\,|4N_3$*,* $d > 0$*. The system of all parabolic generators* S_d *is given by (1.14). The character* χ_{4N_3} *is singular for the group* $\bar{\Gamma}_0(4N_3)$ *with half of the cusps open and the other half closed. The open cusps are* z_d*,* $d\,|n$*,* $d > 0$ $(N_3 = 2n)$*, or* $d = 8d_3$*,* $d_3\,|n$*,* $d_3 > 0$ *(see (1.20)). The closed cusps are* z_d*,* $d = 2d_1$*,* $d_1\,|n$*,* $d_1 > 0$*, or* $d = 4d_2$*,* $d_2\,|n$*,* $d_2 > 0$ *(see (1.21), (1.22), (1.23)).*

2 The Eisenstein Series

We recall the main points of the spectral theory of the automorphic Laplacian on the hyperbolic plane, which we need in this paper (see [Se1], [He], [BV1]).

Let H be the hyperbolic plane. We consider $H = \{z \in \mathbb{C}\,|\,z = x + iy\}$ as the upper half-plane of $\mathbb{C}$ with the Poincaré metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

Let $\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ be the Laplacian associated with the metric ds^2. Then let Γ be a cofinite group of isometries on H and χ a one-dimensional unitary representation (character) of Γ. We define the automorphic Laplacian $A(\Gamma;\chi)$ in the Hilbert space $\mathcal{H}(\Gamma)$ of complex-valued functions f, which are $(\Gamma;\chi)$ automorphic, i.e. $f(\gamma z) = \chi(\gamma)f(z)$ for any $\gamma \in \Gamma$, $z \in H$, and which satisfy

$$\|f\|^2 = \int\limits_F |f(z)|^2 \, d\mu(z) < \infty.$$

It is clear that $\mathcal{H}(\Gamma) = L_2(F; d\mu)$, when F is given. The linear operator $A(\Gamma;\chi)$ is defined on the smooth $(\Gamma;\chi)$ automorphic functions $f \in L_2(F; d\mu)$ by the formula

$$A(\Gamma;\chi)f = -\Delta f.$$

We identify $A(\Gamma;\chi)$ with the restriction $A_F(\Gamma;\chi)$ of $A(\Gamma;\chi)$ to the space of functions $f\,|_F$, where f runs over all smooth (Γ,χ) automorphic functions f. The closure of $A(\Gamma;\chi)$ in $\mathcal{H}_\Gamma$ is a selfadjoint, non-negative operator, also denoted by $A(\Gamma;\chi)$.

We recall that the character χ is regular in the cusps z_j of the fundamental domain F if $\chi(S_j) = 1$ and S_j is the generator of a parabolic subgroup $\Gamma_j \subset \Gamma$, which fixes the cusp z_j. Otherwise $X(S_j) \neq 1$, and X is singular in z_j. It is clear that this property of the character does not depend on the choice of fundamental domain, since in equivalent cusps the character has the same values (that means $\chi(S_j) = \chi(\tilde{S}_j)$, and $S_j, \tilde{S}_j$ correspond to equivalent cusps).

The total degree $k(\Gamma;\chi)$ of singularity of χ relative to Γ is equal to the number of all pairwise non-equivalent cusps of F, in which χ is singular. If Γ is non-compact, which is the only case we consider, and the representation χ is singular, i.e. $h > k(\Gamma;\chi) \geq 1$, then the operator $A(\Gamma;\chi)$ has an absolutely continuous spectrum $\{\lambda \in [1/4, \infty)\}$ of multiplicity $h - k(\Gamma;\chi)$, where h is the number of all inequivalent cusp of F. In other words, the multiplicity $r(\Gamma;\chi)$ of the continuous spectrum is equal to the number of inequivalent cusps where χ is regular, $r(\Gamma;\chi) = h - k(\Gamma;\chi)$.

The continuous spectrum of the operator $A(\Gamma;\chi)$ is related to the generalized eigenfunctions of $A(\Gamma;\chi)$, which are obtained by the analytic continuation of Eisenstein series. We define this as follows. For each cusp z_j of the fundamental domain F, in which the representation χ is regular, we consider again the parabolic subgroup $\Gamma_j \subset \Gamma$, $\Gamma_j = \{\gamma \in \Gamma \,|\, \gamma z_j = z_j\}$. Γ_j is an infinite cyclic subgroup of Γ, generated by a certain parabolic generator $S_j, \chi(S_j) = 1$.

There exists an element $g_j \in PSL(2, \mathbb{R})$ such that

$$g_j \infty = z_j,\ g_j^{-1} S_j g_j z = S_\infty z = z + 1$$

for all $z \in H$. Let $y(z)$ denote $\operatorname{Im} z$. Then the (non-holomorphic Eisenstein (or Eisenstein-Maass) series, is given by

$$E_j(z; s; \Gamma; \chi) = \sum_{\gamma \in \Gamma_j \backslash \Gamma} y^s(g_j^{-1} \gamma z) \overline{\chi(\gamma)}. \tag{2.1}$$

Here χ is the complex conjugate of χ, γ is a coset $\Gamma_j \gamma$ of Γ with respect to Γ_j. The series is absolutely convergent for $\operatorname{Re} s > 1$, and there exists an analytic continuation to the whole complex plane as a meromorphic function of s. We have a system of $r(\Gamma; \chi)$ functions given by (2.1). For $s = 1/2 + it$, $t \in \mathbb{R}$, they constitute the full system of generalized eigenfunctions of the continuous spectrum of the operator $A(\Gamma; \chi)$.

We recall the definition of the automorphic scattering matrix. We have for

$1 \leqq \alpha, \beta \leqq r(\Gamma; \chi)$

$$E_\alpha(g_\beta z; s; \Gamma; \chi) = \sum_{n=-\infty}^{\infty} a_n(y; s; \Gamma; \chi) e^{2\pi i n \chi}. \tag{2.2}$$

This function is periodic under $z \to z + 1$, and moreover

$$a_0(y; s; \Gamma; \chi) = \delta_{\alpha\beta} y^s + \varphi_{\alpha\beta}(s; \Gamma; \chi) y^{1-s}, \tag{2.3}$$

$z = \chi + iy \in H$, where

$$\delta_{\alpha\beta} = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}.$$

The matrix $\phi(s; \Gamma; \chi) = \{\varphi_{\alpha\beta}(s; \Gamma; \chi)\}$ which is of the order $r(\Gamma; \chi)$ is called the automorphic scattering matrix. It is well-known that $\phi(s; \Gamma; \chi)$ is meromorphic in $s \in \mathbb{C}$ and holomorphic in the line $\operatorname{Re} s = 1/2$ and satisfies the functional equation

$$\phi(s; \Gamma; \chi) \phi(1 - s; \Gamma; \chi) = I_r, \tag{2.4}$$

where I_r is the $r(\Gamma;\chi) \times r(\Gamma;\chi)$ identity matrix. The matrix $\phi(s;\Gamma;\chi)$ is important for establishing the analytic continuation and the functional equation for the Eisenstein series given by

$$E_\alpha(z;1-s;\Gamma;\chi) = \sum_{\beta=1}^{r} \varphi_{\alpha\beta}(1-s;\Gamma;\chi) E_\beta(z;s;\Gamma;\chi) \tag{2.5}$$

$1 \leqq \alpha \leqq r = r(\Gamma;\chi)$.

We make now more precise the formulas (2.4), (2.5). We have

$$\begin{aligned} E_\alpha(g_\beta z;s;\Gamma;\chi) = \delta_{\alpha\beta} y^s + \varphi_{\alpha\beta}(s;\Gamma;\chi) y^{1-s} \\ + \sqrt{y} \sum_{n\neq 0} \varphi_{\alpha\beta n}(s;\Gamma;\chi) K_{s-1/2}(2\pi\,|n|\,y) e^{2\pi i n \chi} \end{aligned} \tag{2.6}$$

where $K_{s-1/2}(y)$ is the McDonald-Bessel function. This expression is obtained from the differential equation $\Delta f + 1(1-s)f = 0$ by separation of variables in the strip $-1/2 \leq x \leq 1/2$, $0 < y < \infty$. Let Γ_∞ be the infinite cyclic group generated by $z \to z+1$. Then we construct a double coset decomposition (see [I], p. 163)

$$\Gamma_\infty \backslash g_\alpha^{-1} \Gamma g_\beta / \Gamma_\infty = \delta_{a\beta} \Gamma_\infty \cup \left\{ \bigcup_{c>0} \bigcup_{d \bmod c} \Gamma_\infty \begin{pmatrix} * & * \\ c & d \end{pmatrix} \Gamma_\infty \right\} \tag{2.7}$$

where $\begin{pmatrix} * & * \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in g_\alpha^{-1} g_\beta$.

The general Kloosterman sums are introduced by

$$S_{\alpha\beta}(m,n;c;\Gamma;\chi) = S_{\alpha\beta}(m,n;c) = \sum_{d \text{ mod } c} \bar{\chi} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \exp 2\pi i \frac{ma+nd}{c}. \tag{2.8}$$

Here we have assumed that we can extend the character χ from Γ to $g_\alpha^{-1}\Gamma g_\beta$. Then we have

$$\varphi_{\alpha\beta}(s) = \varphi_{\alpha\beta}(s;\Gamma;\chi) = \sqrt{\pi} \frac{\Gamma(s-1/2)}{\Gamma(s)} \sum_{c>0} \frac{S_{\alpha\beta}(0,0;c)}{c^{2s}} \tag{2.9}$$

$$\varphi_{\alpha\beta n}(s) = \varphi_{\alpha\beta n}(s;\Gamma;\chi) = \frac{2\pi^s}{\Gamma(s)} |n|^{s-1/2} \sum_{c>0} S_{\alpha\beta}(0,n;c). \tag{2.10}$$

where $\Gamma(s)$ is the Euler's gamma function.

The explicit calculation of these series in full generality for our groups $\bar{\Gamma}_0(N_1)$, $\bar{\Gamma}_0(4N_2)$, $\bar{\Gamma}_0(4N_3)$ and the corresponding arithmetical characters in terms of Dirichlet's L-series is rather difficult. The best approach to solve this problem is due to M.N. Huxley (see [Hu]), although he considered congruence groups without characters. We will use his results later to prove the asymptotical Weyl law for discrete eigenvalues of $A(\Gamma;\chi)$ with the Γ and χ considered here.

3 The discrete spectrum of the automorphic Laplacian for $\Gamma_0(N)$ with primitive character

We consider in this paragraph the group $\Gamma = \bar{\Gamma}_0(N)$ with primitive character χ. We will prove here that apart from the continuous spectrum of multiplicity $r(\Gamma;\chi)$ (see Section 2) the operator $A(\Gamma;\chi)$ has an infinite discrete spectrum consisting of eigenvalues of finite multiplicity, satisfying a Weyl asymptotical law

$$N(\lambda;\Gamma;\chi) \underset{\lambda\to\infty}{\sim} \frac{\mu(F)}{4\pi}\lambda \tag{3.1}$$

where $N(\lambda;\Gamma;\chi) = \#\{\lambda_j \leqq \lambda\}$ is the distribution function for eigenvalues of $A(\Gamma;\chi)$, and the λ_j are repeated according to multiplicity, $\mu(F)$ is the area of the fundamental domain F of Γ.

As follows from general results on the spectrum of $A(\Gamma;\chi)$ (see [F], [V]) and the Selberg trace formula, it is enough to prove that the determinant of the automorphic scattering matrix

$$\varphi(s;\Gamma;\chi) = \det \phi(s;\Gamma;\chi) \tag{3.2}$$

is a meromorphic function of order 1. We will prove this indirectly, reducing to the group $\bar{\Gamma}_1(N)$, and then using Huxley's result.

We recall the definitions

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) | c \equiv 0 \bmod N \right\}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) | c \equiv 0,\ a \equiv d \equiv 1 \bmod N \right\}$$

$$\Gamma_2(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) | b \equiv c \equiv 0,\ a \equiv d \equiv 1 \bmod N \right\}$$

and we recall the classical result (see [Mi] p. 104).

Theorem 3.1 1*) Let T_N be the homomorphism of $SL(2,Z)$ into $SL(z;Z/NZ)$*

$$T_N : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a \bmod N, & b \bmod N \\ c \bmod N & d \bmod N \end{pmatrix}$$

then T_N is surjective, $Ker\, T_N = \Gamma(N) = \Gamma_2(N)$.

2) The mapping $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to d \bmod N \in (\mathbb{Z}/N\mathbb{Z})^*$ induces an isomorphism

$$\Gamma_0(N)/\Gamma_1(N) \backsimeq (\mathbb{Z}/N\mathbb{Z})^*$$

and $\Gamma_1(N)$ is a normal subgroup of $\Gamma_0(N)$ of index $\varphi(N)$, where φ is the Euler function.

We recall now the general theorem proved in [V], which we adopt to our situation.

Theorem 3.2 *For a general cofinite Fuchsian group Γ and its normal subgroup Γ' of finite index the following formula for the kernels of the resolvents of $A(\Gamma';1)$ in $H_{\Gamma'}$ and $A(\Gamma;\chi)$ in H_Γ holds:*

$$\frac{1}{[\Gamma:\Gamma']} \sum_{\hat{\chi}\in(\Gamma'\backslash\Gamma)^*} \left[tr_\chi r(z,z';s;\Gamma;\chi)\right] \dim\chi = r(z,z';s;\Gamma';1) \tag{3.3}$$

where $\left[\Gamma:\Gamma'\right]$ denotes the index of Γ' in Γ. Here $\hat{\chi}$ runs over the set of all finite-dimensional, irreducible unitary representations of the factor group $\Gamma'\backslash\Gamma$. We extend the representation $\hat{\chi}$ to a representation χ of the group Γ by the trivial representation, setting for $\gamma=\gamma_1\cdot\gamma_2$, $\gamma_1\in\Gamma'$, $\gamma_2\in\Gamma'\backslash\Gamma$, $\chi(\gamma)=\chi(\gamma_1)\chi(\gamma_2)=\chi(\gamma_1)\hat{\chi}(\gamma_2)=\hat{\chi}(\gamma_2)$. The trace tr_χ is the trace in the space of the representation χ, and $\dim\chi$ is the dimension of χ.

For $\operatorname{Re} s > 1$ the resolvent is defined as

$$R(s;\Gamma;\chi) = (A(\Gamma;\chi) - s(1-s)I)^{-1}, \tag{3.4}$$

where I is the identity operator in H_Γ with norm

$$\|f\|_\Gamma^2 = \int_F |f|_v^2 \, d\mu$$

$f : F \to V$, the finite dimensional space of the representation χ. Then the kernel of the resolvent, considered as an integral operator is given by the absolutely convergent Poincaré series

$$r(z;z^{'};s;\Gamma;\chi) = \sum_{\gamma\in\Gamma} \chi(\gamma) k(z,\gamma z^{'};s), \tag{3.5}$$

where $k(z,z^{'};s)$ is the Green's function for the operator $-\Delta - s(1-s)$ on H.

As the group Γ from Theorem 3.2 we consider the projective group $\bar{\Gamma}_0(N)$ and set $\Gamma^{'} = \bar{\Gamma}_1(N)$. Then from Theorem 3.1 follows that the factor group $\bar{\Gamma}_0(N)/\bar{\Gamma}_1(N)$ is isomorphic to the group of all even Dirichlet characters of $\mathbb{Z}$ mod N. Each of these characters becomes a character of the group $\bar{\Gamma}_0(N)$ if we set

$$\chi(\gamma) = \chi(d), \ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \bar{\Gamma}_0(N) \tag{3.6}$$

since $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} * & * \\ * & dd'+cb' \end{pmatrix}$, $c \equiv 0 \bmod N$.

The identity ((3.3)) becomes

$$\frac{2}{\varphi(N)} \sum_{\substack{\chi \text{ even} \\ \chi \bmod N}} r(z,z';s;\bar{\Gamma}_0(N);\chi) = r(z,z';s;\bar{\Gamma}_1(N);1) \tag{3.7}$$

where 1 means the trivial one-dimensional representation. From (3.7) follows the factorization formula for the Selberg zeta function

$$Z(s;\bar{\Gamma}_0(N);1) = \prod_{\chi \text{ even mod } N} Z(s;\bar{\Gamma}_0(N);\chi) \tag{3.8}$$

and finally we obtain the relation between the distribution functions of discrete eigenvalues of $A(\bar\Gamma_1(N);1)$ and $A(\bar\Gamma_1(N);\chi)$

$$N(\lambda;\bar\Gamma_1(N);1) = \sum_{\chi \text{ even mod } N} N(\lambda;\bar\Gamma_0(N);\chi). \tag{3.9}$$

We have

$$\mu(\bar F_1(N)) = \frac{\varphi(N)}{2}\mu(\bar F_0(N)) \tag{3.10}$$

and the inequality valid for all big enough λ,

$$N(\lambda;\bar\Gamma_0(N);\chi) \le \frac{\mu(\bar F_0(N))}{4\pi}\lambda, \tag{3.11}$$

where $\mu(\bar F_1(N))$, $\mu(\bar F_0(N))$ are the areas of the fundamental domains for $\bar\Gamma_1(N)$ and $\bar\Gamma_0(N)$ respectively. From that follows

Lemma 3.3 *Let the Weyl formula (law) hold for $N(\lambda;\bar\Gamma_1(N);1)$. Then the Weyl formula is true for each summand $N(\lambda;\bar\Gamma_0(N);\chi)$ in (3.9). In particular, the Weyl law is valid for $N(\lambda;\bar\Gamma_0(N);\chi)$ with real primitive character mod N.*

Let us formulate now the result of Huxley (see [Hu], p. 142).

Lemma 3.4 *For the group $\bar\Gamma_1(N)$ the determinant of the scattering matrix $\phi(s;\bar\Gamma_1(N);1)$ is given by*

$$\det\phi(s;\bar\Gamma_1(N);1) = (-1)^{(k-k_0)/2}\left(\frac{\Gamma(1-s)}{\Gamma(s)}\right)^k\left(\frac{A}{\pi^k}\right)^{1-2s}\prod_\chi \frac{L(2-2s,\bar\chi)}{L(2s;\chi)}, \tag{3.12}$$

where k is the number of cusps, $k_0 = tr\phi(1/2,\bar\Gamma_1(N);1)$, A is a positive integer composed of the primes dividing N, and the product has k terms, in each of which χ is a Dirichlet character to some modulus dividing N, $L(s,\chi)$ the corresponding Dirichlet L-series, $\bar\chi$ is the complex conjugated character.

From (3.12) follows

Lemma 3.5 $\det\phi(s;\bar\Gamma_1(N);1)$ *is a meromorphic function of order 1.*

From Lemmas 3.3, 3.4 and 3.5 the Selberg trace formula follows

Theorem 3.6 *For $\Gamma = \bar{\Gamma}_0(N)$ with real primitive character χ mod N the Weyl law (3.1) is valid.*

So we have infinite discrete spectrum of eigenvalues of $A(\bar{\Gamma}_0(N), \chi)$. Actually, having in mind the Selberg eigenvalue conjecture and equality (3.8), it is very likely that the whole spectrum of $A(\bar{\Gamma}_0(N); \chi)$ belongs to $[1/4, \infty)$, since we have nontrivial character χ, coming from the symbol $\left(\frac{N}{\cdot}\right)$.

4 Perturbation of $A(\bar{\Gamma}_0(N), \chi)$ by characters

Let $\omega(z)$ be a holomorphic modular form of weight 2, which belongs to $\bar{\Gamma}_0(N)$. Thus $\omega(\gamma z) = (cz+d)^2\omega(z)$,

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \bar{\Gamma}_0(N).$$

It is well known that the integral

$$\chi_\alpha(\gamma) = \exp 2\pi i\alpha \operatorname{Re} \int_{z_0}^{\gamma z_0} \omega(t)dt,\ \gamma \in \bar{\Gamma}_0(N) \tag{4.1}$$

$\alpha \in \mathbb{R}$, $z \in H$, defines the family of unitary characters for the group $\bar{\Gamma}_0(N)$, which is independent of the choice of the point z_0. We consider the family of self-adjoint operators $A(\bar{\Gamma}_0(N), \chi \cdot \chi_\alpha)$, as we defined in Section 1 by the Laplacian acting on functions $g(z)$ satisfying

$$g(\gamma z) = \chi(\gamma)\chi_a(\gamma)g(z),\ \gamma \in \bar{\Gamma}_0(N). \tag{4.2}$$

We consider α as a small parameter, $|\alpha| < \varepsilon, \varepsilon > 0$. The domain of definition $D(A(\bar{\Gamma}_0(N); \chi \cdot \chi_\alpha))$ is a dense subspace of $L_2(F; d\mu)$, varying with α. We consider then the operator $A_\alpha = A(\bar{\Gamma}_0(N); \chi \cdot \chi_2)$ as a perturbed $A_0 = A(\bar{\Gamma}_0(N), \chi)$, since the character (4.1) becomes trivial when $\alpha = 0$.

In order to make the perturbation correct we have to bring all the operators A_α to the domain of definition of A_0. Then we have to choose the form ω which makes the perturbation regular, and this is very important if we want to get information on eigenvalues and eigenfunction. On the other hand, it is very important to take as $\omega(z)$ the old holomorphic Eisenstein series, coming from the holomorphic Eisenstein series $E_2(z) = P(z)$ which belongs to the modular group. The last condition is crucial for the evaluation of the Phillips-Sarnak integral and to see that it is not zero (see Section 6). We will show that there exists a form $\omega(z)$, which satisfies these two conditions, for exactly the two cases 2), 3) from Theorem 1.1: 2) $\bar{\Gamma}_0(4N_2)$, N_2 a square-free positive integer and $N_2 \equiv 3 \bmod 4$, and its arithmetical character $\chi = \left(\frac{4N_2}{\cdot}\right)$, 3) $\bar{\Gamma}_0(4N_3)$, N_3 a square-free positive integer, $N_3 \equiv 2 \bmod 4$, and its arithmetical character $\left(\frac{4N_3}{\cdot}\right)$. In these two cases the character is always singular, since there exist both open and closed cusps. We construct now this perturbation and then we will find the appropriate form $\omega(z)$.

For a function f, $f(\gamma z) = \chi(\gamma) f(z)$, $\gamma \in \bar{\Gamma}_0(N)$, we define

$$g(z) = f(z) \exp 2\pi i \alpha \operatorname{Re} \int_{z_0}^{z} \omega(t)dt = f(z)\Omega(z, \alpha). \tag{4.3}$$

It is not difficult to see that $g(z)$ satisfies the condition (4.4). Applying the negative Laplacian

$$-\Delta = -4y^2 \frac{\partial^2}{\partial z \partial \bar{z}} \tag{4.5}$$

to the function $g(z)$, we obtain that the operator $A(\bar{\Gamma}_0(N); \chi \cdot \chi_\alpha)$ is unitarily equivalent to the operator

$$L(\alpha) = -\Delta + \alpha M + \alpha^2 N \tag{4.6}$$

where

$$\begin{cases} M = -4\pi i y^2 \left(\omega_1 \frac{\partial}{\partial x} - \omega_2 \frac{\partial}{\partial y}\right) = -4\pi i y^2 \left(\omega \frac{\partial}{\partial \bar{z}} + \bar{\omega} \frac{\partial}{\partial z}\right) \\ N = 4\pi^2 y^2 \left|\omega(z)\right|^2 = 4\pi^2 y^2 \left(\omega_1^2 + \omega_2^2\right) \end{cases} \tag{4.7}$$

and $\omega = \omega_1 + i\omega_2$, $\bar{\omega} = \omega_1 - i\omega_2$. The domain of definition $D(L(\alpha))$ equals $\Omega(z,\alpha)^{-1}D(A_\alpha)$ and

$$L(\alpha) = \Omega(\cdot,\alpha)^{-1}A_\alpha\Omega(\cdot,\alpha). \tag{4.8}$$

Note that M maps odd functions to even and even to odd. Recall that functions satisfying $f(-x+iy) = -f(x+iy)$ are odd and functions satisfying $f(-x+iy) = f(x+iy)$ are even by definition. Note also that a function f, satisfying $f(\gamma z) = \chi(\gamma)f(z)$, $\gamma \in \bar{\Gamma}_0(N)$, with our arithmetical character ω, is allowed to be odd or even. It is also true for the trivial character. It is not difficult to see also that the differential operators M, N map $(\bar{\Gamma}_0(N), \chi\cdot\chi_\alpha)$ automorphic functions to $(\bar{\Gamma}_0(N), \chi\cdot\chi_\alpha)$ automorphic functions.

5 The form $\omega(z)$

We will determine now the form $\omega(z)$. We start with constructing the holomorphic Eisenstein series of weight 2 for $\bar{\Gamma}_0(N)$ without character, using non-holomorphic Eisenstein series of weight zero. This method goes back to Hecke (see also [Sch] p. 15). We consider the series (2.1) for $\Gamma = \bar{\Gamma}_0(N)$, $\chi = 1$. Then we define

$$G_\alpha(z;s;\Gamma;1) = 2i\frac{\partial}{\partial z}E_\alpha(z;s;\Gamma;1) = \left(\frac{\partial}{\partial y} + i\frac{\partial}{\partial x}\right)E_\alpha(z;s;\Gamma;1) \tag{5.1}$$

$1 \le \alpha \le h$, h is the number of all inequivalent cusps of F.

(The function $E_\alpha(z,\cdot)$ depends also on $\bar{z}$, complex conjugate variable, since it is not a z-holomorphic function, so we have to write $E_\alpha(z,\cdot) = E_\alpha(z,\bar{z};\cdot)$ or $E_\alpha(x,y;\cdot)$, $z = x+iy$, $\bar{z} = x - iy$, $\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$, $\frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)$). It is well known that each of the $E_\alpha(z;s;\Gamma;1)$ has a simple pole at $s = 1$ with residue constant, independent of α. That means $G_\alpha(z;s) = G_\alpha(z;s;\Gamma;1)$ is regular at $s = 1$. We set $G_{\alpha,2}(z) = G_\alpha(z,1)$. It is clear then that $G_{\alpha,2}(z)$ transforms as a modular form of weight 2,

$$G_{\alpha,2}(\gamma z) = (z+d)^2 G_{\alpha,2}(z) \text{ for any } \gamma \in \Gamma \tag{5.2}$$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let us denote $cz+d=j(\gamma;z)$. But $G_{\alpha,2}(z)$ is not a holomorphic form.

From (2.8) follows the Fourier decomposition:

$$G_{\alpha,2}(g_\beta z)j^{-2}(g_\beta;z)=\delta_{\alpha\beta}-\frac{2iC}{z-\bar{z}}-4\pi\sum_{n=1}^{\infty}\sqrt{n}\varphi_{\alpha\beta n}(1;\Gamma;1)e^{2\pi inz} \tag{5.3}$$

where $\mathrm{Res}_{s=1}E_\alpha(z;s)=C,\ K_{1/2}(y)=\sqrt{\frac{\pi}{2y}}e^{-y}$.

Let $n_1,...,n_h$ be integers with the condition

$$\sum_{\alpha=1}^{h}n_\alpha=0. \tag{5.4}$$

Then we define

$$\sum_{\alpha=1}^{h}n_\alpha G_{2,\alpha}(z)=G_2(z;n_1,...,n_h). \tag{5.5}$$

From (5.3) follows that $G_2(z;n_1,...,n_h)$ is a holomorphic modular form of weight 2 for Γ. Then it is not difficult to see that all periods

$$\int_{z_0}^{\gamma z_0}G_2(t,n_1,...,n_h)dt \tag{5.6}$$

are real, $\gamma\in\Gamma$, $z_0\in H$.

We construct now our form $\omega(z)$ as one of these functions $G_2(z;n_1,...,n_h)$. Let us go back to Theorem 1.1. We consider there the last two cases. In Case 2) we had $\Gamma=\bar{\Gamma}_0(4N_2)$, N_2 is a square-free positive integer $N_2\equiv 3 \bmod 4$. For our arithmetical character $\left(\frac{4N_2}{\cdot}\right)$ we have open cusps z_d, $d\,|N_2$, $d>0$ and $d=4d_2$, $d_2\,|N_2$, $d_2>0$. We have the closed cusps z_d, $d=2d_1$, $d_1\,|N_2$, $d_1>0$. The total number of all closed cusps is

$$k\left(\bar{\Gamma}_0(4N_2),\left(\frac{4N_2}{\cdot}\right)\right)=d(N_2) \tag{5.7}$$

where $d(\cdot)$ is the divisor function. The total number of open cusps is

$$r\left(\bar{\Gamma}_0(4N_2),\left(\frac{4N_2}{\cdot}\right)\right)=2d(N_2) \tag{5.8}$$

and we certainly have $h=k+r=3d(N_2)=d(4N_2)$. We now define the form $\omega(z)$ by

$$\omega(z)=\sum_{\substack{d_1|N_2\\ d_1>0}}n_{2d_1}G_{2d_1,2}(z) \tag{5.9}$$

where each of the n_{2d_1} is equal to ± 1 with the only condition

$$\sum_{\substack{d_1|N_2 \\ d_1>0}} n_{2d_1} = 0. \tag{5.10}$$

From (5.3) follows that $\omega(z)$ is exponentially small in all open cusps and it is like $j^2(g_\beta, z)$ in each closed cusp $\beta = 2d_1$.

In analogy to this we consider Case 3) of Theorem 1.1. We have $\Gamma = \bar{\Gamma}_0(4N_3)$, N_3 is a square-free positive integer, $N_3 \equiv 2 \bmod 4$, and $\chi = \left(\frac{4N_3}{\cdot}\right)$. The open cusps are z_d, $d\,|n$, $d > 0$ $(N_3 = 2n)$ and $d = 8d_3$, $d_3\,|n$, $d_3 > 0$. The closed cusps are z_d, $d = 2d_1$, $d_1\,|n$, $d_1 > 0$ and $d = 4d_2$, $d_2\,|n$, $d_2 > 0$. We have

$$\begin{cases} k\left(\bar{\Gamma}_0(4N_3), \left(\frac{4N_3}{\cdot}\right)\right) = 2d(N_3/2) \\ r\left(\bar{\Gamma}_0(4N_3), \left(\frac{4N_3}{\cdot}\right)\right) = 2d(N_3/2) \\ h = k + r = 4d(N_3/2) = d(4N_3) \end{cases} \tag{5.11}$$

$$\omega(z) = \sum_{\substack{d_1|N_3/2 \\ d_1>0}} n_{2d_1} G_{2,2d_1}(z) + \sum_{\substack{d_2|N_3/2 \\ d_2>0}} n_{4d_2} G_{2,4d_2}(z) \tag{5.12}$$

where each of $2d_1$, $4d_2$ is equal to ± 1 with the condition

$$\sum_{\substack{d_1|N_3/2 \\ d_1>0}} n_{2d_1} + \sum_{\substack{d_2|N_3/2 \\ d_2>0}} n_{4d_2} = 0. \tag{5.13}$$

From (5.3) again follows that $\omega(z)$ is exponentially small in all open cusps and it is like $j^2(g_\beta; z)$ in each closed cusp $\beta = 2d_1$, $\beta = 4d_2$.

Let us calculate now the parabolic main periods of $\omega(z)$ in the two cases (5.9), (5.12). We consider (5.9) first. Let $S_{d'}$ be a parabolic generator of Case 2), $\bar{\Gamma}_0(4N_2)$ (one of (1.9), (1.10) or (1.11). We have

$$\int\limits_{z_0}^{S_{d'} z_0} \omega(z)dz = \int\limits_{g_{d'}^{-1} z_0}^{g_{d'}^{-1} S_{d'} z_0} \omega(g_{d'} t) \frac{dt}{j^2(g_{d'}; t)} \tag{5.14}$$

where

$$g_{d'}\infty = z_{d'},\ g_{d'}^{-1} S_{d'} g_{d'} z = S_\infty\, z = z + 1. \tag{5.15}$$

The formula (5.14) is equal to

$$\int_{t_0}^{g_{d'}S_{d'}g_{d'}^{-1}t_0} \omega(g_{d'}t)\frac{dt}{j^2(g_{d'};t)} = \int_{t_0}^{S_\infty t_0} \omega(g_{d'}t)\frac{dt}{j^2(g_{d'};t)} \tag{5.16}$$
$$= \sum_{\substack{d_1|N_2\\ d_1>0}} n_{2d_1} \int_{t_0}^{S_\infty t_0} G_{2d_1,2}(g_{d'}t)\frac{dt}{j^2(g_{d'};t)}.$$

We apply formula (5.3) and we obtain finally

$$\int_{z_0}^{S_{d'}z_0} \omega(z)dz = \sum_{\substack{d_1|N_2\\ d_1>0}} n_{2d_1}\delta_{(2d_1)d'}. \tag{5.17}$$

This is zero, if $z_{d'}$ is any open cusp, since the sum in (5.17) is taken over closed cusps only. And if $z_{d'}$ is one of the closed cusps, then (5.17) is equal to $n_{d'}=\pm 1$.

The analogous calculation shows in Case 3) $\bar{\Gamma}_0(4N_3)$ that the $\omega(z)$ from (5.17) has the main parabolic periods equal to

$$\int_{z_0}^{S_{d'}z_0} \omega(z)dz = \begin{cases} 0 & \text{, if } z_{d'} \text{ is an open cusp for } \left(\frac{4N_3}{\cdot}\right) \\ n_{d'} & \text{, if } z_{d'} \text{ is a closed cusp for } \left(\frac{4N_3}{\cdot}\right) \end{cases} \tag{5.18}$$

$n_{d'}=\pm 1$. Thus in Case 2) $\bar{\Gamma}_0(4N_2)$ with $\chi=\left(\frac{4N_2}{\cdot}\right)$ and $\chi_\alpha(\gamma)=\exp 2\pi i\alpha \,\mathrm{Re}\int_{z_0}^{\gamma z_0}\omega(t)dt = \exp 2\pi i\alpha\int_{z_0}^{z}\omega(t)dt$ we obtain

$$\chi(S_{d'})\chi_\alpha(S_{d'}) = \begin{cases} 1 & \text{, if } z_{d'} \text{ is an open cusp for } \left(\frac{4N_2}{\cdot}\right) \\ e^{2\pi i\alpha n_{d'}-\pi i} = e^{\pi i(2\alpha n_{d'}-1)} & \text{, if } z_{d'} \text{ is closed for } \left(\frac{4N_2}{\cdot}\right) \end{cases} \tag{5.19}$$

$n_{d'}=\pm 1$. The same result is valid in Case 3) $\bar{\Gamma}_0(4N_3)$, $\chi=\left(\frac{4N_3}{\cdot}\right)$ and $\chi_\alpha(\gamma)=\exp 2\pi i\alpha\,\mathrm{Re}\int_{z_0}^{\gamma z_0}\omega(t)dt = \exp 2\pi i\alpha\int_{z_0}^{\gamma z_0}\omega(t)dt$. We have

$$\chi(S_{d'})\chi_\alpha(S_{d'}) = \begin{cases} 1 & \text{, if } z_{d'} \text{ is open for } \left(\frac{4N_3}{\cdot}\right) \\ e^{2\pi i\alpha n_{d'}-\pi i} = e^{\pi i(2\alpha n_{d'}-1)} & \text{, if } z_{d'} \text{ is closed for } \left(\frac{4N_3}{\cdot}\right) \end{cases} \tag{5.20}$$

$n_{d'}=\pm 1$. We obtain in both cases that for $\alpha\in(-1/2,1/2)$ the character $\chi\cdot\chi_\alpha$ relative to the group Γ has the same degree of singularity and keeps the same cusps open and closed. For $\bar{\Gamma}_0(4N_2)$ and $\left(\frac{4N_2}{\cdot}\right)$ it is given by (5.7) and for $\bar{\Gamma}_0(4N_3)$, $\left(\frac{4N_3}{\cdot}\right)$ by (5.11). This means that the perturbation (4.6) is regular for the constructed forms $\omega(z)$.

We now consider the case $\bar{\Gamma}_0(4N_2)$, $N_2 = p_1 p_2 ... p_k$.

We want to get an expression for the form $\omega(z)$ of (5.9) as

$$\omega(z) = \sum_{d|4N_2,\, d>0} P(dz)\alpha_d \tag{5.21}$$

with real coefficients α_d and we will prove that there exists a set of integers $n_{2d_1} = \pm 1$ satisfying (5.10) such that the coefficient α_1 which corresponds to $d = 1$ is not zero,

$$\alpha_1 \neq 0. \tag{5.22}$$

Here $P(z)$ is the holomorphic Eisenstein series of weight 2 for the modular group $\bar{\Gamma}_0(1)$. We recall

$$P(z) = E_2(z) = 1 - 24\sum_{n=1}^{\infty} \sigma(n) e^{2\pi i n z}. \tag{5.23}$$

It is not quite a modular form of weight 2. We have the following transformation properties:

$$P\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 P(z) - \frac{6i}{\pi} c(cz+d), \quad \begin{pmatrix} a & b \\ d & d \end{pmatrix} \in \bar{\Gamma}_0(1). \tag{5.24}$$

In particular

$$\begin{cases} P(-1/z) = z^2 P(z) - \frac{6i}{\pi} z \\ P(z+1) = P(z) \end{cases} . \tag{5.25}$$

We consider (5.21) as a system of linear equations with unknown α_d, using well-known asymptotics of $\omega(z)$ and $P(z)$ at cusps of $\Gamma_0(4N_2)$, fundamental domain for $\bar{\Gamma}_0(4N_2)$.

When we defined the non-holomorphic Eisenstein series (2.1) we introduced the elements g_j. We now parametrize these elements by the divisors $d\,|4N_2$, and we will consider all inequivalent cusps of $\Gamma_0(4N_2)$, see (1.3), (1.9), (1.10), (1.11). We have

$$g_d S_\infty g_d^{-1} = S_d \tag{5.26}$$

$$g_d = \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix} \begin{pmatrix} \sqrt{m_d} & 0 \\ 0 & \sqrt{m_d}^{-1} \end{pmatrix} = \begin{pmatrix} \sqrt{m_d} & 0 \\ d\sqrt{m_d} & \sqrt{m_d}^{-1} \end{pmatrix}.$$

As a linear fractional transformation, $g_d z = \frac{m_d z}{d m_d z + 1}$, it has integer coefficients. To calculate the asymptotics of the right hand side of (5.21) at the cusps $1/d'$ we have to find the asymptotics of the functions

$$P(dg_{d'}z) = P\left(\frac{dm_{d'}z}{d'm_{d'}z+1}\right), \ z \to \infty \tag{5.27}$$

for all positive divisors $d\,|4N_2,\ d'\,|4N_2$. We set $m_{d'}z = z'$ and consider $P\left(\frac{dz'}{d'z'+1}\right) = P\left(\frac{d_1 z''}{d_2 z''+1}\right)$ where

$$d_1 = d/(d,d'),\ d_2 = d'/(d,d'),\ z'' = (d,d')z' = (d,d')m_{d'}z$$

where (d,d') is the greatest common divisor of d,d'. The matrix

$$\begin{pmatrix} d_1 & 0 \\ d_2 & 1 \end{pmatrix}$$

does not belong to $\bar{\Gamma}_0(1)$, so we can not directly apply formula (5.24), but since we have $(d_1,d_2) = 1$ we can make the following transformation. We define

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in PSL(2,\mathbb{R})$$

$$P\left(\frac{d_1 z''}{d_2 z''+1}\right) = P\left(g^{-1}\circ g\frac{d_1 z''}{d_2 z''+1}\right)$$

$$g\frac{d_1 z''}{d_2 z''+1} = \frac{\alpha\left(\frac{d_1 z''}{d_2 z''+1}\right)+\beta}{\gamma\left(\frac{d_1 z''}{d_2 z''+1}\right)+\delta} = \frac{(\alpha d_1+\beta d_2)z''+\beta}{(\gamma d_1+\delta d_2)z''+\delta} \tag{5.28}$$

We now choose $\alpha,\beta,\gamma,\delta$ by $\alpha d_1 + \beta d_2 = 1,\ \gamma d_1 + \delta d_2 = 0$.

For example $\gamma = -d_2,\ \delta = d_1$. Since $(d_1,d_2) = 1$ there exist integers α,β with these conditions. That means $g \in SL(2,\mathbb{Z})$. We have

$$g\frac{d_1 z''}{d_2 z''+1} = \frac{z''+\beta}{d_1} = \overset{o}{z},$$

and we apply (5.24) to the function

$$P\left(g^{-1}\overset{o}{z}\right) = P\left(\frac{d_1\overset{o}{z}-\beta}{d_2\overset{o}{z}+\alpha}\right) = \left(d_2\overset{o}{z}+\alpha\right)^2 P\left(\overset{o}{z}\right) - \frac{6i}{\pi}d_2\left(d_2\overset{o}{z}+\alpha\right). \tag{5.29}$$

We have finally from (5.27), (5.28), (5.29) and (5.23)

$$\begin{aligned}\left(d_2\overset{o}{z}+\alpha\right)^2 &= \left(\frac{d_2(z''+\beta)}{d_1}+\alpha\right)^2 = \left(\frac{d_2z''+1}{d_1}\right)^2 && (5.30)\\ &= \frac{1}{d_1^2}\left(d_2(d,d')m_{d'}z+1\right)^2 && (1)\\ &= \frac{1}{d_1^2}\left(d'm_{d'}z+1\right)^2 = \frac{(d,d')^2}{d^2}\left(d'm_{d'}z+1\right)^2.\end{aligned}$$

That means, from (5.25), (5.30) we get

$$\lim_{z\to\infty}(d'm_{d'}z+1)^{-2}P(dg_{d'}z) = \frac{(d,d')^2}{d^2}. \tag{5.31}$$

That gives the desired asymptotics of the right hand side of (5.21).

From (5.3), (5.9) we can see that

$$\lim_{z\to\infty} G_{d,2}(g_{d'}z)\left(d'\sqrt{m_{d'}}z+\sqrt{m_{d'}}^{-1}\right)^{-2} = \delta_{dd'}$$

that means

$$\lim_{z\to\infty} G_{d,2}(g_{d'}z)(d'm_{d'}z+1)^{-2} = \frac{\delta_{dd'}}{m_{d'}}. \tag{5.32}$$

Combining with (5.9) we obtain the following system of linear equations ($d'\,|4N_2$)

$$\sum_{\substack{d''|4N_2\\ d''>0}} \beta_{d''}(d',d'')^2 = \sum_{\substack{d_1|N_2\\ d_1>0}} \delta_{(2d_1)d'}\frac{n_{2d_1}}{m_{d'}} \tag{5.33}$$

where $\alpha_{d''} = \beta_{d''}\cdot d''^2$. From this system we have to determine the coefficients $\alpha_{d''}$ and to see that there exists a form $\omega(z)$ with $\alpha_1 \neq 0$.

Before studying the system (5.33) we will define the analogous system for the case $\bar{\Gamma}_0(4N_3)$. We have

$$\omega(z) = \sum_{d|4N_3\,,d>0} P(dz)\alpha_d. \tag{5.34}$$

Using the definition of $\omega(z)$ in this case (5.12), we obtain in analogy with (5.33) the system

$$\sum_{\substack{d''|4N_3\\ d''>0}} \beta_{d''}(d',d'')^2 = \sum_{\substack{d_1|N_3/2\\ d_1>0}} \delta_{(2d_1)d'} \frac{n_{2d_1}}{m_{d'}} + \sum_{\substack{d_2 \big| N_{3/2}\\ d_2>0}} \delta_{(4d_2)d'} \frac{n_{4d_2}}{m_{d'}} \tag{5.35}$$

where $\alpha_{d''} = \beta_{d''} \cdot d''^2$.

In Appendix 1 we prove the following Theorem about solution of the systems of equations (5.33) and (5.35).

Theorem 5.1 *In both the cases $\bar{\Gamma}_0(4N_2)$ and χ_{4n_2}, $\bar{\Gamma}_0(4N_3)$ and χ_{4N_3} there exist forms $\omega(z)$ given by (5.9), (5.10) and (5.12), (5.13) with the properties that each of them is given by a formula (5.21) with rational coefficients α_d, and the coefficient α_1 is not zero.*

These forms $\omega(z)$ are important for our perturbation (4.6), (4.7) and precisely for these forms we shall prove the Phillips-Sarnak conjecture.

6 Hecke theory and non-vanishing of L-functions on the line $\mathrm{Re}\, s = 1$

We now recall the Hecke theory for Maass forms (see [I]), in application to our cases $\bar{\Gamma}_0(4N_2)$, χ_{4N_2} and $\bar{\Gamma}_0(4N_3)$, χ_{4N_3}. We will write here simply $\bar{\Gamma}_0(N)$ and χ, having in mind these two particular cases (see Section 1 for definitions).

Let f be a continuous $(\bar{\Gamma}_0(N), \chi)$-automorphic function, i.e.

$$f(\gamma z) = \chi(\gamma) f(z)\ \forall \gamma \in \bar{\Gamma}_0(N),\ z \in H$$

and let $n \in \mathbb{Z}_+$ be relatively prime to N $(n, N) = 1$. Then the Hecke operators are defined by

$$T_n f(z) = \frac{1}{\sqrt{n}} \sum_{ad=n} \chi(d) \sum_{b \bmod d} f\left(\frac{az+b}{d}\right) \tag{6.1}$$

and act in the subspace of cusp forms of weight zero

$$\mathcal{H}_0(\bar{\Gamma}_0(N), \chi) \subset \mathcal{H}(\bar{\Gamma}_0(N)) = L_2(F_0(N); d\mu).$$

They are bounded and $\chi(n)$-hermitian, i.e.

$$\langle T_n f, g\rangle = \chi(n)\,\langle f, T_n g\rangle \tag{6.2}$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $\mathcal{H}(\Gamma)$.

We have also very important multiplicative relations for $(mn, N) = 1$

$$T_m \cdot T_n = \sum_{d|(m,n)} \chi(d) T_{mn/d^2}. \tag{6.3}$$

All T_m commute with each other and with the automorphic Laplacian $A(\bar{\Gamma}_0(N); \chi)$. So we can take in the space $\mathcal{H}_0(\bar{\Gamma}_0(N); \chi)$ a common basis of eigenfunctions for all T_m and $A(\bar{\Gamma}_0(N); \chi)$

$$\begin{cases} \left\{v_j(z) = v_j(z; \bar{\Gamma}_0(N); \chi)\right\}_{j=0}^{\infty} \\ Av_j = \lambda_j v_j,\ T_m v_j = \rho_j(m) v_j, \quad (m, N) = 1, \\ \qquad j = 1, 2, ... \end{cases} \tag{6.4}$$

There is a delicate question about the normalization of the eigenfunctions. Clearly for the L$_2$ theory of $A(\bar{\Gamma}_0(N), \chi)$ the normalization

$$\langle v_j, v_k\rangle = \delta_{jk} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$

is important, where, again, $\langle,\rangle$ is the inner product in $\mathcal{H}(\Gamma) = L_2(F_0(N); d\mu),\ F = F_0(N)$ is the fundamental domain of $\bar{\Gamma}_0(N)$ in H:

$$\langle v_j, v_k\rangle = \int_F v_j(z)\overline{v_k(z)} d\mu(z).$$

For Hecke theory there is another normalization which is more natural and which is related to Fourier decomposition of the functions v_j. We explain this after discussing old and new forms for $\bar{\Gamma}_0(N)$. We recall briefly the definition of old and new forms for $\bar{\Gamma}_0(N)$ and χ, generated by a Dirichlet character mod N.

If χ is mod M and $v(z) \in \mathcal{H}_0(\bar{\Gamma}_0(M); \chi)$ then $v(dz) \in \mathcal{H}_0(\bar{\Gamma}_0(N); \chi)$ whenever $dM\,|N$. By definition $\mathcal{H}_0^{\text{old}}(\bar{\Gamma}_0(N); \chi)$ is the subspace of $\mathcal{H}_0(\bar{\Gamma}_0(N); \chi)$ spanned by forms $v(dz)$, where $v(z)$ is defined for $\bar{\Gamma}_0(M)$ with character χ mod M, $M < N$, $dM\,|N$ and v is a common eigenfunction for all Hecke operators T_m with $(m, M) = 1$. Let the space $\mathcal{H}_0^{\mathrm{new}}$ be the orthonormal complement

$$\mathcal{H}_0(\bar{\Gamma}_0(N);\chi) = \mathcal{H}_0^{\mathrm{old}}(\bar{\Gamma}_0(N);\chi) \oplus \mathcal{H}_0^{\mathrm{new}}(\bar{\Gamma}_0(N);\chi).$$

From this definition it is clear that there are no old forms for the pairs $(\bar{\Gamma}_0(4N_2),\ \chi_{4N_2})$ and $(\bar{\Gamma}_0(4N_3),\ \chi_{4N_3})$ we consider, because $\chi_{4N_2},\ \chi_{4N_3}$ are primitive characters mod $4N_2,\ 4N_3$, respectively.

For any function $v_j(z)$ from (6.4) we have a Fourier decomposition

$$v_j(z;\bar{\Gamma}_0(N);\chi) = \sqrt{y}\sum_{n\neq 0}\tilde{\rho}_j(n)K_{s_j-1/2}(2\pi\,|n|\,y)e^{2\pi i n x} \tag{6.5}$$

where $K_s(z)$ is the modified Bessel function, $\tilde{\rho}_j(n) = \tilde{\rho}_j(n;\bar{\Gamma}_0(N);\chi)$, $\lambda_j = s_j(1-s_j)$. For non-trivial new forms we know that $\tilde{\rho}_j(1) \neq 0$. So the second normalization of the functions (6.4), (6.5) is by the condition

$$\tilde{\rho}_j(1) = 1. \tag{6.6}$$

After this normalization we obtain $\tilde{\rho}_j(m) = \rho_j(m)$ (see (6.4), (6.5)) for all j, m. For any $p\,|N$ there is another Hecke operator

$$U_p f(z) = \frac{1}{\sqrt{p}}\sum_{b \bmod p} f\left(\frac{z+b}{p}\right)$$

which acts on $H_0(\bar{\Gamma}_0(N),\chi)$ and commutes with $T_n, (n, N) = 1$. But in general U_p is not hermitian, nor even a normal operator, but it is trivial in our situation of $\bar{\Gamma}_0(4N_2),\ \chi_{4N_2}$ or $\bar{\Gamma}_0(4N_3),\ \chi_{4N_3}$ because all forms are new (trivial in our situation means $U_p = 0$). We have the following result. A normalized new form $v_j(z)$ for $\Gamma_0(N)$ and χ is an eigenfunction for all U_p with $p\,|N$ of eigenvalue $\rho_j(p)$ (we take the normalization (6.6)). Moreover, all its Fourier coefficients satisfy

$$\begin{cases} \rho_j(n)\rho_j(m) = \sum\limits_{d|(m,n)} \chi(d)\rho_j(mn/d^2) & \text{for } (mn, N) = 1 \\ \rho_j(p)\rho_j(n) = \rho_j(pn) & \text{for } p\,|N \end{cases}. \tag{6.7}$$

Then we have

$$U_p v_j(z) = \frac{1}{\sqrt{p}} \sum_{n \neq 0} \rho_j(n) \frac{\sqrt{y}}{\sqrt{p}} K_{s_j - 1/2}\left(2\pi |n| \frac{y}{p}\right) \sum_{b \bmod p} \exp 2\pi i n \frac{x+b}{p} \tag{6.8}$$

$$= \rho_j(p) v_j(z) = \rho_j(p) \sqrt{y} \sum_{n \neq 0} \rho_j(n) K_{s_j-1/2}(2\pi |n| y) \exp 2\pi i n x.$$

From that follows for all j

$$\rho_j(dn) = 0,\ d > 1,\ d\,|N\ ,\ n \in \mathbb{Z}. \tag{6.9}$$

We proceed to the construction of a Dirichlet series corresponding to (6.4), (6.6). Because of the specific properties of the perturbation (4.6) we have to consider odd eigenfunctions, i.e. with the condition

$$\rho_j(n) = -\rho_j(-n). \tag{6.10}$$

For the Hecke theory of Euler products, we will consider next, it is not an important restriction. The difference between odd and even eigenfunctions becomes visible, when one derives the functional equation for the corresponding Dirichlet series.

We will define now some important Dirichlet L-series and we will prove their non-vanishing on the line $\operatorname{Re} s = 1$. We attach to v_j an L-function

$$L(s; v_j) = \sum_{n=1}^{\infty} \frac{\rho_j(n)}{n^s}. \tag{6.11}$$

We will assume the corresponding eigenvalue $\lambda_j \geq 1/4$ (see (6.4)). The above series is absolutely convergent for $\operatorname{Re} s > 1$. This follows from the fact that the Rankin-Selberg convolution

$$\sum_{n=1}^{\infty} \frac{|\rho_j(n)|^2}{n^s} \tag{6.12}$$

is a holomorphic function for $\operatorname{Re} s > 1$ and has a simple pole at $s = 1$. We will show this in Appendix 2, and we will also show there that $L(s; v_j)$ has analytic continuation to C and satisfying the functional equation. We mention here the Ramanujan-Petersson conjecture for Maass forms v_j, which is not proved yet,

$$|\rho_j(n)| \leq d(n) \tag{6.13}$$

where $d(n)$ as usual is the number of positive divisors of n. We can write now the first equation (6.7) in the form

$$\begin{cases} (mn, N) = 1 \\ \rho_j(m)\rho_j(n) = \rho_j(m \cdot n) & (m,n) = 1 \\ \rho_j(p^{k+1)} = \rho_j(p^k)\rho_j(p) - \chi(p)\rho_j(p^{k-1}), & k \geq 0, \text{ by definition} \rho_j(p^{-1}) = 0 \ \cdot \\ p \text{ is a prime, } p \nmid N \\ \rho_j(1) = 1 \end{cases} \tag{6.14}$$

Using this we can obtain the Euler product

$$L(s; v_j) = \prod_{p \nmid N} \frac{1}{1 - \rho_j(p)p^{-s} + \chi(p)p^{-2s}} \tag{6.15}$$

where the product is taken over all primes p except $p\,|N$, $\operatorname{Re} s > 1$. In order to make our perturbation theory efficient we have to establish the following result

Theorem 6.1 *$L(s; v_j)$ is regular for $s = 1 + it$ and*

$$L(1 + it; v_j) \neq 0 \text{ for } t \in \mathbb{R}, \ j = 1, 2, \dots \ . \tag{6.16}$$

Proof. The case $t = 0$ will be proved elsewhere. We now consider the case $t \neq 0$. Clearly, (6.16) is analogous to the prime number theorem, $\zeta(1 + it) \neq 0$ for the Riemann zeta function. In order to prove Theorem 6.1 we will use the following general criterion proved in [M-M] (Theorem 1.2). ■

Lemma 6.2 *Let $f(s)$ be a function satisfying*

1. *f is a holomorphic and $f(s) \neq 0$ in $\{s\,|\operatorname{Re} s = \sigma > 1\}$*
2. *f is holomorphic on the line $\sigma = 1$ except for a pole of order $e \geq 1$ at $s = 1$*
3. *$\log f(s)$ can be written as a Dirichlet series*

$$\sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

with $b_n \geq 0$ for $\sigma > 1$.

Then if f has a zero on the line $\sigma = 1$, the order of the zero is bounded by $e/2$.

In order to apply this result it is very important to study the Rankin-Selberg convolution series (6.12). We investigate the analytic properties of this series in Appendix 2, but here we find the Euler product for this series in the spirit of Rankin (see [R]). The difference from Rankin's case is that our coefficients $\rho_j(n)$ may be complex numbers. It follows from (6.2) that

$$\rho_j(n) = \chi(n)\bar{\rho}_j(n) \ j = 1, 2, ... \tag{6.17}$$

and for $\chi(n) = -1$, $\rho_j(n)$ is purely imaginary (if $\rho_j(n) \neq 0$). In both cases $\chi(n) = \pm 1$ we have

$$|\rho_j(n)|^2 = \chi(n)\rho_j^2(n). \tag{6.18}$$

From (6.14) follows

$$\begin{cases} \rho_j^2(p^n) = (\rho_j(p)\rho_j(p^{n-1}) - \chi(p)\rho_j(p^{n-2}))^2 \\ (\chi(p)\rho_j(p^{n-3}))^2 = (-\rho_j(p^{n-1}) + \rho_j(p)\rho_j(p^{n-2}))^2 \end{cases}. \tag{6.19}$$

Then multiplying the second line of (6.19) by $\chi(p)$ and taking the difference, we obtain

$$\begin{aligned} \rho_j^2(p^n) - \rho_j^2(p)\rho_j^2(p^{n-1}) + \chi(p)\rho_j^2(p^{n-1}) + \chi(p)\rho_j^2(p)\rho_j^2(p^{n-2}) \\ -\rho_j^2(p^{n-2}) - \chi(p)\rho_j^2(p^{n-3}) = 0. \end{aligned} \tag{6.20}$$

Multiplying now (6.20) by $\chi(p^n)$ and using (6.18) we obtain

$$\begin{aligned} |\rho_j(p^n)|^2 - \left|\rho_j(p^{n-1})\right|^2 |\rho_j(p)|^2 + \left|\rho_j(p^{n-1})\right|^2 + |\rho_j(p)|^2 \left|\rho_j(p^{n-2})\right|^2 \\ - \left|\rho_j(p^{n-2})\right|^2 - \left|\rho_j(p^{n-3})\right|^2 = 0. \end{aligned} \tag{6.21}$$

Using this fundamental recurrence relation it is not difficult to see that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{|\rho_j(n)|^2}{n^{2s}} &= \prod_{p\nmid N} \left(1 + \frac{\rho_j^2(p)}{p^{2s}} + \frac{\rho_j^2(p^2)}{p^{4s}} + \frac{\rho_j^2(p^3)}{p^{6s}} + ...\right) \\ &= \prod_{p\nmid N} \frac{1 + p^{-2s}}{1 - |\rho_j(p)|^2 p^{-2s} + p^{-2s} + |\rho_j(p)|^2 p^{-4s} - p^{-4s} - p^{-6s}} \\ &= \prod_{p\nmid N} \left(1 + p^{-2s}\right) \left(1 - p^{-2s}\right)^{-1} \left(1 + (2 - |\rho_j(p)|^2)p^{-2s} + p^{-4s}\right)^{-1} \\ &= \prod_{p\nmid N} \left(1 - p^{-4s}\right) \left(1 - p^{-2s}\right)^{-2} \left(1 + (2 - |\rho_j(p)|^2)p^{-2s} + p^{-4s}\right)^{-1} \\ &= \frac{L^2(2s; \hat{\chi})}{L(4s; \hat{\chi})} \prod_{p\nmid N} \left(1 + (2 - |\rho_j(p)|^2)p^{-2s} + p^{-4s}\right)^{-1} \end{aligned} \tag{6.22}$$

where $L(s;\hat{\chi})$ is the Dirichlet L-series with principal character mod N.

$$L(s;\hat{\chi}) = \prod_{p>0} \left(1-\hat{\chi}(p)p^{-s}\right)^{-1} = \zeta(s)\prod_{p|N}(1-p^{-s}).$$

The products in (6.22) are taken over all primes except $p\,|N$.

We now introduce new functions $\alpha_j(p),\ \beta_j(p)$, which are important to define symmetric power L-series, by

$$\begin{cases} \alpha_j(p)+\beta_j(p)=\rho_j(p) \\ \alpha_j(p)\beta_j(p)=\chi(p) \end{cases}. \tag{6.23}$$

We have $(\alpha_j(p)+\beta_j(p))^2 = \rho_j^2(p) = \alpha_j^2(p)+2\chi(p)+\beta_j^2(p)$, and

$$\begin{cases} \chi(p)\alpha_j^2(p)+\chi(p)\beta_j^2(p)=|\rho_j(p)|^2-2 \\ \alpha_j^2(p)\beta_j^2(p)=1 \end{cases}. \tag{6.24}$$

Applying (6.24) to (6.22) we obtain

$$\prod_{p\nmid N}(1+(2-|\rho_j(p)|^2)p^{-2s}+p^{-4s})^{-1} \tag{6.25}$$

$$= \prod_{p\nmid N}\left(1-\frac{\chi(p)\alpha_j^2(p)}{p^{2s}}\right)^{-1}\left(1-\frac{\chi(p)\beta_j^2(p)}{p^{2s}}\right)^{-1} = L_2(2s;v_j).$$

Combining with (6.22) we finally obtain

$$L(s;v_j\times\bar{v}_j) = \frac{L^2(s;\hat{\chi})}{L(2s;\hat{\chi})}L_2(s;v_j) \tag{6.26}$$

where $L(s;v_j\times\bar{v}_j)$ is the Rankin-Selberg convolution (6.12).

We will prove now Theorem 6.1. From the Appendix we can see that the function $L(1+it;v_j)$ is regular for $s=1+it,\ t\in\mathbb{R},\ j=1,2,\ldots$. This follows from the fact that the function

$$\Omega(s) = \pm\pi^{-s}N^{s/2}\Gamma\left(\frac{s+s_j+1/2}{2}\right)\Gamma\left(\frac{s-s_j+3/2}{2}\right)L(s;v_j)$$

is an entire function, $\Gamma(s)$ is the Euler function, $s=1/2+ir_j,\ r_j\in\mathbb{R}$.

We will prove now that $L(s;v_j)\neq 0,\ s=1+it,\ t\neq 0$.

Let us denote

$$L(s;\hat{v}_j)=\sum_{n=1}^{\infty}\frac{\overline{\rho_j(n)}}{n^s} \tag{6.27}$$

with complex conjugated $\rho_j(n)$. We consider now the following product

$$f(s)=L^2(s;\hat{\chi})L_2(s;v_j)L(s;v_j)L(s;\hat{v}_j). \tag{6.28}$$

Then we have, using (6.25), (6.15), (6.24)

$$\log f(s)=2\log L(s;\hat{\chi})+\log L_2(s;v_j)+\log L(s;v_j)+\log L(s;\hat{v}_j)$$

$$\begin{aligned}=-\sum_{p\nmid N}\big\{&2\log(1-p^{-s})+\log(1-\chi(p)\alpha_j^2(p)p^{-s})+\log(1-\chi(p)\beta_j^2(p)p^{-s})\\ &+\log(1-\alpha_j(p)p^{-s})+\log(1-\beta_j(p)p^{-s})+\log(1-\bar{\alpha}_j(p)p^{-s})\\ &+\log(1-\bar{\beta}_j(p)\cdot p^{-s})\big\}\end{aligned} \tag{6.29}$$

since we can write

$$\begin{aligned}L(s;v_j)&=\prod_{p\nmid N}(1-\rho_j(p)p^{-s}+\chi(p)p^{-2s})^{-1}\\ &=\prod_{p\nmid N}(1-\alpha_j(p)p^{-s})^{-1}(1-\beta_j(p)p^{-s})^{-1}.\end{aligned} \tag{6.30}$$

Then for $|x|<1$ we have $\log(1-x)=-\sum\limits_{n=1}^{\infty}x^n/n$. Using this we continue (6.29)

$$\begin{aligned}\log f(s)&=\sum_{p\nmid N}\sum_{n=1}^{\infty}\frac{1}{np^{ns}}(2+\chi(p)^n\alpha_j^{2n}(p)+\chi(p)^n\beta_j^{2n}(p)+\alpha_j^n(p)\\ &\quad+\beta_j^n+\bar{\alpha}_j^n(p)+\bar{\beta}_j^n(p))\\ &=\sum_{p\nmid N}\sum_{n=1}^{\infty}\frac{a_{n,p}}{np^{ns}}.\end{aligned} \tag{6.31}$$

We will show now $a_{n,p}\geq 0$.

We consider two cases: $\chi(p)=1,\ \chi(p)=-1$. In the first case

$$a_{n,p}=2+2\alpha_j^n(p)+2\beta_j^n(p)+\alpha_j^{2n}(p)+\beta_j^{2n}(p)=(1+\alpha_j^n(p))^2+(1+\beta_j^n(p))^2\geq 0 \tag{6.32}$$

because in that case $\alpha_j(p), \beta_j(p)$ are real numbers. In the second case we have that $\alpha_j(p) = i\tilde{\alpha}_j(p)$, $\beta_j(p) = i\tilde{\beta}_j(p)$, and $\tilde{\alpha}_j(p)$, $\tilde{\beta}_j(p)$ are real numbers. We have

$$a_{n,p} = 2 + \tilde{\alpha}_j^{2n} + \tilde{\beta}_j^{2n} + \tilde{\alpha}_j^n(p)(i)^n((-1)^n + 1) + \tilde{\beta}_j^n(p)(i)^n((-1)^n + 1) \tag{6.33}$$

and this is real and ≥ 0 if $n = 2m - 1$, $m = 1, 2, \ldots$. We consider $n = 2m$, $m = 1, 2, \ldots$

$$\begin{aligned} a_{n,p} &= 2 + \tilde{\alpha}_j^{4m} + \tilde{\beta}_j^{4m} + (-1)^m \cdot 2\tilde{\alpha}_j^{2m} + (-1)^m \cdot 2\tilde{\beta}_j^{2m} \\ &= (1 + (-1)^m \tilde{\alpha}_j^{2m})^2 + (1 + (-1)^m \tilde{\beta}_j^{2m})^2 \geq 0 \end{aligned}$$

and we have proved that $a_{n,p} \geq 0$ for all $p \nmid N$, $n = 1, 2, \ldots$.

We can apply now Lemma 6.2 about non-vanishing of the function $f(1 + it)$. If $L(1 + it, v_j) = 0$ for some $t \neq 0$, then $f(s)$ has a zero of order ≥ 1 on the line $\operatorname{Re} s = 1$, $s \neq 1$. Here we use the facts that $L(s; \hat{v}_j)$ is regular everywhere and $L(s; v_j \times \bar{v}_j)$ has only the pole of first order at $s = 1$, which can not be compensated by the other factors, since $L(1, v_j) \neq 0$ and $L(1, \hat{v}_j) \neq 0$, as proved elsewhere. So we have a contradiction to Lemma 6.2. Thus, $L(s, v_j)$ (and $L(s; \hat{v}_j)$) has no zero on the line $\operatorname{Re} s = 1$. It then follows from the functional equation (see Appendix 3) that $L(s; v_j)$ has no zeros on the line $\operatorname{Re} s = 0$.

7 The Phillips-Sarnak integral

In this section we study the Phillips-Sarnak integral, adapted to our perturbation (4.7). For any odd eigenfunction (6.4), which corresponds to an embedded eigenvalue $\lambda_j > 1/4$ (actually, according to the Selberg eigenvalue conjecture, reduced to our case of congruence character, all $\lambda_j \geq 1/4$) we define the integral over the fundamental domain $F_0(N)$ of $\bar{\Gamma}_0(N)$. We use the notations of Section 5 (see the beginning of Section 5). The cusp $1/N$ is equivalent to ∞. So we have $F_0(N)$, containing ∞, and we define the Eisenstein series

$$E_\infty(z, s) = E_\infty(z, s; \bar{\Gamma}_0(N); \chi) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} y^s(\gamma z)\chi(\gamma). \tag{7.1}$$

The integral is the following one

$$I_j(s) = \int\limits_{F_0(N)} (Mv_j)(z) E_\infty(z, s) d\mu(z). \tag{7.2}$$

Theorem 7.1 *$I_j(s)$ is well defined and $I_j(s) \neq 0$ for $s = 1/2 + it,\ t \in R,\ t \neq 0,\ j = 1, 2, \ldots$.*

Proof. We take first $\mathrm{Re}\, s > 1$. It is not difficult to see that $v_j(z)$ is a function of exponential decay in all parabolic cusps of $F_0(N)$. This follows from the fact that $v_j(z)$ is an eigenfunction of the Laplacian, which is $(\bar{\Gamma}_0(N), \chi)$ automorphic. In open cusps z_m it is a cusp form and it has Fourier decomposition

$$v_j(g_m z) = \sqrt{y} \sum_{n \neq 0} \rho_j^{(m)}(n) K_{s_j - 1/2}(2\pi |n| y) e^{2\pi i n x} \tag{7.3}$$

where g_m is defined in (2.1). In closed cusps z_l it has Fourier decomposition

$$v_j(g_l z) = \sqrt{y} \sum_{n=-\infty}^{\infty} \rho_j^{(l)}(n) K_{s_j - 1/2}(2\pi |n + 1/2| y) e^{2\pi i (n+1/2) x}. \tag{7.4}$$

Then it is obvious that

$$\begin{cases} v_j(\gamma_l z) = \chi(\gamma) v_j(z),\ (M v_j)(\gamma z) = \chi(\gamma)(M v_j)(z) \\ E_\infty(\gamma z; s) = \chi(\gamma) E_\infty(z, s) \\ \chi^2(\gamma) = 1,\ \gamma \in \bar{\Gamma}_0(N) \end{cases} .$$

That means that the integral (7.2) is well defined and we can unfold the Eisenstein series $E_\infty(z, s)$, obtaining

$$I_j(s) = \int_0^\infty \frac{dy}{y^2} \int_{-1/2}^{1/2} dx (M v_j(z)) y^s \tag{7.5}$$

where $y^{-2}(M v_j)(z) = -4\pi i(\omega_1 v_{jx} - \omega_2 v_{jy})$ (see (4.7)),

$$v_{jx} = \frac{\partial v_j}{\partial x},\ v_{jy} = \frac{\partial v_j}{\partial y}.$$

Then we have

$$\int_{-1/2}^{1/2} \omega_1(x, y) v_{jx}(x, y) dx = \omega_1 v_j \Big|_{-1/2}^{1/2} - \int_{-1/2}^{1/2} \omega_{1x}(x, y) v_j(x, y) dx$$
$$= - \int_{-1/2}^{1/2} \omega_{1x}(x, y) v_j(x, y) dx$$

because ω and v_j are periodic in x with period 1. Similarly

$$\int_0^\infty y^s \omega_2 v_{jy} dy = y^s \omega_2 v_j \Big|_0^\infty - s\int_0^\infty y^{s-1}\omega_2 v_j dy - \int_0^\infty y^s v_j \omega_{2y} dy \tag{7.7}$$
$$= -s\int_0^\infty y^{s-1}\omega_2 v_j dy - \int_0^\infty y^s v_j \omega_{2y} dy.$$

Also we have Fourier decompositions

$$\begin{cases} \omega_1(x,y) = \sum_{n=1}^\infty a_n e^{-2\pi ny}\cos 2\pi nx \\ \omega_2(x,y) = \sum_{n=1}^\infty a_n e^{-2\pi ny}\sin 2\pi nx \end{cases}. \tag{7.8}$$

Using (7.3)-(7.8) we obtain

$$I_j(s) = 4\pi i \int_0^\infty y^s dy \int_{-1/2}^{1/2} dx(\omega_{1x} - \omega_{2y})v_j - 4\pi i s\int_0^\infty y^{s-1}dy \int_{-1/2}^{1/2} \omega_2 v_j dx \tag{7.9}$$
$$= -4\pi i s \int_0^\infty y^{s-1} dy \int_{-1/2}^{1/2} \omega_2(x,y)v_j(x,y)dx.$$

Then we apply (6.5), (6.7) to (7.9). We obtain

$$I_j(s) = 4\pi s\int_0^\infty y^{s-1/2}\sum_{n=1}^\infty a_n\rho_j(n)e^{-2\pi ny}K_{s_j-1/2}(2\pi ny)dy \tag{7.10}$$
$$= 4\pi s\frac{1}{(2\pi)^{s+1/2}}\left(\int_0^\infty t^{s-1/2}e^{-t}K_{s_j-1/2}(t)dt\right)\sum_{n=1}^\infty \frac{a_n\rho_j(n)}{n^{s+1/2}}.$$

The standard integral in brackets is equal to

$$\sqrt{\pi}\cdot 2^{-s-1/2}\frac{\Gamma(s+s_j)\Gamma(s-s_j+1)}{\Gamma(s+1)} \tag{7.11}$$

and we finally obtain

$$I_j(s) = \frac{s}{2^{2s-1}\pi^{s-1}}\cdot\frac{\Gamma(s+s_j)\Gamma(s-s_j+1)}{\Gamma(s+1)}\sum_{n=1}^\infty \frac{a_n\rho_j(n)}{n^{s+1/2}}. \tag{7.12}$$

We have now $s = 1/2 + i\tau,\ \tau \neq 0,\ s_j = 1/2 + i\tau_j,\ \tau_j \neq 0\ (\tau, \tau_j \in \mathbb{R})$.

With these conditions the factor to the Dirichlet series in (7.12) is never equal to the zero. So we have to study the Dirichlet series there in more detail. From Theorem 5.1 and the very important vanishing property (6.9) follows then (see also (5.23), (5.21), (5.34))

$$R(s) = -\frac{1}{24\alpha_1} \sum_{n=1}^{\infty} \frac{a_n \rho_j(n)}{n^{s+1/2}} = \sum_{n=1}^{\infty} \frac{\sigma(n)\rho_j(n)}{n^{s+1/2}}. \tag{7.13}$$

Applying the multiplicative properties of $\sigma(n)$ and $\rho_j(n)$ (see (6.7)) we have

$$\begin{aligned} R(s) &= \prod_{p \nmid N} \sum_{k=1}^{\infty} \frac{\rho_j(p^k)\sigma(p^k)}{p^{k(s+1/2)}} \qquad (7.14)\\ &= \prod_{p \nmid N} \sum_{k=0}^{\infty} \frac{\rho_j(p^k)}{p^{k(s+1/2)}} \cdot \frac{p^{k+1}-1}{p-1} \\ &= \prod_{p \nmid N} \sum_{k=0}^{\infty} \frac{1}{p-1} \left(\frac{p\rho_j(p^k)}{p^{k(s-1/2)}} - \frac{\rho_j(p^k)}{p^{k(s+1/2)}} \right). \end{aligned}$$

Then we get

$$\begin{aligned} R(s) = &\prod_{p \nmid N} \frac{1}{p-1} p \left(1 - \rho_j(p) p^{-(s-1/2)} + \chi(p) p^{-2(s-1/2)}\right)^{-1} \qquad (7.15)\\ &- (1 - \rho_j(p) p^{-(s+1/2)} + \chi(p) p^{-2(s+1/2)})^{-1}) \\ &= \prod_{p \nmid N} (1 - \chi(p) p^{-2s})(1 - \rho_j(p) p^{-(s-1/2)} + \chi(p) p^{-2s+1)})^{-1} \\ &\cdot (1 - \rho_j(p) p^{-(s+1/2)} + \chi(p) p^{-2s-1})^{-1} \\ &= L^{-1}(2s; \chi) L(s+1/2; v_j) \cdot L(s-1/2; v_j). \end{aligned}$$

The product in formulas (7.14), (7.15) is taken over all primes except $p\,|N$. From the non-vanishing result of Theorem 6.1 follows that the right hand side of (7.15) is not zero for $s = 1/2 + it$, $t \in \mathbb{R}$, $t \neq 0$, and Theorem 7.1 is proved. ■

8 Perturbation of embedded eigenvalues

Definition. Suppose that F has h cusps $z, ..., z_h$ and that under the character χ_α the cusps $z_1, ..., z_k$ are open and $z_{k+1}, ..., z_h$ are closed. Let $\gamma_i z_i = \infty$, $i = 1, ..., k$, where $\gamma_i = g_i^{-1}$. The Banach spaces $C_{\mu,\nu} = C_{\mu,\nu}(F)$ are defined as the spaces of continuous functions f on F such that

$$|f(\gamma_i z)| \leq C\,|\mathrm{Im}\,\gamma_i z|^{\mu} \ \text{ for } i = 1, ..., k$$

$$|f(\gamma_i z)| \le C\,|\mathrm{Im}\,\gamma_i z|^{\nu} \;\text{ for } i = k+1, ..., h$$

with the norm

$$\|f\|_{\mu,\nu} = \max\left\{ \max_{1\le i\le k} \sup_{\substack{z\in F \\ \mathrm{Im}\,\gamma_i z \ge 1}} |f(\gamma_i z)|\,(\mathrm{Im}\,\gamma_i z)^{-\mu},\ \max_{k+1\le i\le h} \sup_{\substack{z\in F \\ \mathrm{Im}\,\gamma_i z \ge 1}} |f(\gamma_i z)|\,(\mathrm{Im}\,\gamma_i z)^{-\nu} \right\}.$$

We utilize mainly the spaces $C_{1,-2}$, $C_1 = C_{1,1}$ and $C_{-1,0}$.

We make use of results of [F] on estimates and mapping properties of the resolvent kernel of the Laplacian $A(\Gamma)$ extended by [V] to operators $A(\Gamma, \chi)$ with character χ. From the results of [F] and [V] we obtain the following theorem.

Theorem 8.1 *For any* $\alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ *the resolvent* $R(s, \alpha)$ *of* $L(\alpha) = L + \alpha M + \alpha^2 N$ *has an analytic continuation* $\tilde{R}(s, \alpha)$ *to* $\{s\,|0 < \mathrm{Re}\, s < 2\}$ *as an operator in* $B(C_{-1,0}, C_{1,-2})$. *For* $\mathrm{Re}\, s > 1$, $R(s, \alpha) \in B(C_{1,-2})$.

We set $\tilde{R}(s) = \tilde{R}(s, 0)$.

We consider the mapping properties of the operators M and N. In the open cusps the coefficients of M and N are exponentially decreasing, in the closed cusps they go like y^2. It follows that $V(\alpha) \in B(C_{1,-2}), C_{-1,0})$, where $V(\alpha) = \alpha M + \alpha^2 N$. This implies

Lemma 8.2 $V(\alpha)\tilde{R}(s) \in B(C_{-1,0})$ *for* $0 < s < 2$,

$$\left\|V(\alpha)\tilde{R}(s)\right\|_{B(C(-1,0))} \underset{\alpha\to 0}{\to} 0$$

and for $|\alpha| < \varepsilon$

$$\tilde{R}(s, \alpha) = \tilde{R}(s)(1 + V(\alpha)\tilde{R}(s))^{-1} \in B(C_{-1,0}, C_{1,-2}).$$

$\tilde{R}(s, \alpha)$ is analytic with values in $B(C_{-1,0}, C_{1,-2})$ for $0 < \mathrm{Re}\, s < 2$ as a function of α for $|\alpha| < \varepsilon$.

We now consider the operator $L_1(\alpha) = L + \alpha M + \alpha^2 N$ acting in the Banach space. $C_{1,-2}$ with maximal domain $\mathcal{D}(L_1(\alpha))$.

By Theorem 8.1 the resolvent $R_1(s)$ of $L_1 = L_1(0)$ exists as an operator in $B(C_{1,-2})$ for $\operatorname{Re} s > 1$, hence $V(\alpha)R_1(s) \in B(C_{1,-2})$ and $\|V(\alpha)R_1(s)\| \underset{\alpha\to 0}{\to} 0$ for $\operatorname{Re} s > 1$. Moreover, for $|\alpha| < \varepsilon$ and $\operatorname{Re} s > 1$

$$R_1(s,\alpha) = R_1(s)(1 + V(\alpha)R_1(s))^{-1} \in B(C_{1,-2}).$$

It follows that $L_1(\alpha)$ is closed on the domain $\mathcal{D}(L_1(\alpha)) = \mathcal{D}(L_1)$ for $|\alpha| < \varepsilon$. We have established

Lemma 8.3 *$L_1(\alpha)$ is analytic for $|\alpha| < \varepsilon$ as a family of closed operators in $C_{1,-2}$ with domain $D(L_1)$.*

We shall analyze now the perturbation of embedded eigenvalues. This was investigated by [Ho] for Schrödinger operators $-\Delta + \alpha V$ with multiplicative potential V. In our case the form of the perturbation requires a somewhat different approach.

Let $\lambda_0 = s_0(1 - s_0) > \frac{1}{4}$ be an eigenvalue of $L = A(\Gamma, \chi)$, $s_0 = \frac{1}{2} + it_0$, $t_0 \neq 0$, with eigenspace $\mathcal{N} = \mathcal{N}(L - \lambda_0)$ of dimension m.

Let $K = K(s_0, \delta)$ be a circle with center s_0 and radius δ separating s_0 from other points s_i corresponding to eigenvalues $\lambda_i = s_i(1 - s_i)$ of L, and choose $\varepsilon > 0$ such that $\tilde{R}(s,\alpha) \in B(C_{-1,0}, C_{1,-2})$ for $s \in K$ and $|\alpha| < \varepsilon$. The operators $\tilde{P}(\alpha) \in B(C_{-1,0}, C_{1,-2})$ are defined for $|\alpha| < \varepsilon$ by

$$\tilde{P}(\alpha) = -\frac{1}{2\pi i}\int_K \tilde{R}(s,\alpha)(2s-1)ds.$$

$\tilde{P}(\alpha)$ is analytic in α for $|\alpha| < \varepsilon$, and $\tilde{P}(0)$ coincides with the orthogonal projection P_0 of H on $N(L - \lambda_0)$, restricted to $C_{-1,0}$.

We consider the operators

$$\begin{aligned} P_0\tilde{P}(\alpha)P_0 &= -\frac{1}{2\pi i}P_0\int_K \tilde{R}(s,\alpha)(2s-1)dsP_0 \\ &= -\frac{1}{2\pi i}P_0\int_K \tilde{R}(s)(2s-1)dsP_0 \\ &\quad - \frac{1}{2\pi i}P_0\int_K \tilde{R}(s,\alpha)(\alpha M + \alpha^2 N)\tilde{R}(s)(2s-1)dsP_0 \\ &= P_0 - \alpha P_0\frac{1}{2\pi i}\int_K \tilde{R}(s,\alpha)(M + \alpha N)\tilde{R}(s)(2s-1)dsP_0. \end{aligned}$$

Here we use that $\mathcal{R}(P_0) = \mathcal{N} \subset C_{-1,0} = D(\tilde{P}(\alpha))$. Since the eigenfunctions $\phi \in N$ decay exponentially, we can also consider P_0 as an operator in $B(C_{1,-2}, \mathcal{H})$, so $P_0\tilde{P}(\alpha)P_0 \in B(\mathcal{H})$. For $\alpha \to 0$, the second term converges in norm to zero.

It follows that $\dim R(P_0\tilde{P}(\alpha)P_0) = \dim R(P_0) = m$ for $|\alpha| < \varepsilon$.

The circle K contains for each α with $|\alpha| < \varepsilon$ a finite number of poles $s_1(\alpha), ..., s_k(\alpha)$ of the meromorphic function $\tilde{R}(s, \alpha)$ with values in $B(C_{-1,0}, C_{1,-2})$.
Let $\tilde{P}_i(\alpha) = -\mathrm{Res}\Big\{\tilde{R}(s, \alpha)\Big\}_{s=s_i(\alpha)}$. Then

$$\tilde{P}(\alpha) = \sum_{i=1}^{k} \tilde{P}_i(\alpha).$$

For $|\alpha| < \varepsilon$ we have

$$m = \dim \mathcal{R}(\tilde{P}(\alpha)P_0) = \sum_{i=1}^{k} \dim \mathcal{R}(\tilde{P}_i(\alpha)P_0. \tag{8.1}$$

This implies that for $|\alpha| < \varepsilon$ all the poles $s_i(\alpha)$ of $\tilde{R}(\alpha, s)$ inside K are simple.

Then we have with $\lambda_i(\alpha) = s_i(\alpha)(1 - s_i(\alpha))$

$$(L_1(\alpha) - \lambda_i(\alpha))\tilde{P}_i(\alpha) = 0,\ i = 1, ..., k$$

and hence for $\lambda = s(1 - s)$

$$\begin{aligned}(L_1(\alpha) - \lambda)\tilde{P}(\alpha) &= \sum_{i=1}^{k}(L_1(\alpha) - \lambda_i(\alpha))\tilde{P}_i(\alpha) + \sum_{i=1}^{k}(\lambda_i(\alpha) - \lambda)\tilde{P}_i(\alpha) \\ &= \sum_{i=1}^{k}(\lambda_i(\alpha) - \lambda)\tilde{P}_i(\alpha).\end{aligned} \tag{8.2}$$

Consider now the function $\tilde{L}(\alpha, \lambda)$ with values in $B(N)$, defined by

$$\tilde{L}(\lambda, \alpha) = P_0(L_1(\alpha) - \lambda)\tilde{P}(\alpha)P_0.$$

Let $\{a_{ij}(\lambda, \alpha)\}_{i,j=1}^{m}$ be the matrix of $\tilde{L}(\alpha, \lambda)$ in the basis $\{\phi_1, ..., \phi_m\}$,

$$a_{ij}(\lambda, \alpha) = (\phi_i, \tilde{L}(\lambda, \alpha)\phi_j).$$

The functions $\alpha_{ij}(\lambda, \alpha)$ are analytic in α for $|\alpha| < \varepsilon$ and in λ. The determinant $d(\lambda, \alpha) = \det \{a_{ij}(\lambda, \alpha)\}_{i,j=1}^{m}$ is for each α a polynomium in λ of degree m, whose coefficients are analytic in α for $|\alpha| < \varepsilon$. It can be written, using (8.2), as

$$d(\lambda, \alpha) = \det \left\{ \left(\phi_i, \sum_{l=1}^{k} (\lambda_l(\alpha) - \lambda) \tilde{P}_l(\alpha) \phi_j \right) \right\}_{i,j=1}^{m} . \tag{8.3}$$

For each l and $|\alpha| < \varepsilon$ the pole $\lambda_l(\alpha)$ is simple, but $\dim \mathcal{R}(\tilde{P}_l(\alpha)) = m_l(\alpha)$ may be larger than one. For each α, $|\alpha| < \varepsilon$, $d(\lambda, \alpha)$ has m roots, counted with multiplicity, and lying within the curve $K_1 = \{\lambda = s(1-s) \,|\, s \in K\}$.

We have shown that each pole $\lambda_l(\alpha)$ of $\tilde{R}(\lambda, \alpha)$ is a root in the polynomial $d(\lambda, \alpha)$, and it is clear from (8.3) that $\lambda_l(\alpha)$ is a root of multiplicity $m_l(\alpha)$. From (8.1) follows then that the roots of $d(\lambda, \alpha)$ are precisely the poles $\lambda_l(\alpha)$ of $\tilde{R}(\lambda, \alpha)$ and that the multiplicity of $\lambda_l(\alpha)$ as a root of $d(\lambda, \alpha)$ equals $\dim \mathcal{R}(\tilde{P}_l(\alpha)) = \dim \mathcal{N}(L_1(\alpha) - \lambda_l(\alpha))$.

Using general theory of analytic functions of two variables, we conclude that for $|\alpha| < \varepsilon$ the m-dimensional eigenvalue λ_0 splits into analytic branches and Puiseux cycles. Because of the special circumstance, that λ_0 can not move into the resolvent set, but has to move either along the spectrum or onto the second sheet, as α moves into non-zero real numbers, we obtain the following general result on perturbation of embedded eigenvalues, as stated and proved by [Ho] in the case of Schrödinger operators $-\Delta + \alpha V$.

Theorem 8.4 *The poles $\lambda_1(\alpha), ..., \lambda_k(\alpha)$ of $\tilde{R}(\alpha, s)$ inside the curve K_1 can for $|\alpha| < \varepsilon$ be divided into groups forming Puiseux cycles of order $p \geq 1$. If $p = 1$, the corresponding $\lambda_j(\alpha)$ is analytic for $|\alpha| < \varepsilon$. If $p \geq 2$, the Puiseux cycle consists of p branches $\lambda_{j1}(\alpha)...\lambda_{jp}(\alpha)$ of a function having a branch point of order p at $\alpha = 0$. In the first case $p = 1$ we have the following possibilities,*

1. *$\lambda_1(\alpha)$ is real for all real α, and $\lambda_i(\alpha)$ is an embedded eigenvalue of $L(\alpha)$ for $\alpha \in (-\varepsilon, \varepsilon)$.*

2. *$\lambda_i(\alpha) = \lambda_0 + a_1\alpha + ... + a_{2l-1}\alpha^{2l-1} + a_{2l}\alpha^{2l} + \sum_{m \geq 2l+1} a_m \alpha^m$, $a_1, ..., a_{2l-1}$ are real, $\operatorname{Im} a_{2l} > 0$ for $s_0 = \frac{1}{2} + it_0$, $\operatorname{Im} a_{2l} < 0$ for $s_0 = \frac{1}{2} - it_0$, $t_0 > 0$.*

In the case $p \geq 2$, the functions $\lambda_{j1}(\alpha), ..., \lambda_{jp}(\alpha)$ have expansions of the form

$$\lambda_{jl}(\alpha) = \lambda_0 + b_1\alpha + ... + b_{2m-1}\alpha^{2m-1} + b_{2m}\alpha^{2m} + b_{2m+1}\omega^l\alpha^{(2m+1)/p} + ...,$$

$l = 1, ..., p$, where $b_1, ..., b_{2m-1}$ are real and $\operatorname{Im} b_{2m} > 0$ for $s_0 = \frac{1}{2} + it_0$, $\operatorname{Im} b_{2m} < 0$ for $s_0 = \frac{1}{2} - it_0$, $t_0 > 0$.

The multiplicity of each $\lambda_i(\alpha)$ and $\lambda_j(\alpha)$ is constant and is the same for all elements of a Puiseux cycle.

We shall now derive explicit formulas for the perturbation of the eigenvectors ϕ_l to first order and the eigenvalue $\lambda_0 = s_0(1 - s_0)$ to second order.

Let $\phi \in \mathcal{N}(L - \lambda_0) = P_0 H$. Then $\phi(\alpha) = \tilde{P}(\alpha)\phi$ is an analytic function with values in $C_{1,-2}$ for $|\alpha| < \varepsilon$. We calculate $\phi_1 = \phi'(0)$ as follows.

$$\begin{aligned}
\phi_1 &= \lim_{\alpha \to 0} \frac{1}{\alpha} \left\{ \tilde{P}(\alpha) - \tilde{P}(0) \right\} \phi \qquad (8.4)\\
&= \lim_{\alpha \to 0} \frac{1}{\alpha} \frac{(-1)}{2\pi i} \int_K \left\{ \tilde{R}(\alpha, s) - \tilde{R}(s) \right\} (2s - 1) ds \\
&= \lim_{\alpha \to 0} \frac{1}{2\pi i} \int_K \frac{1}{\alpha} \tilde{R}(\alpha, s)(\alpha M + \alpha^2 N)\tilde{R}(s)\phi(2s - 1) ds \\
&= \lim_{\alpha \to 0} \frac{1}{2\pi i} \int_K \tilde{R}(\alpha, s)(M + \alpha N)(L - s(1 - s))^{-1}\phi(2s - 1) ds \\
&= \frac{1}{2\pi i} \int_K \tilde{R}(s)M\phi \left\{ s_0(s - s_0) - s(1 - s) \right\}^{-1} (2s - 1) ds.
\end{aligned}$$

Setting $\psi = M\phi$, we derive an expression for $\tilde{R}(0, s)\psi$. Let $|s - s_0| < \delta$, $\operatorname{Re} s > \frac{1}{2}$, $\operatorname{Im}(s - s_0) > 0$. Then by the spectral theorem,

$$\begin{aligned}
R(s)\psi = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{1}{\frac{1}{4} + r^2 - s(1 - s)} \sum_{j=1}^{m} |E_j\left(\tfrac{1}{2} + ir\right)\rangle\langle E_j\left(\tfrac{1}{2} - ir\right)|\psi\rangle \qquad (8.5)\\
+ R_l(s)\psi + R_d(s)\psi + \left(s_0(1 - s_0) - s(1 - s)\right)^{-1} P_0\psi,
\end{aligned}$$

where h is the number of open cusps,

$$R_l(s)\psi = \sum_{k=1}^{\infty} |\phi_k\rangle\langle\phi_k|\ \psi\rangle \left(s_k(1 - s_k) - s(1 - s)\right)^{-1}$$

$$R_d(s)\psi = \sum_{l=1}^{\infty} |\phi'_k\rangle\langle\phi'_k|\ \psi\rangle \left(s'_k(1 - s'_k) - s(1 - s)\right)^{-1}$$

and $s_k(1-s_k)$ are the embedded eigenvalues different from $s_0(1-s_0)$ with eigenfunctions ϕ_k and $s_l'(1-s_l')$ are the small, discrete eigenvalues with eigenfunctions ϕ_k'.

The integrand is analytic in r, and we can deform the contour $\mathbb{R}$ to a contour Γ_R, $|s-s_0|<R\le\delta$, obtained by replacing $[t_0-R,t_0+R]$ by the semicircle $\left\{-\mathrm{Re}^{i\varphi}\ \ |0\le\varphi\le\pi\right\}$, see Fig. 1.

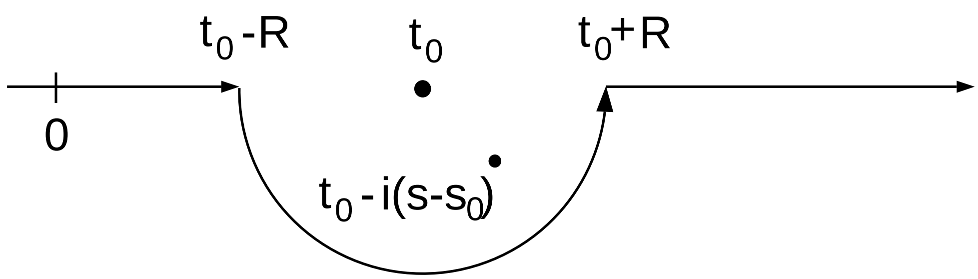

Figure 1: Deformation of the spectrum of L.

For a fixed s the poles of the function $\left(\frac{1}{4}+r^2-s(1-s)\right)^{-1}$ are

$$\rho_\pm=\pm i\left(s-\tfrac{1}{2}\right)=\pm i\left(it_0+s-s_0\right)=\mp t_0\pm i\left(s-s_0\right).$$

We have chosen to focus on $s_0=\frac{1}{2}+it_0$. The root $\rho_-=t_0-i(s-s_0)$ lies inside the above semicircle. The residue of the integrand at the simple pole ρ_- is

$$\operatorname{Res}\left\{\frac{1}{r-i\left(s-\frac{1}{2}\right)}\frac{1}{r+i\left(s-\frac{1}{2}\right)}\sum_{j=1}^{h}|E_j(\tfrac{1}{2}+ir)\rangle\langle E_j\left(\tfrac{1}{2}-ir\right)\ \psi\rangle\right\}_{r=-i\left(s-\frac{1}{2}\right)}$$
$$=\frac{1}{-2i\left(s-\frac{1}{2}\right)}\sum_{j=1}^{h}|E_j\left(s\right)\rangle\langle E_j\left(1-s\right)|\ \psi\rangle$$

so the first term $R_c(s)\psi$ of $R(0,s)\psi$ equals

$$R_c(s)\psi=\frac{1}{4\pi}\int_{\Gamma_R}\frac{1}{\frac{1}{4}+r^2-s(1-s)}\sum_{j=1}^{h}|E_j\left(\tfrac{1}{2}+ir\right)\rangle\langle E_j\left(\tfrac{1}{2}-ir\right)\ \psi\rangle\,dr \qquad (8.6)$$
$$+\frac{1}{4\left(s-\frac{1}{2}\right)}\sum_{j=1}^{h}|E_j\left(s\right)\rangle\langle E_j\left(1-s\right)|\ \psi\rangle\,.$$

Both terms of $R_c(0,s)$ have analytic continuations to $\{s\,|s-s_0|<R\}$, and we obtain $\tilde{R}_c(0,s)\psi$ expressed by the same equation (8.6).

We calculate the first term at $s = s_0$. Replacing R by any smaller radius $\rho > 0$ we obtain

$$\frac{1}{4\pi}\int\limits_{\Gamma_\rho} \frac{1}{\frac14 + r^2 - s_0(1-s_0)} \sum_{j=1}^{h} \left|E_j\left(\tfrac12 + ir\right)\right\rangle\left\langle E_j\left(\tfrac12 - ir\right)\right| \psi\rangle\, dr$$

$$= \frac{1}{4\pi}\lim_{\rho\downarrow 0}\left\{\int\limits_{-\infty}^{t_0-\rho} + \int\limits_{t_0+\rho}^{\infty}\right\} \frac{1}{r^2 - t_0^2} \sum_{j=1}^{h} \left|E_j\left(\tfrac12 + ir\right)\right\rangle\left\langle E_j\left(\tfrac12 - ir\right)\right| \psi\rangle\, dr$$

$$+ \frac{1}{4\pi}\lim_{\rho\downarrow 0}\int\limits_{\tilde C_\rho} \frac{1}{\frac14 + r^2 - s_0(1-s_0)} \sum_{j=1}^{h} \left|E_j\left(\tfrac12 + ir\right)\right\rangle\left\langle E_j\left(\tfrac12 - ir\right)\right| \psi\rangle\, dr$$

$$= \frac{1}{4\pi} PP \int\limits_{-\infty}^{\infty} \frac{1}{r-t_0}\frac{1}{r+t_0} \sum_{j=1}^{h} \left|E_j\left(\tfrac12 + ir\right)\right\rangle\left\langle E_j\left(\tfrac12 - ir\right)\right| \psi\rangle\, dr$$

$$+ \frac{1}{4\pi}\frac12 \cdot 2\pi i \ \mathrm{Res}\left\{\frac{1}{\frac14 + r^2 - s_0(1-s_0)} \sum_{j=1}^{h} |E_j(\tfrac12 + ir)\rangle\langle E_j\left(\tfrac12 - ir\right) \mid \psi\rangle\, dr\right\}_{s(s_0)}$$

where $\tilde C_\rho$ is the semicircle $\{s = \rho e^{i\varphi}\, |{-\pi} \le \varphi \le 0\}$. Thus, half of the previously subtracted residue is added, and we obtain

$$\tilde R_c(s_0)\psi = \frac{1}{4\pi} PP \int\limits_{-\infty}^{\infty} \frac{1}{r-t_0}\frac{1}{r+t_0} \sum_{j=1}^{h} |E_j\left(\tfrac12 + ir\right)\rangle\langle E_j\left(\tfrac12 - ir\right)\ \psi\rangle\, dr \tag{8.7}$$

$$+ \frac{1}{8it_0}\sum_{j=1}^{h} \left|E_j\left(\tfrac12 + it_0\right) >\right\rangle\left\langle E_j\left(\tfrac12 - it_0\right)\right| \psi\rangle\, .$$

We can now introduce (8.5) in (8.4), using (8.6) and (8.7). We obtain the following expression for ϕ_1, using that all the terms of $\tilde R(s)\psi$ have a simple pole at $s = s_0$ except possibly the last term of (8.5) which contains a double pole if $\frac14$ is an eigenvalue of L.

$$\phi_1 = \frac{1}{4\pi} PP \int\limits_{-\infty}^{\infty} \frac{1}{r-t_0}\frac{1}{r+t_0} \sum_{j=1}^{h} \left|E_j\left(\tfrac12 + ir\right)\right\rangle\left\langle E_j\left(\tfrac12 - ir\right)\right| M\phi\rangle\, dr \tag{8.8}$$

$$+ \frac{1}{8it_0}\sum_{j=1}^{h} E_j\left(\tfrac12 + it_0\right)\rangle\langle E_j\left(\tfrac12 - it_0\right)\ M\phi\rangle$$

$$+ R_e(s_0)M\phi + R_d(s_0)M\phi.$$

From this expression for ϕ_1 and Theorem 7.1 it is clear that for every odd eigenfunction ϕ, $\phi(\alpha)$ can not be an eigenfunction corresponding to an embedded eigenvalue $\lambda(\alpha)$, since this would imply $\phi_1 \in H$, whereas at least one of the terms $|E_j\left(\frac{1}{2}+it_0\right)\rangle\langle E_j\left(\frac{1}{2}-it_0\right)|M\phi\rangle$ is not in H, and the functions $E_j\left(\frac{1}{2}+it_0\right)$ are linearly independent.

To complete the picture we prove, using (8.8), the expression for the coefficients in the 2nd order expansion of the eigenvalues $\lambda_i(\alpha)$ known as Fermi's Golden Rule.

1) Assume that $s_0 \neq \frac{1}{2}$ and that $\mathcal{N} = \mathcal{N}(L - s_0(1-s_0))$ consists entirely of odd eigenfunctions. Consider the sesquilinear form $\mathcal{F}$ on $\mathcal{N} \times \mathcal{N}$, defined by

$$\mathcal{F}_\alpha(\phi,\psi) = (P_0(L_1(\alpha)-\lambda_0)P(\alpha)P_0\phi,\psi),\ \phi,\psi \in \mathcal{N}.$$

Expanding $\phi(\alpha) = P(\alpha)\phi$ to second order, we have

$$\begin{aligned}(L_1(\alpha)-\lambda_0)\phi(\alpha) &= \left\{(L_1-\lambda_0)+\alpha M+\alpha^2 N\right\}(\phi+\alpha\phi_1+\alpha^2\phi_2+o(\alpha^2))\\ &= \alpha\left[(L-\lambda_0)\phi_1+M\phi\right]+\alpha^2\left[(L-\lambda_0)\phi_2+M\phi_1+N\phi\right]+o(\alpha^2).\end{aligned}$$

where $\phi_1 = \phi'(0)$, $\phi_2 = \frac{1}{2}\phi'(0)$.

For $\psi \in \mathcal{N}$ this yields

$$\mathcal{F}_\alpha(\phi,\psi) = \langle L_1(\alpha)-\lambda_0)\phi(\alpha),\psi\rangle = \alpha^2(M\phi_1,\psi)+(N\phi,\psi)]+o(\alpha^2) \tag{8.9}$$

since $\langle L_1(\alpha)-\lambda_0)\phi_1,\psi\rangle = \langle L-\lambda_0)\phi_2,\psi\rangle = 0$ and $(M\phi,\psi) = 0$ by symmetry.

From the expression (8.8) for ϕ_1 we obtain

$$\begin{aligned}(M\phi_1,\phi) = &\frac{1}{4\pi}PP\int_{-\infty}^{\infty}\frac{1}{r-t_0}\frac{1}{r+t_0}\sum_{j=1}^{h}\left|\left\langle E_j\left(\tfrac{1}{2}-ir\right)|M\phi\right\rangle\right|^2 \\ &+\frac{1}{8it_0}\sum_{j=1}^{h}\left|\left\langle E_j\left(\tfrac{1}{2}-it_0\right)|M\phi\right\rangle\right|^2 \\ &+\sum_{\substack{\lambda_i\neq\lambda_0\\ \lambda_i\geq\frac{1}{4}}}(\lambda_i-\lambda_0)^{-1}|(\phi_i,M\phi)|+\sum_{\lambda_j<\frac{1}{4}}(\lambda_j-\lambda_0)^{-1}|(\phi_j,M\phi)|^2.\end{aligned} \tag{8.10}$$

Since all terms in (8.9) are real except the second one, which is purely imaginary, we obtain

$$\operatorname{Im}\mathcal{F}_a(\phi,\phi) = \frac{-1}{8t_0}\sum_{j=1}^{h}\left|\left\langle E_j\left(\tfrac{1}{2}-it_0\right)|M\phi\right\rangle\right|^2+o(\alpha^2) \tag{8.11}$$

and hence by Theorem 7.1 $\operatorname{Im} \mathcal{F}_\alpha(\phi,\phi) < 0$ for $|\alpha| < \varepsilon_0$ and all $\phi \in P_0 H$.

The sesquilinear form

$$\mathcal{G}(\phi,\psi) = \sum_{j=1}^{h} \left\langle E_j\left(\tfrac{1}{2} - it_0\right)|M\phi\right\rangle \overline{\left\langle E_j\left(\tfrac{1}{2} - it_0\right)|M\psi\right\rangle} \tag{8.12}$$

is positive definite on $\mathcal{N} \times \mathcal{N}$.

Let $\phi_1, ..., \phi_m$ be an orthonormal basis of $\mathcal{N}$, which diagonalizes $\mathcal{G}(\phi,\psi)$, let G be the self-adjoint operator on $P_0 H$ defined by

$$(G\phi,\psi) = \mathcal{G}(\phi,\psi),$$

and let $0 < \mu_1 \leq ... \leq \mu_m$ be the eigenvalues of G, repeated according to multiplicity, with orthonormal eigenvectors $\phi_1 ... \phi_m$. Then if $\phi = \sum\limits_{j=1}^{m} \beta_j \phi_j$,

$$\mathcal{G}(\phi,\phi) = \sum_{j=1}^{m} \mu_j \beta_j^2.$$

We have shown that for $\phi = \sum\limits_{j=1}^{m} \beta_i \phi_i$

$$\operatorname{Im} \langle (L(\alpha) - \lambda_0) P(\alpha)\phi, \phi\rangle = -\alpha^2 \sum_{i=1}^{m} \mu_i \beta_i^2 + o(\alpha^2).$$

Thus, the total odd eigenspace transforms into resonance functions for $0 < |\alpha| < \varepsilon_0$.

2) Assume that $s \neq \frac{1}{2}$ and that $\mathcal{N} = \mathcal{K} \oplus \mathcal{L}$, where $\mathcal{K}$ consists of odd eigenfunction and $\mathcal{L}$ of even eigenfunctions.

We consider again the sesquilinear form $\mathcal{F}_\alpha(\phi,\psi)$ on $\mathcal{N} \times \mathcal{N}$ given by (8.9) and derive the formula (8.10) for $\operatorname{Im} \mathcal{F}_\alpha(\phi,\phi)$. By Theorem 7.1, $\operatorname{Im} \mathcal{F}_\alpha(\phi,\phi) < 0$ for $\phi \in \mathcal{K}$ and $|\alpha| < \varepsilon$, and since $M\phi$ is odd for $\phi \in \mathcal{L}$, we have $\mathcal{F}_\alpha(\phi,\phi) = o(\alpha^2)$ for $\phi \in \mathcal{L}$.

We now diagonalize the form $\mathcal{G}(\phi,\psi)$ given by (11) on $\mathcal{K} \times \mathcal{K}$ and obtain

$$(\phi,\phi) = \sum_{i=1}^{k} \mu_k \beta_j^2, \ \phi = \sum_{i=1}^{k} \beta_j \phi_j$$

where $0 < \mu_1 \leq ... \leq \mu_k$ are the corresponding eigenvalues with eigenvectors $\phi_1 ... \phi_k$. Choosing any basis $\psi_1 ... \psi_l$ of the null space $\mathcal{L}$ of G, we have

$$\mathcal{G}(\phi,\phi) = \sum_{j=1}^{k} \mu_k \beta_j^2 \text{ for } \phi = \sum_{j=1}^{k} \beta_j \phi_j + \sum_{i=1}^{l} \gamma_j \psi_l.$$

This shows that the odd subspace $\mathcal{K}$ of $\mathcal{N}$ is transformed entirely into resonance functions, and the eigenvalue λ_0 splits into resonances corresponding to odd eigenfunctions and eigenvalues or resonances corresponding to even eigenfunctions. The latter do not split off from the former until fourth order in α and it remains an open question whether even eigenvalues stay in the spectrum as eigenvalues or turn into resonances for $\alpha \neq 0$.

We have proved the following result:

Theorem 8.5 *Let $\lambda_0 > \frac{1}{4}$ be an eigenvalue of $L = L(\Gamma_0(N), \chi)$ with eigenspace N of $\dim N = m$. Let $K \subset N$ be the subspace of odd eigenfunctions, $\dim K = k$, $0 < k \leq m$. Under the perturbation $\alpha M + \alpha^2 N$ of L corresponding to the character $\chi(\alpha)$ all the eigenfunctions ϕ in K are transformed into resonance functions for $0 < |\alpha| < \varepsilon$.*

The eigenvalue λ_0 splits into at most m eigenvalues and resonances of total multiplicity m. Eigenfunctions in K give rise to resonance functions associated with resonances $\lambda_i(\alpha)$, $i = 1, ..., k$. The functions $\lambda_i(\alpha)$ are either analytic or form branches of Puiseux series of order $p \geq 2$. In both cases they are of the form

$$\lambda_i(\alpha) = \lambda_0 + a_2\alpha^2 + o(\alpha^2)$$

where

$$\operatorname{Im} a_2 = \pm \sum_{j=1}^{h} |\langle M\phi_i \, | E_j \rangle|^2 \, for t_0 \lessgtr 0 \text{ and } \lambda_0 = s_0(1 - s_0), s_0 = \tfrac{1}{2} + it_0, \phi_i \in \mathcal{K}.$$

E*igenfunctions in $N \ominus K$ give rise to poles $\lambda_j(\alpha)$, $j = k+1, ..., m$, where the functions $\lambda_j(\alpha)$ are either analytic and remain embedded eigenvalues, or $\lambda_j(\alpha)$ becomes a resonance for small $\alpha \neq 0$ and is of the form*

$$\lambda_j(\alpha) = \lambda_0 + a_2\alpha^2 + ... + a_{2n-1} + a_{2n}\alpha^{2n} + o(\alpha^{2n})$$

where $\alpha_2, ..., a_{2n-1}$ are, real, $n \geq 2$, $\operatorname{Im} a_{2n} \neq 0$ and $\lambda_j(\alpha)$ is either analytic or a branch of a Puiseux series of order $p \geq 2$.

Remark 8.6 *For even eigenfunctions the Phillips-Sarnak integral is zero, since $M\phi$ is odd for even ϕ. It is therefore not known whether even eigenfunctions leave or stay under* *this perturbation. There is another perturbation obtained by replacing* $\mathrm{Re}\,\omega$ *by* $\mathrm{Im}\,\omega$ *in the definition of the characters* $\omega(\alpha)$,

$$\chi(\alpha)(\gamma) = e^{2\pi i\alpha\,\mathrm{Im}\int_{z_0}^{\gamma z_0}\omega(t)dt},\ \gamma \in \Gamma_0(N).$$

The family $A(\Gamma_0(N), \chi \cdot \chi^{(\alpha)})$ *corresponds by unitary equivalence via the operator* $e^{2\pi i\alpha\int_{z_0}^{\gamma z_0}\omega(t)dt}$ *to the family of operators in* $H(\Gamma_0(N), \chi)$

$$\tilde{L}(\alpha) = L + \alpha\tilde{M} + \alpha^2 N$$

where

$$L = A(\Gamma_0(N), \chi)$$

$$\tilde{M} = -4\pi i y^2\left(\omega_2\frac{\partial}{\partial x} + \omega_1\frac{\partial}{\partial y}\right)$$

$$N = 4\pi^2 y^2(\omega_1^2 + \omega_2^2).$$

It turns out that the operator $\tilde{M}$ *is not* L*-bounded, and therefore the perturbation theory developed for* M *does not apply. Although the Phillips-Sarnak integrals are in fact given by the same Rankin-Selberg convolution and are proved to be non-zero, this does not imply that eigenvalues with even eigenfunctions become resonances under this perturbation. Indeed,* $\mathrm{Im}\int_{z_0}^{\gamma z_0}\omega(t)dt = 0$ *for* $\gamma \in \Gamma_0(N)$*, which implies that* $\chi\cdot\chi(\alpha) = \chi$ *for all* α *and the functions* $\Omega(\alpha) = \exp\left\{2\pi i\alpha\,\mathrm{Im}\int_{z_0}^{z}\omega(t)dt\right\}$ *are* $\Gamma_0(N)$*-automorphic. Thus, the operators* $\tilde{L}(\alpha)$ *are unitarily equivalent to* L *for all* α *via* $\tilde{L}(\alpha) = \Omega^{-1}(\alpha)L\Omega(\alpha)$*, and all eigenvalues stay. The domain* $D(\tilde{L}(\alpha))$ *equals* $\Omega(\alpha)D(L)$*, which changes with* α.

Remark 8.7 *The proof that the Phillips-Sarnak integral in not zero is based on the non-vanishing of the Dirichlet L-series of eigenfunctions, which is proved using Hecke theory. This is therefore specific for the operators* $A(\Gamma_0(N), \chi)$*. However, we can draw the following conclusions about embedded eigenvalues of* $A(\Gamma_0(N), \chi\cdot\chi^{(\alpha)})$ *based on general perturbation theory. Due to the analyticity in* α*, each embedded eigenvalue* $\lambda(\alpha_0)$ *of* $L(\alpha_0)$

under the perturbation $\alpha M + \alpha^2 N$ *either stays as an embedded eigenvalue for* $\alpha \neq \alpha_0$, *analytic in* α, *or leaves as a resonance. When* α_0 *is not a congruence point, it is not known whether there exist embedded eigenvalues, but for each congruence point there are embedded eigenvalues obeying a Weyl law. if* $\lambda(\alpha)$ *remains an eigenvalue and does not go to* ∞ *as* $\alpha \to \alpha$, *for some* α_1, *then for* $\alpha = 0$ *it becomes an eigenvalue* $\lambda(0)$, *which stays in the spectrum, hence the corresponding eigenfunction is even.*

Appendix 1.

We will study the matrices of $(d', d'')^2$, which correspond to the systems (5.3). We want to prove that the coefficient β_1 is not zero, for some choices of coefficients n_{2d_1}, n_{4d_2}. We start from (5.3). We have $4N_2 = 4_{p_1 p_2 \dots p_k}$, where p_i are different primes not equal to two. To see the matrix

$$(d', d'')^2,\ d' \,|4N_2,\ \ d', d'' > 0, \tag{A1.1}$$

we consider the following primitive matrices:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 16 \end{pmatrix},\ B_i = \begin{pmatrix} 1 & 1 \\ 1 & p_i^2 \end{pmatrix}\ 1 \le i \le k. \tag{A1.2}$$

It is not difficult to see that the inverse matrices are

$$A^{-1} = \frac{1}{36}\begin{pmatrix} 48 & -12 & 0 \\ -12 & 15 & -3 \\ 0 & -3 & 3 \end{pmatrix},\ B_i^{-1} = \begin{pmatrix} p_i^2 & -1 \\ -1 & 1 \end{pmatrix}\frac{1}{p_i^2 - 1} \tag{A1.3}$$

We define the tensor product $A \otimes B_1 \otimes B_2 ... \otimes B_k$ by recurrence relations as the block matrix

$$C_1 = \begin{pmatrix} A & A \\ A & p_1^2 A \end{pmatrix},\ C_2 = \begin{pmatrix} C_1 & C_1 \\ C_1 & p_2^2 C_1 \end{pmatrix},\ C_k = \begin{pmatrix} C_{k-1} & C_{k-1} \\ C_{k-1} & p_k^2 C_{k-1} \end{pmatrix}. \tag{A1.4}$$

It is not difficult to see that the matrix C_k coincides with the matrix (A1.1), if we take the following order of divisors d' and d''

$$1, 2, 4, p_1(1, 2, 4), p_2\,[1, 2, 4, p_1(1, 2, 4)]\,, ...\ . \tag{A1.5}$$

It is easy to see now that the inverse matrix to C_k is coming from the recurrence relation

$$\begin{cases} c_1^{-1} = \frac{1}{p_1^2-1}\begin{pmatrix} p_1^2 A^{-1}, & -A^{-1} \\ -A^{-1}, & A^{-1} \end{pmatrix},\ c_2^{-1} = \frac{1}{p_2^2-1}\begin{pmatrix} p_2^2 C_1^{-1}, & -C_1^{-1} \\ -C_1^{-1}, & C_1^{-1} \end{pmatrix}, ... \\ ...c_k^{-1} = \frac{1}{p_k^2-1}\begin{pmatrix} p_k^2 C_{k-1}^{-1}, & -C_{k-1}^{-1} \\ -C_{k-1}^{-1}, & C_{k-1}^{-1} \end{pmatrix}. \end{cases} \tag{A1.6}$$

From this follows that C_k^{-1} exists, and we can determine the coefficients $\beta_{d''}$ from (5.3) explicitly. Actually, it is important to see now only the first row in the inverse matrix C_k^{-1}, since we want to prove $\beta_1 \neq 0$. Let us denote this first row of C_l^{-1} by e_l, $1 \leq l \leq k$. From (A1.6) follows

$$\begin{cases} e_1 = \frac{1}{36(p_1^2-1)} \left(p_1^2(48,-12,0) - (48,-12,0)\right) \\ e_{m+1} = \frac{1}{p_{m+1}^2-1} \left(p_{m+1}^2 e_m, -e_m\right) \; 1 \leq m \leq k-1. \end{cases} \tag{A1.7}$$

Let us see now the right hand side of (6.33). When d' runs through all positive divisors of $4N_2$ in the order of (A1.5), we get the column vector, which has non-zero components only on places $d' = 2d_1,\ d_1\,|N_2,\ d_1 > 0$ equal to n_{2d_1}/m_{2d_1}. From (1.8), (1.10) follows $m_{2d_1} = N_2/d_1$. We remind that the coefficients $n_{2d_1} = \pm 1$ with the only condition (5.10). Applying e_k to this vector we obtain up to the common multiple

$$\frac{1}{36(p_1^2-1)(p_2^2-1)...(p_k^2-1)} \tag{A1.8}$$

that β_1 is equal to

$$\sum_{\substack{d_1|N_2\\ d_1>0}} n_{2d_1} x_{2d_1} \tag{A1.9}$$

where x_{2d_1} are pairwise different integers with equal number of positives and negatives. From that follows that there exists the choice of coefficients $n_{2d_1} = \pm 1$ with condition (5.10) which makes (5.6) not equal to zero.

The investigation of the system (5.33) is completely analogous. We have $4N_3 = 8n$, $n = p_1 \cdot p_2 \cdots p_k$ is the product of different odd primes. Instead of matrix A from we take

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 4 & 4 & 4 \\ 1 & 4 & 16 & 16 \\ 1 & 4 & 16 & 64 \end{pmatrix},\ A^{-1} = \begin{pmatrix} 4/3 & -1/3 & 0 & 0 \\ -1/3 & 5/12 & -1/12 & 0 \\ 0 & -1/12 & 5/48 & -1/48 \\ 0 & 0 & -1/48 & 1/48 \end{pmatrix} \tag{A1.10}$$

and then repeat the proof. We obtain that up to the common multiple

$$\frac{1}{48(p_1^2-1)(p_2^2-1)...(p_k^2-1)} \tag{A1.11}$$

the coefficient p_1 is equal to

$$\sum_{\substack{d_1|n \\ d_1>0}} \frac{2_{2d_1}}{m_{2d_1}} x_{2d_1} + \sum_{\substack{d_1|n \\ d_2>0}} \frac{n_{4d_2}}{m_{d2}} x_{4d_2} \tag{A1.12}$$

where x_{2d_1}, x_{4d_2} are integers with equal number of positives and negatives. There exists a choice of coefficients n_{2d_1}, n_{4d_2} which makes (A1.12) not equal to zero. We have proved Theorem 5.1.

Appendix 2. The Rankin-Selberg convolution

For $\operatorname{Re} s > 1$ we consider the following integral

$$\int_{F_0(N)} |v_j(z)|^2 \, E_\infty(z; s; \bar{\Gamma}_0(N); 1) d\mu(z) = A(s)$$

where

$$E_\infty(z; s) = E_\infty(z; s; \bar{\Gamma}_0(N); 1) = \sum_{\gamma \in \Gamma_\infty \backslash \bar{\Gamma}_0(N)} y^s(\gamma z)$$

and $v_j(z) = v_j(z; s; \bar{\Gamma}_0(N); \chi)$ is defined in (6.5). Using the unfolding of the Eisenstein series we obtain

$$\begin{aligned} A(s) &= \int_0^\infty y^{s-1} \sum_{n \neq 0} |\rho_j(n)|^2 \, K^2_{ir_j}(2\pi \, |n| \, y) dy = \\ &= \frac{\Gamma^2(s/2)\Gamma\left(\frac{s}{2} + ir_j\right)\Gamma(s/2 - ir_j)}{4\pi^s \Gamma(s)} \sum_{n=1}^{\infty} \frac{|\rho_j(n)|^2}{n^s}. \end{aligned}$$

It is well known that $E(z, s; \Gamma_0(N); 1)$ has analytic continuation to the whole s-plane, and at $\operatorname{Re} s > 1/2$ it has only a simple pole at $s = 1$ with residue equal to $\mu(F_0(N))^{-1}$ (inverse μ-area of the fundamental domain of $\bar{\Gamma}_0(N)$). From that follows that the Rankin-Selberg convolution (6.12) is a regular function in $\operatorname{Re} s > 1/2$ except for a simple pole at $s = 1$.

Appendix 3. Functional equation for $L(s; v_j)$

Let W_N be the Atkin-Lehner involution for $\bar\Gamma_0(N)$.

$$W_N \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}.$$

We have $W_N \bar\Gamma_0(N) W_N^{-1} = \bar\Gamma_0(N)$ because

$$\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \begin{pmatrix} 0 & 1/N \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} d & -c \\ -Nb & a \end{pmatrix}.$$

Then it is clear that W_N commutes with the automorphic Laplacian $A(\bar\Gamma_0(N); \chi)$ and W_N preserves the subspace of cusp-forms

$$\mathcal{H}_0(\bar\Gamma_0(N); \chi) \subset \mathcal{H}(\bar\Gamma_0(N))$$

and $W_N^2 f = f$, $f \in \mathcal{H}_0(\bar\Gamma_0(N); \chi)$. From this follows that W_N preserves also the subspace of odd cusp-forms. Moreover, we can assume that the common basis of eigenfunctions v_j (see (6.4)) is also common for the involution W_N. So we obtain

$$W_N v_j(z) = v_j(-1/Nz) = \pm v_j(z).$$

We will write $v_j(z) = v_j(x, y)$, where $z = x + iy$. We have

$$v_j(x, y) = \sum_{n \neq 0} \rho_j(n) \sqrt{y} K_{s_j - 1/2}(2\pi |n| y) e^{2\pi i n x}, \; s_j(1 - s_j) = \lambda_j.$$

We have also $v_j(-x, y) = -v_j(x, y)$ or $\rho_j(-n) = -\rho_j(n)$. We keep the normalization $\rho_j(1) = 1$. The involution action on v_j can be written as follows

$$v_j(u, v) = \pm v_j(x, y)$$

$$u = \frac{-x}{N(x^2 + y^2)}, \; v = \frac{y}{N(x^2 + y^2)}.$$

We take the partial derivative $\frac{\partial}{\partial x}$ and we obtain

$$-\frac{1}{Ny^2} \frac{\partial v_j}{\partial u}\bigg|_{x=0} = \pm \frac{\partial v_j}{\partial x}\bigg|_{x=0}.$$

That is equivalent to

$$\pm N^{3/2} y^3 \sum_{n=1}^{\infty} \rho_j(n) n K_{s_i-1/2}(2\pi n y) = \sum_{n=1}^{\infty} \rho_j(n) n K_{s_j-1/2}(2\pi n/Ny). \tag{A3.1}$$

If we multiply the left hand side of this equality, which we denote $\pm B(y)$, by $4\pi \cdot N^{s/2-3/2} y^{s-3}$ and integrate it from 0 to ∞ in y, we obtain the expression

$$\pm \pi^{-s} N^{s/2} \Gamma\left(\frac{s+s_j+1/2}{2}\right) \Gamma\left(\frac{s-s_j+3/2}{2}\right) L(s;v_j) = \pm\Omega(s;v_j)$$

because

$$\int_0^{\infty} y^s K_{s_j-1/2}(y) dy = 2^{s-1} \Gamma\left(\frac{s+s_j+1/2}{2}\right) \Gamma\left(\frac{s-s_j+3/2}{2}\right).$$

We can now write the integral obtained as a sum of two integrals

$$\pm 4\pi \cdot N^{s/2-3/2} \int_0^{\infty} B(y) y^{s-3} dy = \pm 4\pi \cdot N^{s/2-3/2} \left(\int_0^{1/\sqrt{N}} dy + \int_{1/\sqrt{N}}^{\infty} dy \right). \tag{A3.2}$$

In the first integral we use (A3.1) for $B(y)$ and then $y \to 1/N_y$. Then we obtain that (A3.2) is equal to

$$\pm\Omega(s) = \pm C(s) + C(1-s)$$

where

$$\begin{aligned} C(s) &= 4\pi \cdot N^{s/2-3/2} \int_{1/\sqrt{N}}^{\infty} B(y) y^{s-3} dy \\ &= 4\pi \cdot N^{s/2} \int_{1/\sqrt{N}}^{\infty} y^s \sum_{n=1}^{\infty} n\rho_j(n) K_{s_j-1/2}(2\pi n y) dy. \end{aligned}$$

From this follows that $\Omega(s)$ is an entire function, satisfying the functional equation

$$\Omega(1-s) = \pm\Omega(s).$$

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