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Fermionisation of the Spin-S Uimin-Lai-Sutherland Model: Generalisation of the Supersymmetric t-J Model to Spin-S[1]

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Abstract

The *spin*1 Uimin-Lai-Sutherland (ULS) isotropic chain model is expressed in terms of fermions and the equivalence of the fermionic representation to the supersymmetric $t-J$ model is established directly at the level of Hamiltonians.The spin-S ULS model is fermionized and the Hamiltonian of the corresponding generalisation of the $t-J$ model is written down.

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1 Introduction

The low dimensional ($d = 0, 1, 2$) integrable models, for a long time a subject of interest in the mathematical physics, are now attracting much attention also in connection with problems in condensed matter physics. The development of nanostructure technology nowadays allows us to consider the problems in zero(quantum dots), one(quantum wires, organic polymers) and two(hall effect, high temperature superconductivity) spatial dimensions.

The well known one dimensional Heisenberg model of a nearest-neighbour interacting chain of $\frac{1}{2}$ spins,

$$H = \sum_{i=1}^{N} H_{ii+1} = \sum_{i=1}^{N} \vec{\sigma}_i \vec{\sigma}_{i+1}, \qquad N+1 = 1, \tag{1}$$

was solved in 1931 by Bethe [1] using what is now called the Bethe Ansatz. During the last 25 years this method has been developed into what is now called the Quantum Inverse Scattering Method(QISM) [3, 4] (or the Algebraic Bethe Ansatz(ABA)). The method allows one to find a complete set of eigenvalues and eigenstates of the Hamiltonian of a one dimensional integrable model.

The origin of integrability is a set of equations, first derived in the article by Yang [2], and which, after the works of Baxter [3], appeared to be equivalent to existence of an infinite number conservation laws in the model, and called Yang-Baxter Equations (YBE) as a consequence.

However, the ABA technique was mainly developed in the context of the XXZ spin chain model and its application to models formulated originally in fermionic operator language was complicated. For example, in the Hubbard model, solved in the very beginning by a Coordinate Bethe Ansatz (CBA) method [5], the implementation of ABA to find a full solution was made only recently [6] after remarkable step done by Shastry [7, 8, 9]. He used the Jordan- Wigner transformation to pass from fermionic to spin language, in order to apply QISM, formulated in matrix form.

Recently the interest of formulating spin chain models in terms of Fermi operators by use of Jordan-Wigner transformation [10, 11], and by use some new method developed in [12, 13, 14], has increased. It appeared that the explicit construction of integrable models and application of QISM in terms of Fermi operators becomes easy. Because the Fock space of fermions is finite dimensional the investigation of the complete set of eigenstates and eigenvalues in this language becomes relatively simple.

It is well known that the trivial generalisation of the Heisenberg model to higher spins is nonintegrable and gap-full [18]. The first integrable model for the spin-S was established by G.Uimin [15] and later by J.K.Lai [16] and B.Sutherland

[17]. They have considered the Hamiltonian

$$H = J\sum_{i=1}^{N}(Id_{i,i+1} + P_{i,i+1}), \qquad\qquad N+1 = 1, \tag{2}$$

where $P_{i,i+1}$ is the operator permuting spins at the sites i and $i{+}1$ and $Id_{i,i+1}$ is the identity operator, shown its integrability and found a solution by the Coordinate Bethe Ansatz. Another integrable model for arbitrary spin-S was constructed in the articles [19, 20]. As it appeared, the integrable Hamiltonian is a polynomial Q_{2S} of degree $2S$ of the product of nearest-neighbour spins $\vec{S}_i\vec{S}_{i+1}$

$$H_S = J\sum_{i=1}^{N} Q_{2S}(\vec{S}_i\vec{S}_{i+1}) \qquad N+1 = 1, \tag{3}$$

where

$$\begin{aligned} Q_{2S}(x) &= -\sum_{j=1}^{2S}(\sum_{k=1}^{j})\prod_{\substack{l=0\\ l\neq j}}^{2S}\frac{x-x_l}{x_j-x_l}, \\ x_l &= \frac{1}{2}[l(l+1) - 2S(S+1)] \end{aligned} \tag{4}$$

and has a critical point, near which the model is equivalent to Wess-Zimino-Witten-Novikov (WZWN) model with corresponding coupling constant $k = 2S$ [22]. However in the article [19] only 2^L (L-is the length of the chain) states from all 3^L amount of states were found together with corresponding Bethe equations for them.

The full set of $SU(2)$ invariant, integrable, isotropic quantum spin chain models for spins $S \leq 6$ was found in [21].

In this article we will use the technique developed in [12, 13, 14], which is alternative to the Jordan-Wigner transformation, to fermionize the integrable spin-1 ULS isotropic chain model and find its Hamiltonian in terms of fermionic fields, which will demonstrate its equivalence to integrable supersymmetric (SUSY) $t{-}J$ model [23]. After that we generalise this construction to spin-S and carry out the fermionization of the spin-S ULS chain model. The resulting Hamiltonian can be considered as spin-S analogy for the $t - J$ model.

The article is organised as follows. In the Section 2 we describe briefly the fermionization scheme for integrable models and QISM.

The fermionization of the spin-1 Uimin-Lai-Sutherland (ULS) model will be carried out in the Section 3. The Hamiltonian will be written in Fermi terms and coincidence with SUSY $t - J$ model will be shown.

In the section 4 we will construct the fermionic representation of the $SU(2)$ algebra for spins $(2S + 1) \leq 8$ and apply that construction to find the R-matrix and corresponding Hamiltonian in terms of fermions in section 5.

2 Yang-Baxter equation and structure of the quantum spaces of the integrable model.

The key of integrability of the model is the Yang-Baxter equation (YBE), which implies some restrictions on R_{aj}-matrix - the constituent of the transfer matrix $T(u)$, defined as

$$T\left(u\right)=tr_{a}\prod_{j=1}^{N}R_{aj}\left(u\right). \tag{5}$$

The YBE ensures the necessary and sufficient conditions of commutativity of the transfer-matrices for different values of the spectral parameter u, which has a physical meaning of the rapidity of pseudoparticles of the model

$$\left[T\left(u\right),T\left(v\right)\right]=0. \tag{6}$$

The Hamiltonian of the model can be obtained by logarithmic derivative of the transfer matrix at zero spectral parameter

$$H=-\frac{\partial\ln T\left(u\right)}{\partial u}\mid_{u=0}. \tag{7}$$

By definition R_{aj} acts as a intertwining operator on the space of direct product of the so called auxiliary $V_a\left(v\right)$ and quantum $V_j\left(u\right)$ spaces

$$R_{aj}\left(u,v\right): \quad V_a(u)\otimes V_j(v)\rightarrow V_j(v)\otimes V_a(u) \tag{8}$$

and can be represented graphically as

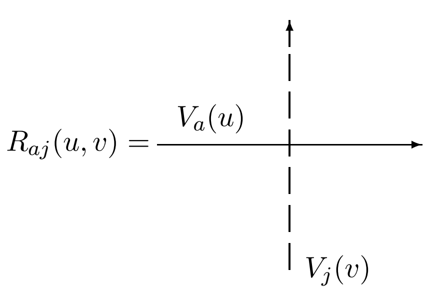

Fig.1.

The spaces $V_a(u)$ and $V_j(v)$ with spectral parameters u and v are irreducible representations of the affine quantum group $U_q\widehat{g}$, which is the symmetry group of the integrable model under consideration. Provided that the states $\mid a\rangle\in V_a$

and $| j\rangle \in V_j$ form a basis for the spaces V_a and V_j, following [12, 13] we can represent the action of the operator R_{aj} as follows

$$R_{aj} \mid j\rangle \otimes \mid a\rangle = R^{aj}_{a'j'} \mid a'\rangle \otimes \mid j'\rangle, \tag{9}$$

where the summation is over the repeating indices a' and j' (but not over a and j).

By introducing the operators

$$X^a_{a'} =\mid a'\rangle\langle a \mid; \qquad X^j_{j'} =\mid j'\rangle\langle j \mid \tag{10}$$

in the graded spaces V_a and V_j correspondingly, one can easily rewrite (9) as follows

$$R_{aj} = R_{aj} \mid j\rangle \mid a\rangle\langle a \mid \langle j \mid= R^{aj}_{a'j'} \mid a'\rangle \mid j'\rangle\langle a \mid \langle j \mid= (-1)^{p(a)p(j')}\, R^{a'j'}_{aj} X^a_{a'} X^j_{j'} \tag{11}$$

where the sign factor takes into account the possible grading of the states $| a\rangle$ and $| j\rangle$, $p(a)$ and $p(j)$ denote the corresponding parities and the summation over the repeating indices. It is easy to see from the expression (10) that the operators $X^{a'}_a$ have the projection property

$$X^b_a X^{b'}_{a'} = \delta^b_{a'} X^{b'}_a \tag{12}$$

In the conventional form the YBE looks like

$$R^{ab}_{a'b'}(u,v) R^{a'j}_{a''j'}(u,w) R^{b'j'}_{b''j''}(v,w) = R^{bj}_{b'j'}(v,w) R^{aj'}_{a'j''}(u,w) R^{a'b'}_{a''b''}(u,v). \tag{13}$$

By use of equations (11) and (12) the matrix-valued YBE eq. (13) one can easily be transformed to the following operator valued equation

$$R_{ab}(u,v) R_{aj}(u,w) R_{bj}(v,w) = R_{bj}(v,w) R_{aj}(u,w) R_{ab}(u,v). \tag{14}$$

Using the graphical representation of the R-operator presented in Fig.1, one can draw the YB equation graphically as follows

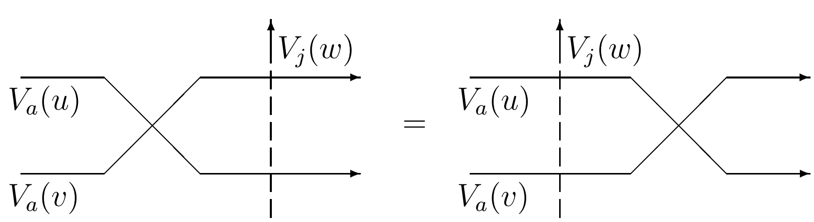

Fig.2

An alternative form of the YBE above is the commutativity of the R-operator (11) with the co-product Δ of the corresponding quantum group $U_q\widehat{g}$ [12, 13].

Let us introduce now the L-operator by taking matrix element of the R-operator (11) in the auxiliary space V_a with the basic vectors $\mid a\rangle$ [14]

$$L_{a'}^{a}\,(u,v) = \langle a' \mid R(u,v) \mid a\rangle = (-1)^{p(a)p(j)}\, R_{a'j'}^{aj}\,(u,v)\, X_j^{j'} \tag{15}$$

Then, following [14], it is easy to show that YBE takes the form

$$\check{R}_{ab}^{a'b'}\,(L\otimes_s L)_{a'b'}^{a''b''} = (L\otimes_s L)_{ab}^{a'b'}\,\check{R}_{a'b'}^{a''b''} \tag{16}$$

where $\check{R}_{cd}^{ab} = R_{cd}^{ba}$ and graded direct product is defined as follows

$$(A\otimes_s B)_{bd}^{ac} = (-1)^{(p(a)+p(b))p(d)}\, A_b^a B_d^c, \tag{17}$$

which is conventional in the graded Quantum Inverse Scattering Method [25].

3 The *spin* 1 ULS-chain in terms of fermions

Now we would like to apply the approach, described above, to fermionize the *spin* 1 integrable isotropic ULS chain with the Hamiltonian (2).

To begin with we should realize the *spin* 1 algebra in terms of fermions. At least two sort of fermions are needed in order to express three basic states $\mid +\rangle$, $\mid 0\rangle$, $\mid -\rangle$ of the *spin*1 particle with the z component of the spin equal to $1, 0, -1$ respectively. Let us define $c_\sigma^+,\;\; c_\sigma,\;\;\; \sigma = \uparrow, \downarrow$ as a creation -annihilation operators of fermions with the up and down spins correspondingly, together with their Fock space $\mid 0\rangle, \mid \sigma\rangle$.

The generators of *spin*1 representation of the $su(2)$ algebra can be built by use of this Fermi operators as follows:

$$\begin{aligned} S^+ &= (1-n_\uparrow)\, c_\downarrow + (1-n_\downarrow) c_\uparrow^+, \\ S^- &= (1-n_\uparrow)\, c_\downarrow^+ + (1-n_\downarrow) c_\uparrow, \\ S_z &= n_\uparrow - n_\downarrow, \end{aligned} \tag{18}$$

where n_σ is the fermion number operators.

By this definition of $su(2)$ generators, the states with definite third projection are realized through fermionic Fock space as follows

$$\mid +\rangle \equiv \mid \uparrow\rangle \mid 0\rangle, \quad \mid 0\rangle \equiv \mid 0\rangle \mid 0\rangle, \quad \mid -\rangle \equiv \mid 0\rangle \mid \downarrow\rangle. \tag{19}$$

Indeed, easy to check that $S_z|+\rangle = |+\rangle$, $S_z|0\rangle = 0$, $S_z|-\rangle = -|-\rangle$, $S_+|+\rangle = 0$, $S_+|0\rangle = |+\rangle$, $S_+|-\rangle = |0\rangle$, $S_-|+\rangle = |0\rangle$, $S_-|0\rangle = |-\rangle$, $S_-|-\rangle = 0$.

As it is obvious from formulas (19), we have constructed a graded space with the following parities for the basic vectors

$$p(\mid +\rangle) = p(\mid -\rangle) = 1, \quad p(\mid 0\rangle) = 0. \tag{20}$$

The sum of the square of spin generators, which is the Casimir operator

$$S^+S^- + S^-S^+ + S_z^2 \equiv \mathcal{C} = 2(1 - n_\uparrow n_\downarrow), \qquad S_a\mathcal{C} = 2S_a = \mathcal{C}S_a \tag{21}$$

determines the projection operator

$$\Pi = 1 - n_\uparrow n_\downarrow, \qquad (1 - n_\uparrow n_\downarrow)^2 = 1 - n_\uparrow n_\downarrow, \tag{22}$$

which maps 4-dimensional space of direct product of two *spin*1/2 spaces onto 3-dimensional subspace *spin*1. $\frac{1}{2}\mathcal{C}$ acts as an identity operator on eigenstates of the S_z:

$$\frac{1}{2}\mathcal{C} \mid m\rangle = \mid m\rangle \qquad m = +, -, 0, \tag{23}$$

and annihilates the fourth state

$$\mathcal{C}\ |\uparrow\rangle\ |\downarrow\rangle = 0, \tag{24}$$

which corresponds to scalar state in the expansion of direct product of two half spins.

In the *spin*1 case the operator X takes the form

$$\begin{aligned} X_{im}^k \equiv |m\rangle_i\langle k|_i &= \begin{pmatrix} |-\rangle\langle-| & |-\rangle\langle 0| & |-\rangle\langle+| \\ |0\rangle\langle-| & |0\rangle\langle 0| & |0\rangle\langle+| \\ |+\rangle\langle-| & |+\rangle\langle 0| & |+\rangle\langle+| \end{pmatrix}_i \\ &= \begin{pmatrix} (1-n_\uparrow)n_\downarrow & (1-n_\uparrow)c_\downarrow^+ & c_\downarrow^+ c_\uparrow \\ (1-n_\uparrow)c_\downarrow & (1-n_\uparrow)(1-n_\downarrow) & c_\uparrow(1-n_\downarrow) \\ c_\uparrow^+ c_\downarrow & c_\uparrow^+(1-n_\downarrow) & n_\uparrow(1-n_\downarrow) \end{pmatrix}_i, \end{aligned} \tag{25}$$

where i denotes the chain site. Trace of this operator is an identity operator due to the completeness of set of the states.

The operator Π_{ij}, which permutes the states between spaces V_i and V_j has the form

$$\Pi_{ij} = \sum_{m,k} \mid m_j\rangle \mid k_i\rangle\langle k_j \mid \langle m_i \mid = (-1)^{p(k)} X_{ik}^m X_{jm}^k \tag{26}$$

The sign in this expression reflects the grading of the spaces V_i

One can find the R-matrix of the *spin*1 model in its bosonic form in [15, 16, 17], but it can easily be guessed also from the expression of the Hamiltonian (2),

$$H = \frac{1}{4}\sum_{i=1}^{N}\left[\vec{S}_i\vec{S}_{i+1} + (\vec{S}_i\vec{S}_{i+1})^2\right], \tag{27}$$

which can be represented also as Unity + Permutation operators in the 3-dimensional *spin*1 space. It has a following form

$$\check{R}_{ms}^{kq}(u) = \delta_m^k\delta_s^q + u\ \delta_s^k\delta_m^q, \qquad m, k, s, q = +, 0, -, \tag{28}$$

where first term is unit and second term bosonic permutation matrices.

The L-operator of the model, defined in fermionic representation by the formula (15), takes the form

$$\begin{aligned} L_i(u) &= \begin{pmatrix} u-(1-n_\uparrow)(1-n_\downarrow) & (1-n_\uparrow)\,c_\downarrow^+ & -c_\uparrow^+ c_\downarrow^+ \\ (1-n_\uparrow)\,c_\downarrow & u+(1-n_\uparrow)\,n_\downarrow & c_\uparrow^+ n_\downarrow \\ c_\uparrow^+ c_\downarrow & c_\uparrow n_\downarrow & u-n_\uparrow n_\downarrow \end{pmatrix}_i \\ &= \begin{pmatrix} u+\frac{1}{2}\left(S_z^2-S_z\right) & -S_+S_z & S_+^2 \\ -S_zS_- & u-1+S_z^2 & S_zS_+ \\ -S_-^2 & S_-S_z & u+\frac{1}{2}\left(S_z^2+S_z\right) \end{pmatrix}_i \end{aligned} \tag{29}$$

and the Yang-Baxter equation (16) takes form

$$\begin{aligned} & L_m^k(u)\,L_s^q(v)+(u-v)\,(-1)^{p(m)p(q)+p(k)p(q)+p(m)p(k)}\,L_s^k(u)\,L_m^q(v) \\ = \; & L_m^k(v)\,L_s^q(u)+(u-v)\,(-1)^{p(q)p(s)+p(m)p(q)+p(s)p(m)}\,L_m^q(v)\,L_s^k(u)\,. \end{aligned} \tag{30}$$

¿From the equations above one can be easily read off the invariance of the YBE under the transformations

$$L_m^k(u) \rightarrow L_m^{\prime k}(u) = M_q^k L_p^q(u)\, N_m^p \tag{31}$$

where M and N are arbitrary numerical matrices of the same grading as $L(u)$.

According to the prescription of the previous section, by use of formulas (11), (28) and the definition (7) as a logarithmic derivative of transfer matrix, it is easy to find the following expression for the Hamiltonian via graded permutation (26) and identity operators

$$H=\sum_{i=1}^N H_{ii+1}=\sum_{i=1}^N\left(I_{ii+1}+\Pi_{ii+1}\right), \tag{32}$$

due to the fact that $R_{ik}(0)=P_{ik}$.

In terms of Fermi operators this Hamiltonian can be represented in very simple way as

$$H_{ij}=\Delta_i\Delta_j\left(I_{ij}+\mathcal{P}_{ij,\uparrow}\mathcal{P}_{ij,\downarrow}\right)\Delta_i\Delta_j, \tag{33}$$

where Δ_i is projection operator defined in (22) and $\mathcal{P}_{ij,\sigma} \equiv 1-(c_{i,\sigma}^+-c_{j,\sigma}^+)(c_{i,\sigma}-c_{j,\sigma})$, $\sigma=\uparrow,\downarrow$ are the fermionic permutation operators.

It is now straightforward to recognise in the expression (33) the Hamiltonian of supersymmetric $t-J$ [13, 14, 23, 24] model.

The symmetry (31) mentioned above is generated by the operator

$$\begin{aligned} Q_{ib}^a &= \begin{pmatrix} (1-n_\uparrow)\,n_\downarrow & (1-n_\uparrow)\,c_\downarrow^+ & c_\downarrow^+ c_\uparrow \\ (1-n_\uparrow)\,c_\downarrow & (1-n_\uparrow)(1-n_\downarrow) & c_\uparrow(1-n_\downarrow) \\ c_\uparrow^+ c_\downarrow & c_\uparrow^+(1-n_\downarrow) & n_\downarrow n_\downarrow \end{pmatrix}_i, \\ Q_b^a &= \sum_{i=1}^N Q_{ib}^a, \qquad [H,Q_b^a]=0. \end{aligned} \tag{34}$$

In fact Q_i coincides with the operator X_i (25). Being defined as global quantity along the chain, the matrix Q is inert under permutation of the site indices, i.e. it commute with the local Hamiltonians H_{ii+1}.

In the unrestricted Fock space of spin-up and spin-down fermions one can construct the X-operator (10), which will be a graded direct product of two X's in the one-fermion space

$$(X)_{ab}^{a'b'} = (X_\uparrow \otimes_s X_\downarrow)_{ab}^{a'b'}, \qquad a, b = 0, 1. \tag{35}$$

It is easy to see, that the *spin*-1 X-operator, defined as in eq.(25), can be obtained from the expression (35) by deleting the row and column, corresponding to the exclusion of state $|\uparrow\rangle\,|\downarrow\rangle$.

4 Spin-S Representation of the SU(2) algebra in the Fock space of Fermi fields

In order to describe the $2S+1$ dimensional space of the $spin-S$ representation by fermions we will consider the Fock space of r copies of the fermionic creation-annihilation operators $c_\mu^+, c_\mu, \mu = 1, 2...r$.

$$\mathcal{V}_r = \bigotimes_{\mu=1}^{r} V_\mu, \tag{36}$$

where V_μ is the Fock space of μ-type fermions. The dimension of this space is 2^r and hence the necessary condition to have a enough degrees of freedom for the $spin - S$ states is $(2S+1) \le 2^r$.

As usual let us mark the basic elements of the fermionic Fock space by their filling numbers $n_\mu = 0, 1; \mu = 1, ...r$

$$\begin{aligned} |n_1, ...n_r\rangle &= |n_1\rangle \cdot |n_2\rangle \cdots |n_r\rangle = (c_1^+)^{n_1} (c_2^+)^{n_2} \cdots (c_r^+)^{n_r} |0\rangle \\ |n_1, ...n_r\rangle &\in \mathcal{V}_r \end{aligned} \tag{37}$$

It is necessary to order this basic vectors by integers which can be achieved, for example, by use of binary system of numbers. But let us define m as a map of the set of sequences $(n_1, n_2, ...n_r)$ onto integers $1, 2, ..2^r$

$$m : (n_1, n_2, ...n_r) \to 1, 2...2^r \tag{38}$$

Then it is clear, that

$$\mathcal{V}_r = \bigotimes_{m=1}^{2^r} W_m \tag{39}$$

where W_m is the space defined by the basic vector $|n_1, n_2, ...n_r\rangle$.

There is a natural grading in the Fock space of fermions. Zero and one particle states in V_μ define a graded homogeneous subspaces $V_\mu = V_{0\mu} \oplus V_{1\mu}$ with the parity p as a function $V_{j\mu} \rightarrow Z_2$

$$p(|n_\mu\rangle) = n_\mu, \qquad |n_\mu\rangle \in V_{n_\mu,\mu}, \qquad n_\mu = 0, 1 \tag{40}$$

Grading of the tensor product space $\mathcal{V}_r = \otimes_{\mu=1}^r V_\mu$ is defined by parities of its constituents, namely, by parities of the basic elements of $\mathcal{V}_r$, as follows

$$p(|n_1, ...n_r\rangle) = \sum_{\mu=1}^{r} p(|n_\mu\rangle). \tag{41}$$

Now, following section 2 let us define the operators

$$\begin{aligned} X_m^{m'} &= X_{n_1,...n_r}^{n'_1,...n'_r} = |n_1, ...n_r\rangle\langle n'_r, ...n'_1| = (-1)^{(p(n_r)+p(n'_r))\sum\limits_{l\neq r} p(n'_l)} \cdot \cdot \\ &\cdot \cdot \cdot (-1)^{(p(n_2)+p(n'_2))p(n'_1)} X_{n_1}^{n'_1} \cdots X_{n_r}^{n'_r}, \end{aligned} \tag{42}$$

where

$$X_{n_\mu}^{n'_\mu} = |n_\mu\rangle\langle n'_\mu|, \qquad \mu = 1, ...r \tag{43}$$

is the corresponding X-operator in the Fock space V_μ of the μ-type fermions. Operators $X_{n_\mu}^{n'_\mu}$ can be considered as a matrix X_μ with the operator valued entities of the form

$$X_\mu = \begin{pmatrix} 1 - n_\mu & c_\mu \\ c_\mu^+ & n_\mu \end{pmatrix}. \tag{44}$$

By use of definition (17) the expression (42) can be represented as a graded direct product of the operators $X_{n_\mu}^{n'_\mu}$

$$X_{n_1,...n_r}^{n'_1,...n'_r} = (X_1 \otimes_s \cdots \otimes_s X_r)_{n_1,...n_r}^{n'_1,...n'_r} \tag{45}$$

It is now meaningful to define also the operators which projects an arbitrary state from the representation space $\mathcal{V}_r$ on to the particular one-dimensional subspace defined by the state $|n_1, ...n_r\rangle$. Obviously it is

$$\begin{aligned} P_{m(n_1,...n_r)} &= P_{n_1,...n_r} = |n_1, ...n_r\rangle\langle n_r, ...n_1| \\ P_{m(n_1,...n_r)}^2 &= P_{m(n_1,...n_r)} \end{aligned} \tag{46}$$

Therefore the operator

$$\Delta_{m(n_1,...n_r)} = 1 - P_{n_1,...n_r} \tag{47}$$

excludes the one dimensional subspace, defined by the basic vector $|n_1, ...n_r\rangle$, from the whole Fock space $\mathcal{V}_r$, reducing the dimension by one. Hence, in order to construct a $(2S + 1)$ dimensional representation space $\mathcal{V}_{2S+1}$ for the $spin - S$

we should act by corresponding amount of projectors Δ_m of the type (47) on the whole 2^r-dimensional space $\mathcal{V}_r$

$$\mathcal{V}_{2S+1} = \Delta_1 \cdots \Delta_{2^r-(2S+1)} \mathcal{V}_r. \tag{48}$$

The definition (48) means that any operator A which acts on the space $\mathcal{V}_r$, will have a form

$$A_S = \Delta_1 \cdots \Delta_{2^r-(2S+1)} A \Delta_1 \cdots \Delta_{2^r-(2S+1)} \tag{49}$$

on the space $\mathcal{V}_{(2S+1)}$.

Again, let's now numerate the remaining basic directions in $\mathcal{V}_(2S+1)$ by $l(n_1, ...n_r) = 1, 2...(2S+1)$ and denote the corresponding X-operator as

$$X_l^{l'} = \Delta_1 \cdots \Delta_{2^r-(2S+1)} X_{n_1,...n_r}^{n'_1,...n'_r} \Delta_1 \cdots \Delta_{2^r-(2S+1)}. \tag{50}$$

Then, as it is easy to check by use of formula (12) and some simple algebraic calculations, that the operators $S^B, B = 1, 2, 3$ defined by

$$S^B = \left(S^B\right)_l^{l'} \cdot X_{l'}^l, \qquad l, l' = 1, ...(2S+1), \tag{51}$$

fulfill the commutation relations of the $SU(2)$-algebra generators. In (51) $\left(S^B\right)_l^{l'}$ are the ordinary number valued matrix elements of the $spin - S$ generators of the $SU(2)$-algebra, the only nonzero elements of which are

$$\begin{aligned} \left(S^+\right)_l^{l-1} &= \left(S^+\right)_M^{M-1} = \sqrt{(S+M)(S-M+1)}, \qquad S^- = \left(S^+\right)^+, \\ (S_3)_l^l &= (S_3)_M^M = M, \qquad\qquad M = -S + l - 1. \end{aligned} \tag{52}$$

¿From now on we will consider the case $r = 3$ and the spins $S \leq (2^3 - 1)/2 = 7/2$ for simplicity.

For the convenience let us numerate the even elements of basic vectors $|n_1, ...n_r\rangle$ as $\alpha, \beta = 1, 2, 3, 4$ in the following way

$$|0,1,1\rangle \equiv 1, \qquad |1,0,1\rangle \equiv 2, \qquad |1,1,0\rangle \equiv 3, \qquad |0,0,0\rangle \equiv 4. \tag{53}$$

and the odd elements $a, b = 5, 6, 7, 8$ as

$$|1,0,0\rangle \equiv 5, \qquad |0,1,0\rangle \equiv 6, \qquad |0,0,1\rangle \equiv 7, \qquad |1,1,1\rangle \equiv 8, \tag{54}$$

Hence we have

$$\mathcal{V}_3 = \bigoplus_{\alpha=1}^{4} W_\alpha \bigoplus_{a=5}^{8} W_a = V_{odd} \oplus V_{even} \tag{55}$$

Now by use of definition (42) and (44) one can find the following expression for the $X_m^{m'}$-operator $(m, m' = 1, ...8)$ in the unrestricted space $\mathcal{V}_3$

$$X_m^{m'} = \begin{pmatrix} X_\alpha^{\alpha'} & X_\alpha^{a'} \\ X_a^{\alpha'} & X_a^{a'} \end{pmatrix}, \tag{56}$$

where

$$X_{\alpha}^{\alpha'} = \begin{pmatrix} (1-n_1)n_2n_3 & -c_1c_2^+n_3 & c_1n_2c_3^+ & (1-n_1)c_2^+c_3^+ \\ c_1^+c_2n_3 & n_1(1-n_2)n_3 & -n_1c_2c_3^+ & c_1^+(1-n_2)c_3^+ \\ -c_1^+n_2c_3 & n_1c_2^+c_3 & n_1n_2(1-n_3) & c_1^+c_2^+(1-n_3) \\ -(1-n_1)c_2c_3 & -c_1(1-n_2)c_3 & -c_1c_2(1-n_3) & (1-n_1)(1-n_2)(1-n_3) \end{pmatrix}, \quad (57)$$

$$X_{a}^{a'} = \begin{pmatrix} n_1(1-n_2)(1-n_3) & -c_1^+c_2(1-n_3) & c_1^+(1-n_2)c_3 & -n_1c_2c_3 \\ -c_1c_2^+(1-n_3) & (1-n_1)n_2(1-n_3) & (1-n_1)c_2^+c_3 & c_1n_2c_3 \\ -c_1(1-n_2)c_3^+ & -(1-n_1)c_2c_3^+ & (1-n_1)(1-n_2)n_3 & -c_1c_2n_3 \\ n_1c_2^+c_3^+ & -c_1^+n_2c_3^+ & c_1^+c_2^+n_3 & n_1n_2n_3, \end{pmatrix}, \quad (58)$$

$$X_{a}^{\alpha} = \begin{pmatrix} -c_1^+c_2c_3 & -n_1(1-n_2)c_3 & -n_1c_2(1-n_3) & c_1^+(1-n_2)(1-n_3) \\ -(1-n_1)n_2c_3 & c_1c_2^+c_3 & c_1n_2(1-n_3) & (1-n_1)c_2^+(1-n_3) \\ (1-n_1)c_2n_3 & c_1(1-n_2)n_3 & -c_1c_2c_3^+ & (1-n_1)(1-n_2)c_3^+ \\ c_1^+n_2n_3 & -n_1c_2^+n_3 & n_1n_2c_3^+ & c_1^+c_2^+c_3^+, \end{pmatrix}, \quad (59)$$

and

$$X_{\alpha}^{a} = (X_{a}^{\alpha})^+ . \quad (60)$$

It is easy to see from the expressions (57) and (58) that $X_{\alpha}^{\alpha'}$ can be obtained from the $X_{a}^{a'}$ by particle-hole transformation of all fermions in a following way

$$c_1 \leftrightarrow c_1^+, \qquad c_2 \leftrightarrow -c_2^+, \qquad c_3 \leftrightarrow c_3^+ \quad (61)$$

5 Fermionic representation of the spin-S ULS model: Spin-S generalisation of the supersymmetric t-J model

In this section we will fermionize the Hamiltonian of the ULS model defined by the formula (2) for spins $S \leq \frac{7}{2}$, which can be described by three Fermi fields. The procedure is based on the approach described in the section 2 and is an alternative to the usual Jordan-Wigner transformation. First we consider spin $\frac{7}{2}$.

We start with the R-matrix of the ULS model

$$\check{R}_{ms}^{kq}(u) = \delta_m^k \delta_s^q + u\, \delta_s^k \delta_m^q, \qquad m,k,s,q = 1,2,...8. \quad (62)$$

and in the same way as for *spin*-1 case described in the section 3, we obtain the Hamiltonian

$$H_{\frac{7}{2}} = \sum_{i=1}^{N} H_{i,i+1} = \sum_{i=1}^{N} (I_{i,i+1} + \Pi_{i,i+1}) = \sum_{i=1}^{N} \left(I_{i,i+1} + \prod_{\mu=1}^{3} \mathcal{P}_{i,i+1;\mu} \right) \quad (63)$$

Our goal is now to rewrite this expression of the Hamiltonian in a way which is similar to $spin1\ t-J$ model.

According to formula (49) the Hamiltonian for spins less that $\frac{7}{2}$ will be

$$H_S = \Delta_1 \cdots \Delta_{8-(2S+1)} H_{\frac{7}{2}} \Delta_{8-(2S+1)} \cdots \Delta_1, \tag{64}$$

where the projector $\Delta_1 \cdots \Delta_{8-(2S+1)}$ cuts off some subspace from the full 8-dimensional Fock space of representation $\mathcal{V}_3$, reducing the degrees of freedom to $(2S+1)$. The different choices of the cutted subspaces (projectors Δ) corresponds to isomorphic representations in a mathematical sense and simply means particle-hole transformations for some of three fermions from a physical point of view.

Let us now expand the expression (63) for the Hamiltonian $H_{i,i+1}$ with the signs, as it should be according to eq.(11) and substitute the expressions for the X-operators (57 -60) . Then after some algebra and by use of definitions (46) of the projectors we obtain

$$H_{j,j+1} = H^{1,hop}_{j,j+1} + H^{2,hop}_{j,j+1} + \bar{H}^{2,hop}_{j,j+1} + H^{diag}_{j,j+1} + H^{3}_{j,j+1} - \bar{H}^{3}_{j,j+1}. \tag{65}$$

The different terms in this Hamiltonian is defined as follows. The terms in $H^{1,hop}_{j,j+1}$ are the one and three particle hopping terms

$$\begin{aligned} H^{1,hop}_{j,j+1} &= (X_j)^{\alpha}_{m}(X_{j+1})^{m}_{\alpha} - (X_j)^{m}_{\alpha}(X_{j+1})^{\alpha}_{m} \\ &= c^+_{j,1}\left(P_{j,4}P_{j+1,4} + P_{j,6}P_{j+1,6} + P_{j,7}P_{j+1,7} + P_{j,1}P_{j+1,1}\right) c_{j+1,1} \\ &+ c^+_{j,2}\left(P_{j,5}P_{j+1,5} + P_{j,4}P_{j+1,4} + P_{j,7}P_{j+1,7} + P_{j,2}P_{j+1,2}\right) c_{j+1,2} \\ &+ c^+_{j,3}\left(P_{j,5}P_{j+1,5} + P_{j,6}P_{j+1,6} + P_{j,4}P_{j+1,4} + P_{j,3}P_{j+1,3}\right) c_{j+1,3} \\ &- \Big(c^+_{j,1}c_{j,2}c_{j,3}c_{j+1,1}c^+_{j+1,2}c^+_{j+1,3} - c_{j,1}c^+_{j,2}c_{j,3}c^+_{j+1,1}c_{j+1,2}c^+_{j+1,3} \\ &- c_{j,1}c_{j,2}c^+_{j,3}c^+_{j+1,1}c^+_{j+1,2}c_{j+1,3} - c^+_{j,1}c^+_{j,2}c^+_{j,3}c_{j+1,1}c_{j+1,2}c_{j+1,3} - h.c.\Big) \end{aligned} \tag{66}$$

We have extracted the terms, which contains $\alpha = 4$ and $m = 8$ indices into separate $H^{2,hop}_{j,j+1}$ and $\bar{H}^{2,hop}_{j,j+1}$ hopping terms, correspondingly.

$$\begin{aligned} H^{2,hop}_{j,j+1} &= (X_j)^{\alpha}_{4}(X_{j+1})^{4}_{\alpha} + (X_j)^{4}_{\alpha}(X_{j+1})^{\alpha}_{4} \\ &= -P_{j,8}\left(c_{j,2}c_{j,3}c^+_{j+1,2}c^+_{j+1,3} + c_{j,1}c_{j,3}c^+_{j+1,1}c^+_{j+1,3} + c_{j,1}c_{j,2}c^+_{j+1,1}c^+_{j+1,2}\right) P_{j+1,8} \end{aligned} \tag{67}$$

As it is clear from the formulas (59) and (60) the $\bar{H}^{2,hop}_{j,j+1}$ term can be obtained from the $H^{2,hop}_{j,j+1}$ by particle hole transformation (61) These terms are responsible for the hopping of the Fermi pairs.

H^{diag} is the diagonal term containing coulomb interaction of fermions

$$\begin{aligned} H^{diag} &= (X_j)^{\alpha}_{\alpha}(X_{j+1})^{a}_{a} + (X_j)^{4}_{4}(X_{j+1})^{\alpha}_{\alpha} - (X_j)^{8}_{8}(X_{j+1})^{a}_{a} + (j \leftrightarrow j+1) \\ &= (1 - P_{j,1} - P_{j,2} - P_{j,3})\left(P_{j+1,1} + P_{j+1,2} + P_{j+1,3} + P_{j+1,4}\right) \\ &- P_{j,8}\left(P_{j+1,5} + P_{j+1,6} + P_{j+1,7} + P_{j+1,8}\right) + (j \leftrightarrow j+1). \end{aligned} \tag{68}$$

The last two terms in (65) are analogous to the spin-spin interaction term in the ordinary $t-J$ model and, as in previous case, $\bar{H}^3_{j,j+1}$ can be obtained from the $H^3_{j,j+1}$ by the particle hole transformation (61)

$$\begin{aligned} H^3_{j,j+1} &= -(X_j)^{\bar{a}}_{\bar{a}}(X_{j+1})^{\bar{b}}_{\bar{b}} - (X_j)^{\bar{b}}_{\bar{a}}(X_{j+1})^{\bar{a}}_{\bar{b}} = & \bar{a},\bar{b}=1,2,3 \\ &= \hat{S}^B_j\hat{S}^B_{j+1} + (\hat{S}^B_j\hat{S}^B_{j+1})^2, & B=1,2,3 \end{aligned} \tag{69}$$

where

$$\begin{aligned} \hat{S}^B_j &= (S^B)^{\bar{a}}_{\bar{b}}X^{\bar{b}}_{\bar{a}} = \psi^+_{\bar{b}}(S^B)^{\bar{a}}_{\bar{b}}\psi_{\bar{a}}, \\ \psi_{\bar{a}} &= c_{\bar{a}}(1-n_1)(1-n_2)(1-n_3), \qquad \bar{a}=1,2,3. \end{aligned} \tag{70}$$

and $(S^B)^{\bar{a}}_{\bar{b}}$ is the $spin-1$ representation of the SU(2)-algebra.

This expression can be obtained easily from

$$X^{\bar{b}}_{\bar{a}} = \psi^{+\bar{b}}\psi_{\bar{a}}. \tag{71}$$

which follows from formula (57). By appropriate an choice of projectors $\Delta_\sigma,\ \sigma = 1,...8-(2S+1)$ we can simplify the expression for the Hamiltonian, for example by deleting the terms $H^{2,hop}$ and $\bar{H}^{2,hop}$.

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