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Stability of Character Resonances

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1 Introduction

We consider the selfadjoint Laplacian $L = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ in $L^2\left(F_{\Gamma(2)}\right)$, where $\Gamma(2)$ is the main congruence subgroup of the modular group of level 2 and $F_{\Gamma(2)}$ a fundamental domain of $\Gamma(2)$. In [B-V] we studied the family of operators $L(\alpha)$ associated with the group of characters χ_α on $\Gamma(2)$ defined by a singular modular form of weight 2 (the holomorphic Eisenstein series). This perturbation closes 2 cusps, and the operators $L(\alpha)$ exhibit for $\alpha \neq 0$ a new set of resonances $\rho_i(\alpha)$ which for $\alpha \to \frac{1}{4}$ go to -∞, i.e.Re $\rho_i(\alpha) \to -\infty$ [S]. For fixed α there are $K(\alpha) \cdot T$ of these resonances up to height T, asymptotically as $T \to \infty$. We call these resonances unstable for $\alpha \to \frac{1}{4}$. We further analyze this result, showing that in the limit $\alpha \to \frac{1}{4}$ the resonances $\rho(\alpha)$ asymptotically lie on a vertical line and are equidistant, moving horizontally to $-\infty$ with $C(\alpha) \to C$ for $\alpha \to \frac{1}{4}$.

The question arises, whether instability of resonances is a specific property of $\Gamma(2)$ or it may occur for other congruence groups. To study this we consider first the group Γ', which has only one cusp form of weight 2. This group has 2 hyperbolic generators and 1 parabolic with one relation, thus a 2-parameter family of characters $\chi(\alpha, \beta)$. It turns out that even in this case there is a one-parameter family of points (α, β) of instability, given by $f(\alpha, \beta) = 0$, such that approach to a point on this curve causes resonances to go $-\infty$. Apart from these points the system is stable, and there is a naturally defined curve of maximal stability, obtained by letting the hyperbolic generators have

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opposite phase. However, a character circle starting from the group Γ' with trivial character has to pass through a point of instability.

We finally investigate the group $\Gamma^{2'} = \Gamma' \cap \Gamma(2)$. This group has 2 hyperbolic and 3 parabolic generators with one relation, thus a 3-parameter family of characters $\chi(\alpha_1, \alpha_2, \alpha_3)$ which leave one cusp open. Of these a 2-parameter family given by $g(\alpha_1, \alpha_2, \alpha_3) = 0$ are points of instability. Approach to such a point causes resonances to go to $-\infty$. Otherwise the system is stable, also for characters which close 2 cusps, which happens except on another surface in parameter space. There is a natural circle of maximal stability. However, a circle starting at $\Gamma^{2'}$ with trivial character must pass through a point of instability.

These results are based on explicit calculation of the zeros of the first two terms of the series for the scattering function $\varphi(s)$ given by (1), (2) of Section 2 in the cases of $\Gamma(2)$, Γ and $\Gamma^{2'}$. The formulas for the zeros are given in Lemmas 1, 2, 4 of Section 3 and summarized in Theorem 1. Then the rest of the series is estimated to obtain our result on the asymptotic behaviour of the poles of the function $\varphi(s)$ formulated in Theorem 2 of Section 4.

Our results indicate that the existence of an unstable family of resonances, $C(\alpha)T$ in number for $T \to \infty$, corresponding to a character circle $\chi(\alpha)$ for the operator $L(\Gamma)$ with $L(\Gamma, 0) = L(\Gamma)$ is a general feature of Laplacians $L(\Gamma)$ for congruence subgroups Γ of the modular group. For the groups Γ' and $\Gamma^{2'}$ there are also stable character circles, which however do not contain $L(\Gamma')$ and $L(\Gamma^{2'})$ respectively. We expect this to hold for general congruence groups, which have a cusp form of weight 2. In the case of $\Gamma(2)$, the unstable resonances $\varrho_k(\alpha)$ of $L(\Gamma(2), \chi_\alpha)$ are asymptotically given by $\varrho_k(\alpha) = -\sigma(\alpha) + i\left(k\frac{\pi}{\log 2} + \pi\right)$ for $\alpha \to \frac{1}{4}$, where $\sigma(\alpha) \to -\infty$ as $\alpha \to \frac{1}{4}$. The Selberg resonances [S] of $L(\Gamma(2))$, which condense of every point of the continuous spectrum of $L(\Gamma(2))$, although asymptotically $K'(\alpha)T$ in number, behave very differently and seem to represent a distinct phenomenon of a different nature, related to the continuous spectrum.

2 General theory

Let Γ be a subgroup of the modular group $\Gamma_{\mathbb{Z}}$ of finite index, acting on the upper half-plane H. A general such group Γ is given by a canonical system of generators

$$\begin{array}{ll} A_1, A_2, ..., A_g, B_1, B_2, ..., B_g & \text{(hyperbolic)} \\ S_1, S_2, ..., S_h & \text{(parabolic)} \\ E_1, E_2, ..., E_k & \text{(elliptic)} \end{array}$$

with relations

$$[A_1, B_1] \cdots [A_g, B_g]\, E_1 \cdots E_k S_1 \cdots S_h = 1$$

$$E_1^{n_1} = E_2^{n_2} = ... = E_k^{n_k}, \text{ where } n_j = 2 \text{ or } 3, \ j = 1, ..., k,$$

$g \geq 0$ is the genus of Γ, $h \geq 1$ is the number of cusps of the corresponding fundamental domain. We recall that $[A, B] = ABA^{-1}B^{-1}$.

We now introduce a one-dimensional unitary representation χ of the group Γ, which we also call a character. We recall the definition of a singular character. The character χ is singular at the parabolic generator ϱ_j (or in the cusp of the fundamental domain, which corresponds to S_j) if $\chi(S_j) \neq 1$, it is non-singular at S_j if $\chi(Sj) = 1$. A character χ is singular if it is singular in at least one cusp. In this paper we will mostly consider singular characters χ of Γ, which are non-singular in just one cusp. Let us assume that $\chi(S_1) = 1$. Then there exists a fractional linear transformation $g_1 \in P - SL(2, \mathbb{R})$ such that

$$g_1^{-1} S_1 g_1 z = z + 1 \text{ for } \operatorname{Im} z > 0.$$

Let us denote by $\tilde{\Gamma}$ the group $\tilde{\Gamma} = g_1^{-1} \Gamma g_1$.

We recall the definition of the scattering matrix for the automorphic Laplacian $A(\Gamma, \chi)$. Let $H = \{z \in \mathbb{C}\,|z = x + iy \mid y > 0\}$ be the upper half-plane of $\mathbb{C}$. We consider H as the hyperbolic plane with the Poincaré metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

Let $\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$ be the Laplacian associated with the metric ds^2. We define as usual the automorphic Laplacian $A(\Gamma, \chi)$ in the Hilbert-space $\mathcal{H}(\Gamma)$ of complex-valued functions f, which are $(\Gamma, \chi)-$automorphic (i.e. $f(\gamma z) = \chi(\gamma) f(z)$ for all $\gamma \in \Gamma$ and $z \in H$) and which satisfy

$$\parallel f \parallel^2 \int_F |f(z)|^2 \, d\mu(z) < \infty.$$

Here F is the fundamental domain of Γ in H and $d\mu$ is the invariant Riemannian measure on H defined by the metric ds and given by

$$d\mu(z) = \frac{dxdy}{y^2}.$$

$A'(\Gamma, \chi)$ is defined on the space of smooth, $(\Gamma, \chi)-$automorphic functions $f \in \mathcal{H}(\Gamma)$ by the formula

$$A'(\Gamma, \chi)f = -\Delta f.$$

We identify $A'(\Gamma, \chi)$ with the restriction $A'_F(\Gamma, \chi)$ of $A'(\Gamma, \chi)$ to the space of functions $f\mid_F$, where f runs over all smooth, (Γ, χ)-automorphic functions f. The closure of $A'(\Gamma, x)$ in $\mathcal{H}(\Gamma)$ is a non-negative, selfadjoint operator, which we denote by $A(\Gamma, \chi)$.

In the case of (Γ, χ) considered here, where χ leaves open only one cusp S, $A(\Gamma, \chi)$ has the absolutely continuous spectrum $\left[\frac{1}{4}, \infty\right]$ of multiplicity one. This continuous spectrum is related to the generalized eigenfunctions of $A(\Gamma, x)$, which are defined by analytic continuation of the Eisenstein series $E(z, s)$ given for $\operatorname{Re} s > 1$ by

$$E(z, s) = E(z, s, \Gamma, x) = \sum_{\gamma \in \Gamma_1 \backslash \Gamma} y^s(g_1^{-1}\gamma z)\overline{x(\gamma)}$$

where $y(z)$ denotes $\operatorname{Im} s, \Gamma_1 \subset \Gamma$ is the group generated by S_1 and γ is the coset $\Gamma_1\gamma$ of Γ with respect to Γ_1. The series is absolutely convergent for $\operatorname{Re} s > 1$, and there exists an analytic continuation to the whole complex plane as a meromorphic function of s. For $s = \frac{1}{2} + it$, $t \in \mathbb{R}$, it constitutes the full system of generalized eigenfunctions of the continuous spectrum of the operator $A(\Gamma, x)$. We recall the definition of the automorphic scattering matrix (function) in this case. We have

$$E(g_1 z, s) = \sum_{k=-\infty}^{\infty} a_j(y, s)e^{2\pi ikx},\ z = x + iy$$

$$a_0(y, s) = y^s + \varphi(s)y^{1-s},\ \varphi(s, \Gamma, x).$$

The function $\varphi(s)$ is called the automorphic scattering function ($n \times n$ matrix if we have n open cusps). It is well known that $\varphi(s)$ is meromorphic in $\mathbb{C}$ and holomorphic in the line $\operatorname{Re} s = \frac{1}{2}$ and satisfies the functional equation

$$\varphi(s)\varphi(1-s) = 1.$$

This function is important for establishing the analytic continuation and the functional equation for the Eisenstein series given by

$$E(z, 1-s) = E(z, s)\varphi(1-s).$$

For our purposes another representation of $\varphi(s)$ is important, given by the series over double cosets. Let Γ_∞ be the infinite cyclic group generated by the map $z \to z+1$. Then we have

$$\Gamma_\infty \backslash g_1^{-1} \Gamma g_1 / \Gamma_\infty = \Gamma_\infty \backslash \tilde{\Gamma} / \Gamma_\infty = \Gamma_\infty \cup \left\{ \bigcup_{c>0} \bigcup_{d (\operatorname{mod} c)} \Gamma_\infty \begin{pmatrix} * & * \\ c & d \end{pmatrix} \Gamma_\infty \right\}$$

where

$$\begin{pmatrix} * & * \\ c & d \end{pmatrix} = \begin{pmatrix} a & b & \in \tilde{\Gamma}. \\ c & d & \end{pmatrix}$$

The general Kloosterman sums are introduced by

$$S(m, n; c; \Gamma; x) = S(m, n; c) = \sum_{d (\operatorname{mod} c)} \bar{\chi} \begin{pmatrix} a & b \\ c & d \end{pmatrix} e^{2\pi i \frac{ma+nd}{c}}, \ m, n \in \mathbb{Z}.$$

Then we have

$$\varphi(s) = \sqrt{\pi} \frac{\Gamma(s - \frac{1}{2}}{\Gamma(s)} \sum_{c>0} \frac{S(0, 0; c}{c^{2s}} \tag{1}$$

The series (1) is absolutely convergent for $\operatorname{Re} s > 1$ and has an analytic continuation to the whole of $\mathbb{C}$ as a mesomorphic function. We make an ordering of the set of coefficients c from $\Gamma_\infty \backslash \tilde{\Gamma} / \Gamma_\infty$, $c_1 < c_2 < \cdots$. Then we can write the series of (1) in the form

$$\sum_{c>0} \frac{S(0, 0; c)}{c^{2s}} = \frac{l_1}{c_1^{2s}} + \frac{l_2}{c_2^{2s}} + \cdots \tag{2}$$

where $l_j = l_j(\Gamma, \chi)$, $j = 1, 2$.

In order to study the problem of instability of resonances, which are the poles to the left of $\{\operatorname{Re} s = \frac{1}{2}\}$, we will calculate these two first coefficients for several instructive examples of Γ and χ. What we can say generally about this problem now is the following. For any group with one cusp of F only and for the trivial representation $\chi(\gamma) = 1$, all zeros of the series (2) are located in a strip $\{\frac{1}{2} < \operatorname{Re} s < a\}$ for some a. That follows from (2) because for $\operatorname{Re} s >> 1$ the dominant term is $\frac{l_1}{c_1^{2s}} (l_1 \neq 0)$. Since the trivial factor $\Gamma(s - \frac{1}{2}) / \Gamma(s)$ has no zeros in $\{\operatorname{Re} s > \frac{1}{2}\}$, the zeros of $\varphi(s)$ in $\{\operatorname{Re} s > \frac{1}{2}\}$ are precisely the zeros of the series (2). From the functional equation for $\varphi(s)$ it then follows that all resonances are located in the strip $\{1 - a < \operatorname{Re} s < \frac{1}{2}\}$.

In the presence of non-trivial characters the situation becomes more complicated. For any fixed character $\chi(\alpha)$ the same holds, and the proof is similar. When we consider a family of characters, usually a group, the question arises whether the sets of resonances of $A(\Gamma, \chi(\alpha))$ remain bounded below, uniformly for all α.

In order to analyze this problem we consider three particularly important and interesting examples, the groups $\Gamma(2)$ (a singular perturbation closing 2 cusps), Γ' (one cusp form) and $\Gamma^{2\prime}$ (a mixture of singular perturbation and cusp form).

3 Explicit calculations of the coefficients l_1 and l_2 for the groups $\Gamma(2)$, Γ' and $\Gamma^{2'}$

a) The group $\Gamma(2)$

Let $\Gamma(2)$ be the principal congruence subgroup of $P-SL(2,\mathbb{Z})$ of level 2. $\Gamma(2)$ is a normal subgroup of the modular group of index 6. By definition, for $n \in \mathbb{Z}_+$

$$\Gamma(n) = \left\{ \gamma \in P-SL(2,\mathbb{R}) \mid \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\mathrm{mod}\, n) \right\}$$

$\Gamma(2)$ is a cofinite group of genus zero, generated by the three parabolic generators A, B, S with one relation,

$$ABS = 1 \text{ (the unity of the group).}$$

We have

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix},\ AB = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix} = S^{-1}\,, S = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}.$$

We identify elements of $\Gamma(2)$, which differ by a factor -1. The family of characters χ_α of $\Gamma(2)$ is defined on the generators A, B, S as follows,

$$\chi_\alpha(A) = 1,\ \chi_\alpha(B) = e^{2\pi i\alpha},\ \chi_\alpha(S) = e^{-2\pi i\alpha},\ 0 \le \alpha \le 1 \tag{3}$$

This family of characters is nonsingular only in the cusp, which corresponds to the generator A.

Let us calculate now the two first terms $\frac{l_1}{c_1^{2s}}$ and $\frac{l_2}{c_1^{2s}}$ of the series (2). If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma = \Gamma(2)$, then $\tilde{\gamma} = \begin{pmatrix} a & b/2 \\ 2c & d \end{pmatrix} \in \tilde{\Gamma}$.

The minimal positive value of $\tilde{c} = 2c$ when $\tilde{\gamma} \in \tilde{\Gamma}$ is $\tilde{c} = c_1 = 4$, and the next value of $\tilde{c}$ is $c_2 = 8$ by definition of the group $\Gamma(2)$. In order to calculate the coefficients l_1, l_2 in the series (2) we have to find representatives of all double cosets $\Gamma_\infty \backslash \tilde{\Gamma} / \Gamma_\infty$ with $\tilde{c} = 4$ and $\tilde{c} = 8$, respectively. Necessary conditions for this on the matrix $\tilde{\gamma} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\Gamma}$ are

$$\text{for } l_1(\tilde{c} = 4):\ a \text{ is mod } 4,\ d \text{ is mod } 4$$

$$\text{for } l_2(\tilde{c} = 8):\ a \text{ is mod } 8,\ d \text{ is mod } 8.$$

This corresponds to the following conditions on $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

$$\text{for } l_1 : c = 2,\ a \text{ mod } 4,\ d \text{ mod } 4$$

$$\text{for } l_2 : c = 4,\ a \text{ mod } 8,\ d \text{ mod } 8.$$

A simple calculation shows that in case 1 a full set of representatives is given by

$$\gamma_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},\ \gamma_2 = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix},$$

and we obtain also

$$\gamma_1 = B^{-1},\ A^{-1}\gamma_2 A^{-1} = B.$$

Then

$$\chi(\gamma_1) = e^{-2\pi i\alpha},\ \chi(\gamma_2) = e^{2\pi i\alpha}$$

and the coefficient l_1 is given by

$$l_1 = \bar{\chi}(\gamma_1) + \bar{\chi}(\gamma_2) = 2\cos 2\pi\alpha.$$

In case 2 we have 4 representatives

$$\gamma_1 = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix},\ \gamma_2 = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix},\ \gamma_3 = \begin{pmatrix} 5 & 6 \\ 4 & 5 \end{pmatrix},\ \gamma_4 = \begin{pmatrix} 7 & 12 \\ 4 & 7 \end{pmatrix}.$$

Then it is not difficult to obtain the relations

$$\gamma_1 = B^{-2},\ B\gamma_2 B = A^{-1},\ A^{-1}\gamma_3 A^{-1} = \gamma_2^{-1},\ A^{-1}\gamma_4 A^{-1} = B^2.$$

From this follows

$$\chi(\gamma_1) = e^{-4\pi i\alpha},\ \chi(\gamma_2) = e^{-4\pi i\alpha},\ \chi)\gamma_3) = e^{4\pi i\alpha},\ \chi(\gamma_4) = e^{4\pi i\alpha}.$$

Thus the second coefficient l_2 is given by

$$l_2 = 4\cos 4\pi\alpha.$$

We have proved the following lemma.

Lemma 1 *For the group* $\Gamma(2)$ *with character* $\chi = \chi_a$ *given by (3) we have the following formulae for the coefficients* $l_1 = l_1(\Gamma,\chi)$ *and* $l_2 = l_2(\Gamma,\chi)$,

$$l_1 = 2\cos 2\pi\alpha,\ l_2 = 4\cos 4\pi\alpha.$$

b) The group Γ'.

Let Γ' be the commutator subgroup of the modular group $\Gamma_{\mathbb{Z}} = P-SL(2,\mathbb{Z})$. Like $\Gamma(2)$ it is a normal subgroup of $\Gamma_{\mathbb{Z}}$ of index 6. Γ' is a group of genus 1, generated by two hyperbolic generators X, Y and one parabolic $\tilde{S}$, with one relation

$$[X,Y]\,\tilde{S} = 1.$$

$$X = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix},\ Y = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},\ \tilde{S} = A^3 = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}.$$

The two parameter family of character $\chi = \chi_{\alpha,\beta}$ is defined on the generators $X, Y, \tilde{S}$ as follows

$$\chi_{\alpha,\beta}(X) = e^{2\pi i\alpha},\ \chi_{\alpha,\beta}(Y) = e^{2\pi i\beta},\ \chi_{\alpha,\beta}(\tilde{S}) = 1,$$

$\alpha,\beta \in [0,1]$. This family of characters is always non-singular in the cusp, which corresponds to generator $\tilde{S}$.

Let us find now the first two terms of the series (2). If

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma = \Gamma',\ \text{then}\ \tilde{\gamma} = \begin{pmatrix} a & b/6 \\ 6c & d \end{pmatrix} \in \tilde{\Gamma}$$

becomes the width of the only cusp for the canonical fundamental domain for Γ' is 6. The minimal positive value of $\tilde{c} = bc$ when $\tilde{\gamma} \in \tilde{\Gamma}$ is $c_1 = 6$ and the next is $c_2 = 12$. Again, in order to calculate the coefficients l_1, l_2 in the series (2) we have to find a full set of representatives of all double co-sets $\Gamma_\infty\backslash\tilde{\Gamma}/\Gamma_\infty$ with given $\tilde{c} = 6$ and $\tilde{c} = 12$. Necessary conditions for that are

1) a and d are mod 6. 2) a and d are mod 12.

For $\gamma \in \Gamma$ $\left(\tilde{\gamma} \in \Gamma_\infty \backslash \tilde{\Gamma}/\Gamma_\infty\right)$ we obtain

1) $c = 1$, a (mod 6), d (mod 6). 2) $c = 2$, a (mod 12), d (mod 12).

There is an important classical characterization of the elements of $\Gamma = \Gamma'$ (see [R],

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma' \subset \Gamma_{\mathbb{Z}} \Leftrightarrow \begin{cases} ad - bc = 1 \\ ab + 3bc + cd \equiv 0 \pmod 6 \end{cases} \tag{4}$$

Case 1. We have $c = 1$, $b = ad - 1$, and the second condition of (4) becomes $a(ad-1) + 3(ad-1) + d \equiv 0 \pmod 6$ where $a = 0, 1, 2, 3, 4, 5$ and $d = 0, 1, 2, 3, 4, 5$. We obtain only 6 matrices which satisfy these conditions

$$\left.\begin{array}{lll} \gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}, & \gamma_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, & \gamma_3 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \\ \gamma_4 = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}, & \gamma_5 = \begin{pmatrix} 4 & 19 \\ 1 & 5 \end{pmatrix}, & \gamma_6 = \begin{pmatrix} 5 & 19 \\ 1 & 4 \end{pmatrix} \end{array}\right\} \tag{5}$$

Let us calculate now the values of the character $\chi = \chi_{\alpha,\beta}$ on these matrices. We obtain

$$\gamma_1 = (XY)^{-1},\ \gamma_2 = X^{-1},\ \gamma_3 = Y,\ \gamma_4 = YX,$$

$$\gamma_5 \tilde{S}^{-1} X^{-1} = \tilde{S},\ \ X^{-1}\tilde{S}^{-1}\gamma_6 = \gamma_1.$$

Then,

$$\left.\begin{array}{lll} \chi(\gamma_1) = e^{-2\pi i(\alpha+\beta)}, & \chi(\gamma_2) = e^{-2\pi i\alpha}, & \chi(\gamma_3) = e^{2\pi i\beta} \\ \chi(\gamma_4) = e^{2\pi i(\alpha+\beta)}, & \chi(\gamma_5) = e^{2\pi i\alpha}, & \chi(\gamma_6) = e^{-2\pi i\beta} \end{array}\right\} \tag{6}$$

We obtain the value of l_1 as the sum of all these values of characters,

$$l_1 = 2(\cos 2\pi\alpha + \cos 2\pi\beta + \cos 2\pi(\alpha+\beta)) \tag{7}$$

Case 2. We have $c = 2$, a (mod 12), d(mod 12), $ad - 2b = 1$, $ab + 6b + 2d \equiv 0 \pmod 6$, or $ab + 2d \equiv 0 \pmod 6$. Then we can see from $ad - 2b = 1$ that both a and d are odd. So we reduce the numbers a, d to $a = 1, 2, 3, 5, 7, 9, 11$ $d = 1, 3, 5, 7, 9, 11$. Finally we obtain only 6 matrices

$$\left.\begin{array}{lll} \gamma_1 = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} & \gamma_2 = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} & \gamma_3 \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \\ \gamma_4 = \begin{pmatrix} 7 & 38 \\ 2 & 22 \end{pmatrix} & \gamma_5 = \begin{pmatrix} 9 & 40 \\ 2 & 9 \end{pmatrix} & \gamma_6 \begin{pmatrix} 11 & 38 \\ 2 & 7 \end{pmatrix} \end{array}\right\} \tag{8}$$

We now calculate the values of $\chi = \chi_{\alpha,\beta}$ on the matrices (8). It is not difficult to see the following relations:

$$\gamma_1 = X^{-1}Y^{-1}X^{-1},\ \gamma_2 = YX^{-1},\ \gamma_3 = YXY$$

$$\gamma_4 = \tilde{S}XYX\tilde{S},\ X^{-1}Y^{-1}\gamma_5 XY = \gamma_2^{-1},\ YX^{-1}Y^{-1}\gamma_6 XY^2 = X^{-1}$$

From this we obtain

$$\left.\begin{array}{ccc} \chi(\gamma_1) = e^{-2\pi i(2\alpha+\beta)}, & \chi(\gamma_2) = e^{2\pi i(\beta-\alpha)}, & \chi(\gamma_3) = e^{2\pi i(\alpha+2\beta)}, \\ \chi(\gamma_4) = e^{2\pi i(2\alpha+\beta)}, & \chi(\gamma_5) = e^{2\pi i(\alpha-\beta)}, & \chi(\gamma_6) = e^{-2\pi i(\alpha+2\beta)} \end{array}\right\} \tag{9}$$

To obtain the coefficient l_2 we sum all these values of characters,

$$l_2 = 2\left[\cos 2\pi(\alpha-\beta) + \cos 2\pi(\alpha+2\beta) + \cos 2\pi(2\alpha+\beta)\right].$$

We have proved

Lemma 2 *For the group $\Gamma = \Gamma'$ with character $\chi = \chi_{\alpha,\beta}$ given by (9) we have the following explicit formulae for the coefficients $l_1 = l_1(\Gamma,\chi)$, $l_2 = l_2(\Gamma,\chi)$,*

$$l_1 = 2\left(\cos 2\pi\alpha + \cos 2\pi\beta + \cos 2\pi(2\alpha+\beta)\right)$$

$$l_2 = 2\left[\cos 2\pi(\alpha-\beta) + \cos 2\pi(\alpha+2\beta) + \cos 2\pi(2\alpha+\beta)\right]$$

For the problem of stability of resonances it is important to analyze the set of points (α,β), where $l_1(\alpha,\beta) = 0$. This analysis is simplified by introducing a new set of generators,

$$Z = XY = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}, W = X^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$
$$\tilde{S} = XYX^{-1}Y^{-1} = ZWZ^{-1}W^{-1} = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}.$$

Set $a = \frac{\alpha+\beta}{2}$, $b = \frac{\alpha-\beta}{2}$, $\alpha = ab$, $\beta = a - b$.
Then

$$\chi_{a,b}(Z) = e^{4\pi i a},\ \chi_{a,b}(W) = e^{-2\pi i(a+b)},\ \chi_{a,b}(\tilde{S}) = 1.$$

The functions $l_1(\alpha,\beta)$ and $l_2(\alpha,\beta)$ of Lemma 2 become

$$l_1(a,b) = 2\left[\cos 2\pi(a+b) + \cos 2\pi(a-b) + \cos 2\pi\cdot 2a\right] = 2\left[2\cos a\cos b + 2\cos^2 a - 1\right]$$

$$l_2(a,b) = 2\left[\cos 2\pi(3a+b) + \cos 2\pi(3a-b) + \cos 2\pi \cdot 2b\right] = 2\left[2\cos 3a \cos b + 2\cos^2 b - 1\right].$$

Let $u = \cos 2\pi a,\ v = \cos 2\pi b$. Then

$$l_1(a,b) = 2(2uv + 2u^2 - 1),\ l_2(a,b) = 2[\cos 3a \cdot v + 2v^2 - 1].$$

We have $l_1(a,b) = 0$ for $v(u) = \frac{\frac{1}{2}-u^2}{u},\ u \neq 0$.

In order to obtain $v \in [-1,1]$ u must satisfy $-1 \leq \frac{\frac{1}{2}-u^2}{u} \leq 1$. Since $v(u)$ is odd, it suffices to consider $0 \leq u \leq 1$. The function $v(u)$ is decreasing and maps $\left[\frac{\sqrt{3}-1}{2}, 1\right]$ onto $\left[-\frac{1}{2}, 1\right]$.

It follows that for $u \in \left(-\frac{\sqrt{3}-1}{2}, \frac{\sqrt{3}-1}{2}\right)$ and $v \in [-1,1]\,, l_1(a,b) \neq 0$. For $u \in \left[-1, -\frac{\sqrt{3}-1}{2}\right] \cup \left[\frac{\sqrt{3}-1}{2}, 1\right]$ and $v = \frac{\frac{1}{2}-u^2}{u}, l_1(a,b) = 0$.

The points on this curve C where $l_2(a,b) = 0$ can be determined from the expression $l_2(a,b) = wv + v^2 + -\frac{1}{2},\ w = \cos 2\pi 3a$. We have $l_1(a,b) = l_2(a,b) = 0$ if $w = \frac{\frac{1}{2}-v^2}{v},\ v = \frac{\frac{1}{2}-u^2}{u}$. Expressing w algebraically by u we obtain (at most) a finite number of points (a,b) on the above curve where also $l_2(a,b) = 0$.

c) The group $\Gamma^{2'}$. This is a more difficult example of a subgroup of the modular group $\Gamma_{\mathbb{Z}}$. By definition, $\Gamma^{2'}$ is the commutator subgroup of Γ^2, which in turn is the subgroup of $\Gamma_{\mathbb{Z}}$ generated by all squares $\gamma^2 \in \Gamma_{\mathbb{Z}}$. The canonical fundamental domain of $\Gamma^{2'}$ has genus 1, three cusps and no elliptic singularities. This group can also be written as the intersection of the two groups considered above, $\Gamma^{2'} \in \Gamma(2) \cap \Gamma'$. Because of the importance of this fact we remind of the proof of this.

Lemma 3 $\Gamma^{2'} = \Gamma(2) \cap \Gamma'$.

Proof. It is known that $\Gamma^{2'}$ is a normal subgroup in Γ' of index 3 and also $\Gamma^{2'}$ is normal in $\Gamma(2)$ of index 3. It is easy to see that the quotients $\Gamma'/\Gamma^{2'}$ and $\Gamma(2)/\Gamma^{2'}$ can be given by the matrices

$$\Gamma'/\Gamma^{2'} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \gamma_1, \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \gamma_2 \right\}$$

$$\Gamma(2)/\Gamma^{2'} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \delta_1, \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} = \delta_2 \right\}.$$

Then we have $\Gamma(2) = \Gamma^{2'} \cup \Gamma^{2'}\gamma_1 \cup \Gamma^{2'}\gamma_2,\ \Gamma' = \Gamma^{2'} \cup \Gamma^{2'}\delta_1 \cup \Gamma^{2'}\delta_2$. In order to prove Lemma 3 it is enough to prove that all intersections

$$\Gamma^{2'}\delta_i\gamma_j^{-1} \cap \Gamma^{2'} \quad i = 1,2\ j = 1,2$$

are empty sets. We obtain

$$\delta_1\gamma_1^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \notin \Gamma^{2'}$$

$$\delta_1\gamma_2^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ -1 & 2 \end{pmatrix} \notin \Gamma^{2'}$$

$$\delta_2\gamma_1^{-1} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix} \notin \Gamma^{2'}$$

$$\delta_2\gamma_2^{-1} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -3 & 7 \\ -1 & 2 \end{pmatrix} \notin \Gamma^{2'}$$

because all of these products are elliptic elements if third order, and $\Gamma^{2'}$ is free from elliptic elements. We have proved lemma 3. ■

The group $\Gamma^{2'}$ is generated by 4 hyperbolic (non-canonical) generators (see [R]),

$$\begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}, \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}, \begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix}, \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}.$$

The arithmetical description of matrices γ from $\Gamma^{2'}$ is the following: $\gamma \in \Gamma^{2'} \Leftrightarrow \Gamma \in \Gamma_{\mathbb{Z}}$ and at least one of the following relations is valid,

$$\gamma \equiv \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}, \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 6 \qquad (10)$$

For our purposes it is better to take as generators the inverse matrices

$$H_1 = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix},\ H_2 = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix},\ H_3 = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix},\ H_4 = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}. \quad (11)$$

We can find now the canonical system of generators of $\Gamma^{2'}$, related to the generators chosen for $\Gamma(2)$ and Γ'. It is easy to see that A^3, B^3 and S^3 belong to $\Gamma^{2'}$. Then we find the following relation.

Let

$$S_1 = (H_3 H_1 H_2^{-1}) S^{-3} (H_3 H_1 H_2^{-1})^{-1}$$
$$S_2 = H_3 B^{-3} H_3^{-1}$$
$$S_3 = H_1 A^{-3} H_1^{-1}$$

Then we have one relation between the 2 hyperbolic and 3 parabolic generators,

$$[H_1, H_3]\, S_1 S_2 S_3 = 1. \tag{12}$$

The following formulas are useful,

$$A^3 = H_3 H_4^{-1} H,\ B^3 = H_3^{-1} H_2 H_1^{-1},\ S^3 = H_4 H_2^{-1}.$$

The canonical generators A^3, B^3, S^3, H_1, H_3 are important if we like to define a singular representation of $\Gamma^{2'}$. We define the family of characters χ, non-singular at the cusp which corresponds to A^3. By definition, on the generators H_j

$$\chi_{\alpha_j}(H_j) = e^{2\pi\alpha_j},\ \alpha_j \in [0,1]\,,\ j = 1,2,3,4. \tag{13}$$

Then χ is extended to the whole group by the multiplicative property.

The condition of non-singularity of A^3 is $\chi(A^3) = 1$. We have

$$\chi(A^3) = \chi(H_3 H_4^{-1} H_1) = e^{2\pi i(\alpha_1 - \alpha_4 + \alpha_1)} = 1 \Leftrightarrow \alpha_4 = \alpha_1 + \alpha_3.$$

This defines a three parameter family of characters $\chi_{\alpha_1,\alpha_2,\beta_3}$ on $\Gamma^{2'}$which keeps the cusp corresponding to A^3 open.

Also we have

$$\chi(B^3) = e^{2\pi i(\alpha_2 - \alpha_1 - \alpha_3)}.$$

Thus, on the two-dimensional surface in parameter space given by $\alpha_2 = \alpha_1 + \alpha_3$ all three cusps remain open.

Let us calculate now the two first terms of the series (2) for $\Gamma^{2'}$ and the three parameter family of characters $\chi_{\alpha_1,\alpha_2,\alpha_3}$ defined by (13) and $\alpha_4 = \alpha_1 + \alpha_3$. If

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma = \Gamma^{2'},\ \text{then } \tilde{\gamma} = \begin{pmatrix} a & b/6 \\ 6c & d \end{pmatrix} \in \tilde{\Gamma} \text{ (see Section 2).}$$

The minimal positive value of $\tilde{c} = 6c$ for $\tilde{\gamma} \in \tilde{\Gamma}$ is $\tilde{c} = c_1 = 12$, and the next one is $c_2 = 24$ by the definitions of the groups $\Gamma(2)$ and Γ. We shall now find representatives of all double cosets $\Gamma_\infty \backslash \tilde{\Gamma} / \Gamma_\infty$ with given $\tilde{c} = 12$ and $\tilde{c} = 24$. Necessary conditions on the matrix elements are

1) $\tilde{c} = 12$, a is mod 12, d is mod 12

2) $\tilde{c} = 24$, a is mod 24, d is mod 24.

For $\gamma \in \Gamma$ ($\tilde{\gamma} \in \Gamma_\infty \backslash \tilde{\Gamma} / \Gamma_\infty$) we obtain

1) $c = 2$, a (mod 12), d (mod 12)

2) $c = 4$, a (mod 24), d (mod 24).

Finally we have to find integer matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } c, a, d \text{ like in 1) and 2) and } ad - bc = 1,$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod 2 \text{ and } ab + 3bc + cd \equiv 0 \pmod 6).$$

where we have used Lemma 3. It is not difficult to see that in case 1) the desired set of matrices coincides with set (8) because all of these clearly belong also to the group $\Gamma(2)$. In case 2) the calculation is longer, because we have to move the choice from 576 integer matrices. The set of all matrices which satisfy our conditions are

$$\gamma_1 = \begin{pmatrix} 17 & 4 \\ 4 & 1 \end{pmatrix} \; \gamma_2 = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \; \gamma_3 = \begin{pmatrix} 13 & 16 \\ 4 & 5 \end{pmatrix} \; \gamma_4 = \begin{pmatrix} 23 & 40 \\ 4 & 7 \end{pmatrix}$$

$$\gamma_5 = \begin{pmatrix} 9 & 20 \\ 4 & 5 \end{pmatrix} \; \gamma_6 = \begin{pmatrix} 19 & 52 \\ 4 & 11 \end{pmatrix} \; \gamma_7 = \begin{pmatrix} 5 & 16 \\ 4 & 13 \end{pmatrix} \; \gamma_8 = \begin{pmatrix} 15 & 56 \\ 4 & 15 \end{pmatrix} \tag{14}$$

$$\gamma_9 = \begin{pmatrix} 1 & 4 \\ 4 & 17 \end{pmatrix} \; \gamma_{10} = \begin{pmatrix} 11 & 52 \\ 4 & 19 \end{pmatrix} \; \gamma_{11} = \begin{pmatrix} 21 & 110 \\ 4 & 21 \end{pmatrix} \; \gamma_{12} = \begin{pmatrix} 7 & 40 \\ 4 & 23 \end{pmatrix}.$$

We now find the value of the characters $\chi = \chi_{\alpha_1,\alpha_2,\alpha_3}, \alpha_4 = \alpha_1 + \alpha_3$, on the matrices (8) which correspond to case 1). We have

$$\gamma_1 = H_1,\ \gamma_2 = H_2,\ \gamma_3 = H_3.$$

Then it is not difficult to see that

$$H_3^{-1}\gamma_4 A^{-3} = H_4^{-1},\ H_3^{-1}\gamma_5 A^{-3} H_2 A^{-3} = H_3^{-1},\ A^{-3}\gamma_6 A^{-3} = H_3^{-1}.$$

We can calculate now the values of the characters χ,

$$\chi(\gamma_1) = e^{2\pi i\alpha_1}\ \chi(\gamma_2) = e^{2\pi i\alpha_2}\ \chi(\gamma_3) = e^{2\pi i\alpha_3}$$

$$\chi(\gamma_4) = e^{-2\pi i(\alpha_1+\alpha_3)+2\pi i\alpha_3} = e^{-2\pi i\alpha_1}$$

$$\chi(\gamma_5) = e^{-2\pi i\alpha_3+2\pi i\alpha_3-2\pi i\alpha_2} = e^{-2\pi i\alpha_2}$$

$$\chi(\gamma_6) = e^{-2\pi i\alpha_3}.$$

From this follows that

$$l_1(\Gamma^{2'}, \chi) = l = 2\left[\cos 2\pi\alpha_1 + \cos 2\pi\alpha_2 + \cos 2\pi\alpha_3\right].$$

For case 2) we have to see the values of χ on the 12 matrices from (14). We have $(\alpha_4 = \alpha_3 + \alpha_1)$

$$\begin{array}{ll}
H_4H_3^{-1}\gamma_1H_3^{-1}H_2 = H_1 & \chi(\gamma_1) = e^{2\pi i(\alpha_3-\alpha_2)} \\
\gamma_2 = H_4 & \chi(\gamma_2) = e^{2\pi i(\alpha_1+\alpha_3)} \\
H_3^{-1}\gamma_3H_2^{-1} = H_4^{-1} & \chi(\gamma_3) = e^{2\pi i(\alpha_2-\alpha_1)} \\
\gamma_4H_2^{-1}H_3 = A^3 & \chi(\gamma_4) = e^{2\pi i(\alpha_2-\alpha_3)} \\
H_3^{-1}\gamma_5 = H_1 & \chi(\gamma_5) = e^{2\pi i(\alpha_1+\alpha_3)} \\
\gamma_6H_1^{-1}H_2 = A^3 & \chi(\gamma_6) = e^{2\pi i(\alpha_1-\alpha_2)} \\
H_4H_2^{-1}\gamma_7 = H_1 & \chi(\gamma_7) = e^{2\pi i(\alpha_2-\alpha_3)} \\
H_4^2H_3^{-1}\gamma_8 = H_1 & \chi(\gamma_8) = e^{-2\pi i(\alpha_1+\alpha_3)} \\
H_2H_1^{-1}\gamma_9 = A^3 & \chi(\gamma_9) = e^{2\pi i(\alpha_1-\alpha_2)} \\
H_3^{-1}\gamma_{10}A^{-3} = H_2^{-1} & \chi(\gamma_{10}) = e^{2\pi i(\alpha_3-\alpha_2)} \\
A^{-3}\gamma_{11}A^{-3} = H_4^{-1} & \chi(\gamma_{11}) = e^{-2\pi i(\alpha_1+\alpha_3)} \\
H_3^{-1}\gamma_{12}A^{-3} = B^3 & \chi(\gamma_{12}) = e^{2\pi i(\alpha_2-\alpha_1)}
\end{array} \tag{15}$$

From (15) follows

$$l_2 = 4\left[\cos 2\pi(\alpha_1+\alpha_3) + \cos 2\pi(\alpha_1-\alpha_2) + \cos 2\pi(\alpha_2-\alpha_3)\right].$$

We have proved the following lemma.

Lemma 4 *For the group $\Gamma^{2'}$ with character $\chi = \chi_{\alpha_1,\alpha_2,\alpha_3}$ given by (13) and $\alpha_4 = \alpha_1 + \alpha_3$ we have the following explicit formulae for the coefficients $l_1 = l_1(\Gamma, x),\ l_2 = l_2(\Gamma, x)$,*

$$l_1 = 2\left[\cos 2\pi\alpha_1 + \cos 2\pi\alpha_2 + \cos 2\pi\alpha_3\right]$$

$$l_2 = 4\left[\cos 2\pi(\alpha_1+\alpha_3) + \cos 2\pi(\alpha_1-\alpha_2) + \cos 2\pi(\alpha_2-\alpha_3)\right].$$

For the analysis of stability it is important to consider the set of points $(\alpha_1, \alpha_2, \alpha_3)$ such that $l_1(\alpha_1, \alpha_2, \alpha_3) = 0$, a surface in parameter space.

The set of points $(\alpha_1, \alpha_2, \alpha_3)$, such that $l_2(\alpha_1, \alpha_2, \alpha_3) = 0$ is a surface intersecting the set where $l_1(\alpha_1, \alpha_2, \alpha_3) = 0$ in a curve. To see this we note that

$$l_1\left(\frac{1}{6}, \frac{1}{3}, \frac{1}{4}\right) = l_1\left(\frac{1}{2}, 0, \frac{1}{4}\right) = l_1\left(\frac{1}{8}, \frac{3}{8}, \frac{1}{4}\right) = 0$$

while

$$l_2\left(\frac{1}{6}, \frac{1}{3}, \frac{1}{4}\right) = \frac{\sqrt{3}}{2},\ l_2\left(\frac{1}{2}, 0, \frac{1}{4}\right) = -1,\ l_2\left(\frac{1}{8}, \frac{3}{8}, \frac{1}{4}\right) = 0.$$

From the point of view of stability it is of interest to introduce as characters on $\Gamma^{2'}$ the products of the characters defined on groups $\Gamma(2)$ and Γ', which are both well defined on the intersection $\Gamma(2) \cap \Gamma' = \Gamma^{2'}$.

We now introduce new parameters and new notation, setting $\chi_\alpha = \chi_1, \chi_{\alpha,\beta} = \chi_2$. We have for $\Gamma(2)$ that $\chi_1 = \chi_1(a)$ is defined by its values on generators,

$$\chi_1(A) = 1,\ \chi_1(B) = e^{2\pi i a},\ \chi_1(S) = e^{-2\pi i a},\ a \in [0,1]$$

and correspondingly for Γ', $\chi_2 = \chi_2(a,b)$ is defined by

$$\chi_2(A^3) = 1,\ \chi_2(X) = e^{2\pi i b},\ \chi_2(Y) = e^{2\pi i c},\ b, c \in [0,1]\,.$$

Then we define $\chi_3 = \chi_3(a,b,c)$ on $\Gamma^{2'}$ by

$$\chi_3(\gamma) = \chi_1(\gamma)\chi_2(\gamma),\ \gamma \in \Gamma^{2'},\ a, b, c \in [0,1]\,.$$

We have five generators for $\Gamma^{2'}$: H_1, H_3, A^3, B^3, S^3 with one relation.

We obtain $H_1 = B^{-1}A,\ H_3 = AB^{-1}$. Thus

$$\left.\begin{array}{lll} \chi_1(H_1) = e^{-2\pi i a}, & \chi_1(H_3) = e^{-2\pi i a}, & \chi_1(A^3) = 1 \\ \chi_1(B^3) = e^{6\pi i a}, & \chi_1(S^3) = e^{-6\pi i a} & \end{array}\right\}. \tag{16}$$

Also, because all three cusps for $\Gamma^{2'}$ are equivalent in the overgroup Γ', we have

$$\chi_2(A^3) = \chi_2(B^3) = \chi_2(S^3) = 1. \tag{17}$$

Moreover, $H_1 = X^{-1}Y^{-1}X^{-1}$, $H_3 = YXY$, and we obtain

$$\chi_2(H_2) = e^{-2\pi i(2b+c)},\ \chi_2(H_3) = e^{-2\pi i(b+2c)}. \tag{18}$$

Using (16), (17), (18) we obtain

$$\left.\begin{array}{lll} \chi_3(H_1) = e^{-2\pi i(a+2b+c)} & & \chi_3(H_3) = e^{-2\pi i(a-b-2c)} \\ \chi_3(A^3) = 1 & \chi_3(B^3) = e^{6\pi ia} & \chi_3(S^3) = e^{-6\pi ia} \end{array}\right\}. \tag{19}$$

Using these new parameters we collect the results of Lemma 1, Lemma 2, Lemma 4, applying to characters χ_1, χ_3, χ_3, in the following theorem.

Theorem 1 *We have the following explicit expressions of the two first terms in the series (2) of the scattering function $\varphi(s;\Gamma;\chi)$ for the pairs $(\Gamma(2),\chi_1)$, (Γ',χ_i), $(\Gamma^{2'},\chi_3)$,*

1)

$$\sum_{c>0} \frac{S(0,0;c;\Gamma(2);\chi_1)}{c^{2s}} = \frac{2\cos 2\pi a}{4^{2s}} + \frac{4\cos 4\pi a}{8^{2s}} + ...$$

2)

$$\sum_{c>0} \frac{S(0,0;c;\Gamma';\chi_2)}{c^{2s}} = \frac{2(\cos 2\pi b + \cos 2\pi c + \cos 2\pi(b+c))}{6^{2s}}$$

$$+\frac{2(\cos 2\pi(b-c) + \cos 2\pi(b+2c) + 2\cos 2\pi(2b+c))}{12^{2s}} + ...$$

3)

$$\sum_{c>0} \frac{S(0,0;c;\Gamma';\chi_3)}{c^{2s}} = \frac{2(\cos 2\pi(a+2b+c) + \cos 2\pi(a-b+c) + \cos 2\pi(a-b-2c))}{12^{2s}}$$

$$+\frac{4(\cos 2\pi(2a+b-c) + \cos 2\pi(2a+b+2c) + 2\pi(2a-2b-c))}{24^{s}} + ...$$

4 Stability and instability of resonances

We shall now apply Theorem 1 of Section 3 to discuss stable and unstable resonances for the groups $\Gamma(2)$, Γ' and $\Gamma^{2'}$. We write the result in the common form of (1), (2) of Section 2, valid for Res >1,

$$\varphi(s,\chi) = \sqrt{\pi}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\left(\frac{l_1(\chi)}{c_1^{2s}} + \frac{l_2(\chi)}{c_2^{2s}} + \frac{l_3(x)}{c_3^{2s}} + ...\right)$$

where $\chi = \chi(\alpha) = \chi(a)$ for $\Gamma(2)$, $\chi = \chi(\alpha_1, \alpha_2) = \chi(b, c)$ for Γ', $\chi = \chi(\alpha_1, \alpha_2, \alpha_3) = \chi(a, b, c)$ for $\Gamma^{2'}$, $i = 1, 2$, and l_1, l_2 are given in Lemmas 1, 2 and 4 expressed by the parameters α, β and α_i and in Theorem 1 by a, b, c.

In order to investigate the behaviour of resonances as the parameters approach the set of values where $l_1 = 0$, but $l_2 \neq 0$, we study the zeros of $\varphi(s, \chi)$ using the functional equation. For $l_1 \neq 0, l_2 \neq 0$ we write $\varphi(s, \chi)$ for Re >1 in the form

$$\varphi(s,\chi) = \sqrt{\pi}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\frac{l_1}{c_1^{2s}}\left\{1+\frac{1}{2^{2s}}\frac{l_2}{l_1}\left[1+\frac{1}{\left(\frac{3}{2}\right)^{2s}}+\frac{1}{2^{2s}}\frac{l_4}{l_2}+...\right]\right\}. \tag{20}$$

The zeros of $\varphi(s, \chi)$ coincide with the zeros of the function $f(s, \chi)$ given by the series in brackets. To study the zeros of this series we consider the first two terms

$$g(s,\chi) = 1 + \frac{1}{2^{2s}}\frac{l_2(\chi)}{l_1(\chi)}.$$

The zeros of $g(s, \chi)$ are given, setting $s = \sigma + i\tau$, by $s = s(\chi, k) = \sigma(\chi) + i\tau(k)$, where

$$\begin{aligned} \sigma(\chi) &= \tfrac{1}{2\log 2}\log\left|\tfrac{l_2(\chi)}{l_1(\chi)}\right| \\ \tau(k) &= \begin{cases} \tfrac{1}{2\log 2}(\pi + k\cdot 2\pi) & \text{when } \tfrac{l_2(\chi)}{l_1(\chi)} < 0 \\ \tfrac{1}{2\log 2}\cdot k2\pi & \text{when } \tfrac{l_2(\chi)}{l_1(\chi)} < 0 \end{cases} \end{aligned} \tag{21}$$

Let $C(s(\chi, k), \rho)$ be the circle with center $s(\chi, k)$ and radius ρ, $0 < \rho < \frac{\pi}{2\log 2}$. We shall now prove that for σ large enough $\varphi(x, s)$ has precisely one zero inside $C(s(\chi, k))$ for all k and no other zeros.

By definition of $s(\chi, k)$,

$$\frac{1}{2^{2s(\chi,k)}}\frac{l_2(\chi)}{l_1(\chi)} = -1$$

so for $s = s(\chi, k) + z$,

$$\frac{1}{2^{2s}}\frac{l_2(\chi)}{l_1(\chi)} = -\frac{1}{2^{2z}}.$$

Introducing this in (20), we get

$$f(s,\chi) = 1 - \frac{1}{2^{2z}}\left[1+\frac{1}{\left(\frac{3}{2}\right)^{2s}}\frac{l_3(\chi)}{l_2(\chi)}+\frac{1}{2^{2s}}\frac{l_4(\chi)}{l_2(\chi)}+...\right].$$

We estimate the remainder as follows

$$\left| \frac{1}{\left(\frac{3}{2}\right)^{2s}} \frac{l_3(\chi)}{l_2(\chi)} + \frac{1}{2^{2s}} \frac{l_4(\chi)}{l_2(\chi)} + \ldots \right| \leqq \frac{K}{\left(\frac{3}{2}\right)^{2\sigma}}$$

where $K = \frac{2\varphi(k)}{|l_2(\chi)|}$ and φ is the Euler function, k equals 4 for $\Gamma(2)$, 6 for Γ', 12 for $\Gamma^{2'}$.

Let ρ be given with $0 < \rho \leqq \frac{\pi}{2\log 2}$ and let

$$\delta_0 = \min_{|z|=\rho} \left|2^{2z} - 1\right| = \min \left\{ \left|2^{2z} - 1\right| \,\middle|\, |z| \geq \rho, |\operatorname{Im} z| \leqq \frac{\pi}{2\log 2} \right\}.$$

Then for $\delta < \delta_0$

$$\left| \frac{\delta}{2^{2z}} \right| < \left| -\frac{1}{2^{2z}} \right| 1 \text{ for } |z| \geq \rho, \ |\operatorname{Im} z| \leqq \frac{\pi}{2\log 2}.$$

Choose σ_0 such that

$$\frac{K}{\left(\frac{3}{2}\right)^{2\sigma_0}} < \delta_0.$$

Then for $\sigma > \sigma_0$ and $|z| \geq \rho$,

$$|\operatorname{Im} z| \leq \frac{\pi}{2\log 2}$$

$$\left| \frac{1}{2^{2z}} \left[\frac{1}{\left(\frac{3}{2}\right)^{2s}} \frac{l_3(\chi)}{l_2(\chi)} + \frac{1}{2^{2s}} \frac{l_4(\chi)}{l_2(\chi)} + \ldots \right] \right| < \left| 1 - \frac{1}{2^{2z}} \right|.$$

It follows that for $\sigma > \sigma_0 = \sigma_0(\rho)$, $\varphi(\chi, s)$ has no zeros on or outside the circles $C(s(\chi, k), \rho)$.

Also, by Rouché's Theorem, $\varphi(\chi, s)$ has precisely one simple zero inside $C(s(\chi, k), \rho)$ for $\sigma > \sigma_0$.

To obtain a good estimate of σ_0 excluding other zeros than the unstable ones, let $\rho = \frac{\pi}{2\log 2}$.

Then

$$\delta_0 = \min_{|z|=\rho} \left|2^{2z} - 1\right| \geq e^{-\pi\sqrt{2}},$$

and

$$\sigma_0 = \frac{\pi\sqrt{2} + \log K}{2\log\frac{3}{2}}$$

so for $\sigma > \sigma_0$ there are no zeros on or outside the circles $C\left(s(\chi,k), \frac{\pi}{2\log 2}\right)$ and one inside each such circle.

If we replace circles by squares, we can improve this to $\sigma_0 = \frac{\pi+\log K}{2\log\frac{3}{2}}$.

Using the functional equation for $\varphi(\chi, s)$, we obtain the following Theorem about resonances.

Theorem 2 *Let* $\chi(\alpha) = \{a(\alpha)\}$ *for* $\Gamma(2)$, $\chi(\alpha) = \{b(\alpha), c(\alpha)\}$ *for* Γ', $\chi(\alpha) = \{a(\alpha), b(\alpha), c(\alpha)\}$ *for* $\Gamma^{2'}$, $\alpha \in I$, *be a one-parameter family of characters on the group* $\Gamma(2)$, Γ' *or* $\Gamma^{2'}$. *Assume that* $\chi(\alpha_0) \in \mathcal{F} = \{\chi \,|l_1(P) = 0, l_2(P) \neq 0\}$, *where* $P = a$ *for* $\Gamma(2)$, $P = (a,b)$ *for* Γ', $P = (a,b,c)$ *for* $\Gamma^{2'}$.

Then for every ρ, $0 < \rho \leq \frac{\pi}{2\log 2}$ there exists $\sigma_0 = \sigma_0(\rho)$ such that for $\sigma > \sigma_0$ each circle $C(p(\chi,k),\rho)$ contains precisely one simple pole (resonance) of $\varphi(\chi, s)$ and there are no resonances outside these circles. Here $p(\chi, k) = 1 - \sigma(\chi) - i\tau(k)$, where $\sigma(\chi)$ and $\tau(k)$ are defined by (21). The condition on σ is satisfied if $|l_1(P) < 2^{-2\sigma_0}\,|l_2(P)\,|$, which holds for $|\alpha - \alpha_0| < \eta$ where η is to be determined in each case.

Remark 1 *In the case of* $\Gamma(2)$ *there is only one circle of characters which close two cusps and keep one open. This has an unstable point at* $\frac{1}{4}$. *This value of* α *corresponds to a congruence group and is therefore very important.*

In the case of Γ' *a circle of characters starting from* Γ' *with trivial character at* $\alpha = 0$ *has to cross the curve* $l_1(\alpha,\beta) = 0$, *and it is important to identify the points in the space (torus) of parameters* (α,β), *which corresponds to congruence subgroups of* Γ'. *We expect that points of intersection of a character circle with the curve* $l_1(\alpha,\beta) = 0$ *will be congruence points. Other interesting character circles are the maximally stable ones. Such a circle is obtained by setting* $\alpha + \beta = \frac{1}{2}$. *Then*

$$l_1 \equiv 1,\ \chi(X) = e^{2\pi i\alpha},\ \chi(Y) = e^{-2\pi i\alpha},\ \chi(\tilde{S}) = 1.$$

In the case of $\Gamma^{2'}$ *likewise a character circle starting from* $\Gamma^{2'}$ *with trivial character and closing 2 cusps will have to intersect the surface* $l_1(\alpha_1,\alpha_2,\alpha_3) = 0$, *and again we expect the point of intersection to the particularly important and connected with a congruence subgroup. Because the surface* $l_1(\alpha_1,\alpha_2,\alpha_3) = 0$ *separates the parameter space in two disjoint parts, we can not expect to find stable circles. If we set for example*

$$a - b + c = (a + 2b + c) + (a - b - 2c) = \frac{1}{2}$$

we find $a = \frac{1}{3}$, $c = b + \frac{1}{6}$, $l_1(a,b,c) \equiv -1$, $\chi_3(A^3) = \chi_3(B^3) = \chi_3(S^3) = 1$, *so we have maximal stability, but all cusps remain open. This is close to* Γ'.

Setting

$$a - b + c = (a + 2b + c) + (a - b - 2c) = \frac{1}{4}$$

we get $a = \frac{1}{6}$, $c = b + \frac{1}{12}$, $\chi_3(B^3) = \chi_3(S^3) = -1$, *but* $l_1(\alpha, b, c) = 0$ *for* $b = \frac{1}{24}$, $c = \frac{1}{8}$.

This is like the case of $\Gamma(2)$.

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