# Spectral Theory of Laplacians for Hecke Groups with Primitive Character

E. Balslev and A. Venkov

MaPhySto*

Department of Mathematical Sciences

University of Aarhus

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**Abstract**

We study embedded eigenvalues of automorphic Laplacians $A(\bar{\Gamma}_0(N), \chi)$, for Hecke groups with primitive character $\chi$ and develop the corresponding Hecke theory for cusp forms. We prove that $A(\bar{\Gamma}_0(N), \chi)$ with the Hecke operators $T(p)$, $p \nmid N$, determine common eigenfunctions uniquely (multiplicity one theorem). The Hecke operators $U(q)$, $q\,|N$, are proved to be unitary with eigenvalues $\pm 1$, and the continuous spectrum equals the unit circle. From this follows the analogue of Selberg's small eigenvalue conjecture for the exceptional Hecke operators. Utilizing this theory we prove that for a large class of regular character perturbations defined by holomorphic Eisenstein series of weight 2, at least one eigenfunction from each odd eigenspace becomes a resonance function except for possible eigenvalues of the form

$$\lambda_n = \frac{1}{4} + r_n^2,\ r_n = n\frac{\pi}{\log q},\ n \in \mathbb{Z},\ q \text{ is a prime, } q\,|N.$$

This indicates that the Phillips-Sarnak conjecture about violation of the Weyl law holds true, provided the dimensions of eigenspaces remain bounded.

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## Introduction

It was proved by [Se1] that the Laplacian $A(\Gamma)$ for congruence subgroups $\Gamma$ of the modular group $\Gamma_{\mathbb{Z}}$ has an infinite sequence of embedded eigenvalues $\{\lambda_i\}$ satisfying a Weyl law $\#\{\lambda_i \leq \lambda\} \sim \frac{|F|}{4\pi}\lambda$ for $\lambda \to \infty$. Here $|F|$ is the area of the fundamental domain $F$ of the group $\Gamma$, and the eigenvalues $\lambda_i$ are counted according to multiplicity. The same holds true for the Laplacian $A(\Gamma;\chi)$, where $\chi$ is a character on $\Gamma$ and $A(\Gamma;\chi)$ is associated with a congruence subgroup $\Gamma_1$ of $\Gamma$. It is an important question whether this is a characteristic of congruence groups or it may hold also for some non-congruence subgroups of $\Gamma_{\mathbb{Z}}$. To investigate this problem Phillips and Sarnak studied perturbation theory for Laplacians $A(\Gamma)$ with regular perturbations derived from modular forms of weight 4 [P-Sa1] and singular perturbations by characters derived from modular forms of weight 2 [P-Sa2]. Their work on singular perturbations was inspired by work of Wolpert [W1], [W2]. See also [DIPS] for a short version of these ideas and related conjectures. Central to their approach was the application of perturbation theory and in that connection the evaluation of the integral of the product of the Eisenstein series $E_k(s_j)$ at an eigenvalue $\lambda_j = s_j(1-s_j)$ with the first order perturbation $M$ applied to the eigenfunction $v_j$. If this integral $I_k(s_j)$, which we call the Phillips-Sarnak integral, is non-zero for at least one of the Eisenstein series $E_k(s_j)$, then the eigenvalue $\lambda_j$ disappears under the perturbation $aM+\alpha^2 N$ for small $\alpha \neq 0$ and becomes a resonance. This follows from the fact that $\operatorname{Im} \lambda_j''(0)$ is proportional to the sum over $k$ of $|I_k(s_j)|^2$, a fact known as Fermi's Golden Rule. The strategy of Phillips and Sarnak is on the one hand to prove this rule for Laplacians $A(\Gamma)$ and on the other hand to prove that $I_k(s_j) \neq 0$ for some $k$ under certain conditions. For congruence groups with singular character perturbation closing two or more cusps a fundamental difficulty presents itself due to the appearance of new resonances of $A(\Gamma, \alpha)$ for $\alpha \neq 0$, which condense at every point of the continuous spectrum of $A$ as $\alpha \to 0$. These resonances (poles of the $S$-matrix) were discovered by Selberg [Se2] for the group $\Gamma(2)$ with singular character perturbation closing 2 cusps, so we call them the Selberg resonances. Any method of proving that eigenvalues become resonances or remain eigenvalues has to deal with these resonances, which arise from the continuous spectrum of the cusps, which are closed by the perturbation. This makes the problem very difficult in that case. This is also illustrated by the example of $\Gamma_0(p)$ with trivial character, where $p$ is a prime. Here the Riemann surface has two cusps, which are both closed by a singular perturbation defined by a holomorphic Eisenstein series of weight 2. The Phillips-Sarnak integral is non-zero for all new odd cusp forms, but the spectra of the perturbed operators are purely discrete, condensing in the limit on the original continuous spectrum. Se also Remark 8.9 for another example. In the case of regular perturbations derived from cusp forms it is not too difficult to prove Fermi's Golden Rule, but it is very hard to prove that the integral is not zero.

We consider instead as our basic operator $A(\Gamma, \chi)$, where $\Gamma = \bar{\Gamma}_0(N)$ is the Hecke group of level $N$ and $\chi$ is the one-dimensional unitary representation of $\bar{\Gamma}_0(N)$ defined by a real, even, primitive Dirichlet character mod $N$. These characters are fundamental in number theory, since they are related to real quadratic fields. We study basic spectral properties of the operators $A(\bar{\Gamma}_0(N), \chi)$ and related Hecke theory. We prove that the multiplicity one conjecture holds for $A(\bar{\Gamma}_0(N),\chi)$ and the set of non-exceptional Hecke operators $T(p)$, $p \nmid N$. The exceptional Hecke operators $U(q)$, $q\,|N$, are shown to be unitary and have only the eigenvalues $\pm 1$. This implies that the analogue of Selberg's small eigenvalue conjecture is valid for the exceptional Hecke operators $U(q)$ with primitive character. Moreover, we prove that the Hecke $L$-functions are regular and non-zero on the boundary of the critical strip.

Based on this theory we develop the perturbation theory for embedded eigenvalues of the operators $A(\bar{\Gamma}_0(N),\chi)$. We prove that for a class of regular perturbations defined by holomorphic Eisenstein series of weight 2 the Phillips-Sarnak integral $I_j(s_j) \neq 0$ for all odd Hecke eigenfunctions of $A(\bar{\Gamma}_0(N),\chi)$ with eigenvalues $\lambda_j = s_j(1-s_j)$ except if $s_j$ takes one of the values $s(n,q) = \frac{1}{2} + in\frac{\pi}{\log q}$, $q$ a prime, $q\,|N$.

Consequently, at least one eigenfunction from each odd eigenspace of $A(\bar{\Gamma}_0(N),\chi)$ with $s_j \neq s(n,q)$ for all $n$ and $q\,|N$ becomes a resonance function under this perturbation, the corresponding eigenvalue giving rise to a resonance. We notice that in the case of $A(\bar{\Gamma}_0(N))$ with trivial character the embedded eigenvalues are discrete in the space of new forms, whereas in the case of $A(\bar{\Gamma}_0(N))$ with primitive character they are genuinely embedded, since both cusp forms and Eisenstein series are new.

We now describe in more detail the contents of the paper. It consists of the following sections:

1. The group $\bar{\Gamma}_0(N)$ with real, primitive character $\chi$

2. The Eisenstein series

3. The discrete spectrum of the automorphic Laplacian $A(\bar{\Gamma}_0(N), \chi)$

4. Hecke theory for Maass cusp forms

5. Non-vanishing of Hecke $L$-functions

6. The form $\omega(z)$

7. The Phillips-Sarnak integral

8. Perturbation of embedded eigenvalues

In Section 1 we introduce the Hecke groups $\bar{\Gamma}_0(N)$ to be considered and define their primitive character mod $N$. We consider precisely those sequences of values of the level $N$ for which such a character exists: 1) $N = N_1 \equiv 1 \bmod 4$, 2) $N = 4N_2$, $N_2 \equiv 3 \bmod 4$, 3) $N = 4N_3$, $N_3 \equiv 2 \bmod 4$, where $N_1$, $N_2$, $N_3$ are square-free integers. The Riemann surfaces associated with the groups $\bar{\Gamma}_0(N)$ have $d(N)$ cusps, where $d(N)$ is the number of divisors in $N$.

The primitive character $\chi$ keeps all cusps open in case 1 and closes one third of the cusps in case 2 and one half of the cusps in case 3 (Theorem 1.1). Closing of a cusp means that the continuous spectrum associated with that cusp disappears.

In Section 2 we discuss the Eisenstein series, and in Section 3 we prove the Weyl law for eigenvalues of $A(\bar{\Gamma}_0(N), \chi)$ (Theorem 3.6), using the factorization formula for the Selberg zeta function [V1] and Huxley's explicit formula for the scattering matrix of $\bar{\Gamma}_1(N)$ [Hu].

In Section 4 we develop the Hecke theory for Maass wave cusp forms of $A(\bar{\Gamma}_0(N), \chi)$. We prove that there is a unique common eigenfunction of $A(\bar{\Gamma}_0(N), \chi)$ and the Hecke operators $T(p)$, $p \nmid N$, with given eigenvalues and first Fourier coefficient 1 (the multiplicity one theorem) (Theorem 4.2). The exceptional Hecke operator $U(q)$, $q \,|N$ , are unitary (Theorem 4.1) and have only the eigenvalues $\pm 1$ (Theorem 4.3). This is in contrast with the case of $A(\bar{\Gamma}_0(N))$ with trivial character, where the operators $U(q)$ are not normal in the whole Hilbert space and normal but not unitary in the space of new forms.

In Section 5 we study the Dirichlet $L$-series $L(s; v_j)$ and $L(s, \hat{v}_j)$ associated with the eigenfunctions $v_j$ of $A(\bar{\Gamma}_0(N), \chi)$ and their conjugates $\hat{v}_j$. They have an Euler product representation (Theorem 5.1) and analytic continuation to all of $\mathbb{C}$, connected by a functional equation (Theorem 5.2). Based on this together with a general criterion proved in [M-M] (Lemma 5.3) we prove that $L(s; v_j)$ and $L(s; \hat{v}_j)$ are regular and non-zero on the boundary of the critical strip (Theorem 5.4).

In Section 6 we introduce perturbation of $A(\bar{\Gamma}_0(N), \chi)$ by characters $\chi(\alpha)$. This is equivalent to perturbation by a family of operators $\alpha M + \alpha^2 N$, where $M = -4\pi i y^2 \left(\omega_1 \frac{\partial}{\partial x} - \omega_2 \frac{\partial}{\partial y}\right)$, $N = 4\pi^2 \left|\omega\right|^2$ and $\omega = \omega(z) = \omega_1 + \omega_2$ is a modular form of weight 2 derived from the classical holomorphic Eisenstein series $E_2(z)$. The basic result here is in each of the cases 2) and 3) the existence of such a form $\omega(z)$, which keeps the same cusps open and closed, which are already open and closed by the primitive character $\chi$ (Theorem 6.1). In Theorem 6.2 an explicit $(k-1)$-parameter family of such forms $\omega$ is constructed, where $k$ is the number of prime factors in $N_2$ or $N_3$. This makes the perturbation defined by $\omega$ regular relative to $A(\bar{\Gamma}_0(N))$ and thereby accessible to analysis of embedded eigenvalues. In case 1) this is not possible, and the remaining part of the paper deals with the cases 2) and 3).

In Section 7 we prove for this class of perturbations, using the non-vanishing of the Dirichlet $L$-series for eigenfunctions, that for some $k$ the Phillips-Sarnak integral $I_k\left(\frac{1}{2}+ir_j\right)$ is different from zero for all odd eigenfunction except for $r_j = n\frac{\pi}{\log q}$, $q = 2$ or $q$ an odd prime, $q\,|N$, $n \in \mathbb{Z}$ (Theorem 7.1).

Section 8 contains the general perturbation theory (Theorem 8.4), which allows to conclude from the non-vanishing of the Phillips-Sarnak integral that at least one eigenfunction from each odd eigenspace turns into a resonance function (Theorem 8.5). Using this result, the proportion of odd eigenfunctions which leave as resonance functions can be estimated depending on the growth of the dimensions $m(s_j)$ of the eigenspaces (Theorem 8.6). The estimate $m(\lambda_j) << \frac{\lambda_j}{\log \lambda_j}$, which can be obtained using Selberg's trace formula, gives at least the proportion $c\sqrt{\lambda_j}\log\lambda_j$, while boundedness $m(\lambda_j) \leq m$, which has been conjectured, implies that a positive proportion leaves (Corollary 8.7).

The operator $M$, which is derived from the real part of the form $\omega$, maps odd functions into even functions and even into odd. Therefore, the Phillips-Sarnak integral is always zero for even eigenfunctions, and it remains an open question whether some of these leave under this perturbation.

There is another perturbation $\alpha\tilde{M} + \alpha^2 N$, where $\tilde{M}$ is derived from $\mathrm{Im}\,\omega$ and $\tilde{M}$ preserves parity. This perturbation, however, is completely different. It is singular, but in some sense very simple. Although the Phillips-Sarnak integral is non-zero for all even Hecke eigenfunctions, it does not follow that corresponding eigenvalues give rise to resonances. Quite the contrary happens. All eigenvalues and resonances remain constant, because the Laplacians $L(\alpha)$ are conjugate to $L$ via multiplication by an automorphic phase function (Remark 8.8).

The proof that the Phillips-Sarnak integral is not zero utilizes strong arithmetical properties based on Hecke theory and is specific for the operators $A(\Gamma_0(N),\chi)$. General perturbation theory makes it possible, however, to draw some conclusions about the eigenvalues more globally (Remark 8.9). Thus, eigenvalues of $A(\Gamma_0(N),\chi)$ with odd eigenfunctions which leave the spectrum for $\alpha \neq 0$ can then only become eigenvalues for isolated values of $\alpha \in \left(-\frac{1}{2},\frac{1}{2}\right)$.

This preprint is a revised version of previous preprints entitled "The Phillips-Sarnak conjecture for Hecke groups with primitive character" and "The Phillips-Sarnak conjecture for $\Gamma_0(8)$ with primitive character". Section 4 gives a considerably expanded account of the Hecke theory, which was incomplete in the previous versions. In Section 8 we give the correct version of Theorems 8.5 and 8.4 from the previous preprints.

# 1 The group $\bar{\Gamma}_0(N)$ with primitive character

We consider the Hecke congruence group $\bar{\Gamma}_0(N)$ together with its one-dimensional unitary representation $\hat{\chi}$, also called a character of the group. We are interested here only in arithmetically important characters, coming from real primitive Dirichlet characters $\chi$

mod $N$. We have, following Hecke,

$$\chi(\gamma) = \chi_N(n), \ \gamma = \left( \begin{array}{cc} a & b \\ Nc & n \end{array} \right) \in \bar{\Gamma}_0(N).\ an - bcN = 1.$$

It is well-known (see [D]) that the real primitive characters mod $N = |d|$ are identical with the symbols

$$\left(\frac{d}{n}\right)$$

where $d$ is a product of relatively prime factors of the form

$$-4, 8, -8, (-1)^{(p-1)/2}p, \ p > 2. \tag{1.1}$$

We have

$$\left(\frac{d_1 d_2}{n}\right) = \left(\frac{d_1}{n}\right)\left(\frac{d_2}{n}\right)$$

provided $(d_1, d_2) = 1$.

By definition

$$\left(\frac{-4}{n}\right) = \chi_4(n) = \begin{cases} 1, & n \equiv 1 \mod 4 \\ -1, & n \equiv 3 \mod 4 \\ 0, & \text{otherwise} \end{cases}$$

$$\left(\frac{8}{n}\right) = \chi_8(n) = \begin{cases} 1, & n \equiv \pm 1 \mod 8 \\ -1, & n \equiv \pm 3 \mod 8 \\ 0, & \text{otherwise} \end{cases}$$

$$\left(\frac{-8}{n}\right) = \chi_4(n)\chi_8(n).$$

We have also, by the law of quadratic reciprocity for the Legendre symbol

$$\left(\frac{n}{p}\right) = \left(\frac{p'}{n}\right) \text{ where } p' = (-1)^{(p-1)/2}p \tag{1.2}$$

provided $n$ is an odd square-free integer. By Kronecker's extension of Legendre's symbol we have

$$\left(\frac{p'}{2}\right) = \left(\frac{2}{p}\right)$$

or, more generally,

$$\left(\frac{p'}{2^k m}\right) = \left(\frac{2^k m}{p}\right) \quad k \in \mathbb{Z},\ k \geq 1.$$

Thus, relation (1.2) holds whether $n$ is odd or even. It holds also in the more general form

$$\left(\frac{n}{l}\right) = \left(\frac{l'}{n}\right)$$

where

$$l' = (-1)^{(l-1)/2} l$$

and $l = p_1 p_2, ...,$ that is if $l$ is any square-free odd positive integer, and $l' = p_1' p_2' ...$ (see (1.2)).

Finally, we recall that

$$\left(\frac{n}{p}\right) = \begin{cases} +1 & \text{if} \quad nRp \\ -1 & \text{if} \quad nNp \end{cases}$$

for $p$ an odd prime and $(p, n) = 1$. We also define

$$\left(\frac{n}{p}\right) = 0 \quad p\,|n\,.$$

Here by definition, $nRp$ just means that there exists an integer $x$, such that $x^2 \equiv n$ mod $p$, in the case of $nNp$ such integer does not exist.

For odd $n$ we also have

$$\left(\frac{2}{n}\right) = \left(\frac{8}{n}\right), \left(\frac{-2}{n}\right) = \left(\frac{-8}{n}\right).$$

This explicit definition of the symbol $\left(\frac{d}{n}\right)$ is important in order to calculate the values of the character $\chi$ on the parabolic generators of the Hecke group.

The numbers

$$(-1)^{(p-1)/2}p, \ p > 2,$$

are all congruent to 1 mod 4, and the products of relatively prime factors, i.e. distinct factors, each of this form, comprise all square-free integers, positive and negative, that are congruent to 1 mod 4. In addition, we get all such numbers, multiplied by -4, that is, all numbers $4N$, where $N$ is square free and congruent to 3 mod 4. Finally, we get all such numbers, multiplied by $\pm 8$, which is equivalent to saying all numbers $4N$ where $N$ is congruent to 2 mod 4 (see [D]).

By this we have obtained all real primitive Dirichlet characters. But we need only even characters here, since we consider the projective Hecke group $\bar{\Gamma}_0(N) \subset PSL(2, \mathbb{R})$, that means we identify two matrices

$$\begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \sim \begin{pmatrix} -a & -b \\ -Nc & -d \end{pmatrix}$$

and $\chi(d) = \chi(-d)$. According to this classification of primitive even real characters, we will consider three different choices of $N$ in $\bar{\Gamma}_0(N)$.

1) $\bar{\Gamma}_0(N_1)$, $N_1 = \Pi_{p>2}(-1)^{(p-1)/2}p$, $N_1 > 0$.

That means $N_1$ is any positive square-free integer and $N_1 \equiv 1 \bmod 4$.

2) We take $M_2^{'} = \Pi_{p>2}(-1)^{(p-1)/2}p$, $M_2^{'} < 0$, and we consider $\bar{\Gamma}_0(4N_2)$, where $N_2 = -M_2^{'}$. That means, $N_2$ is any square-free positive integer $N_2 \equiv 3 \bmod 4$.

3) We take $M_3^{'} = \Pi_{p>2}(-1)^{(p-1)/2}p$, and we define $N_3 = 2\left|M_3^{'}\right|$.

We have, $N_3$ is any square-free positive integer, and $N_3 \equiv 2 \bmod 4$.

Then we consider $\bar{\Gamma}_0(4N_3)$.

We now recall the basic properties of the group $\bar{\Gamma}_0(N)$, having in mind our choices 1), 2), 3) for $N$. It is well-known [Sh] that for any $N$

$$\left[\bar{\Gamma}_0(1) : \bar{\Gamma}_0(N)\right] = N \prod_{p|N}(1 + 1/p) = m$$

$$n_2 = \begin{cases} 0 & 4\,|N \\ \prod_{p|N}\left(1 + \left(\frac{-1}{p}\right)\right) & \text{otherwise,} \end{cases} \quad \left(\frac{-1}{p}\right) = \begin{cases} 0, & p = 2 \\ 1, & p \equiv 1 \mod 4 \\ -1, & p \equiv 3 \mod 4 \end{cases}$$

$$n_3 = \begin{cases} 0 & 9\,|N \\ \prod_{p|N}\left(1 + \left(\frac{-3}{P}\right)\right) & \text{otherwise,} \end{cases} \quad \left(\frac{-3}{p}\right) = \begin{cases} 0, & p = 3 \\ 1, & p \equiv 1 \mod 3 \\ -1, & p \equiv 2 \mod 3 \end{cases}$$

$$h = \sum_{\substack{d|N \\ d>0}} \varphi((d, N/d))$$

$$g = 1 + \frac{m}{12} - \frac{n_2}{4} - \frac{n_3}{3} - \frac{h}{2}.$$

Here $m$ is the index of $\bar{\Gamma}_0(N)$ in the modular group, $n_2$ is the number of $\bar{\Gamma}_0(N)$ inequivalent elliptic points of order 2 ($n_3$, of order 3), $h$ is the number of $\bar{\Gamma}_0(N)$ inequivalent cusps, $g$ is the genus, $\varphi(n)$ is the Euler function, $\varphi(1) = 1$, $\varphi(n) = n(1-1/p_1)(1-1/p_2)...(1-1/p_k)$, $n = p_1^{l_1} p_2^{l_2} ... p_k^{l_k}$.

For our purposes it is important to see the parabolic generators of our groups and corresponding cusps of the canonical fundamental domains.

Case 1). For $\bar{\Gamma}_0(N_1)$ we have $N_1 = p_1 p_2 ... p_k$, a product of odd different primes. Then $h_1 = \sum_{\substack{d|N_1 \\ d>0}} \varphi((d, N_1/d)) = d(N_1)$, the number of positive divisors of the positive integer $N_1$. Let $\sigma_d \in \Gamma_0(1)$, $d\,|N$, $d > 0$

$$\sigma_d = \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix}. \tag{1.3}$$

We can take a complete set of inequivalent cusps for $\bar{\Gamma}_0(N_1)$ the set of points $z_d = 1/d \in \mathbb{R}$, $d\,|N_1$, $d > 0$. We define then $\bar{\Gamma}_d = \left\{\gamma \in \bar{\Gamma}_0(N_1)\,|\gamma z_d = z_d\right\}$. Let $S_d$ be the generator of $\bar{\Gamma}_d$. We can find $S_d$ from the condition $S_d^{'} \in \bar{\Gamma}_0(N_1)$

$$S_d^{'} = \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix} \begin{pmatrix} 1 & m_d^{'} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -d & 1 \end{pmatrix} = \begin{pmatrix} 1 - dm_d^{'} & m_d^{'} \\ -d^2 m_d^{'} & 1 + dm_d^{'} \end{pmatrix} \tag{1.4}$$

where we have to take the minimum $\left|m_d^{'}\right|$ (width). That gives $m_d = N_1/d$, and we obtain

$$S_d = \begin{pmatrix} 1 - N_1 & N_1/d \\ -dN_1 & 1 + N_1 \end{pmatrix},\ d > 0,\ d\,|N_1\ . \tag{1.5}$$

Since our character $\chi = \chi_{N_1}$ is mod $N_1$, we obtain

$$\chi_{N_1}(S_d) = \chi_{N_1}(1 + N_1) = \chi_{N_1}(1) = 1 \text{ for any } d\,|N_1\ ,\ d > 0. \tag{1.6}$$

Case 2) $\bar{\Gamma}_0(4N_2)$, $N_2$ is the product of different odd primes. Then we have

$$h_2 = \sum_{\substack{d|4N_2 \\ d>0}} \varphi((d, 4N_2/d)).$$

Since $\varphi(2) = 1$ we obtain $h_2 = d(4N_2)$, the divisor function of $4N_2$. For any $d\,|4N_2$ we introduce the matrix $\sigma_d \in \Gamma_0(1)$ (1.3). Again we take as a complete set of inequivalent cusps for $\Gamma_0(4N_2)$ the set of points $z_d = 1/d \in \mathbb{R}$, $d\,|4N_2$, $d > 0$. We define in analogy with the first case

$$\bar{\Gamma}_d = \left\{\gamma \in \bar{\Gamma}_0(4N_2)\,|\gamma z_d = z_d\right\} \tag{1.7}$$

and for the generator $S_d = S_d^{(2)}$ we have

$$S_d = \begin{pmatrix} 1 - dm_d & m_d \\ -d^2 m_d & 1 + dm_d \end{pmatrix} \tag{1.8}$$

where for $m_d$ we have the minimum $\left|m_d^{'}\right|$, when $S_d^{'} \in \Gamma_0(4N_2)$ (see (1.4)). We have three possibilities now. In case (i) $d\,|N_2$, then we have $4N_2\,|d^2 m_d$ and $m_d = 4N_2/d$. We obtain

$$S_d = \begin{pmatrix} 1 - 4N_2 & 4N_2/d \\ -4N_2 d & 1 + 4N_2 \end{pmatrix},\ d\,|N_2\,,\ d > 0. \tag{1.9}$$

In case (ii) $d = 2d_1$, $d_1\,|N_2$, then we have $4N_2\,|d^2 m_d$ and $m_d = N_2/d_1$. We get

$$S_d = \begin{pmatrix} 1 - 2N_2 & N_2/d_1 \\ -4d_1 N_2 & 1 + 2N_2 \end{pmatrix},\ d_1\,|N_2\,,\ d_1 > 0,\ d = 2d_1. \tag{1.10}$$

Finally in case (iii) we have $d = 4d_2$, $d_2\,|N_2$. Then $4N_2\,|16d_2^2 m_d$, and we get $m_d = N_2/d_2$. We obtain

$$S_d = \begin{pmatrix} 1 - 4N_2 & N_2/d_2 \\ -16d_2 N_2 & 1 + 4N_2 \end{pmatrix},\ d_2\,|N_2\,,\ d_2 > 0,\ d = 4d_2. \tag{1.11}$$

Case 3) For $\bar{\Gamma}_0(4N_3)$ we have $N_3 = 2n$, $n = p_1, p_2 ... p_k$ is the product of different odd primes. We get

$$h_3 = \sum_{\substack{d|4N_3 \\ d>0}} \varphi((d, 4N_3/d)) = d(4N_3). \tag{1.12}$$

We take as the set of all $\bar{\Gamma}_0(4N_3)$ inequivalent cusps the set of points

$$z_d = 1/d,\ d\,|4N_3\ ,\ d > 0, \tag{1.13}$$

then we define in analogy to (1.7)

$$\bar{\Gamma}_d = \left\{\gamma \in \bar{\Gamma}_0(4N_3)\,|\gamma z_d = z_d\right\}$$

and its generator $S_d$, given by (1.8) with $d\,|4N_3,\ d > 0$. Similar to (1.4) for $m_d$ we take the minimum $\left|m_d'\right|$, when $S_d' \in \bar{\Gamma}_0(4N_3)$. We have 4 possibilities now,

$$\text{(i) } d\,|n \ \ \text{(ii) } d = 2d_1, d_1\,|n \ \ \text{(iii) } d = 4d_2, d_2\,|n \ \ \text{(iv) } d = 8d_3, d_3\,|n\ .$$

Analogous to (1.8), (1.10) we obtain

$$S_d = \begin{cases} \begin{pmatrix} 1-4N_3 & 4N_3/d \\ -8dn & 1+4N_3 \end{pmatrix} & d\,|n\ ,\ d>0 \\ \begin{pmatrix} 1-2N_3 & N_3/d_1 \\ -4d_1N_3 & 1+2N_3 \end{pmatrix} & d = 2d_1,\ d_1\,|n\ ,\ d_1 > 0 \\ \begin{pmatrix} 1-2N_3 & n/d_2 \\ -16d_2n & 1+2N_3 \end{pmatrix} & d = 4d_2,\ d_2\,|n\ ,\ d_2 > 0 \\ \begin{pmatrix} 1-4N_3 & n/d_3 \\ -64d_2n & 1+4N_3 \end{pmatrix} & d = 8d_3,\ d_3\,|n\ ,\ d_3 > 0. \end{cases} \tag{1.14}$$

In (1.6) we calculated the values of $\chi_{N_1}(S_d),\ d\,|N_1,\ d > 0$. Now we do that for all other cases. We have in Case 2) of $4N_2$ with either (i) $d\,|N_2$ or (iii) $d = 4d_2,\ d_2\,|N_2$

$$\chi_{4N_2}(S_d) = \chi_{4N_2}(1+4N_2) = \chi_{4N_2}(1) = 1,\ d\,|N_2,\ d > 0 \tag{1.15}$$

$$\chi_{4N_2}(S_d) = 1,\; d = 4d_2,\; d_2\,|N_2\;,\; d_2 > 0 \text{ (see (1.11))} \tag{1.16}$$

For the case (ii) $d = 2d_1,\; d_1\,|N_2\,,\; d_1 > 0$ we have to calculate

$$\chi_{4N_2}(S_d) = \chi_{4N_2}(1+2N_2) \text{ (see (1.10))}. \tag{1.17}$$

We obtain

$$\begin{aligned}\chi_{4N_2}(1+2N_2) &= \left(\frac{4N_2}{1+2N_2}\right) = \left(\frac{-4}{1+2N_2}\right)\left(\frac{-N_2}{1+2N_2}\right) \\ &= \chi_4(1+2N_2)\left(\frac{M_2'}{1+2N_2}\right) \\ &= \chi_4(1+2N_2)\left(\frac{1+2N_2}{N_2}\right) = \chi_4(1+2N_2).\end{aligned} \tag{1.18}$$

Since $N_2 \equiv 3 \bmod 4$ we get $\chi_4(1+2N_2) = -1$ and then

$$\chi_{4N_2}(S_d) = \chi_{4N_2}(1+2N_2) = -1,\quad d = 2d_1,\; d_1\,|N_2\;,\; d_1 > 0. \tag{1.19}$$

In Case 3) $\bar{\Gamma}_0(4N_3)$ we have (see (1.14))

$$\chi_{\substack{4N_3\\ d|n}}(S_d) = \chi_{\substack{4N_3\\ d_3|n}}(S_{8d_3}) = 1 \tag{1.20}$$

$$\chi_{\substack{4N_3\\ d_1|n}}(S_{2d_1}) = \chi_{\substack{4N_3\\ d_2|n}}(S_{4d_2}) = \chi_{4N_3}(1+2N_3) = -1. \tag{1.21}$$

From the basic properties of the symbol $\left(\frac{d}{n}\right)$ (see the beginning of this section) follows

$$\chi_{4N_3}(1+2N_3) = \left(\frac{4N_3}{1+2N_3}\right) = \begin{cases} \chi_8(1+2N_3)\left(\frac{1+2N_3}{M_3'}\right), & M_3' > 0 \\ \chi_8(1+2N_3)\chi_4(1+2N_3)\left(\frac{1+2N_3}{M_3'}\right), & M_3' < 0 \end{cases}$$

where $M_3' = \prod\limits_{p>2}(-1)^{(p-1)/2}p$. Then we have

$$\left(\frac{1+2N_3}{M_3'}\right) = \left(\frac{M_3}{1+2N_3}\right) = 1$$

where $M_3 = \prod_{p>2} p$, corresponding to the product $M_3'$. Next $2N_3 = 4\left|M_3'\right|$ and we have

$$\chi_4(1 + 2N_3) = 1. \tag{1.22}$$

Since $N_3 \equiv 2 \bmod 4$ we have finally

$$\chi_8(1 + 2N_3) = -1. \tag{1.23}$$

We have proved the following theorem.

**Theorem 1.1** *1. For the group $\bar{\Gamma}_0(N_1)$, $N_1$ a square-free positive integer, $N_1 \equiv 1 \bmod 4$, and its arithmetical character $\hat{\chi}_{N_1} = \left(\frac{N_1}{\cdot}\right)$ we have a complete system of $\bar{\Gamma}_0(N_1)$ inequivalent cusps $z_d$ given by $z_d = \frac{1}{d}$, $d\,|N_1$, $d > 0$. The system of all parabolic generators $S_d$ is given by (1.5). Then all the above-mentioned cusps are open relative to this character. That precisely means that $\chi_{N_1}(S_d) = 1$. We are also saying in this case that the character $\chi$ is regular for the group $\bar{\Gamma}_0(N_1)$ (see Section 2).*

*2. For the group $\bar{\Gamma}_0(4N_2)$, $N_2$ a square-free positive integer, $N_2 \equiv 3 \bmod 4$, and its arithmetical character $\chi_{4N_2} = \left(\frac{4N_2}{\cdot}\right)$ we have the complete system of $\bar{\Gamma}_0(4N_2)$ inequivalent cusps $z_d = 1/d$, $d\,|4N_2$, $d > 0$. The system of all parabolic generators $S_d$ is given by (1.9), (1.10), (1.11). The character $\chi_{4N_2}$ is singular for the group $\bar{\Gamma}_0(4N_2)$, two third of the cusps $z_d$ are open and one third is closed by the character $\chi$. That precisely means, that for open cusps $z_d$, $d\,|N_2$, $d > 0$, or $d = 4d_2$, $d_2 > 0$, $d_2\,|N_2$, we have $\chi(S_d) = 1$ (see (1.15), (1.16)). For closed cusps $z_d$, $d = 2d_1$, $d_1 > 0$, $d_1\,|N_2$, $\chi(S_d) = -1$ (see (1.19)).*

*3. For the group* $\bar{\Gamma}_0(4N_3)$, $N_3$ *a square-free positive integer,* $N_3 \equiv 2 \bmod 4$, *and its arithmetical character* $\chi_{4N_3} = \left(\frac{4N_3}{\cdot}\right)$ *we have the complete system of* $\bar{\Gamma}_0(4N_3)$ *inequivalent cusps* $z_d = 1/d,\ d\,|4N_3\,,\ d > 0$. *The system of all parabolic generators* $S_d$ *is given by* (1.14). *The character* $\chi_{4N_3}$ *is singular for the group* $\bar{\Gamma}_0(4N_3)$ *with half of the cusps open and the other half closed. The open cusps are* $z_d,\ d\,|n\,,\ d > 0$ $(N_3 = 2n)$, *or* $d = 8d_3,\ d_3\,|n\,,\ d_3 > 0$ *(see* (1.20)*). The closed cusps are* $z_d,\ d = 2d_1,$ $d_1\,|n,\ d_1 > 0$, *or* $d = 4d_2,\ d_2\,|n\,,\ d_2 > 0$ *(see* (1.21), (1.22), (1.23)*).*

## 2 The Eisenstein Series

We recall the main points of the spectral theory of the automorphic Laplacian on the hyperbolic plane, which we need in this paper (see [Se1], [He], [BV], [V2]).

Let $H$ be the hyperbolic plane. We consider $H = \{z \in \mathbb{C}\,|\,z = x + iy\}$ as the upper half-plane of $\mathbb{C}$ with the Poincaré metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

Let $\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ be the Laplacian associated with the metric $ds^2$. Then let $\Gamma$ be a cofinite group of isometries on $H$ and $\chi$ a one-dimensional unitary representation (character) of $\Gamma$. We define the automorphic Laplacian $A(\Gamma;\chi)$ in the Hilbert space $\mathcal{H}(\Gamma)$ of complex-valued functions $f$, which are $(\Gamma;\chi)$ automorphic, i.e. $f(\gamma z) = \chi(\gamma)f(z)$ for any $\gamma \in \Gamma$, $z \in H$, and which satisfy

$$\|f\|^2 = \int\limits_F |f(z)|^2\, d\mu(z) < \infty.$$

It is clear that $\mathcal{H}(\Gamma) = L_2(F; d\mu)$, when $F$ is given. The linear operator $A(\Gamma;\chi)$ is defined on the smooth $(\Gamma;\chi)$ automorphic functions $f \in L_2(F; d\mu)$ by the formula

$$A(\Gamma;\chi)f = -\Delta f.$$

We identify $A(\Gamma;\chi)$ with the restriction $A_F(\Gamma;\chi)$ of $A(\Gamma;\chi)$ to the space of functions $f\,|_F$, where $f$ runs over all smooth $(\Gamma,\chi)$ automorphic functions $f$. The closure of $A(\Gamma;\chi)$ in $\mathcal{H}_\Gamma$ is a selfadjoint, non-negative operator, also denoted by $A(\Gamma;\chi)$.

We recall that the character $\chi$ is regular in the cusps $z_j$ of the fundamental domain $F$ if $\chi(S_j) = 1$ and $S_j$ is the generator of a parabolic subgroup $\Gamma_j \subset \Gamma$, which fixes the cusp $z_j$. Otherwise $\chi(S_j) \neq 1$, and $\chi$ is singular in $z_j$. It is clear that this property of the character does not depend on the choice of fundamental domain, since in equivalent cusps the character has the same values (that means $\chi(S_j) = \chi(\tilde{S}_j)$, and $S_j, \tilde{S}_j$ correspond to equivalent cusps).

The total degree $k(\Gamma;\chi)$ of singularity of $\chi$ relative to $\Gamma$ is equal to the number of all pairwise non-equivalent cusps of $F$, in which $\chi$ is singular. If $\Gamma$ is non-compact, which is the only case we consider, and the representation $\chi$ is singular, i.e. $h > k(\Gamma;\chi) \geq 1$, then the operator $A(\Gamma;\chi)$ has an absolutely continuous spectrum $\{\lambda \in [1/4, \infty)\}$ of multiplicity $h - k(\Gamma;\chi)$, where $h$ is the number of all inequivalent cusp of $F$. In other words, the multiplicity $r(\Gamma;\chi)$ of the continuous spectrum is equal to the number of inequivalent cusps where $\chi$ is regular, $r(\Gamma;\chi) = h - k(\Gamma;\chi)$.

The continuous spectrum of the operator $A(\Gamma;\chi)$ is related to the generalized eigenfunctions of $A(\Gamma;\chi)$, which are obtained by the analytic continuation of Eisenstein series. We define this as follows. For each cusp $z_j$ of the fundamental domain $F$, in which the representation $\chi$ is regular, we consider again the parabolic subgroup $\Gamma_j \subset \Gamma$, $\Gamma_j = \{\gamma \in \Gamma \,|\, \gamma z_j = z_j\}$. $\Gamma_j$ is an infinite cyclic subgroup of $\Gamma$, generated by a certain parabolic generator $S_j, \chi(S_j) = 1$.

There exists an element $g_j \in PSL(2,\mathbb{R})$ such that

$$g_j \infty = z_j,\; g_j^{-1} S_j g_j z = S_\infty z = z + 1$$

for all $z \in H$. Let $y(z)$ denote $\operatorname{Im} z$. Then the non-holomorphic Eisenstein (or Eisenstein-Maass) series, is given by

$$E_j(z; s; \Gamma; \chi) = \sum_{\gamma \in \Gamma_j \backslash \Gamma} y^s(g_j^{-1}\gamma z)\overline{\chi(\gamma)}. \tag{2.1}$$

Here $\chi$ is the complex conjugate of $\chi$, $\gamma$ is a coset $\Gamma_j \gamma$ of $\Gamma$ with respect to $\Gamma_j$. The series is absolutely convergent for $\operatorname{Re} s > 1$, and there exists an analytic continuation to the whole complex plane as a meromorphic function of $s$. We have a system of $r(\Gamma; \chi)$ functions given by (2.1). For $s = 1/2 + it$, $t \in \mathbb{R}$, they constitute the full system of generalized eigenfunctions of the continuous spectrum of the operator $A(\Gamma; \chi)$.

We recall the definition of the automorphic scattering matrix. We have for

$1 \leqq \alpha, \beta \leqq r(\Gamma; \chi)$

$$E_\alpha(g_\beta z; s; \Gamma; \chi) = \sum_{n=-\infty}^{\infty} a_n(y; s; \Gamma; \chi) e^{2\pi i n \chi}. \tag{2.2}$$

This function is periodic under $z \to z + 1$, and moreover

$$a_0(y; s; \Gamma; \chi) = \delta_{\alpha\beta} y^s + \varphi_{\alpha\beta}(s; \Gamma; \chi) y^{1-s}, \tag{2.3}$$

$z = x + iy \in H$, where

$$\delta_{\alpha\beta} = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}.$$

The matrix $\phi(s;\Gamma;\chi) = \{\varphi_{\alpha\beta}(s;\Gamma;\chi)\}$ which is of the order $r(\Gamma;\chi)$ is called the automorphic scattering matrix. It is well-known that $\phi(s;\Gamma;\chi)$ is meromorphic in $s \in \mathbb{C}$ and holomorphic in the line $\operatorname{Re} s = 1/2$ and satisfies the functional equation

$$\phi(s;\Gamma;\chi)\phi(1-s;\Gamma;\chi) = I_r, \tag{2.4}$$

where $I_r$ is the $r(\Gamma;\chi) \times r(\Gamma;\chi)$ identity matrix. The matrix $\phi(s;\Gamma;\chi)$ is important for establishing the analytic continuation and the functional equation for the Eisenstein series given by

$$E_\alpha(z;1-s;\Gamma;\chi) = \sum_{\beta=1}^{r} \varphi_{\alpha\beta}(1-s;\Gamma;\chi)E_\beta(z;s;\Gamma;\chi) \tag{2.5}$$

$1 \leqq \alpha \leqq r = r(\Gamma;\chi)$.

We make now more precise the formulas (2.4), (2.5). We have

$$\begin{aligned} E_\alpha(g_\beta z;s;\Gamma;\chi) &= \delta_{\alpha\beta}y^s + \varphi_{\alpha\beta}(s;\Gamma;\chi)y^{1-s} \\ &\quad + \sqrt{y}\sum_{n\neq 0} \varphi_{\alpha\beta n}(s;\Gamma;\chi)K_{s-1/2}(2\pi\,|n|\,y)e^{2\pi i n \chi} \end{aligned} \tag{2.6}$$

where $K_{s-1/2}(y)$ is the McDonald-Bessel function. This expression is obtained from the differential equation $\Delta f + 1(1-s)f = 0$ by separation of variables in the strip $-1/2 \leq x \leq 1/2$, $0 < y < \infty$. Let $\Gamma_\infty$ be the infinite cyclic group generated by $z \to z+1$. Then we construct a double coset decomposition (see [I], p. 163)

$$\Gamma_\infty \backslash g_\alpha^{-1}\Gamma g_\beta / \Gamma_\infty = \delta_{\alpha\beta}\Gamma_\infty \cup \left\{ \bigcup_{c>0} \bigcup_{d \bmod c} \Gamma_\infty \begin{pmatrix} * & * \\ c & d \end{pmatrix} \Gamma_\infty \right\} \tag{2.7}$$

where $\begin{pmatrix} * & * \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in g_\alpha^{-1} g_\beta$.

The general Kloosterman sums are introduced by

$$S_{\alpha\beta}(m,n;c;\Gamma;\chi) = S_{\alpha\beta}(m,n;c) = \sum_{d \bmod c} \bar{\chi} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \exp 2\pi i \frac{ma+nd}{c}. \tag{2.8}$$

Here we have assumed that we can extend the character $\chi$ from $\Gamma$ to $g_\alpha^{-1}\Gamma g_\beta$. Then we have

$$\varphi_{\alpha\beta}(s) = \varphi_{\alpha\beta}(s;\Gamma;\chi) = \sqrt{\pi}\frac{\Gamma(s-1/2}{\Gamma(s)} \sum_{c>0} \frac{S_{\alpha\beta}(0,0;c)}{c^{2s}} \tag{2.9}$$

$$\varphi_{\alpha\beta n}(s) = \varphi_{\alpha\beta n}(s;\Gamma;\chi) = \frac{2\pi^s}{\Gamma(s)} |n|^{s-1/2} \sum_{c>0} S_{\alpha\beta}(0,n;c) c^{-2s}. \tag{2.10}$$

where $\Gamma(s)$ is the Euler's gamma function.

The explicit calculation of these series in full generality for our groups $\bar{\Gamma}_0(N_1)$, $\bar{\Gamma}_0(4N_2)$, $\bar{\Gamma}_0(4N_3)$ and the corresponding arithmetical characters in terms of Dirichlet $L$-series is rather technical and will be presented it in a separate paper. The approach to solve this problem is by developing an idea due to M.N. Huxley (see [Hu]), although he considered congruence groups without characters. We will use his results later in this paper to prove the asymptotical Weyl law for discrete eigenvalues of $A(\Gamma;\chi)$ with the $\Gamma$ and $\chi$ considered here.

# 3 The discrete spectrum of the automorphic Laplacian for $\Gamma_0(N)$ with primitive character

We consider in this paragraph the group $\Gamma = \bar{\Gamma}_0(N)$ with primitive character $\chi$. We will prove here that apart from the continuous spectrum of multiplicity $r(\Gamma;\chi)$ (see Section 2) the operator $A(\Gamma;\chi)$ has an infinite discrete spectrum consisting of eigenvalues of finite multiplicity, satisfying a Weyl asymptotical law

$$N(\lambda;\Gamma;\chi) \underset{\lambda\to\infty}{\sim} \frac{\mu(F)}{4\pi}\lambda \tag{3.1}$$

where $N(\lambda;\Gamma;\chi) = \#\{\lambda_j \leqq \lambda\}$ is the distribution function for eigenvalues of $A(\Gamma;\chi)$, and the $\lambda_j$ are repeated according to multiplicity, $\mu(F) = |F|$ is the area of the fundamental domain $F$ of $\Gamma$.

As follows from general results on the spectrum of $A(\Gamma;\chi)$ (see [F] ,p. 382, [V1], p. 77) and the Selberg trace formula, it is enough to prove that the determinant of the automorphic scattering matrix

$$\varphi(s;\Gamma;\chi) = \det\phi(s;\Gamma;\chi) \tag{3.2}$$

is a meromorphic function of order 1. We will prove this indirectly, reducing to the group $\bar{\Gamma}_1(N)$, and then using Huxley's result.

We recall the definitions

$$\Gamma_0(N) = \left\{\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in SL(2,\mathbb{Z}) | c \equiv 0 \bmod N\right\}$$

$$\Gamma_1(N) = \left\{\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in SL(2,\mathbb{Z}) | c \equiv 0,\ a \equiv d \equiv 1 \bmod N\right\}$$

$$\Gamma(N) = \Gamma_2(N) = \left\{\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in SL(2,\mathbb{Z}) | b \equiv c \equiv 0,\ a \equiv d \equiv 1 \bmod N\right\}$$

and we recall the classical result (see [Mi] p. 104).

**Theorem 3.1** *1. Let $T_N$ be the homomorphism of $SL(2,Z)$ into $SL(z;Z/NZ)$*

$$T_N : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a \; mod \; N, & b \; mod \; N \\ c \; mod \; N & d \; mod \; N \end{pmatrix}$$

*then $T_N$ is surjective, $Ker\, T_N = \Gamma(N) = \Gamma_2(N)$.*

*2. The mapping $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to d \; mod \; N \in (\mathbb{Z}/N\mathbb{Z})^*$ induces an isomorphism*

$$\Gamma_0(N)/\Gamma_1(N) \backsimeq (\mathbb{Z}/N\mathbb{Z})^*$$

*and $\Gamma_1(N)$ is a normal subgroup of $\Gamma_0(N)$ of index $\varphi(N)$, where $\varphi$ is the Euler function.*

We recall now the general theorem proved in [V1], which we adopt to our situation.

**Theorem 3.2** *For a general cofinite Fuchsian group $\Gamma$ and its normal subgroup $\Gamma'$ of finite index the following formula for the kernels of the resolvents of $A(\Gamma';1)$ in $H_{\Gamma'}$ and $A(\Gamma;\chi)$ in $H_\Gamma$ holds:*

$$\frac{1}{[\Gamma : \Gamma']} \sum_{\hat{\chi} \in (\Gamma' \backslash \Gamma)^*} \Big[ tr_\chi r(z,z';s;\Gamma;\chi) \Big] \dim \chi = r(z,z';s;\Gamma';1) \tag{3.3}$$

*where $\big[\Gamma : \Gamma'\big]$ denotes the index of $\Gamma'$ in $\Gamma$. Here $\hat{\chi}$ runs over the set of all finite-dimensional, irreducible unitary representations of the factor group $\Gamma' \backslash \Gamma$. We extend the representation $\hat{\chi}$ to a representation $\chi$ of the group $\Gamma$ by the trivial representation, setting for $\gamma = \gamma_1 \cdot \gamma_2$, $\gamma_1 \in \Gamma'$, $\gamma_2 \in \Gamma' \backslash \Gamma$, $\chi(\gamma) = \chi(\gamma_1)\chi(\gamma_2) = \chi(\gamma_1)\hat{\chi}(\gamma_2) = \hat{\chi}(\gamma_2)$. The trace $tr_\chi$ is the trace in the space of the representation $\chi$, and $\dim \chi$ is the dimension of $\chi$.*

For $\operatorname{Re} s > 1$ the resolvent is defined as

$$R(s;\Gamma;\chi) = (A(\Gamma;\chi) - s(1-s)I)^{-1}, \tag{3.4}$$

where $I$ is the identity operator in $H_\Gamma$. We recall

$$\|f\|_\Gamma^2 = \int_F |f|_v^2 \, d\mu$$

$f : F \to V$, the finite dimensional space of the representation $\chi$. Then the kernel of the resolvent, considered as an integral operator is given by the absolutely convergent Poincaré series

$$r(z;z';s;\Gamma;\chi) = \sum_{\gamma\in\Gamma} \chi(\gamma) k(z,\gamma z';s), \tag{3.5}$$

where $k(z,z';s)$ is the Green's function for the operator $-\Delta - s(1-s)$ on $H$.

As the group $\Gamma$ from Theorem 3.2 we consider the projective group $\bar{\Gamma}_0(N)$ and set $\Gamma' = \bar{\Gamma}_1(N)$. Then from Theorem 3.1 follows that the factor group $\bar{\Gamma}_0(N)/\bar{\Gamma}_1(N)$ is isomorphic to the group of all even Dirichlet characters of $\mathbb{Z}$ mod $N$. Each of these characters becomes a character of the group $\bar{\Gamma}_0(N)$ if we set

$$\chi(\gamma) = \chi(d), \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \bar{\Gamma}_0(N) \tag{3.6}$$

since $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} * & * \\ * & dd' + cb' \end{pmatrix}$, $c \equiv 0 \bmod N$.

The identity (3.3) becomes

$$\frac{2}{\varphi(N)} \sum_{\substack{\chi \text{ even} \\ \chi \bmod N}} r(z,z';s;\bar{\Gamma}_0(N);\chi) = r(z,z';s;\bar{\Gamma}_1(N);1) \tag{3.7}$$

where $\chi = 1$ means the trivial one-dimensional representation. From (3.7) follows the factorization formula for the Selberg zeta function

$$Z(s;\bar{\Gamma}_1(N);1) = \prod_{\chi \text{ even mod } N} Z(s;\bar{\Gamma}_0(N);\chi) \tag{3.8}$$

and finally we obtain the relation between the distribution functions of discrete eigenvalues of $A(\bar{\Gamma}_1(N);1)$ and $A(\bar{\Gamma}_1(N);\chi)$

$$N(\lambda;\bar{\Gamma}_1(N);1) = \sum_{\chi \text{ even mod } N} N(\lambda;\bar{\Gamma}_0(N);\chi). \tag{3.9}$$

We have

$$\mu(\bar{F}_1(N)) = \frac{\varphi(N)}{2}\mu(\bar{F}_0(N)) \tag{3.10}$$

and the inequality valid for all big enough $\lambda$,

$$N(\lambda;\bar{\Gamma}_0(N);\chi) \leq \frac{\mu(\bar{F}_0(N))}{4\pi}\lambda, \tag{3.11}$$

where $\mu(\bar{F}_1(N))$, $\mu(\bar{F}_0(N))$ are the areas of the fundamental domains for $\bar{\Gamma}_1(N)$ and $\bar{\Gamma}_0(N)$ respectively. From that follows

**Lemma 3.3** *Let the Weyl formula (law) hold for $N(\lambda;\bar{\Gamma}_1(N);1)$. Then the Weyl formula is true for each summand $N(\lambda;\bar{\Gamma}_0(N);\chi)$ in (3.9). In particular, the Weyl law is valid for $N(\lambda;\bar{\Gamma}_0(N);\chi)$ with real primitive character mod $N$.*

Let us formulate now the result of Huxley (see [Hu], p. 142).

**Lemma 3.4** *For the group $\bar{\Gamma}_1(N)$ the determinant of the scattering matrix $\phi(s;\bar{\Gamma}_1(N);1)$*

*is given by*

$$\det\phi(s;\bar\Gamma_1(N);1)=(-1)^{(k-k_0)/2}\left(\frac{\Gamma(1-s)}{\Gamma(s)}\right)^k\left(\frac{A}{\pi^k}\right)^{1-2s}\prod_{\chi}\frac{L(2-2s,\bar\chi)}{L(2s;\chi)}, \tag{3.12}$$

*where $k$ is the number of cusps, -$k_0 = tr\phi(1/2,\bar\Gamma_1(N);1)$, $A$ is a positive integer composed of the primes dividing $N$, and the product has $k$ terms, in each of which $\chi$ is a Dirichlet character to some modulus dividing $N$, $L(s,\chi)$ the corresponding Dirichlet L-series, $\bar\chi$ is the complex conjugated character.*

From (3.12) follows

**Lemma 3.5** $\det\phi(s;\bar\Gamma_1(N);1)$ *is a meromorphic function of order 1.*

From Lemmas 3.3-3.5 and the Selberg trace formula follows

**Theorem 3.6** *For $\Gamma=\bar\Gamma_0(N)$ with real primitive character $\chi$ mod $N$ the Weyl law (3.1) is valid.*

So we have infinite discrete spectrum of eigenvalues of $A(\bar\Gamma_0(N),\chi)$. Actually, having in mind the Selberg eigenvalue conjecture and equality (3.8), it is very likely that the whole spectrum of $A(\bar\Gamma_0(N);\chi)$ belongs to $[1/4,\infty)$, since we have a nontrivial congruence character $\chi$, coming from the symbol $\left(\frac{N}{\cdot}\right)$.

The even and odd subspaces $\mathcal{H}_e$ and $\mathcal{H}_0$ of $\mathcal{H}$ are defined by

$$\mathcal{H}_e=\{f\in\mathcal{H}\,|\,f(-\bar z)=f(z)\}\,,\ \mathcal{H}_0=\{f\in\mathcal{H}\,|\,f(-\bar z)=-f(z)\}\,.$$

The spaces $\mathcal{H}_e$ and $\mathcal{H}_0$ are invariant under $A(\bar{\Gamma}_0(N);\chi)$, giving rise to operators $A_e(\bar{\Gamma}_0(N);\chi)$ and $A_0(\bar{\Gamma}_0(N);\chi)$. The spectrum of $A_0(\bar{\Gamma}_0(N);\chi)$ is purely discrete.

**Corollary 3.7** *The Weyl law for $A_e(\bar{\Gamma}_0(N);\chi)$ and $A_0(\bar{\Gamma}_0(N);\chi)$ is given by*

$$\#\{\lambda_j\,|\lambda_j \leq \lambda\} \simeq \frac{|F|}{8\pi}\lambda,$$

*where $\{\lambda_j\}$ is the sequence of eigenvalues counted with multiplicity for either $A_e(\bar{\Gamma}_0(N);\chi)$ or $A_0(\bar{\Gamma}_0(N);\chi)$.*

**Proof.** We refer to [V3], which deals with the modular group without character. This can be extended to $\bar{\Gamma}_0(N)$ with primitive character. ■

# 4 Hecke theory for Maass cusp forms

We now recall the Hecke theory for Maass cusp forms in application to all our cases $\bar{\Gamma}_0(N_1), \chi_{N_1}$, $\bar{\Gamma}_0(4N_2)$,$\chi_{4N_2}$ and $\bar{\Gamma}_0(4N_3)$, $\chi_{4N_3}$. There is no published account of this theory except for the short review by H. Iwaniec (see [I], p.70-72). But for holomorphic forms the corresponding results are well known and published in [Ogg], [A-L], [Li]. We make this transfer to the case of Maass forms specifically for the form with real primitive character in the style of [I] supplying more details about exceptional Hecke operators. We will write here simply $\bar{\Gamma}_0(N)$ and $\chi$, having in mind the three particular cases we consider.

Let $f$ be a continuous $(\bar{\Gamma}_0(N),\chi)$-automorphic function, i.e.

$$f(\gamma z) = \chi(\gamma)f(z),\ \forall\gamma \in \bar{\Gamma}_0(N),\ z \in H$$

and let $n \in \mathbb{Z}_+$. Then the Hecke operators are defined by

$$T(n)f(z) = \frac{1}{\sqrt{n}} \sum_{ad=n} \chi(a) \sum_{b \bmod d} f\left(\frac{az+b}{d}\right) \tag{4.1}$$

and $T(n)f(z)$ is again a continuous $(\bar{\Gamma}_0(N), \chi)$ automorphic function.

It is not easy to see immediately this property from the definition (4.1). We have to bear in mind the more general definition

$$T_g f(z) = \sum_{j=1}^{M} \chi(\gamma_j) f(g^{-1}\gamma_j z) \tag{4.2}$$

for an arithmetical cofinite group $\Gamma$ acting on $H$ and for some isometrical transformations $g$ of $H$ with the property that the intersection $\Gamma' = g^{-1}\Gamma g \cap \Gamma$ has a finite index both in $g^{-1}\Gamma g$ and $\Gamma$. Then we define $\gamma_j$ from the right coset decomposition

$$\Gamma = \bigcup_{j=1}^{M} \Gamma' \gamma_j .$$

In this definition we assume that we can define the character $\chi$ of $\Gamma$ for the group $g^{-1}\Gamma g$. Then the definition (4.1) follows from (4.2) if we take $\Gamma = \bar{\Gamma}_0(N)$ with our character $\chi$ and

$$g = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}.$$

It is easy to check this in the simplest case $n = p$ prime, $p \nmid N$. We have

$$\Gamma' = \Gamma_0(N, p),\ \Gamma' \backslash \Gamma = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},\ b \bmod p,\ \begin{pmatrix} p\beta & 1 \\ N\gamma & 1 \end{pmatrix} \right.$$

(see [A-L], p. 137) where $\beta, \gamma$ are any integers with the property $p\beta - N\gamma = 1$.

Then from (4.1) directly follows the basic relation

$$T(m)T(n) = \sum_{d|(m,n)} \chi(d) T(mn/d^2). \tag{4.3}$$

It is easy to check (4.3) for $(m,n)=1$ and then to consider the case $n=p^k$, $m=p^{k'}$, different powers of the same prime $p$. From (4.3) follows that all $T(n)$ commute with each other and also commute with the automorphic Laplacian $A(\bar{\Gamma}_0(N);\chi)$.

From (4.3) follows also that the most fundamental are the Hecke operators $T(p)$ which correspond to primes. Here we have to distinguish two cases, 1) $p\nmid N$, 2) $p\,|N$. For convenience we introduce the notation $U(q)=T(q)$ for $q\,|N$, while $T(p)$ is reserved for $p\nmid N$. We can see from (4.1) and the definition of $\chi$ :

$$\begin{cases} T(p)f(z)=\frac{1}{\sqrt{p}}\chi(p)f(pz)+\frac{1}{\sqrt{p}}\sum\limits_{b \bmod p} f\left(\frac{z+b}{p}\right) & p\nmid N \\ U(q)f(z)=T(q)f(z)=\frac{1}{\sqrt{q}}\sum\limits_{b \bmod q} f\left(\frac{z+b}{q}\right) & q\,|N \end{cases} \tag{4.4}$$

All the operators $T(p)$, $U(q)$ are bounded in the Hilbert space $\mathcal{H}(\bar{\Gamma}_0(N))$, also they map the subspace of cusp forms $\mathcal{H}(\Gamma_0(N))$ into itself. The operators $T(p)$ are $\chi(p)$-hermitian in $\mathcal{H}(\Gamma_0(N))$:

$$\langle T(p)f,g\rangle=\chi(p)\,\langle f,T(p)g\rangle \tag{4.5}$$

or

$$T(p)^*=\chi(p)T(p).$$

The equality (4.5) is similar to Lemma 13 of [A-L], where the corresponding fact was proved for holomorphic forms without character in relation to the Petersson inner product (on the subspace of cusp-forms).

We introduce next two involutions, $Kf(z)=\overline{f(z)}$ is the complex conjugation, $H_Nf(z)=f\left(-1/Nz\right)$. It is easy to see that they map $(\bar{\Gamma}_0(N),\chi)$-automorphic functions to themselves because, in particular, we have

$$\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \begin{pmatrix} 0 & 1/N \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} d & -c \\ -Nb & a \end{pmatrix},$$

$H_N \bar{\Gamma}_0(N) H_N^{-1} = \bar{\Gamma}_0(N)$. Then we have obviously

$$\begin{cases} KA(\bar{\Gamma}_0(N), \chi) = A(\bar{\Gamma}_0(N), \chi)K \\ KT(p) = T(p)K, \ U(p)K = KU(p) \\ KH_N = H_N K \end{cases} . \tag{4.6}$$

Less trivial facts are

$$T^*(p) = H_N T(p) H_N \tag{4.7}$$

$$U^*(q) = H_N U(q) H_N \tag{4.8}$$

where $T^*(p)$ and $U^*(q)$ are the adjoint operators of $T(p)$, $U(q)$ in $\mathcal{H}(\Gamma)$ respectively. From (4.5), (4.7) follows

$$H_N T(p) = \chi(p) T(p) H_N, \ p \nmid N. \tag{4.9}$$

Then we can see that all Hecke operators have only point spectrum in the space of all cusp forms $\mathcal{H}_0(\Gamma_0(N); \chi)$, and we want to find the common basis of eigenfunctions for $A(\Gamma_0(N), \chi)$ and all Hecke operators $T(p)$, $U(q)$ in this space. And actually it is possible, because we consider primitive characters $\chi$, which make all cusp forms "new". We recall briefly the definition of old and new forms for $\bar{\Gamma}_0(N)$ and $\chi$, generated by a Dirichlet character mod $N$.

If $\chi$ is mod $M$ and $v(z) \in \mathcal{H}_0(\bar{\Gamma}_0(N); \chi)$ then $v(dz) \in \mathcal{H}_0(\bar{\Gamma}_0(N); \chi)$ whenever $dM \,|\, N$. By definition $\mathcal{H}_0^{\mathrm{old}}(\bar{\Gamma}_0(N); \chi)$ is the subspace of $\mathcal{H}_0(\bar{\Gamma}_0(N); \chi)$ spanned by all forms $v(dz)$, where $v(z)$ is defined for $\bar{\Gamma}_0(M)$ with character $\chi$ mod $M$, $M < N$, $dM\,|N$ and $v$ is a common eigenfunction for all Hecke operators $T(m)$ with $(m, M) = 1$. Let the space $\mathcal{H}_0^{\text{new}}$ be the orthogonal complement

$$\mathcal{H}_0(\bar{\Gamma}_0(N); \chi) = \mathcal{H}_0^{\text{old}}(\bar{\Gamma}_0(N); \chi) \oplus \mathcal{H}_0^{\text{new}}(\bar{\Gamma}_0(N); \chi).$$

From this definition it is clear that there are no old forms for the pairs $(\bar{\Gamma}_0(N_1); \chi_{N_1})$, $(\bar{\Gamma}_0(4N_2); \chi_{4N_2})$ and $(\bar{\Gamma}_0(4N_3); \chi_{4N_3})$ we consider, because $\chi_{N1}$, $\chi_{4N_2}$, $\chi_{4N_3}$ are primitive characters mod $N_1$, $4N_2$, $4N_3$, respectively. The existence of the above-mentioned common basis of eigenfunctions follows from the following important theorem.

**Theorem 4.1** *Each Hecke exceptional operator $U(q)$, $q\,|N$ , is a unitary operator in the space $\mathcal{H}(\bar{\Gamma}_0(p))$, $U(q)U^*(q) = U_q^* U_q = I$, where $I$ is the identity operator in $\mathcal{H}(\bar{\Gamma}_0(N))$.*

**Proof.** The proof is a transfer of Theorem 4 and Corollary 1 of [Ogg] to our case of non-holomorphic forms with primitive character. The case $q = 2$ is the simplest, because $2^2\,|N$. We consider the more difficult case where $q$ is a prime, $q\,|N$, $q \neq 1$. By (4.8) we have to prove $U(q)H_m U(q)H_m = I$. We have

$$\begin{aligned} U(q)H_N U(q)H_N f(z) &= \frac{1}{q} \sum_{b \bmod q} \; \sum_{b' \bmod q} f\left(\frac{z+b'}{-N/q\; bz + 1 - N/q\; bb'}\right) \\ &= \frac{1}{q} \sum_{\substack{b' \bmod q \\ b=0}} f\left(\frac{z+b'}{-N/q\; bz + 1 - N/q\; bb'}\right) \\ &\quad + \frac{1}{q} \sum_{\substack{b' \bmod q \\ b\neq 0}} f\left(\frac{z+b'}{-N/q\; bz + 1 - N/q\; bb'}\right) \\ &= f(z) + \frac{1}{q} \sum_{\substack{b \bmod q \\ b\neq 0}} \; \sum_{b' \bmod q} f\left(\frac{z+b'}{-N/q\; bz + 1 - N/q\; bb'}\right). \end{aligned} \tag{4.10}$$

We want to prove that the double sum on the right hand side of (4.10) is equal to zero. Then we get that for each pair $b, b'$ mod $q$, $b \neq 0$ there exists a unique matrix depending on $a'$ mod $q$,

$$\begin{pmatrix} \alpha & \beta \\ N\gamma & \delta \end{pmatrix} \in \Gamma_0(N), \text{ such that } \begin{pmatrix} 1 & b' \\ -N/q\, b & 1 - N/q\, bb' \end{pmatrix}$$
$$= \begin{pmatrix} \alpha & \beta \\ N\gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & a' \\ -N/q & 1 - N/q\, a' \end{pmatrix}$$

and $\delta \equiv b \bmod q$, $\delta \equiv 1 \bmod N/q$. That means $\chi(\delta) = \chi_q(\delta)\chi_{N/q}(\delta) = \chi_q(b)$. Here we use a notation for the $\chi_q(\delta)$ part of the character symbol $\left(\frac{N}{\delta}\right)$, which corresponds to the period $q$ (see Section 1).

Then we get that the double sum considered is equal to

$$\frac{1}{q} \sum_{a' \bmod q} f\left(\frac{z + a'}{-N/q\, z + 1 - N/q\, z'}\right) \sum_{b \bmod q} \chi_q(b) = 0$$

and that proves the first part of the theorem. The proof of the identity $U_q^* U_q = I$ is similar. ■

From Theorem 4.1 follows that all operators $U^*(q)$ also commute with all Hecke operators and $A(\Gamma_0(N), \chi)$ and that is the reason why there exists the common basis of eigenfunctions for all these operators in the space $\mathcal{H}_0(\Gamma_0(N); \chi)$. In fact, it is possible to prove a much stronger result about the existence of the common basis of eigenfunctions, the so-called "multiplicity one" theorem. Unitarity of $U(q)$ does not follow from this theorem, however. It is analogous to Theorem 3 of [A-L] and to Theorem 3 of [Li]. This theorem is about the following. We take first the common basis of all eigenfunctions $v_j(z)$ for all $T(n)$, $(n, N) = 1$, and $A(\bar{\Gamma}_0(N); \chi)$ in the space $\mathcal{H}_0(\bar{\Gamma}_0(N); \chi)$ of cusp forms. Let us introduce

$$T'(n) = iT(n) \text{ if } \chi(n) = -1 \text{ and } T'(n) = T(n) \text{ if } \chi(n) = 1 \tag{4.11}$$

We can see then that all $T'(n)$ are selfadjoint operators (see (4.5)).

Since $v_j$ is an eigenfunction of $A(\bar{\Gamma}_0(N);\chi)$, $Av_j = \lambda_j v_j$, $\lambda_j = s_j(1-s_j)$ we have for $j = 1, 2, ...$

$$v_j(z) = \sum_{n\neq 0} \rho_j(n)\sqrt{y}K_{s_j-1/2}(2\pi\,|n|\,y)e^{2\pi inx} \tag{4.12}$$

(similar to (2.6)) with $\rho_j(n) \in \mathbb{C}$. We have also

$$T(n)v_j(z) = \Lambda_j(n)v_j(z)\ (n, N) = 1. \tag{4.13}$$

From (4.1), (4.12), (4.13) follows that if $\rho_j(1) = 0$, then $\rho_j(n) = 0$ for all $n$, $(n, N) = 1$. If $\rho_j(1) \neq 0$ we obtain for all $n$, $(n, N) = 1$

$$\Lambda_j(n) = \frac{\rho_j(n)}{\rho_j(1)}. \tag{4.14}$$

Before talking about the proof of this theorem we make the following remark.

There is also the important involution

$$J : z \to -\bar{z},\ z \in H.$$

This involution acts on the space of all continuous $(\Gamma_0(N), \chi)$ automorphic functions and splits this space into the sum of subspaces of even and odd functions given by $f(Jz) = f(z)$ or $f(Jz) = -f(z)$. This $J$ commutes with $A(\Gamma_0(N), \chi)$ and with all Hecke operators. The conditions for the eigenfunction $v_j(z)$ of $A(\bar{\Gamma}_0(N), \chi)$, we consider in (4.12) to be even or odd are the following, respectively

$$\rho_j(n) = \rho_j(-n),\ \rho_j(-n) = -\rho_j(n). \tag{4.15}$$

That means, in particular, that the Fourier coefficients $\rho_j(n)$ with negative numbers $n$ are determined in both cases by $\rho_j(n)$ with positive numbers $n$. We have also

$$H_N J = J H_N,\ KJ = JK. \tag{4.16}$$

Let us consider the case $\rho_j(1) = 0$ first. Then we will show that the whole function $v_j(z)$ is zero. From that follows that only the eigenvalues $\Lambda_j(n)$ for $(n, N) = 1$ determine completely the function $v_j(z)$ up to multiplication by a constant, of course. In that case $v_j(z)$ has to be an eigenfunction of all $U(q)$, $U^*(q)$, $q\,|N$. And that is the multiplicity one theorem in our case.

Very briefly the idea of the proof of the multiplicity one theorem is the following. We consider an eigenfunction (4.12) with (4.13) and we assume that $\rho_j(1) = 0,\ \rho_j(n) = 0$, $(n, N) = 1$. Then we see that the series (4.12) can be written as a sum of terms

$$v_j(z) = \sum_{q|N} w_{jq}(z). \tag{4.17}$$

Each $w_{jq}$ is associated with a subgroup of $\Gamma_0(N)$ with character $\chi$ and level $q$, where the numbers $q$ are mutually prime. Then, since the whole sum (4.17) belongs to $(\Gamma_0(N), \chi)$ it follows that each $w_{jq} \in (\Gamma_0(N), \chi)$. The last step of the proof is to see from the structure of $w_{jq}$ as a Fourier series similar to (4.12) that each $w_{jq}$ belongs to some overgroup of $\bar{\Gamma}_0(N)$ with trivial extension of $\chi$. Since the character $\chi$ is primitive it is only possible if each $w_{jq}(z) = 0$. Now we have for any nontrivial $v_j(z)$ from (4.12) with (4.13) that $\rho_j(1) \neq 0$. Let us introduce the normalization

$$\rho_j(1) = 1 \tag{4.18}$$

for all $f_j(z)$. This normalization is different from the Hilbert space theory normalization

$$\|v_j\| = 1 \tag{4.19}$$

but it is more natural when we are talking about Hecke theory. From the previous argument follows that $v_j(z)$ from (4.12) with (4.13), (4.18) is completely determined, in other words for the eigenvalues $\{\lambda_j, \Delta_j(n)\}$, $(n, N) = 1$, (4.14), there is only one eigenfunction $v_j(z)$. This is the idea of the proof of the "multiplicity one" theorem. We formulate this theorem as follows.

**Theorem 4.2** *1. There exists a unique common basis of eigenfunctions for all operators $A(\Gamma_0(N); \chi)$, $T(n)$, $T^*(n)$, $n \geq 1$ in the space of cusp forms $\mathcal{H}_0(\Gamma_0(N); \chi)$.*

*2. Each eigenfunction $v_j(z)$ (4.12) of this basis can be taken with normalization (4.18) and is uniquely determined by the eigenvalues $\lambda_j$, $\Lambda_j(n)$, $(n, N) = 1$, (4.14).*

*3. We have also (see (4.3))*

$$U(q)v_j(z) = \rho_j(q)v_j(z),\ U^*(q)v_j(z) = \overline{\rho_j(q)}v_j(z)$$

*and*

$$4) \qquad \rho_j(n)\rho_j(m) = \sum_{d|(m,n)} \chi(d)\rho_j(mn/d^2),$$

*in particular* $\rho_j(q)\rho_j(n) = \rho_j(qn),\ q\,|N,$

$$\rho_j(p^{k+1}) = \rho_j(p^k)\rho_j(p) - \chi(p)\rho_j(p^{k-1}),\ p \nmid N,\ k \geq 0,$$

*where by definition* $\rho_j(p^{-1}) = 0$, $p, q$ *are primes.*

On the basis of these two theorems we can prove

**Theorem 4.3** *For any* $q\,|N$ *we have* $\rho_j(q) = \pm 1,\ j = 1, 2, \ldots$ *, see 3) of Theorem 4.2.*

**Proof.** We consider the involution $H_N K$ (see (4.6)). We have

$$T(p)(H_N K)v_j = \chi(p)KH_N T(p)v_j = \chi(p)\bar{\Lambda}_j(p)(H_N K)v_j$$

$$= \Lambda_j(p)(H_N K)v_j.$$

From Theorem 4.2 follows then that $H_N K v_j = \nu_j v_j$ with $\nu_j \in \mathbb{C}$. Since $(H_N K)^2 = 1$ we have $\nu_j = \pm 1$. So we obtain

$$H_N K v_j = \pm v_j \tag{4.20}$$

for any $j = 1, 2, \ldots$ . Then from Theorem 4.1 follows

$$H_N U(q) H_N U(q) = I$$

is equivalent to

$$(H_N K)U(q)(H_N K)U(q) = K \cdot K = I \text{ (see (4.6))}. \tag{4.21}$$

Applying (4.21) to the function $v_j(z)$ and using (4.20), we obtain the claim of the theorem $\Lambda_j^2(q) = \rho_j(q)^2 = 1$. ■

**Remark 4.4** *The Selberg small eigenvalue conjecture for $A(\bar{\Gamma}_0(N), \chi)$ says that all eigenvalues are embedded in the continuous spectrum $[\frac{1}{4}, \infty)$. It is not difficult to see that for $q\,|N$ the continuous spectrum of $U(q)$ is the whole unit circle. Since the only eigenvalues are $\pm 1$, the analogue of Selberg's small eigenvalue conjecture holds true for the exceptional Hecke operator.*

# 5 Non-vanishing of Hecke $L$-functions

For each function $v_j(z)$ from (4.12) with (4.13), (4.18) we define the Dirichlet series

$$L(s; v_j) = \sum_{n=1}^{\infty} \frac{\rho_j(n)}{n^s}. \tag{5.1}$$

From studying the Rankin-Selberg convolution we can see that the series (5.1) is absolutely convergent for $\operatorname{Re} s > 1$.

From Theorems 4.2 and 4.3 also follows

**Theorem 5.1** *Let $L(s, v_j)$ be the series* (5.1) *and the function $v_j(z)$ be as in the Theorem 4.2. Then for* $\operatorname{Re} s > 1$ *we have an Euler product representation for $L(s; v_j)$*

$$L(s; v_j) = \prod_p \left(1 - \rho_j(p) p^{-s} + \chi(p) p^{-2s}\right)^{-1}. \tag{5.2}$$

*The product is taken over all primes.*

We can also write (5.2) in the form

$$L(s; v_j) = \prod_{q|N} \left(1 - \rho_j(q) q^{-s}\right)^{-1} \prod_{p\nmid N} \left(1 - \rho_j(p) p^{-s} + \chi(p) p^{-2s}\right)^{-1} \tag{5.3}$$

since $\chi(q) = 0$, $q\,|N$. From Theorem 4.3 we know $\rho_j(q) = \pm 1$, $q\,|N$, $j = 1, 2, 3, \ldots$ .

We derive now the functional equation for the pair of the Dirichlet series

$$\begin{cases} L(s; v_j) = \sum\limits_{n=1}^{\infty} \frac{\rho_j(n)}{n^s} \\ L(s; \hat{v}_j) = \sum\limits_{n=1}^{\infty} \frac{\overline{\rho_j(n)}}{n^s}, \quad \operatorname{Re} s > 1. \end{cases} \tag{5.4}$$

We only consider the case of odd eigenfunctions since that is important for this paper. We have together with (4.12) by definition

$$\hat{v}_j(z) = \sum_{n \neq 1}^{\infty} \overline{\rho_j(n)} \sqrt{y} K_{s_j - 1/2}(2\pi\,|n|\,y) e^{2\pi i n x}. \tag{5.5}$$

If $s_j - 1/2 \in i\mathbb{R}$ or $s_j \in \left(\frac{1}{2}, 1\right)$ then $K_{s_{j-\frac{1}{2}}}(2\pi\,|n|\,y)$ is a real-valued function and for odd $v_j$ we have $\bar{v}_j(z) = -\hat{v}_j(z)$. We will write $v_j(z) = v_j(x, y)$, where $z = x + iy$. We have $v_j(-x, y) = -.v_j(x, y)$. The action of the involutions $H_N K$ (4.20) can be written as follows

$$\begin{cases} \overline{v_j(u, v)} = \pm v_j(x, y) \\ u = -\frac{x}{N(x^2+y^2)}, \qquad v = \frac{y}{N(x^2+y^2)} \end{cases} . \tag{5.6}$$

We apply the partial derivative $\frac{\partial}{\partial x}$ and obtain

$$-\frac{1}{Ny^2} \frac{\partial \overline{v_j}}{\partial u}\bigg|_{x=0} = \pm \frac{\partial v_j}{\partial x}\bigg|_{x=0} .$$

This is equivalent to

$$\pm B(y) = \pm N^{3/2} y^3 \sum_{n=1}^{\infty} \overline{\rho_j(n)} n K_{s_{j-\frac{1}{2}}}(2\pi n y) = \sum_{n=1}^{\infty} \rho_j(n) n K_{s_j - 1/2}(2\pi n / N y). \tag{5.7}$$

We multiply the left hand side of (5.7) by $4\pi N^{s/2 - 3/2} y^{s-3}$ and integrate it from 0 to

$\infty$ in $y$. We obtain

$$\int_0^\infty 4\pi^{s/2-3/2} y^{s-3} B(y) dy \tag{5.8}$$
$$= \pi^{-s} m^{s/2} \Gamma\left(\frac{s+s_j}{2} + 1/4\right) \Gamma\left(\frac{s-s_j}{2} + 3/4\right) \cdot L(s; \hat{v}_j)$$
$$= \Omega(s; \hat{v}_j).$$

That is because

$$\int_0^\infty y^s K_{s_j-1/2}(y) dy = 2^{s-1} \Gamma\left(\frac{s+s_j}{2} + 1/4\right) \Gamma\left(\frac{s-s_j}{2} + 3/4\right).$$

We can now write the integral obtained as a sum of two integrals

$$4\pi^{s/2-3/2} \int_0^\infty B(y) y^{s-3} dy = 4\pi m^{s/2-3/2} \left( \int_0^{1/\sqrt{N}} + \int_{1/\sqrt{N}}^\infty \right). \tag{5.9}$$

In the first integral we use (5.7) for $B(y)$ and then map $y \to 1/Ny$.

Then we obtain that (5.9) is equal to

$$4\pi \left( N^{3/2} \int_{1/\sqrt{N}}^\infty y^s \sum_{n=1}^\infty \overline{\rho_j(n)} n K_{s_j-1/2}(2\pi n y) dy \pm N^{\frac{1-s}{2}} \int_{1/\sqrt{N}}^\infty y^{1-s} \sum_{n=1}^\infty \rho_j(n) n K_{s_j-1/2}(2\pi n y) dy \right)$$
$$= C(s; \hat{v}_j) \pm C(1-s; v_j). \tag{5.10}$$

It is clear that $C(s; v_j)$, $C(s; \hat{v}_j)$ are entire functions of $s$. Then we have

$$\Omega(s; \hat{v}_j) = C(s; \hat{v}_j) \pm C(1-s; v_j). \tag{5.11}$$

The analogous calculation shows

$$\Omega(s; v_j) = C(s; v_j) \pm C(1-s; \hat{v}_j)$$

$$\Omega(1-s; v_j) = C(1-s; v_j) \pm C(s; \hat{v}_j)$$

and we finally obtain

$$\pm\Omega(1-s;v_j)=\Omega(s;\hat{v}_j). \tag{5.12}$$

We have proved the

**Theorem 5.2** *Any* $L(s;v_j)$ (5.1) *which is defined by the odd eigenfunction* $v_j(z)$ *by* (4.12) *with* (4.13), (4.18) *has an analytic continuation to all* $s\in\mathbb{C}$. *The same property has the Dirichlet series* $L(s;\hat{v}_j)$ *which is defined in* (5.4). *Both series are connected by* (5.12) *with* (5.8), *where the functions* $\Omega(s;v_j)$ *and* $\Omega(s;\hat{v}_j)$ *are entire functions of* $s\in\mathbb{C}$.

We shall prove that the functions $L(s;v_j)$ and $L(s,\hat{v}_j)$ are regular and non-vanishing on the boundary of the critical strip.

We start with the Rankin-Selberg convolution. For each eigenfunction $v_j(z)$ from (4.12) we define the series

$$\sum_{n=1}^{\infty}\frac{|\rho_j(n)|^2}{n^s} \tag{5.13}$$

which is absolutely convergent for $\operatorname{Re}s>1$.

For $\operatorname{Re}s>1$ we consider the following Selberg integral

$$\int\limits_{F_0(N)}|v_j(z)|^2\,E_\infty(z;s;\bar{\Gamma}_0(N);1)d\mu(z)=A(s)$$

where

$$E_\infty(z;s)=E_\infty(z;s;\bar{\Gamma}_0(N);1)=\sum_{\gamma\in\Gamma_\infty\backslash\bar{\Gamma}_0(N)}y^s(\gamma z)$$

Using the unfolding of the Eisenstein series we obtain

$$A(s) = \int_0^\infty y^{s-1} \sum_{n\neq 0} |\rho_j(n)|^2 K_{ir_j}^2 (2\pi |n| y) dy =$$
$$= \frac{\Gamma^2(s/2)\Gamma\left(\frac{s}{2}+ir_j\right)\Gamma(s/2-ir_j)}{4\pi^s\Gamma(s)} \sum_{n=1}^\infty \frac{|\rho_j(n)|^2}{n^s}.$$

It is well known that $E(z,s;\Gamma_0(N);1)$ has analytic continuation to the whole $s$-plane, and at $\mathrm{Re}\, s > 1/2$ it has only a simple pole at $s=1$ with residue equal to $\mu(F_0(N))^{-1}$ (inverse $\mu$-area of the fundamental domain of $\bar\Gamma_0(N)$). From that follows that the Rankin-Selberg convolution (5.13) is a regular function in $\mathrm{Re}\, s > 1/2$ except for a simple pole at $s=1$.

We want to see now the Euler product for the Rankin-Selberg convolution (5.13). The method is due to Rankin (see [R]). The main difference from Rankin's case is that our coefficients $\rho_j$ may be complex numbers, and that we have also exceptional primes $q\,|N$ .

First consider the main case $(n,N)=1$. It follows from (4.5) that

$$\rho_j(n) = \chi(n)\bar\rho_j(n) \ \ j = 1,2,... \tag{5.14}$$

and for $\chi(n) = -1$, $\rho_j(n)$ is purely imaginary (it can not be zero). In both the cases $\chi(n) = \pm 1$ we have

$$|\rho_j(n)|^2 = \chi(n)\rho_j^2(n). \tag{5.15}$$

From Theorem 4.2 follows

$$\begin{cases} \rho_j^2(p^n) = (\rho_j(p)\rho_j(p^{n-1}) - \chi(p)\rho_j(p^{n-2}))^2 \\ (\chi(p)\rho_j(p^{n-3}))^2 = (-\rho_j(p^{n-1}) + \rho_j(p)\rho_j(p^{n-2}))^2 \end{cases}. \tag{5.16}$$

Then multiplying the second line of (5.16) by $\chi(p)$ and taking the difference, we obtain

$$\rho_j^2(p^n) - \rho_j^2(p)\rho_j^2(p^{n-1}) + \chi(p)\rho_j^2(p^{n-1}) + \chi(p)\rho_j^2(p)\rho_j^2(p^{n-2}) - \rho_j^2(p^{n-2}) - \chi(p)\rho_j^2(p^{n-3}) = 0. \tag{5.17}$$

Multiplying now (5.17) by $\chi(p^n)$ and using (5.15) we obtain

$$|\rho_j(p^n)|^2 - \left|\rho_j(p^{n-1})\right|^2 |\rho_j(p)|^2 + \left|\rho_j(p^{n-1})\right|^2 + |\rho_j(p)|^2 \left|\rho_j(p^{n-2})\right|^2 - \left|\rho_j(p^{n-2})\right|^2 - \left|\rho_j(p^{n-3})\right|^2 = 0. \tag{5.18}$$

In the case $q\,|N$ we have from Theorems 4.2, 4.3 that

$$|\rho_j(q^n)| = 1,\ n = 1, 2, \dots . \tag{5.19}$$

Then applying Theorem 4.2 again we get

$$\sum_{n=1}^{\infty} \frac{|\rho_j(n)|^2}{n^{2s}} = \prod_{p\nmid N} \left(1 + \frac{|\rho_j(p)|^2}{p^{2s}} + \frac{|\rho_j(p^2)|^2}{p^{4s}} + \frac{|\rho_j(p^3)|^2}{p^{6s}} + \dots\right) \cdot \prod_{q|N} \left(1 + \frac{1}{q^{2s}} + \frac{1}{q^{4s}} + \dots\right) \tag{5.20}$$

$$\begin{aligned}
&= \prod_{q|N} \left(1 - q^{-2s}\right)^{-1} \cdot \prod_{p\nmid N} \frac{1 + p^{-2s}}{1 - |\rho_j(p)|^2 p^{-2s} + p^{-2s} + |\rho_j(p)|^2 p^{-4s} - p^{-4s} - p^{-6s}} \\
&= \prod_{q|N} \left(1 - q^{-2s}\right)^{-1} \cdot \prod_{p\nmid N} \left(1 + p^{-2s}\right) \left(1 - p^{-2s}\right)^{-1} \left(1 + (2 - |\rho_j(p)|^2) p^{-2s} + p^{-4s}\right)^{-1} \\
&= \prod_{q|N} \left(1 - q^{-2s}\right)^{-1} \cdot \prod_{p\nmid N} \left(1 - p^{-4s}\right) \left(1 - p^{-2s}\right)^{-2} \left(1 + (2 - |\rho_j(p)|^2) p^{-2s} + p^{-4s}\right)^{-1} \\
&= \zeta(2s) L(2s;\hat{\chi}) L^{-1}(4s;\hat{\chi}) \prod_{p\nmid N} \left(1 + (2 - |\rho_j(p)|^2) p^{-2s} + p^{-4s}\right)^{-1},
\end{aligned}$$

where $L(s;\hat{\chi})$ is the Dirichlet $L$-series with principal character mod $N$.

$$L(s;\hat{\chi}) = \prod_{p} \left(1 - \hat{\chi}(p) p^{-s}\right)^{-1} = \zeta(s) \prod_{p|N} (1 - p^{-s}).$$

The products in (5.20) are taken over all primes $p \nmid N$, $q\,|N$. For $p \nmid N$ we now introduce new functions $\alpha_j(p)$, $\beta_j(p)$, which are important to define symmetric power $L$-series, by

$$\left\{ \begin{array}{l} \alpha_j(p) + \beta_j(p) = \rho_j(p) \\ \alpha_j(p)\beta_j(p) = \chi(p) \end{array} \right. . \tag{5.21}$$

We have $(\alpha_j(p)+\beta_j(p))^2 = \rho_j^2(p) = \alpha_j^2(p) + 2\chi(p) + \beta_j^2(p)$, and

$$\begin{cases} \chi(p)\alpha_j^2(p) + \chi(p)\beta_j^2(p) = |\rho_j(p)|^2 - 2 \\ \alpha_j^2(p)\beta_j^2(p) = 1 \end{cases} . \tag{5.22}$$

Applying (5.22) to (5.20) we obtain by the definition

$$\prod_{p\nmid N}(1+(2-|\rho_j(p)|^2)p^{-2s}+p^{-4s})^{-1} \tag{5.23}$$

$$= \prod_{p\nmid N}\left(1-\frac{\chi(p)\alpha_j^2(p)}{p^{2s}}\right)^{-1}\left(1-\frac{\chi(p)\beta_j^2(p)}{p^{2s}}\right)^{-1} = L_2(2s;v_j).$$

Combining with (5.20) we finally obtain

$$L(s;v_j\times\bar{v}_j) = \frac{L(s;\hat{\chi})}{L(2s;\hat{\chi})}L_2(s;v_j)\zeta(s) \tag{5.24}$$

where $L(s;v_j\times\bar{v}_j)$ is the Rankin-Selberg convolution (5.13).

We can also write using (5.21) for $\operatorname{Re} s > 1$

$$\begin{cases} L(s;v_j) = \prod_{q|N}(1\pm q^{-s})^{-1}\prod_{p\nmid N}\left(1-\frac{\alpha_j(p)}{p^s}\right)^{-1}\left(1-\frac{\beta_j(p)}{p^s}\right)^{-1} \\ L(s;\hat{v}_j) = \prod_{q|N}(1\pm q^{-s})^{-1}\prod_{p\nmid N}\left(1-\frac{\overline{\alpha_j(p)}}{p^s}\right)^{-1}\left(1-\frac{\overline{\beta_j(p)}}{p^s}\right)^{-1} \end{cases} . \tag{5.25}$$

For the proof of the next theorem we will make use of the following general criterion proved in [M-M] (Theorem 1.2).

**Lemma 5.3** *Let $f(s)$ be a function satisfying*

1. *$f$ is holomorphic and $f(s)\neq 0$ in $\{s\,|\operatorname{Re} s = \sigma > 1\}$*

2. *$f$ is holomorphic on the line $\sigma = 1$ except for a pole of order $e \geq 1$ at $s = 1$*

3. *$\log f(s)$ can be written as a Dirichlet series*

$$\sum_{n=1}^{\infty}\frac{b_n}{n^s}$$

*with $b_n \geq 0$ for $\sigma > 1$.*

*Then if $f$ has a zero on the line $\sigma = 1$, the order of the zero is bounded by $e/2$.*

We now want to prove the following

**Theorem 5.4** *$L(s; v_j)$ and $L(s, \hat{v}_j)$ from (5.4) are regular for $s = 1 + it$, $s = it$, $t \in \mathbb{R}$, and*

$$L(1 + it; v_j) \neq 0,\ L(it; v_j) \neq 0,\ L(1 + it; \hat{v}_j) \neq 0,\ L(it; \hat{v}_j) \neq 0,\ j = 1, 2, \ldots . \tag{5.26}$$

**Proof.** Clearly, (5.26) is analogous to the prime number theorem, $\zeta(1 + it) \neq 0$, for the Riemann zeta function. This kind of property for different zeta-functions is very important in number theory (see, for example, [J-S], [M-M].

From the functional equation (5.12) follows that it is enough to prove the inequalities

$$L(1 + it; v_j) \neq 0,\ L(1 + it; \hat{v}_j) \neq 0 \tag{5.27}$$

because we know all singular points of the Euler $\Gamma$-function from (5.12).

Consider the following product

$$f(s) = L(s; v_j \times \hat{v}_j) L(2s; \hat{\chi}) L(s; v_j) L(s, \hat{v}_j) \prod_{q|N} (1 - q^{-s})(1 - \rho_j(q) q^{-s})(1 - \bar{\rho}_j(q) q^{-s}).$$

Let $\operatorname{Re} s > 1$, then from (5.23), (5.24), (5.25) follows

$$\log f(s) = -\sum_{p \nmid N} \{2 \log(1 - p^{-s}) + \log(1 - \chi(p) \alpha_j^2(p) p^{-s}) + \log(1 - \chi(p) \beta_j^2(p) p^{-s})$$

$$+\log(1-\alpha_j(p)p^{-s})+\log(1-\beta_j(p)p^{-s})+\log(1-\bar{\alpha}_j(p)p^{-s})$$

$$+\log(1-\bar{\beta}_j(p)\cdot p^{-s})\big\}\,. \tag{5.28}$$

For $|x|<1$ we have $\log(1-x)=-\sum\limits_{n=1}^{\infty} x^n/n$. Using this we continue (5.28)

$$\log f(s)=\sum_{p\nmid N}\sum_{n=1}^{\infty}\frac{1}{np^{ns}}(2+\chi(p)^n\alpha_j^{2n}(p)+\chi(p)^n\beta_j^{2n}(p)+\alpha_j^n(p) \tag{5.29}$$

$$+\beta_j^n(p)+\bar{\alpha}_j^n(p)+\bar{\beta}_j^n(p))$$

$$=\sum_{p\nmid N}\sum_{n=1}^{\infty}\frac{a_{n,p}}{np^{ns}}.$$

We will show now $a_{n,p}\geq 0$.

We consider two cases: $\chi(p)=1,\ \chi(p)=-1$. In the first case

$$a_{n,p}=2+2\alpha_j^n(p)+2\beta_j^n(p)+\alpha_j^{2n}(p)+\beta_j^{2n}(p)=(1+\alpha_j^n(p))^2+(1+\beta_j^n(p))\geq 0 \tag{5.30}$$

because in that case $\alpha_j(p),\beta_j(p)$ are real numbers. In the second case we have that $\alpha_j(p)=i\tilde{\alpha}_j(p)$, $\beta_j(p)=i\tilde{\beta}_j(p)$, and $\tilde{\alpha}_j(p)$, $\tilde{\beta}_j(p)$ are real numbers. We have

$$a_{n,p}=2+\tilde{\alpha}_j^{2n}+\tilde{\beta}_j^{2n}+\tilde{\alpha}_j^n(i)^n((-1)^n+1)+\tilde{\beta}_j^n(i)^n((-1)^n+1) \tag{5.31}$$

and this is real and $\geq 0$ if $n=2m-1$, $m=1,2,\ldots$ . We consider $n=2m$, $m=1,2,\ldots$

$$a_{n,p}=2+\tilde{\alpha}_j^{4m}+\tilde{\beta}_j^{4m}+(-1)^m\cdot 2\tilde{\alpha}_j^{2m}+(-1)^m\cdot 2\tilde{\beta}_j^{2m}=(1+(-1)^m\tilde{\alpha}_j^{2m})^2+(1+(-1)^m\tilde{\beta}_j^{2m})^2\geq 0$$

and we have proved that $a_{n,p}\geq 0$ for all $p\nmid N$, $n=1,2,\ldots$ .

Let us assume first that $L(s;v_j)=0$ at $s=1$. That means also $L(s;\hat{v}_j)=0$ at $s=1$. Since $L(s;v_j\times\hat{v}_j)$ has only a simple pole at $s=1$, we see that the function $f(s)$ has a zero at $s=1$. On the other hand, since $\log f(s)$ has the property 3, $\log f(s)>0$ for $s>1$, so $f(s)>1$ for $s>1$, a contradiction. So we have $L(1;v_j)\neq 0$, $L(1,\hat{v}_j)\neq 0$.

Suppose now that $L(1+it;v_j)=0$ for some $t\neq 0$. Then $f(s)$ has a zero of order $\geq 1$ on the line $\operatorname{Re} s>1$, $s\neq 1$, since $L(s;\hat{v}_j)$ and $L(s;v_j\times\hat{v}_j)$ are regular at $s=1+it$. Also, $f(s)$ has a pole of order 1 at $s=1$, since $L(s;v_j\times\hat{v}_j)$ has a simple pole at $s=1$ and $L(1;v_j)\neq 0$, $L(1;\hat{v}_j)\neq 0$. This is in contradiction with Lemma 5.3, and Theorem 5.4 is proved. ■

# 6 The form $\omega(z)$ and perturbation of $A(\bar{\Gamma}_0(N),\chi)$ by characters

Let $\omega(z)$ be a holomorphic modular form of weight 2, which belongs to $\bar{\Gamma}_0(N)$. Thus $\omega(\gamma z)=(cz+d)^2\omega(z)$,

$$\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\bar{\Gamma}_0(N).$$

It is well known that the integral

$$\chi_\alpha(\gamma)=\exp 2\pi i\alpha \operatorname{Re}\int_{z_0}^{\gamma z_0}\omega(t)dt,\ \gamma\in\bar{\Gamma}_0(N) \tag{6.1}$$

$\alpha\in\mathbb{R}$, $z\in H$, defines a family of unitary characters for the group $\bar{\Gamma}_0(N)$, which is independent of the choice of the point $z_0$. We consider the family of self-adjoint operators $A(\bar{\Gamma}_0(N),\chi\cdot\chi_\alpha)$, as we defined in Section 1 by the Laplacian acting on functions $g(z)$ satisfying

$$g(\gamma z)=\chi(\gamma)\chi_a(\gamma)g(z),\ \gamma\in\bar{\Gamma}_0(N). \tag{6.2}$$

We consider $\alpha$ as a small parameter, $|\alpha| < \varepsilon, \varepsilon > 0$. The domain of definition $D(A(\bar{\Gamma}_0(N); \chi \cdot \chi_\alpha))$ is a dense subspace of $L_2(F; d\mu)$, varying with $\alpha$. We consider then the operator $A_\alpha = A(\bar{\Gamma}_0(N); \chi \cdot \chi_\alpha)$ as a perturbed $A_0 = A(\bar{\Gamma}_0(N), \chi)$, since the character (6.1) becomes trivial when $\alpha = 0$.

In order to apply perturbation theory we have to bring all the operators $A_\alpha$ to the domain of definition of $A_0$. Then we have to choose the form $\omega$ which makes the perturbation regular, and this is very important if we want to get information on eigenvalues and eigenfunction. On the other hand, it is very important to take as $\omega(z)$ the old holomorphic Eisenstein series, coming from the holomorphic Eisenstein series $E_2(z) = P(z)$ which belongs to the modular group. The last condition is crucial for the evaluation of the Phillips-Sarnak integral and for proving that it is not zero (see Section 7). We will show that there exists a form $\omega(z)$, which satisfies these two conditions, for exactly the two cases 2), 3) from Theorem 1.1: 2) $\bar{\Gamma}_0(4N_2)$, $N_2$ a square-free positive integer and $N_2 \equiv 3 \mod 4$, and its arithmetical character $\chi = \left(\frac{4N_2}{\cdot}\right)$, 3) $\bar{\Gamma}_0(4N_3)$, $N_3$ a square-free positive integer, $N_3 \equiv 2 \mod 4$, and its arithmetical character $\chi = \left(\frac{4N_3}{\cdot}\right)$. In these two cases the character $\chi$ is always singular, since there exist both open and closed cusps. We construct now this perturbation, and then we will find the appropriate form $\omega(z)$.

For a function $f$, $f(\gamma z) = \chi(\gamma) f(z)$, $\gamma \in \bar{\Gamma}_0(N)$, we define

$$g(z) = f(z) \exp 2\pi i \alpha \operatorname{Re} \int_{z_0}^{z} \omega(t) dt = f(z) \Omega(z, \alpha). \tag{6.3}$$

It is not difficult to see that $g(z)$ satisfies the condition (6.2). Applying the negative

Laplacian

$$-\Delta = -4y^2 \frac{\partial^2}{\partial z \partial \bar{z}} \tag{6.4}$$

to the function $g(z)$, we obtain that the operator $A(\bar{\Gamma}_0(N); \chi \cdot \chi_\alpha)$ is unitarily equivalent to the operator

$$L(\alpha) = -\Delta + \alpha M + \alpha^2 N \tag{6.5}$$

where

$$\begin{cases} M = -4\pi i y^2 \left(\omega_1 \frac{\partial}{\partial x} - \omega_2 \frac{\partial}{\partial y}\right) = -4\pi i y^2 \left(\omega \frac{\partial}{\partial \bar{z}} + \bar{\omega} \frac{\partial}{\partial z}\right) \\ N = 4\pi^2 y^2 \left|\omega(z)\right|^2 = 4\pi^2 y^2 \left(\omega_1^2 + \omega_2^2\right) \end{cases} \tag{6.6}$$

and $\omega = \omega_1 + i\omega_2$, $\bar{\omega} = \omega_1 - i\omega_2$. The domain of definition $D(L(\alpha))$ equals $\Omega(z, \alpha)^{-1} D(A_\alpha)$ and

$$L(\alpha) = \Omega(\cdot, \alpha)^{-1} A_\alpha \Omega(\cdot, \alpha). \tag{6.7}$$

Note that $M$ maps odd functions to even and even to odd. Recall that functions satisfying $f(-x+iy) = -f(x+iy)$ are odd and functions satisfying $f(-x+iy) = f(x+iy)$ are even by definition. Note also that a function $f$, satisfying $f(\gamma z) = \chi(\gamma) f(z)$, $\gamma \in \bar{\Gamma}_0(N)$, with our arithmetical character $\chi$, is allowed to be odd or even. It is also true for the trivial character. It is not difficult to see also that the differential operators $M, N$ map $(\bar{\Gamma}_0(N), \chi \cdot \chi_\alpha)$ automorphic functions to $(\bar{\Gamma}_0(N), \chi \cdot \chi_\alpha)$ automorphic functions.

We will determine now the form $\omega(z)$. We start with constructing the holomorphic Eisenstein series of weight 2 for $\bar{\Gamma}_0(N)$ without character, using non-holomorphic Eisenstein series of weight zero. This method goes back to Hecke (see also [Sch] p. 15). We consider the series (2.1) for $\Gamma = \bar{\Gamma}_0(N)$, $\chi = 1$. Then we define

$$G_g(z;s;\Gamma;1) = 2i\frac{\partial}{\partial z}E_g(z;s;\Gamma;1) = \left(\frac{\partial}{\partial y} + i\frac{\partial}{\partial x}\right)E_g(z;s;\Gamma;1) \tag{6.8}$$

$1 \le g \le h$, where $h$ is the number of all inequivalent cusps of $F$.

(The function $E_g(z,\cdot)$ depends also on $\bar{z}$, complex conjugate variable, since it is not a $z$-holomorphic function, so we have to write $E_g(z,\cdot) = E_g(z,\bar{z};\cdot)$ or $E_g(x,y;\cdot)$, $z = x + iy$, $\bar{z} = x - iy$, $\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$, $\frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)$). It is well known that each of the $E_g(z;s;\Gamma;1)$ has a simple pole at $s = 1$ with residue constant, independent of $g$. That means $G_g(z;s) = G_g(z;s;\Gamma;1)$ is regular at $s = 1$. We set $G_{g,2}(z) = G_g(z,1)$. It is clear then that $G_{g,2}(z)$ transforms as a modular form of weight 2,

$$G_{g,2}(\gamma z) = (z+d)^2 G_{g,2}(z) \text{ for any } \gamma \in \Gamma \tag{6.9}$$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

But $G_{g,2}(z)$ is not a holomorphic form. Let us denote $cz + d = j(\gamma;z)$.

From (2.6) follows the Fourier decomposition:

$$G_{g,2}(g_\beta z) j^{-2}(g_\beta;z) = \delta_{\alpha\beta} - \frac{2iC}{z - \bar{z}} - 4\pi \sum_{n=1}^{\infty} \sqrt{n}\varphi_{\alpha\beta n}(1;\Gamma;1)e^{2\pi i n z} \tag{6.10}$$

where $\mathrm{Res}_{s=1}E_g(z;s) = C,\ K_{1/2}(y) = \sqrt{\frac{\pi}{2y}}e^{-y}$.

Let $n_1, ..., n_h$ be integers with the condition

$$\sum_{\alpha=1}^{h} n_\alpha = 0. \tag{6.11}$$

Then we define

$$\sum_{\alpha=1}^{h} n_\alpha G_{\alpha,2}(z) = G(z; n_1, ..., n_h). \tag{6.12}$$

From (6.10) follows that $G(z; n_1, ..., n_h)$ is a holomorphic modular form of weight 2 for $\Gamma$.

Then it is not difficult to see that all periods

$$\int_{z_0}^{\gamma z_0} G(t, n_1, ..., n_h) dt \tag{6.13}$$

are real, $\gamma \in \Gamma$, $z_0 \in H$.

We construct now our form $\omega(z)$ as one of these functions $G(z; n_1, ..., n_h)$. Let us go back to Theorem 1.1. We consider there the last two cases. In Case 2) we had $\Gamma = \bar{\Gamma}_0(4N_2)$, $N_2$ is a square-free positive integer $N_2 \equiv 3 \bmod 4$. For our arithmetical character $\left(\frac{4N_2}{\cdot}\right)$ we have open cusps $z_d$, $d\,|N_2$, $d > 0$ and $d = 4d_2$, $d_2\,|N_2$, $d_2 > 0$. We have the closed cusps $z_d$, $d = 2d_1$, $d_1\,|N_2$, $d_1 > 0$. The total number of all closed cusps is

$$k\left(\bar{\Gamma}_0(4N_2), \left(\frac{4N_2}{\cdot}\right)\right) = d(N_2) \tag{6.14}$$

where $d(\cdot)$ is the divisor function. The total number of open cusps is

$$r\left(\bar{\Gamma}_0(4N_2), \left(\frac{4N_2}{\cdot}\right)\right) = 2d(N_2) \tag{6.15}$$

and we certainly have $h = k + r = 3d(N_2) = d(4N_2)$. We now define the form $\omega(z)$ by

$$\omega(z) = \sum_{\substack{d_1|N_2 \\ d_1>0}} n_{2d_1} G_{2d_1,2}(z) \tag{6.16}$$

where each of the $n_{2d_1}$ is equal to $\pm 1$ with the only condition

$$\sum_{\substack{d_1|N_2 \\ d_1>0}} n_{2d_1} = 0. \tag{6.17}$$

From (6.10) follows that $\omega(z)$ is exponentially small in all open cusps and it is like $j^2(g_\beta, z)$ in each closed cusp $\beta = 2d_1$.

In analogy to this we consider Case 3) of Theorem 1.1. We have $\Gamma = \bar{\Gamma}_0(4N_3)$, $N_3$ is a square-free positive integer, $N_3 \equiv 2 \bmod 4$, and $\chi = \left(\frac{4N_3}{\cdot}\right)$. The open cusps are $z_d$, $d\,|n$, $d > 0$ $(N_3 = 2n)$ and $d = 8d_3$, $d_3\,|n$ , $d_3 > 0$. The closed cusps are $z_d$, $d = 2d_1$, $d_1\,|n$, $d_1 > 0$ and $d = 4d_2$, $d_2\,|n$, $d_2 > 0$. We have

$$\begin{cases} k\left(\bar{\Gamma}_0(4N_3), \left(\frac{4N_3}{\cdot}\right)\right) = 2d(N_3/2) \\ r\left(\bar{\Gamma}_0(4N_3), \left(\frac{4N_3}{\cdot}\right)\right) = 2d(N_3/2) \\ h = k + r = 4d(N_3/2) = d(4N_3) \end{cases} \tag{6.18}$$

$$\omega(z) = \sum_{\substack{d_1|N_3/2 \\ d_1>0}} n_{2d_1} G_{2,2d_1}(z) + \sum_{\substack{d_2|N_3/2 \\ d_2>0}} n_{4d_2} G_{2,4d_2}(z) \tag{6.19}$$

where each of $n_{2d_1}$, $n_{4d_2}$ is equal to $\pm 1$ with the condition

$$\sum_{\substack{d_1|N_3/2 \\ d_1>0}} n_{2d_1} + \sum_{\substack{d_2|N_3/2 \\ d_2>0}} n_{4d_2} = 0. \tag{6.20}$$

From (6.20) again follows that $\omega(z)$ is exponentially small in all open cusps and it is like $j^2(g_\beta; z)$ in each closed cusp $\beta = 2d_1$, $\beta = 4d_2$.

Let us calculate now the parabolic main periods of $\omega(z)$ in the two cases (6.16), (6.19). We consider (6.16) first. Let $S_{d'}$ be a parabolic generator of Case 2), $\bar{\Gamma}_0(4N_2)$ (one of (1.9), (1.10) or (1.11). We have

$$\int\limits_{z_0}^{S_{d'}z_0} \omega(z)dz = \int\limits_{g_{d'}^{-1}z_0}^{g_{d'}^{-1}S_{d'}z_0} \omega(g_{d'}t)\frac{dt}{j^2(g_{d'};t} \tag{6.21}$$

where

$$g_{d'}\infty = z_{d'},\ g_{d'}^{-1}S_{d'}g_{d'}z = S_\infty\ z = z + 1. \tag{6.22}$$

The right hand side of (6.21) is equal to

$$\int\limits_{t_0}^{g_{d'}^{-1}S_{d'}g_{d'}t_0} \omega(g_{d'}t)\frac{dt}{j^2(g_{d'};t)} = \int\limits_{t_0}^{S_\infty t_0} \omega(g_{d'}t)\frac{dt}{j^2(g_{d'};t)} \tag{6.23}$$

$$= \sum_{\substack{d_1|N_2\\ d_1>0}} n_{2d_1} \int\limits_{t_0}^{S_\infty t_0} G_{2d_1,2}(g_{d'}t)\frac{dt}{j^2(g_{d'};t)}.$$

We apply formula (6.10) and we finally obtain

$$\int\limits_{z_0}^{S_{d'}z_0} \omega(z)dz = \sum_{\substack{d_1|N_2\\ d_1>0}} n_{2d_1}\delta_{(2d_1)d'},\ \delta_{\alpha\beta} = \begin{cases} 1 & \alpha=\beta \\ 0 & \alpha\neq\beta \end{cases} \tag{6.24}$$

This is zero, if $z_{d'}$ is any open cusp, since the sum in (6.24) is taken over closed cusps only. And if $z_{d'}$ is one of the closed cusps, then (6.24) is equal to $n_{d'} = \pm 1$.

The analogous calculation shows in Case 3) $\bar\Gamma_0(4N_3)$ that the $\omega(z)$ from (6.19) has the main parabolic periods equal to

$$\int\limits_{z_0}^{S_{d'}z_0} \omega(z)dz = \begin{cases} 0 & \text{if } z_{d'} \text{ is an open cusp for } \left(\frac{4N_3}{\cdot}\right) \\ n_{d'} & \text{if } z_{d'} \text{ is a closed cusp for } \left(\frac{4N_3}{\cdot}\right) \end{cases} \tag{6.25}$$

$n_{d'} = \pm 1$. Thus in Case 2) $\bar\Gamma_0(4N_2)$ with $\chi = \left(\frac{4N_2}{\cdot}\right)$ and $\chi_\alpha(\gamma) = \exp 2\pi i\alpha \operatorname{Re} \int\limits_{z_0}^{\gamma z_0} \omega(t)dt = \exp 2\pi i\alpha \int_{z_0}^{\gamma z_0} \omega(t)dt$ we obtain

$$\chi(S_{d'})\chi_\alpha(S_{d'}) = \begin{cases} 1 & \text{if } z_{d'} \text{ is an open cusp for } \left(\frac{4N_2}{\cdot}\right) \\ e^{2\pi i\alpha n_{d'} - \pi i} = e^{\pi i(2\alpha n_{d'}-1)} & \text{if } z_{d'} \text{ is closed for } \left(\frac{4N_2}{\cdot}\right) \end{cases} \tag{6.26}$$

$n_{d'} = \pm 1$. The same result is valid in Case 3) $\bar\Gamma_0(4N_3)$, $\chi = \left(\frac{4N_3}{\cdot}\right)$ and $\chi_\alpha(\gamma) = \exp 2\pi i\alpha \operatorname{Re} \int\limits_{z_0}^{\gamma z_0} \omega(t)dt = \exp 2\pi i\alpha \int\limits_{z_0}^{\gamma z_0} \omega(t)dt$. We have

$$\chi(S_{d'})\chi_\alpha(S_{d'}) = \begin{cases} 1 & \text{if } z_{d'} \text{ is open for } \left(\frac{4N_3}{\cdot}\right) \\ e^{2\pi i\alpha n_{d'} - \pi i} = e^{\pi i(2\alpha n_{d'}-1)} & \text{if } z_{d'} \text{ is closed for } \left(\frac{4N_3}{\cdot}\right) \end{cases} \tag{6.27}$$

$n_{d'} = \pm 1$. We obtain in both cases that for $\alpha \in (-1/2, 1/2)$ the character $\chi\cdot\chi_\alpha$ relative to the group $\Gamma$ has the same degree of singularity and keeps the same cusps open and closed. For $\bar{\Gamma}_0(4N_2)$ and $\left(\frac{4N_2}{\cdot}\right)$ it is given by (6.14) and for $\bar{\Gamma}_0(4N_3)$, $\left(\frac{4N_3}{\cdot}\right)$ by (6.18). This means that the perturbation (6.6) is regular for the constructed forms $\omega(z)$.

We now consider the case $\bar{\Gamma}_0(4N_2)$, $N_2 = p_1 p_2 ... p_k$.

We want to get an expression for the form $\omega(z)$ of (6.6) as

$$\omega(z) = \sum_{d|4N_2, d>0} P(dz)\alpha_d \tag{6.28}$$

with real coefficients $\alpha_d$, and we will prove that there exists a set of integers $n_{2d_1} = \pm 1$ satisfying (6.17) such that the coefficient $\alpha_1$ which corresponds to $d = 1$ is not zero,

$$\alpha_1 \neq 0. \tag{6.29}$$

Here $P(z)$ is the holomorphic Eisenstein series of weight 2 for the modular group $\bar{\Gamma}_0(1)$. We recall

$$P(z) = E_2(z) = 1 - 24\sum_{n=1}^{\infty} \sigma(n) e^{2\pi i n z}. \tag{6.30}$$

It is not quite a modular form of weight 2. We have the following transformation properties:

$$P\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 P(z) - \frac{6i}{\pi} c(cz+d), \; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \bar{\Gamma}_0(1). \tag{6.31}$$

In particular

$$\left\{ \begin{array}{l} P(-1/z) = z^2 P(z) - \frac{6i}{\pi} z \\ P(z+1) = P(z) \end{array} \right. . \tag{6.32}$$

We consider (6.28) as a system of linear equations with unknown $\alpha_d$, using well-known asymptotics of $\omega(z)$ and $P(z)$ at cusps of $F_0(4N_2)$, fundamental domain for $\bar{\Gamma}_0(4N_2)$.

When we defined the non-holomorphic Eisenstein series (2.1) we introduced the elements $g_j$. We now parametrize these elements by the divisors $d\,|4N_2$, and we will consider all inequivalent cusps of $\Gamma_0(4N_2)$, see (1.8), (1.9), (1.10), (1.11). We have

$$g_d S_\infty g_d^{-1} = S_d \tag{6.33}$$

$$g_d = \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix} \begin{pmatrix} \sqrt{m_d} & 0 \\ 0 & \sqrt{m_d}^{-1} \end{pmatrix} = \begin{pmatrix} \sqrt{m_d} & 0 \\ d\sqrt{m_d} & \sqrt{m_d}^{-1} \end{pmatrix}.$$

As a linear fractional transformation, $g_d z = \frac{m_d z}{d m_d z+1}$, it has integer coefficients. To calculate the asymptotics of the right hand side of (6.28) at the cusps $1/d'$ we have to find the asymptotics of the functions

$$P(dg_{d'}z) = P\left(\frac{dm_{d'}z}{d'm_{d'}z+1}\right), \; z \to \infty \tag{6.34}$$

for all positive divisors $d\,|4N_2$, $d'\,|4N_2$. We set $m_{d'}z = z'$ and consider $P\left(\frac{dz'}{d'z'+1}\right) = P\left(\frac{d_1 z''}{d_2 z''+1}\right)$ where

$$d_1 = d/(d,d'),\; d_2 = d'/(d,d'),\; z'' = (d,d')z' = (d,d')m_{d'}z$$

where $(d,d')$ is the greatest common divisor of $d, d'$. The matrix

$$\begin{pmatrix} d_1 & 0 \\ d_2 & 1 \end{pmatrix}$$

does not belong to $\bar\Gamma_0(1)$, so we can not directly apply formula (6.31), but since we have $(d_1,d_2) = 1$ we can make the following transformation. We define

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in PSL(2,\mathbb{R})$$

$$P\left(\frac{d_1 z''}{d_2 z''+1}\right) = P\left(g^{-1} \circ g \frac{d_1 z''}{d_2 z''+1}\right)$$

$$g\frac{d_1z''}{d_2z''+1}=\frac{\alpha\left(\frac{d_1z''}{d_2z''+1}\right)+\beta}{\gamma\left(\frac{d_1z''}{d_2z''+1}\right)+\delta}=\frac{(\alpha d_1+\beta d_2)z''+\beta}{(\gamma d_1+\delta d_2)z''+\delta} \tag{6.35}$$

We now choose $\alpha,\beta,\gamma,\delta$ by $\alpha d_1+\beta d_2=1,\ \gamma d_1+\delta d_2=0.$

For example $\gamma=-d_2,\ \delta=d_1$. Since $(d_1,d_2)=1$ there exist integers $\alpha,\beta$ with these conditions. That means $g\in SL(2,\mathbb{Z})$. We have

$$g\frac{d_1z''}{d_2z''+1}=\frac{z''+\beta}{d_1}=\overset{o}{z},$$

and we apply (6.31) to the function

$$P\left(g^{-1}\overset{o}{z}\right)=P\left(\frac{d_1\overset{o}{z}-\beta}{d_2\overset{o}{z}+\alpha}\right)=\left(d_2\overset{o}{z}+\alpha\right)^2P\left(\overset{o}{z}\right)-\frac{6i}{\pi}d_2\left(d_2\overset{o}{z}+\alpha\right). \tag{6.36}$$

We have finally from (6.34), (6.35), (6.36) and (6.30)

$$\begin{aligned}\left(d_2\overset{o}{z}+\alpha\right)^2&=\left(\frac{d_2(z''+\beta)}{d_1}+\alpha\right)^2=\left(\frac{d_2z''+1}{d_1}\right)^2 \\ &=\frac{1}{d_1^2}\left(d_2(d,d')m_{d'}z+1\right)^2 \\ &=\frac{1}{d_1^2}\left(d'm_{d'}z+1\right)^2=\frac{(d,d')^2}{d^2}\left(d'm_{d'}z+1\right)^2.\end{aligned} \tag{6.37}$$

That means, from (6.32), (6.37) we get

$$\lim_{z\to\infty}(d'm_{d'}z+1)^{-2}P(dg_{d'}z)=\frac{(d,d')^2}{d^2}. \tag{6.38}$$

That gives the desired asymptotics of the right hand side of (6.28).

From (6.10), (6.16) we can see that

$$\lim_{z\to\infty}G_{d,2}(g_{d'}z)\left(d'\sqrt{m_{d'}}z+\sqrt{m_{d'}}^{-1}\right)^{-2}=\delta_{dd'}$$

that means

$$\lim_{z\to\infty} G_{d,2}(g_{d'}z)(d'm_{d'}z+1)^{-2} = \frac{\delta_{dd'}}{m_{d'}}. \tag{6.39}$$

Combining with (6.16) we obtain the following system of linear equations where $d'\,|4N_2$

$$\sum_{\substack{d''|4N_2\\ d''>0}} \beta_{d''}(d',d'')^2 = \sum_{\substack{d_1|N_2\\ d_1>0}} \delta_{(2d_1)d'}\frac{n_{2d_1}}{m_{d'}} \tag{6.40}$$

where $\alpha_{d''} = \beta_{d''}\cdot d''^2$. From this system we have to determine the coefficients $\alpha_{d''}$ and to see that there exists a form $\omega(z)$ with $\alpha_1 \neq 0$.

Before studying the system (6.40) we will define the analogous system for the case $\bar{\Gamma}_0(4N_3)$. We have

$$\omega(z) = \sum_{d|4N_3\,,d>0} P(dz)\alpha_d. \tag{6.41}$$

Using the definition of $\omega(z)$ in this case (6.19), we obtain in analogy with (6.40) the system

$$\sum_{\substack{d''|4N_3\\ d''>0}} \beta_{d''}(d',d'')^2 = \sum_{\substack{d_1|N_3/2\\ d_1>0}} \delta_{(2d_1)d'}\frac{n_{2d_1}}{m_{d'}} + \sum_{\substack{d_2|N_3/2\\ d_2>0}} \delta_{(4d_2)d'}\frac{n_{4d_2}}{m_{d'}} \tag{6.42}$$

where $\alpha_{d''} = \beta_{d''}\cdot d''^2$.

In AppendixA we prove the following Theorem about solution of the systems of equations (6.40) and (6.42).

**Theorem 6.1** *In both the cases $\bar{\Gamma}_0(4N_2)$ and $\chi_{4n_2}$, $\bar{\Gamma}_0(4N_3)$ and $\chi_{4N_3}$ there exist forms $\omega(z)$ given by* (6.16), (6.17) *and* (6.19), (6.20) *with the properties that each of them is given by a formula* (6.28) *with rational coefficients $\alpha_d$, and the coefficient $\alpha_1$ is not zero.*

At the end of this paragraph we present some class of forms $\omega(z)$ with explicit coefficients $\alpha_d$. For these forms we can evaluate the Phillips-Sarnak integral (see Section 7). We have to satisfy two necessary conditions for the coefficients $\alpha_d$. Namely 1) $\omega(z)$ has to be a holomorphic form of weight 2 for our group $\bar{\Gamma}_0(N)$ with trivial character. 2) It has to be small in all open cusps for the character $\chi$. We have $\omega(z) = \sum\limits_{d|N} \alpha_d P(dz)$, $\omega(1/d') = 0$ if $1/d'$ is an open cusp. Then we have

$$P\left(dg_{d'}z\right) = P\left(\frac{dm_{d'}z}{Nz+1}\right) = (Nz+1)^2 P\left(\overset{\circ}{z}\right) \frac{(d,d')^2}{d^2} - \frac{6i}{\pi} \cdot \frac{d'}{d}\left(Nz+1\right).$$

The first condition becomes

$$\sum_{d|N} \alpha_d \cdot 1/d = 0 \tag{6.43}$$

since we have $d_2\left(d_2\overset{\circ}{z}+\alpha\right) = d_2 \cdot \frac{(d,d')}{d}(Nz+1) = \frac{d'}{(d,d')} \cdot \frac{(d,d')}{d}(Nz+1)$ (see notations for (6.35)). And the second condition is

$$\sum_{d|N} \frac{(d,d')^2}{d^2} \alpha_d = \gamma_{d'} = 0 \text{ if } 1/d' \text{ is open.} \tag{6.44}$$

We introduce now the notations $d = 2^{\beta_0} \cdot q_1^{\beta_1} ... q_k^{\beta_k}$, $d\,|N$, where $q_j$ are different primes. Then $\beta_0 = 0, 1, 2$ in the second case (see Theorem 1.1) and $\beta_0 = 0, 1, 2, 3$ in the third case. For other $\beta_j$ $j = 1, ..., k$ we have in both cases $\beta_j = 0, 1$. We denote

$$\alpha_d = \alpha_{\beta_0\beta_1...\beta_k}.$$

We will now prove the theorem, considering the cases 2) and 3) separately.

**Theorem 6.2** *Let* $\omega(z) = \sum_{d|N} \alpha_d P(dz), \alpha_d = \alpha_{\beta_0\beta_1...\beta_k}$*. The following systems of coefficients* $\alpha_{\beta_0...\beta_k}$ *define a class of forms* $\omega(z) = \sum\limits_{d|N} \alpha_d P(dz)$ *which satisfy* (6.43), (6.44) *and* *therefore define regular character perturbations of* $A(\bar{\Gamma}_0(N),\chi)$*. For* $N=4N_2$ *we have* $d=2^{\beta_0}q_1^{\beta_1}...q_k^{\beta_k}$*,* $q_j\,|N_2$ *different primes, then* $\alpha_d=\alpha_{\beta_0\beta_1...\beta_k}$*, where*

$$\begin{cases} \alpha_{0\beta_1...\beta_k}=(-1)^{\beta_1+\beta_2...+\beta_k}\varepsilon_1^{\beta_1}...\varepsilon_k^{\beta_k}q_1^{\beta_1}...q_k^{\beta_k},\ \varepsilon_1=1 \\ \alpha_{1\beta_1...\beta_k}=-5\alpha_{0\beta_1...\beta_k} \\ \alpha_{2\beta_1...\beta_k}=4\alpha_{0\beta_1...\beta_k} \\ \beta_j=0,1,\ j=1,...,k. \end{cases} \tag{6.45}$$

*For* $N=8N_3,\ d=2^{\beta_0}q_1^{\beta_1}...q_k^{\beta_k}$

$$\begin{cases} \alpha_{0\beta_1...\beta_k}=(-1)^{\beta_1+...+\beta_k}\varepsilon_1^{\beta_1}...\varepsilon_k^{\beta_k}q_1^{\beta_1}...q_k^{\beta_k},\ \varepsilon_1=1 \\ \alpha_{1\beta_1...\beta_k}=-7\alpha_{0\beta_1...\beta_k} \\ \alpha_{2\beta_1...\beta_k}=14\alpha_{0\beta_1...\beta_k} \\ \alpha_{3\beta_1...\beta_k}=-8_{\alpha_{0\beta_1...\beta_k}} \end{cases} \tag{6.46}$$

*where* $\varepsilon_2,...,\varepsilon_k$ *are any real numbers.*

**Proof.** Let $N=4N_2$. We have

$$\frac{1}{d}\alpha_{0\beta_1...\beta_k}=(-1)^{\beta_1+...+\beta_k}\varepsilon_2^{\beta_2}...\varepsilon_k^{\beta_k} \tag{6.47}$$

and if we sum (6.47) over all $d\,|N_2$ we are obviously getting zero, because of $(-1)^{\beta_1}$ coming with $\beta_1=0$ and $\beta_1=1$. The same is true for the other two lines in (6.45). So the condition (6.43) is satisfied. We check now (6.44). We have

$$\sum_{\beta_0=0}^{2}\sum_{\beta_1=0}^{1}...\sum_{\beta_k=0}^{1}\frac{\alpha_{\beta_0...\beta_k}}{2^{2\beta_0}q_1^{2\beta_1}...q_k^{2\beta_k}}2^{2\min(\beta_0,\beta_0')}q_1^{2\min(\beta_1,\beta_1')}...q_k^{2\min(\beta_k,\beta_k')} \tag{6.48}$$

$$=\sum_{\beta_1=0}^{1}...\sum_{\beta_k=0}^{1}\frac{(-1)^{\beta_1+...\beta_k}}{q_1^{\beta_1}...q_k^{\beta_k}}\varepsilon_1^{\beta_1}...\varepsilon_k^{\beta_k}\cdot q_1^{2\min(\beta_1,\beta_1')}...q_k^{2\min(\beta_k,\beta_k')}$$

$$\left(1-\frac{5}{4}\cdot 2^{2\min(1,\beta_0')}+\frac{4}{16}\cdot 2^{2\min(2,\beta_0')}\right). \tag{6.49}$$

From Theorem 1.1 follows that $\beta_0'=0$ or $\beta_0'=2$ for the open cusp $z_{d'}$. Then the last term in parenthesis of (6.48) is equal to zero in both these cases.

The case $N=4N_3$ is dealt with in the same way. For the last step we have the common multiple

$$1-\frac{7}{4}\cdot 2^{\min(1,\beta_0')}+\frac{14}{16}\cdot 2^{2\min(2,\beta_0')}-\frac{8}{64}2^{2\min(3,p_0')}. \tag{6.50}$$

The open cusps in that case correspond to $\beta_0'=0$ or $\beta_0'=3$, and we obtain in that case that (6.50) is equal to zero. The theorem is proved. ■

These forms $\omega(z)$ are important for our perturbation (6.5), (6.6), and precisely for these forms we will consider the Phillips-Sarnak conjecture.

# 7 The Phillips-Sarnak integral

In this section we study the Phillips-Sarnak integral, adapted to our perturbation (6.6). For any odd eigenfunction of Theorem 4.2, which corresponds to an embedded eigenvalue $\lambda_j>1/4$ (actually, according to the Selberg eigenvalue conjecture, reduced to our case of congruence character, all $\lambda_j\geq 1/4$) we define the integral over the fundamental domain $F_0(N)$ of $\bar{\Gamma}_0(N)$. We use the notations of Section 6 (see the beginning of Section 6). The cusp $1/N$ is equivalent to $\infty$. So we have $F_0(N)$, containing $\infty$, and we define the Eisenstein series

$$E_\infty(z,s)=E_\infty(z,s;\bar{\Gamma}_0(N);\chi)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} y^s(\gamma z)\chi(\gamma). \tag{7.1}$$

The integral is the following one

$$I_j(s)=\int\limits_{F_0(N)}(Mv_j)(z)E_\infty(z,s)d\mu(z). \tag{7.2}$$

**Theorem 7.1** *1) For any Hecke eigenfunction $v_j$ of $A(\bar{\Gamma}_0(N);\chi)$, given by Theorem 4.2, with eigenvalue $\lambda_j = s_j(1-s_j) = \frac{1}{4}+r^2$ and for any form $\omega(z)$ from Section 6 the integral $I_j(s_j)$ is well defined.*

*2) Let $\omega(z)$ be a form given by Theorem 6.2 with real parameters $\varepsilon_{2,...,}\varepsilon_k$ ($\varepsilon_1 = 1$). Let $\varepsilon_2, ..., \varepsilon_l = \pm 1$ and $\varepsilon_m \neq \pm 1$ for $m = l+1, ..., k$. Then $I_j(s_j) \neq 0$ if $s_j = \frac{1}{2} + ir_j$ does not belong to any of the following sequences:*

$$s_n = \frac{1}{2} + ir_n \text{ with } r_n = n\frac{\pi}{\log 2},\ r_n = n\frac{\pi}{\log q_m},\ m = 1...l,\ n \in \mathbb{Z}.$$

**Proof.** The proof contains two parts, the first being the calculation of the integral, and the second the evaluation of the Dirichlet series coming from $v_j$ and $\omega$.

We take first $\mathrm{Re}\, s > 1$. It is not difficult to see that $v_j(z)$ is a function of exponential decay in all parabolic cusps of $F_0(N)$. This follows from the fact that $v_j(z)$ is an eigenfunction of the Laplacian, which is $(\bar{\Gamma}_0(N), \chi)$ automorphic. In open cusps $z_m$ it is a cusp form and it has Fourier decomposition

$$v_j(g_m z) = \sqrt{y}\sum_{n\neq 0} \rho_j^{(m)}(n) K_{s_j-1/2}(2\pi\,|n|\,y)e^{2\pi inx} \tag{7.3}$$

where $g_m$ is defined in (2.1). In closed cusps $z_l$ it has Fourier decomposition

$$v_j(g_l z) = \sqrt{y}\sum_{n=-\infty}^{\infty} \rho_j^{(l)}(n) K_{s_j-1/2}(2\pi\,|n+1/2|\,y)e^{2\pi i(n+1/2)x}. \tag{7.4}$$

Then it is obvious that

$$\begin{cases} v_j(\gamma z) = \chi(\gamma)v_j(z),\ (Mv_j)(\gamma z) = \chi(\gamma)(Mv_j)(z) \\ E_\infty(\gamma z; s) = \chi(\gamma)E_\infty(z, s) \\ \chi^2(\gamma) = 1,\ \gamma \in \bar{\Gamma}_0(N) \end{cases}.$$

That means that the integral (7.2) is well defined and we can unfold the Eisenstein series $E_\infty(z,s)$, obtaining

$$I_j(s) = \int_0^\infty \frac{dy}{y^2} \int_{-1/2}^{1/2} dx(Mv_j(z))y^s \tag{7.5}$$

where $y^{-2}(Mv_j)(z) = -4\pi i(\omega_1 v_{jx} - \omega_2 v_{jy})$ (see (6.6)),

$$v_{jx} = \frac{\partial v_j}{\partial x},\ v_{jy} = \frac{\partial v_j}{\partial y}.$$

Then we have

$$\begin{aligned} \int_{-1/2}^{1/2} \omega_1(x,y) v_{jx}(x,y)dx &= \omega_1 v_j \int_{-1/2}^{1/2} - \int_{-1/2}^{1/2} \omega_{1x}(x,y)v_j(x,y)dx \\ &= - \int_{-1/2}^{1/2} \omega_{1x}(x,y)v_j(x,y)dx \end{aligned} \tag{7.6}$$

because $\omega$ and $v_j$ are periodic in $x$ with period 1. Similarly

$$\begin{aligned} \int_0^\infty y^s \omega_2 v_{jy} dy &= y^s \omega_2 v \Big|_0^\infty - s\int_0^\infty y^{s-1}\omega_2 v_j dy - \int_0^\infty y^s v_j \omega_2 dy \\ &= -s\int_0^\infty y^{s-1}\omega_2 v_j dy - \int_0^\infty y^s v_j \omega_{2y} dy. \end{aligned} \tag{7.7}$$

Also we have Fourier decompositions

$$\begin{cases} \omega_1(x,y) = \sum_{n=1}^\infty a_n e^{-2\pi n y}\cos 2\pi n x \\ \omega_2(x,y) = \sum_{n=1}^\infty a_n e^{-2\pi n y}\sin 2\pi n x \end{cases}. \tag{7.8}$$

Using (7.3)-(7.8) we obtain

$$\begin{aligned} I_j(s) &= 4\pi i \int_0^\infty y^s dy \int_{-1/2}^{1/2} dx(\omega_{1x} - \omega_{2y})v_j - 4\pi i s\int_0^\infty y^{s-1}dy \int_{-1/2}^{1/2} \omega_2 v_j dx \\ &= -4\pi i s \int_0^\infty y^{s-1}dy \int_{-1/2}^{1/2} \omega_2(x,y)v_j(x,y)dx. \end{aligned} \tag{7.9}$$

Then we apply (4.12), (7.8) to (7.9). We obtain

$$
\begin{aligned}
I_j(s) &= 4\pi s \int_0^\infty y^{s-1/2} \sum_{n=1}^\infty a_n \rho_j(n) e^{-2\pi n y} K_{s_j-1/2}(2\pi n y)dy \qquad (7.10)\\
&= 4\pi s \frac{1}{(2\pi)^{s+1/2}} \left( \int_0^\infty t^{s-1/2} e^{-t} K_{s_j-1/2}(t)dt \right) \sum_{n=1}^\infty \frac{a_n \rho_j(n)}{n^{s+1/2}}.
\end{aligned}
$$

The standard integral in brackets is equal to

$$
\sqrt{\pi} \cdot 2^{-s-1/2} \frac{\Gamma(s+s_j)\Gamma(s-s_j+1)}{\Gamma(s+1)} \qquad (7.11)
$$

and we finally obtain

$$
I_j(s) = \frac{s}{2^{2s-1}\pi^{s-1}} \cdot \frac{\Gamma(s+s_j)\Gamma(s-s_j+1)}{\Gamma(s+1)} \sum_{n=1}^\infty \frac{a_n \rho_j(n)}{n^{s+1/2}}. \qquad (7.12)
$$

We have now $s = 1/2 + i\tau,\ \tau \neq 0,\ s_j = 1/2 + i\tau_j,\ \tau_j \neq 0\ (\tau, \tau_j \in \mathbb{R})$.

With these conditions the factor to the Dirichlet series in (7.12) is never equal to the zero. So we have to study the Dirichlet series in more detail.

We have $\omega(z) = \sum\limits_{d|N} \alpha_d P(dz)$. We introduce $\tilde{\omega}(z) = -\frac{1}{24}\omega(z)$, $b(n) = -1/24 a_n$. The series we will study is

$$
R_j(s) = \sum_{n=1}^\infty \frac{b(n)\rho_j(n)}{n^{s+1/2}},\ \operatorname{Re} s > 1. \qquad (7.13)
$$

We have then

$$
\tilde{\omega}(z) = \sum_{d|N} \alpha_d \sum_{n=1}^\infty \sigma(n) e^{2\pi i n d z}. \qquad (7.14)
$$

We arrange the summation in (7.13) in the following way. We write $n = p_0^{r_0} p_1^{r_1} ... p_k^{r_k} \cdot p_{k+1}^{r_{k+1}} ... p_m^{r_m}$ where $r_j \geq 0$, $0 \leq j \leq m$, $m = 0, 1, ...$ and $p_j\,|N$, $0 \leq j \leq k$, $p_0 = 2$. Then we have for $d\,|N$, $d=p_0^{\beta_0}p_1^{\beta_1}...p_k^{\beta_k}$ where $\beta_0=0,1,2$ in the case $N=4N_2$ and $\beta_0=0,1,2,3$ in the case $N=4N_3$. For other primes $\beta_j=0,1,\ 1\le j\le k$ we have

$$\tilde{\omega}(z)=\sum_{\beta_0,...,\beta_k}\alpha_{\beta_0...\beta_k}\sum_n\sigma\left(p_0^{r_0}\right)...\sigma\left(p_m^{r_m}\right)\exp 2\pi iz\left(p_0^{r_0+\beta_0}...p_k^{r_k+\beta_k}\cdot p_{k+1}^{r_{k+1}}...p_{p_m}^{r_m}\right). \quad (7.15)$$

Using (7.14), (7.15) we obtain

$$b\left(p_0^{r_0}...p_m^{r_m}\right)={\sum}'_{\beta_0,...,\beta_k}\alpha_{\beta_0...\beta_k}\sigma\left(p_0^{r_0-\beta_0}\right)...\sigma\left(p_k^{r_k-\beta_k}\right)\sigma\left(p_{k+1}^{r_{k+1}}\right)...\sigma\left(p_m^{r_m}\right). \quad (7.16)$$

The prime means that the sum is taken over $\beta_j$ with the conditions $r_j-\beta_j\ge 0$.

We have from (7.16) and Theorem 4.2, setting

$$n=p_0^{l_0+\beta_0}...p_k^{l_k+\beta_k}\cdot p_{k+1}^{l_{k+1}}...p_m^{l_m}, \quad (7.17)$$

$$\sum_{n=1}^{\infty}\frac{b(n)\rho_j(n)}{n^{s+1/2}}=\sum_{\beta_0,...,\beta_k}\alpha_{\beta_0...\beta_k}\sum_{n=1}^{\infty}\frac{\sigma\left(p_0^{l_0}\right)\sigma\left(p_1^{l_1}\right)...\sigma\left(p_m^{l_m}\right)\rho_j\left(p_0^{l_0+\beta_0}...p_k^{l_k+\beta_k}\cdot p_{k+1}^{l_{k+1}}...p_m^{l_m}\right)}{\left(p_0^{l_0+\beta_0}...p_k^{l_k+\beta_k}\cdot p_{k+1}^{l_{k+1}}...p_m^{l_m}\right)^{s+1/2}} \quad (7.18)$$

$$=\sum_{\beta_0,...,\beta_k}\alpha_{\beta_0...\beta_k}\frac{\rho_j^{\beta_0}(p_0)...\rho_j^{\beta_k}(p_k)}{\left(p_0^{\beta_0}...p_k^{\beta_k}\right)^{s+1/2}}\prod_{q|N}\sum_{n=0}^{\infty}\frac{\sigma(q^n)\rho_j^n(q)}{q^{n(s+1/2)}}\prod_{p\nmid N}\sum_{n=0}^{\infty}\frac{\sigma(p^n)\rho_j(p^n)}{p^{n(s+1/2)}}.$$

The right hand side of (7.18) is the product of three factors $\theta_1,\theta_2,\theta_3$. We consider first $\theta_2$ and $\theta_3$. For $\theta_2$ we have

$$\theta_2=\prod_{q|N}\sum_{n=0}^{\infty}\frac{\sigma(q^n)\rho_j^n(q)}{q^{n(s+1/2)}}=\prod_{q|N}\sum_{n=0}^{\infty}\frac{q^{n+1}-1}{q-1}\cdot\frac{\rho_j^n(q)}{q^{n(s+1/2)}} \quad (7.19)$$
$$=\prod_{q|N}\left(1-\rho_j(q)q^{-s+1/2}\right)^{-1}\left(1-\rho_j(q)q^{-s-1/2}\right)^{-1}.$$

For $\theta_3$ we have

$$\theta_3=\prod_{p\nmid N}\sum_{n=0}^{\infty}\frac{\sigma(p^n)}{p^{n(s+1/2)}}\rho_j(p^n) \quad (7.20)$$
$$=\prod_{p\nmid N}\sum_{n=0}^{\infty}\frac{\rho_j(p^n)}{p^{n(s+1/2)}}\cdot\frac{p^{n+1}-1}{p-1}$$
$$=\prod_{p\nmid N}\sum_{n=0}^{\infty}\frac{1}{p-1}\left(\frac{p\rho_j(p^n)}{p^{n(s-1/2)}}-\frac{\rho_j(p^n)}{p^{n(s+1/2)}}\right).$$

Then we get

$$\theta_3 = \prod_{p\nmid N} \frac{1}{p-1}[p\left(1-\rho_j(p)p^{-(s-1/2)}+\chi(p)p^{-2(s-1/2)}\right)^{-1} \tag{7.21}$$
$$- (1-\rho_j(p)p^{-(s+1/2)}+\chi(p)p^{-2(s+1/2)})^{-1}]$$
$$= \prod_{p\nmid N}(1-\chi(p)p^{-2s})(1-\rho_j(p)p^{-(s-1/2)}+\chi(p)p^{-2s+1})^{-1}$$
$$\cdot (1-\rho_j(p)p^{-(s+1/2)}+\chi(p)p^{-2s-1})^{-1}$$

Then using Theorem 5.1 we obtain that the product of $\theta_2$ and $\theta_3$ from (7.18) is equal to

$$\theta_2 \cdot \theta_3 = \prod_{q|N} \cdot \prod_{p\nmid N} = L^{-1}(2s;\chi)L(s+1/2;v_j)L(s-1/2;v_j). \tag{7.22}$$

From Theorem 5.4 follows that (7.22) is not zero for any $s = 1/2 + ir$, $r \in \mathbb{R}$. In order to prove the theorem we have to study now the first factor $\theta_1$ in (7.18). Here we have to consider separately two different cases $N = 4N_2$, $N = 4N_3$. For $N = 4N_2$ the factor $\theta_1$ is equal to (with notation $p_j = q_j\,|N$)

$$\sum_{\beta_0\dots\beta_k} \alpha_{\beta_0\dots\beta_k}\rho_j(2)^{\beta_0}\rho_j(q_1)^{\beta_1}\dots\rho_j(q_k)^{\beta_k}\left(2^{\beta_0}q_1^{\beta_1}\dots q_k^{\beta_k}\right)^{-(s+1/2)} \tag{7.23}$$
$$= \left(1-5\rho_j(2)\cdot 2^{\frac{1}{2}-s}+4\rho_j^2(2)\cdot 2^{1-2s}\right)\prod_{q|N_2,q \text{ prime}}\left(1-\frac{\varepsilon_q}{q^{s-1/2}}\rho_j(q)\right),\ \varepsilon_q=\varepsilon_{q_j}.$$

We have $s - 1/2 = ir$. To make (7.23) equal to zero we have to satisfy one of the following conditions,

$$\rho_j(2) = 2^{ir_j},\ \rho_j(2) = 1/2\cdot 2^{ir_j},\ \rho_j(q) = q^{ir_j}/\varepsilon_q.$$

We apply now Theorem 4.3 that $\rho_j(q) = \pm 1$, $\rho_j(2) = \pm 1$, and that gives the result in the case $N = 4N_2$. To prove it for $N = 4N_3$ we have to see also the equation

$$1-7\rho_j(2)\cdot 2^{-ir}+14\rho_j^2(2)2^{-2ir}-8\rho_j^3(2)2^{-3ir}=0. \tag{7.24}$$

That gives solutions for $\rho_j(2)$ equal to $2^{ir_j}$, $1/2 \cdot 2^{ir_j}$, $1/4 \cdot 2^{ir_j}$ and the result follows as in the case $N = 4N_2$. The theorem is proved. ■

# 8 Perturbation of embedded eigenvalues

**Definition 1** *Suppose that $F$ has $h$ cusps $z, ..., z_h$ and that under the character $\chi_\alpha$ the cusps $z_1, ..., z_k$ are open and $z_{k+1}, ..., z_h$ are closed. Let $\gamma_i z_i = \infty$, $i = 1, ..., k$, where $\gamma_i = g_i^{-1}$. The Banach spaces $C_{\mu,\nu} = C_{\mu,\nu}(F)$ are defined as the spaces of continuous functions $f$ on $F$ such that*

$$|f(\gamma_i z)| \leq C\,|\mathrm{Im}\,\gamma_i z|^{\mu} \ \textit{for } i = 1, ..., k$$

$$|f(\gamma_i z)| \leq C\,|\mathrm{Im}\,\gamma_i z|^{\nu} \ \textit{for } i = k+1, ..., h$$

*with the norm*

$$\|f\|_{\mu,\nu} = \max\left\{ \max_{1\leq i\leq k} \sup_{\substack{z\in F \\ \mathrm{Im}\,\gamma_i z\geq 1}} |f(\gamma_i z)|\,(\mathrm{Im}\,\gamma_i z)^{-\mu}, \max_{k+1\leq i\leq h} \sup_{\substack{z\in F \\ \mathrm{Im}\,\gamma_i z\geq 1}} |f(\gamma_i z)|\,(\mathrm{Im}\,\gamma_i z)^{-\nu} \right\}.$$

We utilize mainly the spaces $C_{1,-2}$, $C_1 = C_{1,1}$ and $C_{-1,0}$.

We make use of results of [F] on estimates and mapping properties of the resolvent kernel of the Laplacian $A(\Gamma)$ extended by [V1] to operators $A(\Gamma, \chi)$ with character $\chi$. From the results of [F] and [V1] we obtain the following theorem.

**Theorem 8.1** *For any $\alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ the resolvent $R(s,\alpha)$ of $L(\alpha) = L + \alpha M + \alpha^2 N$ has an analytic continuation $\tilde{R}(s,\alpha)$ to $\{s\,|0 < \mathrm{Re}\,s < 2\}$ as an operator in $B(C_{-1,0}, C_{1,-2})$. For $\mathrm{Re}\,s > 1$, $R(s,\alpha) \in B(C_{1,-2})$.*

We set $\tilde{R}(s) = \tilde{R}(s, 0)$.

We consider the mapping properties of the operators $M$ and $N$. In the open cusps the coefficients of $M$ and $N$ are exponentially decreasing, in the closed cusps they go like $y^2$. It follows that $V(\alpha) \in B(C_{1,-2}), C_{-1,0})$, where $V(\alpha) = \alpha M + \alpha^2 N$. This implies

**Lemma 8.2** *$V(\alpha)\tilde{R}(s) \in B(C_{-1,0})$ for $0 < s < 2$,*

$$\left\| V(\alpha)\tilde{R}(s) \right\|_{B(C(-1,0))} \underset{\alpha \to 0}{\longrightarrow} 0$$

*and for $|\alpha| < \varepsilon$*

$$\tilde{R}(s, \alpha) = \tilde{R}(s)(1 + V(\alpha)\tilde{R}(s))^{-1} \in B(C_{-1,0}, C_{1,-2}).$$

$\tilde{R}(s, \alpha)$ is analytic with values in $B(C_{-1,0}, C_{1,-2})$ for $0 < \operatorname{Re} s < 2$ as a function of $\alpha$ for $|\alpha| < \varepsilon$.

We now consider the operator $L_1(\alpha) = L + \alpha M + \alpha^2 N$ acting in the Banach space. $C_{1,-2}$ with maximal domain $\mathcal{D}(L_1(\alpha))$.

By Theorem 8.1 the resolvent $R_1(s)$ of $L_1 = L_1(0)$ exists as an operator in $B(C_{1,-2})$ for $\operatorname{Re} s > 1$, hence $V(\alpha)R_1(s) \in B(C_{1,-2})$ and $\|V(\alpha)R_1(s)\| \underset{\alpha \to 0}{\longrightarrow} 0$ for $\operatorname{Re} s > 1$. Moreover, for $|\alpha| < \varepsilon$ and $\operatorname{Re} s > 1$

$$R_1(s, \alpha) = R_1(s)(1 + V(\alpha)R_1(s))^{-1} \in B(C_{1,-2}).$$

It follows that $L_1(\alpha)$ is closed on the domain $\mathcal{D}(L_1(\alpha)) = \mathcal{D}(L_1)$ for $|\alpha| < \varepsilon$. We have established

**Lemma 8.3** *$L_1(\alpha)$ is analytic for $|\alpha| < \varepsilon$ as a family of closed operators in $C_{1,-2}$ with domain $D(L_1)$.*

We shall analyze now the perturbation of embedded eigenvalues. This was investigated by [Ho] for Schrödinger operators $-\Delta+\alpha V$ with multiplicative potential $V$. In our case the form of the perturbation requires a somewhat different approach, combining the analytic family $\tilde{P}(\alpha)$ derived from Faddeev's analytic continuation of the resolvent with Kato's theory of regular perturbations of isolated eigenvalues for selfadjoint operators [K].

Let $\lambda_0 = s_0(1 - s_0) > \frac{1}{4}$ be an eigenvalue of $L = A(\bar{\Gamma}, \chi)$, $s_0 = \frac{1}{2} + it_0$, $t_0 \neq 0$, with eigenspace $\mathcal{N} = \mathcal{N}(L - \lambda_0)$ of dimension $m$.

Let $K = K(s_0, \delta)$ be a circle with center $s_0$ and radius $\delta$ separating $s_0$ from other points $s_i$ corresponding to eigenvalues $\lambda_i = s_i(1 - s_i)$ of $L$, and choose $\varepsilon > 0$ such that $\tilde{R}(s, \alpha) \in B(C_{-1,0}, C_{1,-2})$ for $s \in K$ and $|\alpha| < \varepsilon$. The operators $\tilde{P}(\alpha) \in B(C_{-1,0}, C_{1,-2})$ are defined for $|\alpha| < \varepsilon$ by

$$\tilde{P}(\alpha) = -\frac{1}{2\pi i} \int_K \tilde{R}(s, \alpha)(2s - 1) ds.$$

$\tilde{P}(\alpha)$ is analytic in $\alpha$ for $|\alpha| < \varepsilon$, and $\tilde{P}(0)$ coincides with the orthogonal projection $P_0$ of $H$ on $N(L - \lambda_0)$, restricted to $C_{-1,0}$.

We consider the operators

$$
\begin{aligned}
P_0\tilde{P}(\alpha)P_0 &= -\frac{1}{2\pi i}P_0\int_K \tilde{R}(s,\alpha)(2s-1)dsP_0 \\
&= -\frac{1}{2\pi i}P_0\int_K \tilde{R}(s)(2s-1)dsP_0 \\
&+ \frac{1}{2\pi i}P_0\int_K \tilde{R}(s,\alpha)(\alpha M+\alpha^2 N)\tilde{R}(s)(2s-1)dsP_0 \\
&= P_0 + \alpha P_0\frac{1}{2\pi i}\int_K \tilde{R}(s,\alpha)(M+\alpha N)\tilde{R}(s)(2s-1)dsP_0.
\end{aligned}
$$

Here we use that $\mathcal{R}(P_0) = \mathcal{N} \subset C_{-1,0} = D(\tilde{P}(\alpha))$. Since the eigenfunctions $\phi \in N$ decay exponentially, we can also consider $P_0$ as an operator in $B(C_{1,-2},\mathcal{H})$, so $P_0\tilde{P}(\alpha)P_0 \in B(\mathcal{H})$. For $\alpha \to 0$, the second term converges in norm to zero.

It follows that $\dim R(P_0\tilde{P}(\alpha)P_0) = \dim R(P_0) = m$ for $|\alpha| < \varepsilon$.

The circle $K$ contains for each $\alpha$ with $|\alpha| < \varepsilon$ a finite number of poles $s_1(\alpha), ..., s_k(\alpha)$ of the meromorphic function $\tilde{R}(s,\alpha)$ with values in $B(C_{-1,0}, C_{1,-2})$. Let $\tilde{P}_i(\alpha) = -\mathrm{Res}\Big\{\tilde{R}(s,\alpha)\Big\}_{s=s_i(\alpha)}$. Then

$$\tilde{P}(\alpha) = \sum_{i=1}^{k} \tilde{P}_i(\alpha).$$

For $|\alpha| < \varepsilon$ we have

$$m = \dim \mathcal{R}(\tilde{P}(\alpha)P_0) = \sum_{i=1}^{k} \dim \mathcal{R}(\tilde{P}_i(\alpha)P_0. \tag{8.1}$$

This implies that for $|\alpha| < \varepsilon$ all the poles $s_i(\alpha)$ of $\tilde{R}(\alpha, s)$ inside $K$ are simple.

Then we have with $\lambda_i(\alpha) = s_i(\alpha)(1 - s_i(\alpha))$

$$(L_1(\alpha) - \lambda_i(\alpha))\tilde{P}_i(\alpha) = 0, \; i = 1, ..., k.$$

We choose a basis $\phi_1, ..., \phi_m$ of $N$ and set $\phi_j(\alpha) = \tilde{P}(\alpha)\phi_j$.

Let now $\phi(\alpha)$ be an eigenfunction or a resonance function with eigenvalue or resonance $\lambda(\alpha)$. Then $\phi(\alpha) \in \tilde{P}(\alpha)N$, hence

$$\phi(\alpha) = \sum_{j=1}^{m} a_j(\alpha)\phi_j(\alpha) = \sum_{j=1}^{m} a_j(\alpha)\tilde{P}(\alpha)\phi_j. \tag{8.2}$$

The condition $(L_1(\alpha) - \lambda(\alpha))\phi(\alpha) = 0$ is equivalent to

$$\langle (L_1(\alpha) - \lambda(\alpha))\phi(\alpha), \phi_k\rangle = 0, \; k = 1...m \tag{8.3}$$

or

$$\sum_{j=1}^{m} a_j(\alpha)\langle (L_1(\alpha) - \lambda(\alpha))\tilde{P}(\alpha)\phi_j, \phi_k\rangle = 0, \; k = 1...m.$$

In order that these equations have non-trivial solutions, it is necessary and sufficient that

$$d(\lambda; \alpha) = \det\left\{\langle (L_1(\alpha) - \lambda(\alpha))\tilde{P}(\alpha)\phi_j, \phi_k\rangle\right\}_{j,k=1}^{m} = 0. \tag{8.4}$$

Since the coefficients of the polynomial $d(\lambda, \alpha)$ are analytic in $\alpha$ for $|\alpha| < \varepsilon$, it follows that the $m-$dimensional eigenvalue $\lambda_0$ splits into analytic functions $\lambda_i(\alpha)$ and branches $\lambda_{jl}(\alpha)$ of Puiseux cycles. Any embedded, real eigenvalue is analytic, while resonances, which are non-real for $\alpha \neq 0$, may be analytic functions or branches of Puiseux cycles.

Due to the special circumstance that a resonance $\lambda_{jl}(\alpha)$ can not move into the resolvent set, but has to move to the second sheet, the Puiseux series for $\lambda_{jl}(\alpha)$ has to begin with a polynomial of the form

$$\sum_{l=0}^{2k-1} a_l\alpha^l + a_{2k}\alpha^{2k},\ a_l \text{ real for } l \leq 2k-1,\ \operatorname{Im} a_{2k} \neq 0,\ k \geq 1.$$

This implies that $\lambda_{jl}(\alpha) \in C^{2k}(-\varepsilon, \varepsilon)$.

Based on this we construct a Kato basis for $N$ and $L(\alpha)$.

Let

$$\lambda_1(\alpha), \lambda_2(\alpha), ..., \lambda_s(\alpha)$$

be the distinct real eigenvalues of $L(\alpha)$ with multiplicity $m_i$, $i = 1, ..., s$, and let

$$\lambda_{s+1}(\alpha), \lambda_{s+2}(\alpha), ..., \lambda_{s+t}(\alpha)$$

be the distinct resonances of $L(\alpha)$ with multiplicity $m_i$, $i = s+1, ..., s+t$. The functions $\lambda_1(\alpha), ..., \lambda_s(\alpha)$ are analytic for $|\alpha| < \varepsilon$, while the functions $\lambda_{s+1}(\alpha), ..., \lambda_{s+t}(\alpha)$ are at least $C^2(-\varepsilon, \varepsilon)$ and may be analytic. The dimensions $m_j$, $j = 1, ..., s+t$, are independent of $\alpha$ for $0 < |\alpha| < \varepsilon$ and $\sum\limits_{j=1}^{s+t} m_j = m$.

For $i = 1, .., s$ we get from (8.3) with $\lambda(\alpha) = \lambda_i(\alpha)$

$$\sum_{j=1}^{m} a_j(\alpha)\langle(L_1(\alpha) - \lambda_i(\alpha))\tilde{P}(\alpha)\phi_j, \phi_k\rangle = 0,\ k = 1, ..., m. \tag{8.5}$$

Since $d(\lambda_i(\alpha), \alpha) = 0$ and $\lambda_i(\alpha)$ has multiplicity $m_i$ as root in $d(\lambda_i(\alpha), \alpha)$, we can obtain $m_i$ linearly independent solutions $\{a_{lj}(\alpha)\}_{j=1}^{m}$ of (8.5), $l = 1, ..., m_i$. Then by (8.2) we obtain $m_i$ linearly independent eigenfunctions $\phi_l(\alpha) = \sum_{j=1}^{m} a_{lj}(\alpha)\phi_j(\alpha)$, analytic for $|\alpha| < \varepsilon$. It follows that for any linear combination $\phi = \sum_{j=1}^{m_i} c_l\phi_l(0)$, $\phi(\alpha) = \tilde{P}(\alpha)\phi = \sum_{j=1}^{m_i} c_l\tilde{P}(\alpha)\phi_l(0) = \sum_{j=1}^{m_i} c_l\phi_l(\alpha)$, so $(L_1(\alpha) - \lambda_i(\alpha))\phi(\alpha) = 0$ for any $\phi$ in the subspace $E_i$ spanned by $\phi_1(0), ..., \phi_l(0)$.

Consider now a resonance $\lambda_i(\alpha)$, where we assume that $\lambda_{li}(\alpha)$ is a b ranch of a Puiseux cycle, $i = s+1, ..., s+t$. Again we insert $\lambda_i(\alpha)$ in (8.3), getting (8.5), but we only know that $\lambda_{li}(\alpha) \in C^2(-\varepsilon, \varepsilon)$. We can again solve (8.5) for $a_{lj}(\alpha)$, since $d(\lambda_{li}(\alpha), \alpha) = 0$, obtaining $C^2(-\varepsilon, \varepsilon)$ functions $a_{li}(\alpha)$, $l = 1, ..., m_i$. Moreover, we get $m_i$ linearly independent vectors $\{a_{li}(\alpha)\}_{l=1}^{m_i}$, since for $|\alpha| < \varepsilon$, $\lambda_{li}(\alpha)$ is a simple pole of $\tilde{R}(\lambda, \alpha)$ so that the range of $\mathrm{Res}_{\lambda_i(\alpha)}\tilde{R}(\lambda, \alpha)$ is $N(L_1(\alpha) - \lambda_i(\alpha))$. This gives by (8.2) $m_i$ linearly independent resonance functions $\psi_{li}(\alpha)$, $l = 1, ..., m_i$, which are $C^2(-\varepsilon, \varepsilon)$ with values in $C_{1,-2}$. As above, for any function $\psi = \sum\limits_{l=1}^{m_i} \beta_l\psi_{li}(0)$

$$\tilde{P}(\alpha)\psi = \psi(\alpha) = \sum_{l=1}^{m_i} \beta_l\psi_{li}(\alpha) \text{ and } (L_1(\alpha) - \lambda_{li}(\alpha))\psi(\alpha) = 0.$$

We can choose an orthonormal basis for $E_{li} = \mathrm{span}\{\psi_{li}(0)\}_{l=1}^{m_i}$, but whereas the subspaces $E_i$ and $E_j$ corresponding to embedded eigenvalues $\lambda_i(\alpha)$ and $\lambda_j(\alpha)$ are orthogonal, $E_{li}$ is not necessarily orthogonal to the spaces $E_{ki}$, $E_{lj}$ and $E_i$. If the resonance $\lambda_i(\alpha)$ is analytic, we onbtain analytic functions $\psi_i(\alpha)$.

We have proved the following general result on perturbation of embedded eigenvalues and eigenfunctions.

**Theorem 8.4** *Let $\lambda_0 = s_0(1-s_0)$ be an eigenvalue of $L$ with eigenspace $N$ of dimension $m$. The poles $\lambda_1(\alpha), ..., \lambda_k(\alpha)$ of $\tilde{R}(\alpha, s)$ inside the curve $K_1$ can for $|\alpha| < \varepsilon$ be divided into groups forming Puiseux cycles of order $p \geq 1$. If $p = 1$, the corresponding $\lambda_j(\alpha)$ is analytic for $|\alpha| < \varepsilon$. If $p \geq 2$, the Puiseux cycle consists of $p$ branches $\lambda_{j1}(\alpha) ... \lambda_{jp}(\alpha)$ of a function having a branch point of order $p$ at $\alpha = 0$. In the first case $p = 1$ we have the following possibilities,*

1. *$\lambda_1(\alpha)$ is real for all real $\alpha$, and $\lambda_i(\alpha)$ is an embedded eigenvalue of $L(\alpha)$ for $\alpha \in (-\varepsilon, \varepsilon)$.*

2. *$\lambda_i(\alpha) = \lambda_0 + a_1\alpha + ... + a_{2l-1}\alpha^{2l-1} + a_{2l}\alpha^{2l} + \sum_{m \geq 2l+1} a_m\alpha^m$, $a_1, ..., a_{2l-1}$ are real, $\operatorname{Im} a_{2l} > 0$ for $s_0 = \frac{1}{2} + it_0$, $\operatorname{Im} a_{2l} < 0$ for $s_0 = \frac{1}{2} - it_0$, $t_0 > 0$.*

*In the case $p \geq 2$, the functions $\lambda_{j1}(\alpha), ..., \lambda_{jp}(\alpha)$ have expansions of the form*

$$\lambda_{jl}(\alpha) = \lambda_0 + b_1\alpha + ... + b_{2m-1}\alpha^{2m-1} + b_{2m}\alpha^{2m} + b_{2m+1}\omega^l\alpha^{(2m+1)/p} + ...,$$

*$l = 1, ..., p$, where $b_1, ..., b_{2m-1}$ are real and $\operatorname{Im} b_{2m} > 0$ for $s_0 = \frac{1}{2} + it_0$, $\operatorname{Im} b_{2m} < 0$ for $s_0 = \frac{1}{2} - it_0$, $t_0 > 0$.*

*The multiplicity of each $\lambda_i(\alpha)$ and $\lambda_j(\alpha)$ is constant and is the same for all elements of a Puiseux cycle.*

*The total dimension of the eigenvalues and resonances $\lambda_i(\alpha)$ and $\lambda_{jl}(\alpha)$ equals $m$.*

*For each eigenvalue function or analytic resonance function $\lambda_i(\alpha)$ of multiplicity $m_i$ there exists an $m_i$-dimensional subspace $N_i$ of $N$, such that for $\phi \in N_i$, $\phi(\alpha) = \tilde{P}(\alpha)\phi \in$*

*$N(L_1(\alpha) - \lambda_i(\alpha))$, and $\phi(\alpha)$ is analytic for $|\alpha| < \varepsilon$ with values in $\mathcal{H}$ for embedded eigenvalues $\lambda_i(\alpha)$ and $C_{1,-2}$ for resonances $\lambda_i(\alpha)$. When $\lambda_{lj}(\alpha)$ with multiplicity $m_j$ is a branch of a Puiseux cycle, there exists an $m_j$-dimensional subspace $N_{lj}$ of $N$, such that for $\phi \in N_{lj}$, $\phi(\alpha) = \tilde{P}(\alpha)\phi \in N(L_1(\alpha) - \lambda_{lj}(\alpha))$, and $\phi(\alpha) \in C^2(-\varepsilon, \varepsilon)$ with values in $C_{1,-2}$. Choosing any orthonormal bases of each of the spaces $N_i$ and $N_{lj}$, we obtain taking their union a Kato basis of $N$, where functions from different subspaces are not necessarily orthogonal unless both consist of eigenfunctions $\phi_i \in N_i, \phi_k \in N_k$ where $\phi_i(\alpha)$ and $\phi_j(\alpha)$ are eigenfunctions of $L(\alpha)$.*

We shall now derive explicit formulas for the perturbation of the eigenvectors $\phi$ to first order and the eigenvalue $\lambda_0 = s_0(1 - s_0)$ to second order.

Let $\phi \in \mathcal{N}(L - \lambda_0) = P_0 H$. Then $\phi(\alpha) = \tilde{P}(\alpha)\phi$ is an analytic function with values in $C_{1,-2}$ for $|\alpha| < \varepsilon$. We calculate $\phi_1 = \phi'(0)$ as follows.

$$\begin{aligned}
\phi_1 &= \lim_{\alpha \to 0} \frac{1}{\alpha} \left\{ \tilde{P}(\alpha) - \tilde{P}(0) \right\} \phi \qquad (8.6) \\
&= \lim_{\alpha \to 0} \frac{1}{\alpha} \frac{(-1)}{2\pi i} \int_K \left\{ \tilde{R}(\alpha, s) - \tilde{R}(s) \right\} (2s - 1) ds \\
&= \lim_{\alpha \to 0} \frac{1}{2\pi i} \int_K \frac{1}{\alpha} \tilde{R}(\alpha, s)(\alpha M + \alpha^2 N)\tilde{R}(s)\phi(2s - 1) ds \\
&= \lim_{\alpha \to 0} \frac{1}{2\pi i} \int_K \tilde{R}(\alpha, s)(M + \alpha N)(L - s(1 - s))^{-1}\phi(2s - 1) ds \\
&= \frac{1}{2\pi i} \int_K \tilde{R}(s) M\phi \left\{ s_0(1 - s_0) - s(1 - s) \right\}^{-1} (2s - 1) ds.
\end{aligned}$$

Setting $\psi = M\phi$, we derive an expression for $\tilde{R}(0,s)\psi$. Let $|s-s_0| < \delta$, $\operatorname{Re} s > \frac{1}{2}$, $\operatorname{Im}(s-s_0) > 0$. Then by the spectral theorem,

$$R(s)\psi = \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{1}{\frac{1}{4}+r^2-s(1-s)} \sum_{j=1}^{m} |\, E_j(\tfrac{1}{2}+ir)\rangle\langle E_j(\tfrac{1}{2}-ir)|\psi\rangle \tag{8.7}$$
$$+ R_l(s)\psi + R_d(s)\psi + \left(s_0(1-s_0) - s(1-s)\right)^{-1} P_0\psi,$$

where $h$ is the number of open cusps,

$$R_l(s)\psi = \sum_{k=1}^{\infty} |\phi_k\,\rangle\,\langle\,\phi_k|\,\psi\rangle\,\left(s_k(1-s_k) - s(1-s)\right)^{-1}$$

$$R_d(s)\psi = \sum_{l=1}^{\infty} |\phi'_k\,\rangle\,\langle\,\phi'_k|\,\psi\rangle\,\left(s'_k(1-s'_k) - s(1-s)\right)^{-1}$$

and $s_k(1-s_k)$ are the embedded eigenvalues different from $s_0(1-s_0)$ with eigenfunctions $\phi_k$ and $s'_l(1-s'_l)$ are the small, discrete eigenvalues with eigenfunctions $\phi'_k$.

Here we use the notation $\langle u,v\rangle = \int_F u(z)\bar{v}(z)\,d\mu(z)$ for any pair of functions on $F$ such that $\int_F |u|\cdot|v|\,d\mu(z) < \infty$. Also, $|u\rangle$ means multiplication by the function $u$.

The integrand is analytic in $r$, and we can deform the contour $\mathbb{R}$ to a contour $\Gamma_R$, $|s-s_0| < R \le \delta$, obtained by replacing $[t_0-R, t_0+R]$ by the semicircle $\left\{-\operatorname{Re}^{i\varphi}\,|0 \le \varphi \le \pi\right\}$, see Fig. 1.

For a fixed $s$ the poles of the function $\left(\frac{1}{4}+r^2-s(1-s)\right)^{-1}$ are

$$\rho_\pm = \pm i(s-\tfrac{1}{2}) = \pm i\,(it_0 + s - s_0) = \mp t_0 \pm i\,(s-s_0)\,.$$

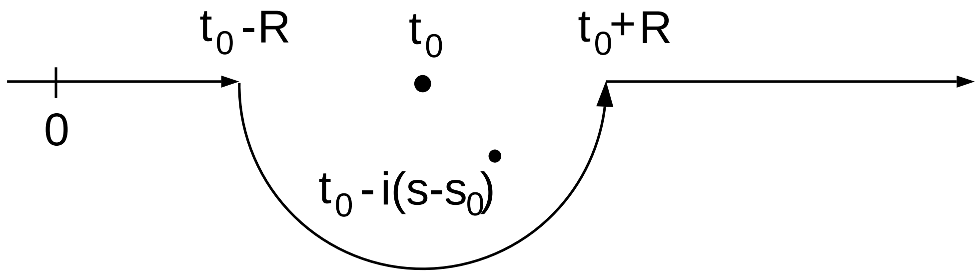


Figure 1: Deformation of the spectrum of $L$.

We have chosen to focus on $s_0 = \frac{1}{2} + it_0$. The root $\rho_- = t_0 - i(s - s_0)$ lies inside the above semicircle. The residue of the integrand at the simple pole $\rho_-$ is

$$\operatorname{Res}\left\{\frac{1}{r - i\left(s - \frac{1}{2}\right)} \frac{1}{r + i\left(s - \frac{1}{2}\right)} \sum_{j=1}^{h} |E_j\left(\tfrac{1}{2} + ir\right)\rangle \langle E_j(\tfrac{1}{2} - ir) \mid \psi\rangle\right\}_{r=-i\left(s-\frac{1}{2}\right)}$$

$$= \frac{1}{-2i\left(s - \frac{1}{2}\right)} \sum_{j=1}^{h} |E_j(s)\rangle \langle E_j(1-s)| \psi\rangle$$

so the first term $R_c(s)\psi$ of $R(s)\psi$ equals

$$R_c(s)\psi = \frac{1}{4\pi} \int_{\Gamma_R} \frac{1}{\frac{1}{4} + r^2 - s(1-s)} \sum_{j=1}^{h} \mid E_j(\tfrac{1}{2} + ir\rangle \langle E_j \mid \tfrac{1}{2} - ir\ \psi\rangle\, dr \qquad (8.8)$$
$$+ \frac{1}{4\left(s - \frac{1}{2}\right)} \sum_{j=1}^{h} |E_j(s)\rangle \langle E_j(1-s)| \psi\rangle\,.$$

Both terms of $R_c(0, s)$ have analytic continuations to $\{s\,|s - s_0| < R\}$, and we obtain $\tilde{R}_c(0, s)\psi$ expressed by the same equation (8.8).

We calculate the first term at $s = s_0$. Replacing $R$ by any smaller radius $\rho > 0$ we obtain

$$
\begin{aligned}
&\frac{1}{4\pi} \int\limits_{\Gamma_\rho} \frac{1}{\frac{1}{4} + r^2 - s_0(1-s_0)} \sum_{j=1}^{h} |\, E_j(\tfrac{1}{2} + ir)\,\rangle \langle\, E_j(\tfrac{1}{2} - ir) |\, \psi \rangle \, dr \\
&= \frac{1}{4\pi} \lim_{\rho \downarrow 0} \left\{ \int\limits_{-\infty}^{t_0-\rho} + \int\limits_{t_0+\rho}^{\infty} \right\} \frac{1}{r^2 - t_0^2} \sum_{j=1}^{h} |\, E_j(\tfrac{1}{2} + ir)\,\rangle \langle\, E_j(\tfrac{1}{2} - ir) |\, \psi \rangle \, dr \\
&+ \frac{1}{4\pi} \lim_{\rho \downarrow 0} \int\limits_{\tilde{C}_\rho} \frac{1}{\frac{1}{4} + r^2 - s_0(1-s_0)} \sum_{j=1}^{h} |\, E_j(\tfrac{1}{2} + ir)\,\rangle \langle\, E_j(\tfrac{1}{2} - ir) |\, \psi \rangle \, dr \\
&= \frac{1}{4\pi} PP \int\limits_{-\infty}^{\infty} \frac{1}{r - t_0} \frac{1}{r + t_0} \sum_{j=1}^{h} |\, E_j(\tfrac{1}{2} + ir)\,\rangle \langle\, E_j(\tfrac{1}{2} - ir) |\, \psi \rangle \, dr \\
&+ \frac{1}{4\pi} \frac{1}{2} \cdot 2\pi i \ \mathrm{Res} \left\{ \frac{1}{\frac{1}{4} + r^2 - s_0(1-s_0)} \sum_{j=1}^{h} |\, E_j(\tfrac{1}{2} + ir)\,\rangle \langle\, E_j(\tfrac{1}{2} - ir) |\, \psi \rangle \, dr \right\}_{s=s_0}
\end{aligned}
$$

where $\tilde{C}_\rho$ is the semicircle $\{ s = \rho e^{i\varphi} \,| -\pi \le \varphi \le 0 \}$. Thus, half of the previously subtracted residue is added, and we obtain

$$
\begin{aligned}
\tilde{R}_c(s_0)\psi = \frac{1}{4\pi} PP \int\limits_{-\infty}^{\infty} \frac{1}{r - t_0} \frac{1}{r + t_0} \sum_{j=1}^{h} |\, E_j\left(\tfrac{1}{2} + ir\right)\rangle \langle\, E_j\left(\tfrac{1}{2} - ir\right) |\, \psi \rangle \, dr \\
+ \frac{1}{8it_0} \sum_{j=1}^{h} |\, E_j\left(\tfrac{1}{2} + it_0\right)\rangle \langle\, E_j\left(\tfrac{1}{2} - it_0\right) |\, \psi \rangle \,.
\end{aligned}
\tag{8.9}
$$

We can now introduce (8.7) in (8.6), using (8.9). We obtain the following expression for $\phi_1$, using that all the terms of $\tilde{R}(s)\psi$ have a simple pole at $s = s_0$ except possibly the last term of (8.7) which contains a double pole if $\frac{1}{4}$ is an eigenvalue of $L$.

$$
\begin{aligned}
\phi_1 = &\frac{1}{4\pi} PP \int\limits_{-\infty}^{\infty} \frac{1}{r - t_0} \frac{1}{r + t_0} \sum_{j=1}^{h} |\, E_j\left(\tfrac{1}{2} + ir\right)\rangle \langle\, E_j\left(\tfrac{1}{2} - ir\right) |\, M\phi \rangle \, dr \\
&+ \frac{1}{8it_0} \sum_{j=1}^{h} | E_j(\tfrac{1}{2} + it_0)\,\rangle \langle\, E_j(\tfrac{1}{2} - it_0) \,| M\phi \rangle \\
&+ R_e(s_0) M\phi + R_d(s_0) M\phi.
\end{aligned}
\tag{8.10}
$$

From this expression for $\phi_1$ and Theorem 7.1 it is clear that for every odd Hecke eigenfunction $v_j$ of Theorem 4.2, $v_j(\alpha)$ can not be a linear combination of eigenfunctions corresponding to embedded eigenvalues $\lambda_i(\alpha)$, since $v_{j1} \notin \mathcal{H}$. Thus, if $N(L-\lambda_0)$ contains odd functions, there exists at least one eigenfunction $\phi$ with eigenvalue $\lambda_0$, such that $\phi(\alpha)$ is resonance function with resonance $\lambda(\alpha)$.

To complete the picture we prove, using (8.10), the expression for the imaginary part of the coefficients in the 2nd order expansion of the resonances $\lambda_i(\alpha)$ known as Fermi's Golden Rule.

Let $\lambda(\alpha) = \lambda_i(\alpha)$, $i = s+1, ..., s+t$, be a resonance of $L(\alpha)$ of multiplicity $m_i$, $0 < |\alpha| < \varepsilon$, $\alpha$ real, $\lambda(0) = \lambda$. Let $\phi$ be a function in the subspace $N_i$ of $N$ of Theorem 8.4, such that

$$L_1(\alpha)\phi(\alpha) = \lambda(\alpha)\phi(\alpha). \tag{8.11}$$

Since $\lambda(\alpha) \in C^2(-\varepsilon, \varepsilon)$ and $\phi(\alpha) \in C^2(-\varepsilon, \varepsilon)$ with values in $C_{1,-2}$, we can expand both sides of the equation (8.11) to second order, obtaining

$$\begin{aligned}&(L + \alpha M + \alpha^2 N)(\phi + \alpha\phi_1 + \tfrac{1}{2}\alpha^2\phi_2 + o(\alpha^2)) \qquad (8.12)\\ &= (\lambda + \alpha\lambda_1 + \tfrac{1}{2}\alpha^2\lambda_2 + o(\alpha^2))(\phi + \alpha\phi_1 + \tfrac{1}{2}\alpha^2\phi_2 + o(\alpha^2)).\end{aligned}$$

The first and second order equations are

$$L\phi_1 + M\phi = \lambda_0\phi_1 + N\phi \tag{8.13}$$

and

$$\tfrac{1}{2}L\phi_2 + M\phi_1 + N\phi = \tfrac{1}{2}\lambda\phi_2 + \lambda_1\phi_1 + \tfrac{1}{2}\lambda_2\phi. \tag{8.14}$$

Integrating (8.13) and (8.14) with $\bar{\phi}$, we get

$$\langle M\phi_1, \phi\rangle = \lambda_1 \tag{8.15}$$

$$\langle M\phi_1, \phi\rangle + \langle N\phi, \phi\rangle = \tfrac{1}{2}\lambda_2. \tag{8.16}$$

Here we have used that $\phi$ is a cusp form and hence

$$\langle (L = \lambda)\phi_i, \phi\rangle = \langle \phi_i, (L - \lambda)\phi\rangle = 0,\ i = 1, 2$$

and $\langle \phi_1, \phi\rangle = 0$, which follows from (8.10).

Introducing (8.10) in (8.16), we obtain

$$\begin{aligned} \operatorname{Re}\lambda_2 = {} & \frac{1}{2\pi} PP \int_{-\infty}^{\infty} \frac{1}{r - t_0} \frac{1}{r + t_0} \sum_{j=1}^{h} \mid \langle E_j(\tfrac{1}{2} - ir) \mid M\phi\rangle^2 dr \\ & + ((R_e(s) + R_d(s))M\phi, \phi) + (N\phi, \phi) \end{aligned} \tag{8.17}$$

$$\operatorname{Im}\lambda_2 = \frac{1}{4t_0} \sum_{j=1}^{h} \mid \langle E_j(\tfrac{1}{2} - it) \mid M\phi\rangle \mid^2 . \tag{8.18}$$

By Theorems 7.1, 8.4 and (8.10), (8.18), we have obtained the following result.

**Theorem 8.5** *Let* $\lambda = \frac{1}{4} + t^2$, $s = \frac{1}{2} + it$, *be an eigenvalue of* $L = A(\bar{\Gamma}_0(N), \chi)$ *with eigenspace* $N$ *of* $\dim N = m$, *and assume that* $N$ *contains a subspace of odd functions. Let* $\phi \in N$ *and* $\phi(\alpha) = \tilde{P}(\alpha)\phi$. *Then* $\phi_1 = \frac{d}{d\alpha}\phi(\alpha) \mid_{\alpha=0}$ *is given by*

$$\phi_1 = \tilde{\phi}_1 + \frac{1}{8it} \sum_{j=1}^{h} \mid E_j(\tfrac{1}{2} + it)\rangle\langle E_j(\tfrac{1}{2} - it) \mid M\phi\rangle$$

*where* $\tilde{\phi}_1 \in \mathcal{H}$ *is given by* (8.10).

*The function* $\phi_1 \notin \mathcal{H}$ *if and only if* $\langle E_j(\frac{1}{2} - it) \mid M\phi\rangle \neq 0$ *for some* $j$.

*For odd Hecke functions* $\phi$*, the function* $\phi_1$ *does not belong to* $\mathcal{H}$.

*There exists at least one eigenfunction* $\phi$ *in* $N$*, such that* $\phi(\alpha)$ *is a resonance function with resonance* $\lambda(\alpha)$, $\lambda(0) = \lambda$.

*For any such eigenfunction,* $\operatorname{Im}\lambda''(0)$ *is given by Fermi's Golden Rule*

$$\operatorname{Im}\lambda_2 = \frac{1}{4t_0}\sum_{j=1}^{h} \mid \langle E_j(\tfrac{1}{2} - it) \mid M\phi\rangle \mid^2 .$$

**Definition 2** *Let*

$$\lambda_1 < \lambda_2 < ... < \lambda_k < ...$$

*be the eigenvalues of* $L$ *whose eigenspaces contain odd subspaces* $\mathcal{K}_k \subset N(L - \lambda_k)$ *with multiplicities*

$$d_k = \dim \mathcal{K}_k.$$

*Let*

$$m_k = \max\{d_j \,|1 \leq j \leq k\}$$

$$m(\lambda) = m_k \textit{ for } \lambda_k \leq \lambda < \lambda_{k+1}.$$

*Let* $S$ *be the union of the exceptional sequences of Theorem 7.1 and*

$$N_1(\lambda) = \#\{\lambda_k \leq \lambda\}$$

$$N_2(\lambda) = \#\{\lambda_k \le \lambda \,|\, \lambda_k \in S\}$$

$$N_3(\lambda) = \#\{\lambda_k \le \lambda \,|\, \lambda_k \notin S\} = N_1(\lambda) - N_2(\lambda).$$

*We write*

$$f_1(\lambda) \gtrsim f_2(\lambda) \text{ if, for every } \varepsilon > 0,\ \frac{f_1(\lambda)}{f_2(\lambda)} \ge 1-\varepsilon,\ \lambda \ge \Lambda(\varepsilon).$$

**Theorem 8.6** *Assume that* $m(\lambda) = o(\lambda)$ *as* $\lambda \to \infty$. *Then*

$$N_3(\lambda) \gtrsim \frac{1}{m(\lambda)} \frac{|F|}{8\pi}\lambda.$$

**Proof.** We consider first $\lambda = \lambda_k$. By Corollary 3.7

$$\frac{N(\lambda_k)}{\frac{|F|}{8\pi}\lambda_k} \ge 1 - \varepsilon_1 \text{ for } \lambda_k > \Lambda(\varepsilon_1). \tag{8.19}$$

But

$$N_1(\lambda_k) \ge \frac{N(\lambda_k)}{m_k} \text{ for all } k \tag{8.20}$$

so

$$N_1(\lambda_k) \ge (1-\varepsilon_1)\frac{|F|}{m_k \cdot 8\pi}\lambda_k.$$

Since $N_2(\lambda_k) \approx c\lambda_k^{\frac{1}{2}}$, we conclude that

$$N_3(\lambda_k) \gtrsim \frac{1}{m_k}\frac{|F|}{8\pi}\lambda_k.$$

To obtain the result for general $\lambda$ we first prove $\frac{N(\lambda_{k+1})}{N(\lambda_k)} \underset{k\to\infty}{\longrightarrow} 1$. We have

$$N(\lambda_{k+1}) = \#\{\mu_i \le \lambda_{k+1}\} = \#\{\mu_i \le \lambda_k\} + d_{k+1} = N(\lambda_k) + d_{k+1} \le N(\lambda_k) + m_{k+1}$$

where

$$\mu_1 \le \mu_2 \le ... \le \mu_k \le ...$$

are the eigenvalues of $L$, counted with multiplicity. Then

$$1 \le \frac{N(\lambda_{k+1})}{N(\lambda_k)} \le 1 + \frac{m_{k+1}}{N(\lambda_k)} = 1 + \frac{N(\lambda_{k+1})}{N(\lambda_k)}\varepsilon(\lambda_{k+1}),$$

hence

$$\frac{N(\lambda_{k+1})}{N(\lambda_k)}(1-\varepsilon(\lambda_{k+1})) \le 1,\ \varepsilon(\lambda_{k+1}) \underset{k\to\infty}{\to} 0,$$

so

$$\lim_{k\to\infty} \frac{N(\lambda_{k+1})}{N(\lambda_k)} = 1. \tag{8.21}$$

From (8.20) and (8.21) we obtain

$$\tilde{N}(\lambda_k) \ge \frac{N(\lambda_k)}{m_k} = \frac{N(\lambda_k)}{N(\lambda_{k+1})} \cdot \frac{N(\lambda_{k+1})}{m_k} \ge (1-\varepsilon_1)\frac{N(\lambda_{k+1})}{m_k},\ \lambda_k > \Lambda(\varepsilon_1). \tag{8.22}$$

By (8.19)

$$\frac{N(\lambda_{k+1})}{\frac{|F|}{4\pi}\lambda_{k+1}} \ge 1-\varepsilon_2,\ \lambda_k > \Lambda(\varepsilon_2). \tag{8.23}$$

From (8.22) and (8.23) follows

$$\frac{\tilde{N}(\lambda_k)}{\frac{1}{m_k}\frac{|F|}{4\pi}\lambda_{k+1}} \ge (1-\varepsilon_1)(1-\varepsilon_2) \ge 1-\varepsilon \text{ for } \lambda_k > \Lambda(\varepsilon),$$

where

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_1\varepsilon_2 < \varepsilon,\ \Lambda(\varepsilon) = \max\{\Lambda(\varepsilon_1), \Lambda(\varepsilon_2)\}.$$

Since $\tilde{N}(\lambda) = \tilde{N}(\lambda_k)$, $m(\lambda) = m(\lambda_k) = m_k$ for $\lambda_k \leq \lambda < \lambda_{k+1}$, this implies

$$\frac{N(\lambda)}{\frac{1}{m(\lambda)}\frac{|F|}{4\pi}\lambda} \geq 1 - \varepsilon, \ \lambda > \Lambda(\varepsilon),$$

and the Theorem is proved. ■

The result on the asymptotic number of eigenvalues, which become resonances under character perturbation, thus depends on bounds on the dimension of eigenspaces, see [Sa]. We obtain the following asymptotics from increasingly strong proved or conjectured bounds.

**Corollary 8.7** *(a) Assume* $m(\lambda) \leq c\frac{\sqrt{\lambda}}{\log \lambda}$, $c > 0$

*This can be proved using estimates of the argument of the Selberg zeta function on* $\operatorname{Re} s = \frac{1}{2}$, *(cf. [Se], [V]).*

*This bound implies by Theorem 8.6*

$$\tilde{N}(\lambda) \gtrsim \frac{|F|}{8\pi c}\lambda^{\frac{1}{2}} \log \lambda.$$

*(b) Assume* $m(\lambda) \leq c\lambda^{\beta}$ *for some* $c > 0$, $0 < \beta < 1$.

*This is a conjecture [Sa], which implies*

$$\tilde{N}(\lambda) \gtrsim \frac{|F|}{8\pi c}\lambda^{1-\beta}.$$

*(c) Assume* $m(\lambda) \leq m$ *for all* $\lambda > \Lambda$.

*This boundedness conjecture implies*

$$\tilde{N}(\lambda) \gtrsim \frac{|F|}{8\pi m}\lambda.$$

This indicates that the Weyl law is violated for small $\alpha \neq 0$. We note that it follows from the Hecke theory of Section 4 that $m \geq 2$.

**Remark 8.8** *For even eigenfunctions the Phillips-Sarnak integral is zero, since $M\phi$ is odd for even $\phi$. It is therefore not known whether even eigenfunctions leave or stay under this perturbation. There is another perturbation obtained by replacing* $\mathrm{Re}\,\omega$ *by* $\mathrm{Im}\,\omega$ *in the definition of the characters $\omega(\alpha)$,*

$$\chi(\alpha)(\gamma) = e^{2\pi i\alpha \,\mathrm{Im} \int_{z_0}^{\gamma z_0} \omega(t)dt}, \ \gamma \in \Gamma_0(N).$$

*The family $A(\Gamma_0(N), \chi \cdot \chi^{(\alpha)})$ corresponds by unitary equivalence via the operator $e^{2\pi i\alpha \int_{z_0}^{z} \omega(t)dt}$ to the family of operators in $H(\Gamma_0(N), \chi)$*

$$\tilde{L}(\alpha) = L + \alpha\tilde{M} + \alpha^2 N$$

*where*

$$L = A(\Gamma_0(N), \chi)$$

$$\tilde{M} = -4\pi i y^2 \left(\omega_2 \frac{\partial}{\partial x} + \omega_1 \frac{\partial}{\partial y}\right)$$

$$N = 4\pi^2 y^2(\omega_1^2 + \omega_2^2).$$

*It turns out that the operator $\tilde{M}$ is not $L$-bounded, and therefore the perturbation theory developed for $M$ does not apply. Although the Phillips-Sarnak integrals are in fact given by the same Rankin-Selberg convolution and can be proved to be non-zero for Hecke* *eigenfunctions, this does not imply that certain eigenvalues with even eigenfunctions become resonances under this perturbation. Indeed,* $\operatorname{Im}\int_{z_0}^{\gamma z_0}\omega(t)dt = 0$ *for* $\gamma \in \Gamma_0(N)$, *which implies that* $\chi \cdot \chi(\alpha) = \chi$ *for all* $\alpha$ *and the functions* $\Omega(\alpha) = \exp\left\{2\pi i\alpha \operatorname{Im}\int_{z_0}^{z}\omega(t)dt\right\}$ *are* $\Gamma_0(N)$*-automorphic. Thus, the operators* $\tilde{L}(\alpha)$ *are unitarily equivalent to* $L$ *for all* $\alpha$ *via* $\tilde{L}(\alpha) = \Omega^{-1}(\alpha)L\Omega(\alpha)$, *and all eigenvalues stay. The domain* $D(\tilde{L}(\alpha))$ *equals* $\Omega(\alpha)D(L)$, *which changes with* $\alpha$.

**Remark 8.9** *The proof that the Phillips-Sarnak integral is not zero is based on the non-vanishing of the Dirichlet L-series for eigenfunctions, which is proved using Hecke theory. This is therefore specific for the operators* $A(\bar{\Gamma}_0(N), \chi)$. *However, we can draw the following conclusions about embedded eigenvalues of* $A(\bar{\Gamma}_0(N), \chi \cdot \chi^{(\alpha)})$ *based on general perturbation theory. Due to the analyticity in* $\alpha$, *each embedded eigenvalue* $\lambda(\alpha_0)$ *of* $L(\alpha_0)$ *under the perturbation* $\alpha M + \alpha^2 N$ *either stays as an embedded eigenvalue for* $\alpha \neq \alpha_0$, *analytic in* $\alpha$, *or leaves as a resonance.*

*Therefore eigenvalues of* $L = A(\bar{\Gamma}_0(N), \chi)$, *which leave the spectrum as resonances for* $\alpha \neq 0$, *can only become eigenvalues for isolated values of* $\alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right)$.

# A Appendix

We will study the matrices of $(d', d'')^2$, which correspond to the systems . We want to prove that the coefficient $\beta_1$ is not zero, for some choices of coefficients $n_{2d_1}, n_{4d_2}$. We start from (5.3). We have $4N_2 = 4p_1...p_k$, where $p_i$ are different primes not equal to two.

To see the matrix

$$(d', d'')^2, \; d' \,|4N_2, \;\; d', d'' > 0, \tag{A.1}$$

we consider the following primitive matrices:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 16 \end{pmatrix}, \; B_i = \begin{pmatrix} 1 & 1 \\ 1 & p_i^2 \end{pmatrix} \; 1 \le i \le k. \tag{A.2}$$

It is not difficult to see that the inverse matrices are

$$A^{-1} = \frac{1}{36} \begin{pmatrix} 48 & -12 & 0 \\ -12 & 15 & -3 \\ 0 & -3 & 3 \end{pmatrix}, \; B_i^{-1} = \begin{pmatrix} p_i^2 & -1 \\ -1 & 1 \end{pmatrix} \frac{1}{p_i^2 - 1} \tag{A.3}$$

We define the tensor product $A \otimes B_1 \otimes B_2 ... \otimes B_k$ by recurrence relations os the block matrix

$$C_1 = \begin{pmatrix} A & A \\ A & p_1^2 A \end{pmatrix}, \; C_2 = \begin{pmatrix} C_1 & C_1 \\ C_1 & p_2^2 C_1 \end{pmatrix}, \; C_k = \begin{pmatrix} C_{k-1} & C_{k-1} \\ C_{k-1} & p_k^2 C_{k-1} \end{pmatrix}. \tag{A.4}$$

It is not difficult to see that the matrix $C_k$ coincides with the matrix (A1.1), if we take the following order of divisors $d'$ and $d''$

$$1, 2, 4, p_1(1, 2, 4), p_2 \left[1, 2, 4, p_1(1, 2, 4)\right], ... \, . \tag{A.5}$$

It is easy to see now that the inverse matrix to $C_k$ is coming from the recurrence relation

$$\begin{cases} c_1^{-1} = \frac{1}{p_1^2 - 1} \begin{pmatrix} p_1^2 A^{-1}, & -A^{-1} \\ -A^{-1}, & A^{-1} \end{pmatrix}, \; c_2^{-1} = \frac{1}{p_2^2 - 1} \begin{pmatrix} p_2^2 C_1^{-1}, & -C_1^{-1} \\ -C_1^{-1}, & C_1^{-1} \end{pmatrix}, ... \\ ... c_k^{-1} = \frac{1}{p_k^2 - 1} \begin{pmatrix} p_k^2 C_{k-1}^{-1}, & -C_{k-1}^{-1} \\ -C_{k-1}^{-1}, & C_{k-1}^{-1} \end{pmatrix}. \end{cases} \tag{A.6}$$

From this follows that $C_k^{-1}$ exists, and we can determine the coefficients $\beta_{d''}$ from (5.3) explicitly. Actually, it is important to see now only the first row in the inverse matrix $C_k^{-1}$, since we want to prove $\beta_1 \neq 0$. Let us denote this first row of $C_l^{-1}$ by $e_l$, $1 \le l \le k$.

From (A.6) follows

$$\begin{cases} e_1 = \frac{1}{36(p_1^2-1)}\left(p_1^2(48,-12,0)-(48,-12,0)\right) \\ e_{m+1} = \frac{1}{p_{m+1}^2-1}\left(p_{m+1}^2 e_m, -e_m\right) \; 1 \le m \le k-1. \end{cases} \tag{A.7}$$

Let us see now the right hand side of (6.40). When $d'$ runs through all positive divisors of $4N_2$ in the order of (A.5), we get the column vector, which has non-zero components only on places $d' = 2d_1,\ d_1\,|N_2,\ d_1 > 0$ equal to $n_{2d_1}/m_{2d_1}$. From (1.8), (1.10) follows $m_{2d_1} = N_2/d_1$. We remind that the coefficients $n_{2d_1} = \pm 1$ with the only condition (6.11). Applying $e_k$ to this vector we obtain up to the common multiple

$$\frac{1}{36(p_1^2-1)(p_2^2-1)...(p_k^2-1)} \tag{A.8}$$

that $\beta_1$ is equal to

$$\sum_{\substack{d_1|N_2 \\ d_1>0}} n_{2d_1} x_{2d_1} \tag{A.9}$$

where $x_{2d_1}$ are pairwise different integers with equal number of positives and negatives. From that follows that there exists the choice of coefficients $n_{2d_1} = \pm 1$ with condition (6.11) which makes (A.9) not equal to zero.

The investigation of the system (6.42) is completely analogous. We have $4N_3 = 8n$, $n = p_1 \cdot p_2 \cdots p_k$ is the product of different odd primes. Instead of the matrix $A$ from we take

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 4 & 4 & 4 \\ 1 & 4 & 16 & 16 \\ 1 & 4 & 16 & 64 \end{pmatrix}, \; A^{-1} = \begin{pmatrix} 4/3 & -1/3 & 0 & 0 \\ -1/3 & 5/12 & -1/12 & 0 \\ 0 & -1/12 & 5/48 & -1/48 \\ 0 & 0 & -1/48 & 1/48 \end{pmatrix} \tag{A.10}$$

and then repeat the proof. We obtain that up to the common multiple

$$\frac{1}{48(p_1^2-1)(p_2^2-1)...(p_k^2-1)} \tag{A.11}$$

the coefficient $\beta_1$ is equal to

$$\sum_{\substack{d_1|n \\ d_1>0}} \frac{n_{2d_1}}{m_{2d_1}} x_{2d_1} + \sum_{\substack{d_2|n \\ d_2>0}} \frac{n_{4d_2}}{m_{4d2}} x_{4d_2} \tag{A.12}$$

where $x_{2d_1}$, $x_{4d_2}$ are integers with equal number of positives and negatives. There exists a choice of coefficients $n_{2d_1}$, $n_{4d_2}$ which makes (A.12) not equal to zero. We have proved Theorem 6.1.

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We would like to thank Jørgen Tornehave for his remark relating the systems $(d', d'')^2$ to tensor products of "primitive" matrices, which correspond to prime divisors.

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