Approximations of small jumps of Lévy processes with a view towards simulation

Søren Asmussen* Jan Rosiński†

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Abstract

Let $X = (X(t) : t \geq 0)$ be a Lévy process and X_ϵ the compensated sum of jumps not exceeding ϵ in absolute value, $\sigma^2(\epsilon) = \mathbf{Var}(X_\epsilon(1))$. In simulation, $X - X_\epsilon$ is easily generated as the sum of a Brownian term and a compound Poisson one, and we investigate here when $X_\epsilon/\sigma(\epsilon)$ can be approximated by another Brownian term. A necessary and sufficient condition in terms of $\sigma(\epsilon)$ is given, and it is shown that when the condition fails, the behaviour of $X_\epsilon/\sigma(\epsilon)$ can be quite intricate. This condition is also related to the decay of terms in series expansions. We further discuss error rates in terms of Berry–Esseen bounds and Edgeworth approximations.

Keywords Berry–Esseen theorem, Brownian motion, cumulants, Edgeworth expansion, functional central limit theorem, Gamma process, infinitely divisible distribution, normal inverse Gaussian process, stable process

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*Mathematical Statistics; Centre of Mathematical Sciences; Lund University; Box 118; S-221 00 Lund; Sweden (asmus@maths.lth.se)

†Mathematics Department; University of Tennessee; Knoxville, TN 37996-1300; USA (rosinski@math.utk.edu). Research partially supported by an NSF grant. The work was completed during the author's visit at the Department of Mathematical Statistics, Lund University, in June 2000. He gratefully acknowledges their hospitality and support..

1 Introduction

Let $X = \{X(t) : t \geq 0\}$ be a Lévy process with characteristic function of the form

$$\mathbb{E} \exp iuX(t) = \exp \left\{ t \left[iau - \frac{b^2u^2}{2} + \int_{-\infty}^{\infty} (e^{iux} - 1 - iuxI(|x| \leq 1))\, Q(dx) \right] \right\} \quad (1.1)$$

where $a \in \mathbb{R}$, $b^2 \geq 0$, and Q is a Lévy measure. With standard terminology, we refer to the case $\int_{|x|\leq 1} |x|Q(dx) < \infty$ as the finite variation case, and to $\int_{|x|\leq 1} |x|Q(dx) = \infty$ as the compensated case. The term corresponding to $xI(|x| \leq 1)$ in (1.1) represents a centering which is necessary for convergence in the compensated case and may be deleted in the bounded variation case (say X is a subordinator). $X(t)$ is the independent sum of a drift term at, a Brownian component $bW(t)$, and a compensated pure jump part with Lévy measure Q, having the interpretation that a jump of size x occurs at rate $Q(dx)$. See e.g. Bertoin [3] or Sato [15] for relevant background on Lévy processes, and Samorodnitsky & Taqqu [14] for the special case of stable processes.

For simulation, the generation of the Brownian part is a standard topic and will not be discussed here. For the jump part, the most straightforward case is the compound Poisson case $\|Q\| < \infty$ where jumps have distribution $Q/\|Q\|$ and can be simulated at the epochs of a Poisson process with rate $\|Q\|$. When $\|Q\| = \infty$, one could attempt to generate a discrete skeleton. This is straightforward if the marginal distribution of $X(t)$ is easily simulated for any t as for the stable case (Chambers, Mallows & Stuck [6]), the Gamma case, or the inverse Gaussian case. However, most often marginal distributions are not easily simulated and in practice, one often as an approximation simulates a Lévy process obtained by neglecting jumps with absolute size smaller than ϵ. In the finite variation case, this may be implemented by simply removing such jumps, leading to

$$X_0^\epsilon(t) \;:=\; X(t) - \sum_{s\leq t} \Delta X(s) I(|\Delta(X(s))| < \epsilon). \quad (1.2)$$

In the compensated case, the idea leads instead to

$$X_1^\epsilon(t) \;:=\; \mu_\epsilon t + bW(t) + N^\epsilon(t) \quad (1.3)$$

where

$$\mu_\epsilon \;:=\; a - \int_{\epsilon\leq|x|\leq 1} x\, Q(dx) \quad (1.4)$$

and

$$N^\epsilon(t) \;:=\; \sum_{s\leq t} \Delta X(s) I(|\Delta(X(s))| \geq \epsilon)$$

is a compound Poisson process with jump measure $Q_{|\{|x|\geq\epsilon\}}$ and independent of the standard Brownian motion W. In the finite variation case, the definition (1.3) means that X_1^ϵ is obtained from X by replacing small jumps by their expected value rather than just only removing them. That is,

$$X_1^\epsilon(t) \;=\; X_0^\epsilon(t) + \mathbb{E}[X(t) - X_0^\epsilon(t)]. \tag{1.5}$$

A further improvement is to incorporate also the contribution from the variation of small jumps, which leads to

$$X_2^\epsilon(t) \;:=\; \mu_\epsilon t + (b^2 + \sigma^2(\epsilon))^{1/2} W(t) + N^\epsilon(t), \tag{1.6}$$

where

$$\sigma^2(\epsilon) \;:=\; \int_{|x|<\epsilon} x^2 \, Q(dx). \tag{1.7}$$

Notice a Brownian term appears in X_2^ϵ even when the original process X does not have one (that is, $b = 0$). This implicitly asssumes, of course, that the error

$$X_\epsilon(t) \;:=\; X(t) - X_1^\epsilon(t) \tag{1.8}$$

is approximately normal, as has been suggested on intuitive grounds in some particular cases (Bondesson [5], Rydberg [13]). The purpose of the present paper is to provide a rigorous discussion of when the functional CLT underlying (1.6) is indeed valid and to study some further related problems including convergence rates and possible non–Brownian limits of $X_\epsilon/\sigma(\epsilon)$.

The discussed problem is closely related to simulation of X based on series representations of the form

$$X(t) \;=\; \sum_{n=1}^{\infty} [H(\Gamma_n, V_n) I(U_n \le t) - c_n t] \tag{1.9}$$

where the U_n, V_n are i.i.d. uniform (0,1) r.v.'s, the Γ_n are the epochs of an independent Poisson process, H a suitable function, and the c_n centering constants. One then needs to truncate either to a finite number of terms of the series or to a finite time span of the Poisson process, and since H is typically decreasing in the first argument, this also means removing small jumps. When the series converges slowly, the normal approximation of the small jump part is advisable. See Rosiński [12] for an overview of such series representations and Bondesson [5] for aspects of the simulation implementation.

In Section 2, we provide a necessary and sufficient condition on the function $\sigma(\epsilon)$ for $\sigma(\epsilon)^{-1} X_\epsilon$ to converge in distribution to W, and we give a sufficient condition of a simpler form which is also necessary in most well–behaved cases. It will be seen that indeed a Brownian limit holds in substantial generality but also that there are important exceptions; this point is further illustrated in Section 4 where we show that the limiting behaviour of $\sigma(\epsilon)^{-1} X_\epsilon$ can be quite intricate. In Section 2, the condition on $\sigma(\epsilon)$ is also put in relation to the decay of the terms in the series (1.9). Error rates in the form of Berry–Esseen bounds and Edgeworth approximations are discussed in Section 3, which also contains some further discussion of simulation aspects.

2 The range of the normal approximation for the small jumps of a Lévy process

Recall the approximation error X_ϵ given by (1.8). X_ϵ is a Lévy process with characteristic function

$$\mathbb{E}\exp iuX_\epsilon(t) = \exp\left\{t\int_{|x|<\epsilon}(e^{iux}-1-iux)\,Q(dx)\right\}. \tag{2.1}$$

Consequently, $\mathbb{E}X_\epsilon = 0$ and $\mathbf{Var}(X_\epsilon(1)) = \sigma^2(\epsilon)$ is given by (1.7). The weak convergence of $\sigma(\epsilon)^{-1}X_\epsilon$ to a standard Brownian motion W will be understood in $D[0,1]$ equipped with the uniform metric (see [11]). Specifically, $\sigma(\epsilon)^{-1}X_\epsilon \xrightarrow{\mathcal{D}} W$ means that for every function $f : D[0,1] \to \mathbb{R}$, that is continuous with respect to the uniform metric, bounded, and measurable with respect to the projection σ-field, one has $\mathbb{E}f(\sigma(\epsilon)^{-1}X_\epsilon) \to \mathbb{E}f(W)$ as $\epsilon \to \infty$.

Theorem 2.1 *$\sigma(\epsilon)^{-1}X_\epsilon \xrightarrow{\mathcal{D}} W$ as $\epsilon \to 0$ if and only if for each $\kappa > 0$*

$$\sigma(\kappa\sigma(\epsilon)\wedge\epsilon) \;\sim\; \sigma(\epsilon), \qquad \textit{as } \epsilon \to 0. \tag{2.2}$$

Proof Put $Y_\epsilon(t) := \sigma(\epsilon)^{-1}X_\epsilon(t)$. We have

$$\mathbb{E}\exp iuY_\epsilon(t) = \exp\left\{t\left[iub_\epsilon + \int_{-\infty}^{\infty}(e^{iux}-1-iux1(|x|\le 1))Q_\epsilon(dx)\right]\right\}$$

where

$$Q_\epsilon(B) \;:=\; Q\left(\sigma(\epsilon)B\cap(-\epsilon,\epsilon)\right), \quad B\in\mathcal{B}(\mathbb{R}), \tag{2.3}$$

and

$$b_\epsilon \;:=\; -\sigma(\epsilon)^{-1}\int_{\sigma(\epsilon)\wedge\epsilon<|x|<\epsilon} x\,Q(dx).$$

Since Y_ϵ have stationary independent increments, it is enough to show the convergence at the endpoints, that is $Y_\epsilon(1) \xrightarrow{\mathcal{D}} W(1)$ (this is an easy application of [11] Th. V.19). Furthermore, $Y_\epsilon(1) \xrightarrow{\mathcal{D}} W(1)$ if and only if $\int_{|x|<\kappa}x^2\,Q_\epsilon(dx) \to 1$ for each $\kappa > 0$, $Q_\epsilon(|x|\ge 1) \to 0$, and $b_\epsilon \to 0$ as $\epsilon \to \infty$ (see, e.g., [9] Th. 13.14). We have

$$\int_{|x|<\kappa}x^2\,Q_\epsilon(dx) = \frac{\sigma^2(\kappa\sigma(\epsilon)\wedge\epsilon)}{\sigma^2(\epsilon)}.$$

Hence (2.2) is necessary for the convergence in distribution. To complete the proof of its sufficiency observe that

$$Q_\epsilon(\{|x|\ge 1\}) = Q(\{\epsilon\wedge\sigma(\epsilon)\le|x|\le\epsilon\}) \le \frac{\sigma^2(\epsilon)-\sigma^2(\epsilon\wedge\sigma(\epsilon))}{\epsilon^2\wedge\sigma^2(\epsilon)} \to 0$$

and that

$$|b_\epsilon| \le \sigma(\epsilon)^{-2} \int_{\sigma(\epsilon)\wedge\epsilon<|x|<\epsilon} x^2\, Q(dx) \le \frac{\sigma^2(\epsilon) - \sigma^2(\epsilon \wedge \sigma(\epsilon))}{\epsilon^2 \wedge \sigma^2(\epsilon)} \to 0.$$

□

The next proposition gives a more intuitive condition for the validity of the normal approximation. It says that the normal approximation holds when the dispersion of the small jump part of a Lévy process converges slower to zero than the level of truncation. Equivalently, the range $\epsilon/\sigma(\epsilon)$ of jumps of $\sigma(\epsilon)^{-1}X_\epsilon$ approaches 0.

Proposition 2.2 *Condition (2.2) is implied by*

$$\lim_{\epsilon\to 0} \frac{\sigma(\epsilon)}{\epsilon} = \infty. \tag{2.4}$$

Moreover, if Q does not have atoms in some neighborhood of the origin, then (2.2) and (2.4) are equivalent.

Proof The fact that (2.4) implies (2.2) is obvious. We will prove the converse under the above assumption on Q. Assume (2.2) and that Q does not have atoms in some interval $[-\epsilon_0, \epsilon_0]$, $\epsilon_0 > 0$. Then $\sigma([0, \epsilon_0])$ is an interval containing 0. Let $u_n \to 0$ be a sequence of positive numbers and let $\kappa > 0$. It follows that for sufficiently large n, there exists $\epsilon_n \in (0, \epsilon_0]$ such that $\kappa^{-1}u_n = \sigma(\epsilon_n)$ and $\epsilon_n \to 0$ as $n \to \infty$. Consequently,

$$\frac{\sigma(u_n)}{u_n} = \frac{\sigma(\kappa\sigma(\epsilon_n))}{\kappa\sigma(\epsilon_n)} \ge \kappa^{-1}\frac{\sigma(\kappa\sigma(\epsilon_n) \wedge \epsilon)}{\sigma(\epsilon_n)} \to \kappa^{-1},$$

which gives (2.4). □

The obvious question is whether (2.2) and (2.4) are equivalent in general. The answer is negative as it shows the following example.

Example 2.3 Let $a_n \searrow 0$ be such that $a_1 = 1$ and $\frac{a_{n+1}}{a_n} \to 0$ as $n \to \infty$. Let Q be a symmetric Lévy measure such that

$$\sigma(\epsilon) = a_n \qquad \text{for} \quad \epsilon \in (a_{n+1}, a_n],$$

and $\sigma(\epsilon) = 1$ for $\epsilon > 1$. Condition (2.4) fails because $\liminf_{\epsilon\to 0} \frac{\sigma(\epsilon)}{\epsilon} = 1$. However, for each $\kappa > 0$ there exists n_0 such that $\kappa a_n > a_{n+1}$ for $n \ge n_0$. Hence

$$\sigma(\kappa\sigma(\epsilon) \wedge \epsilon) = \sigma(\epsilon)$$

for $\epsilon \in (0, a_{n_0}]$. Thus (2.2) holds.

We will now give some examples of Lévy processes that admit or do not admit the approximation described above. They will be described by the Lévy measure Q of $X(1)$.

Example 2.4 Compound Poisson processes. Since $Q(\mathrm{I\!R}) < \infty$, $\sigma(\epsilon) = o(\epsilon)$ and so (2.2) fails.

Example 2.5 Stable processes of index $\alpha \in (0,2)$. In this case $Q(dx) = a|x|^{-1-\alpha} I(x < 0)dx + bx^{-1-\alpha} I(x > 0)dx$, $a, b \geq 0$, $a + b > 0$ and $\sigma(\epsilon) = ((a+b)/(2-\alpha))^{1/2} \epsilon^{1-\alpha/2}$. By Proposition 2.2, the normal approximation holds. The normal approximation is valid also in the more general situation of $Q(dx) = |x|^{-1-\alpha} L(x) dx$, where L is slowly varying at 0. Indeed, applying Karamata's theorem one has $\sigma(\epsilon) \sim ((L(-\epsilon) + L(\epsilon))/(2-\alpha))^{1/2} \epsilon^{1-\alpha/2}$. Since $\epsilon^{-\gamma}(L(-\epsilon) + L(\epsilon)) \to \infty$ for every $\gamma > 0$, we get (2.4) (see also Proposition 2.1 in Asmussen [1]).

Example 2.6 Gamma processes. We have $Q(dx) = ax^{-1} e^{-x/b} dx$, $a, b > 0$, $x > 0$, and $\sigma(\epsilon) \sim (a/2)^{1/2} \epsilon$. Therefore, the normal approximation does not hold. Now consider $Q(dx) = |x|^{-1} I(|x| \leq 1) dx$. We have an interesting case of self–normalization

$$\sigma(\epsilon)^{-1} X_\epsilon \stackrel{\mathcal{D}}{=} \sigma(1)^{-1} X_1 \qquad \text{for every} \quad \epsilon \in (0,1],$$

from which is evident that the normal approximation fails.

Example 2.7 The normal inverse Gaussian Lévy process is a Lévy process with $X(1)$ distributed accordingly to the normal inverse Gaussian distribution, see [2]. Normal approximation for small jumps was used on a intuitive basis and computer simulation provided by Rydberg [13]. The approximation is valid by Proposition 2.2 since $\sigma(\epsilon) \sim (2\delta/\pi)^{1/2} \epsilon^{1/2}$ as $\epsilon \to 0$.

In conclusion of this section, we will relate the normal approximation of Lévy processes to the decay of terms in their series expansions (1.9). For the sake of simplicity, suppose that X is a symmetric Lévy process without Gaussian component and with an infinite Lévy measure Q. Then

$$X(t) \;=\; \sum_{n=1}^{\infty} r_n Q^{\leftarrow}(\Gamma_n) I(U_n \leq t) \tag{2.5}$$

where

$$Q^{\leftarrow}(t) \;:=\; \inf\{x > 0 : 2Q([x,\infty)) < t\}$$

and r_n are i.i.d. symmetric Bernoulli random variables independent of the sequences Γ_n and U_n. These random variables can be defined on the same probability space as X so that representation (2.5) holds a.s. and the series converges a.s. uniformly on $[0,1]$ (see Rosiński [12]). The relation to (1.9) is that $H(\Gamma_n, V_n) = (2I(V_n \leq .5) - 1) Q^{\leftarrow}(\Gamma_n)$.

Proposition 2.8 *Suppose that for every $a > 0$,*

$$\lim_{t\to\infty} \frac{Q^{\leftarrow}(t+a)}{Q^{\leftarrow}(t)} = 1. \tag{2.6}$$

Then (2.4) holds. Consequently,

$$X_\epsilon(t) \ = \sum_{Q^{\leftarrow}(\Gamma_n)<\epsilon} r_n Q^{\leftarrow}(\Gamma_n) I(U_n \le t)$$

is asymptotically equal to a Brownian motion with variance $\sigma^2(\epsilon)$.

Proof We have

$$\sigma^2(\epsilon) = \int_{|x|<\epsilon} x^2\, Q(dx) = \int_{Q^{\leftarrow}(t)<\epsilon} (Q^{\leftarrow}(t))^2\, dt.$$

Observe that the set $\{t > 0 : \ Q^{\leftarrow}(t) < \epsilon\}$ is of the form $(t(\epsilon), \infty)$ and that $t(\epsilon) \to \infty$ as $\epsilon \to 0$. Hence, for each $a > 0$ and sufficiently small ϵ,

$$\sigma^2(\epsilon) \ge \int_{t(\epsilon)}^{t(\epsilon)+a} (Q^{\leftarrow}(t))^2\, dt \ge a(Q^{\leftarrow}(t(\epsilon)+a))^2 \ge (a/2)(Q^{\leftarrow}(t(\epsilon)))^2 \ge (a/2)\epsilon^2.$$

Since a is arbitrary, condition (2.4) is proved. □

Condition (2.6) holds in Examples 2.5 and 2.7 but fails in Example 2.6.

3 Speed of convergence.

We only consider one–dimensional distributions. Note, however, that many of the basic results in the area have functional extensions, see e.g. Götze [8].

We begin with a Berry–Esseen type estimate for the difference of distribution functions of $X_2^\epsilon(1)$ and $X(1)$ (recall (1.6)).

Theorem 3.1

$$\sup_{x\in\mathbb{R}} |\mathbb{P}(X_2^\epsilon(1) \le x) - \mathbb{P}(X(1) \le x)| \ \le \ (0.7975)\sigma^{-3}(\epsilon) \int_{|x|<\epsilon} |x|^3\, Q(dx). \tag{3.1}$$

Proof Write $X_\epsilon(1) = \sum_{k=1}^n [X_\epsilon(k/n) - X_\epsilon((k-1)/n)]$ as a sum of i.i.d. random variables with mean zero and variance $\sigma^2(\epsilon)/n$. Applying the Berry–Esseen theorem (see [4] Chapter 3.12) to this decomposition we obtain

$$\sup_{z\in\mathbb{R}} |\mathbb{P}(\sigma^{-1}(\epsilon) X_\epsilon(1) \le z) - \Phi(z)| \ \le \ (0.7975)\sigma^{-3}(\epsilon) n \mathbb{E}|X_\epsilon(1/n)|^3$$

where Φ denotes the standard normal distribution function. Letting $n \to \infty$ on the right hand side and using Lemma 3.2 given below, we get

$$\sup_{z \in \mathbb{R}} |\mathbb{P}(X_\epsilon(1) \le z) - \Phi(\sigma^{-1}(\epsilon)z)| \ \le \ (0.7975)\sigma^{-3}(\epsilon) \int_{|x|<\epsilon} |x|^3 \, Q(dx). \tag{3.2}$$

Notice now that $X_2^\epsilon \stackrel{\mathcal{D}}{=} X_1^\epsilon + \sigma(\epsilon)W'$, where W' is a Brownian motion independent of X_1^ϵ. Hence, for every $x \in \mathbb{R}$,

$$\begin{aligned} &|\mathbb{P}(X_2^\epsilon(1) \le x) - \mathbb{P}(X(1) \le x)| \\ &\quad = \ |\mathbb{P}(X_1^\epsilon(1) + \sigma(\epsilon)W'(1) \le x) - \mathbb{P}(X_1^\epsilon(1) + X_\epsilon(1) \le x)| \\ &\quad = \ \le \int_{\mathbb{R}} |\mathbb{P}(\sigma(\epsilon)W'(1) \le x - y) - \mathbb{P}(X_\epsilon(1) \le x - y)| \, \mathbb{P}(X_1^\epsilon(1) \in dy). \end{aligned}$$

Applying the bound (3.2) to the last integrand we conclude the proof. □

Lemma 3.2 *Let $Z = \{Z(t) : t \ge 0\}$ be a Lévy process with Lévy measure H. Suppose that for some $p \ge 2$, $\mathbb{E}|Z(1)|^p < \infty$ and $\mathbb{E}Z(1) = 0$. Then*

$$\lim_{n \to \infty} n\mathbb{E}|Z(1/n)|^p \ = \ \int_{-\infty}^{\infty} |x|^p \, H(dx).$$

Proof This lemma is obvious when $p = 2$. Thus assume $p > 2$. By integration by parts we have

$$n\mathbb{E}|Z(1/n)|^p \ = \ p \int_0^\infty s^{p-1} n\mathbb{P}(|Z(1/n)| > s) \, ds. \tag{3.3}$$

We will split this integral into three parts, integrating over disjoint regions. The first integral is

$$\int_0^{s_0} s^{p-1} n\mathbb{P}(|Z(1/n)| > s) \, ds \ \le \ \int_0^{s_0} s^{p-3} n\mathbb{E}|Z(1/n)|^2 \, ds \ = \ (p-2)^{-1} s_0^{p-2} \mathbb{E}|Z(1)|^2.$$

We used Chebyshev's inequality; this integral is small when s_0 is small. Now consider the second integral, $\int_{s_0}^{s_1} s^{p-1} n\mathbb{P}(|Z(1/n)| > s) \, ds$. Write $Z(1) = \sum_{k=1}^n [Z(k/n) - Z((k-1)/n)]$ as the sum of i.i.d. random variables. By the Central Limit Theorem we have

$$n\mathbb{P}(|Z(1/n)| > s) \ = \ \sum_{k=1}^n \mathbb{P}(|Z(k/n) - Z((k-1)/n)| > s) \ \to \ H(\{x : |x| > s\}) \tag{3.4}$$

as $n \to \infty$, for each continuity point $s > 0$, that is, $H(\{x : |x| = s\}) = 0$. Let $0 < s_0 < s_1$ be such that $H(\{x : |x| = s_i\}) = 0$, $i = 1, 2$. Since the set of continuity points is at most countable, we get by (3.4)

$$\lim_{n \to \infty} \int_{s_0}^{s_1} s^{p-1} n\mathbb{P}(|Z(1/n)| > s) \, ds \ = \ \int_{s_0}^{s_1} s^{p-1} H(\{x : |x| > s\}) \, ds. \tag{3.5}$$

Now we consider the third integral, $\int_{s_1}^{\infty} s^{p-1} n\mathbb{P}(|Z(1/n)| > s)\, ds$. Using (3.4) one can choose s_1 sufficiently large such that $n\mathbb{P}(|Z(1/n)| > s_1) < 1$ for all $n \geq 1$. Then, for $s > s_1$,

$$\begin{aligned} e^{-1} n\mathbb{P}(|Z(1/n)| > s) &\leq 1 - (1 - \mathbb{P}(|Z(1/n)| > s))^n \\ &\leq \mathbb{P}(\max_{k \leq n} |Z(k/n)| > s/2) \;\leq\; 9\mathbb{P}(|Z(1)| > s/60). \end{aligned}$$

The last inequality follows from Theorem 1.1.5 in [7]. Hence

$$\int_{s_1}^{\infty} s^{p-1} n\mathbb{P}(|Z(1/n)| > s)\, ds \;\leq\; 9e \int_{s_1}^{\infty} s^{p-1} \mathbb{P}(|Z(1)| > s/60)\, ds$$

and the upper bound goes to 0 as $s_1 \to \infty$, because $\mathbb{E}|Z(1)|^p < \infty$. Consequently, one can choose the continuity points s_0, s_1 that make the first and third integrals, resulting from the integraton by parts formula (3.3), uniformly small with respect to $n \geq 1$. Notice also that $(\int_0^{s_0} + \int_{s_1}^{\infty})|x|^{p-1} H(\{x : |x| > s\})\, ds$ is small for sufficiently small s_0 and large s_1. Now the lemma follows from (3.3) and (3.5). □

Inequality (3.1) gives a base for the use of quantity

$$\delta(\epsilon) \;:=\; \sigma^{-3}(\epsilon) \int_{|x|<\epsilon} |x|^3\, Q(dx) \tag{3.6}$$

as a measure of accuracy of approximation and a criterion for the selection of ϵ. Notice that $\delta(\epsilon) \leq \epsilon/\sigma(\epsilon)$, which is consistent with (2.4). In the case of Lévy stable processes (see Example 2.5) we have

$$\delta(\epsilon) \;=\; \frac{(2-\alpha)^{3/2}}{(3-\alpha)(a+b)^{1/2}}\, \epsilon^{\alpha/2}.$$

Notice an improvement of the accuracy as α approaches 2 and also as $a + b$ becomes large. In the more general case of regularly varying tails described in Example 2.5 we have

$$\delta(\epsilon) \;\sim\; \frac{(2-\alpha)^{3/2}}{(3-\alpha)(L(-\epsilon) + L(\epsilon))^{1/2}}\, \epsilon^{\alpha/2}.$$

In the case of normal inverse Gaussian Lévy processes we have

$$\delta(\epsilon) \;\sim\; \sqrt{\frac{\pi}{8\delta}}\, \epsilon^{1/2}$$

where δ on the right hand side is a parameter of the normal inverse Gaussian distribution appearing also in Example 2.7.

One would expect the exact convergence rate to be given in terms of Edgeworth expansions (Petrov [10] or Skovgaard [16]), which would lead to

$$\mathbb{P}(X_\epsilon(1) \leq x/\sigma(\epsilon)) - \Phi(x) \;=\; F_k(x; \kappa_{3,\epsilon}, \ldots, \kappa_{k,\epsilon}) + o(\kappa_{k,\epsilon}) \tag{3.7}$$

where $F_k(\cdot;\kappa_3,\dots,\kappa_k)$ is the correction term in the $(k-1)$ Edgeworth expansion of a distribution with mean 0, unit variance and cumulants $\kappa_3,\kappa_4,\dots$ (recall in particular that $F_3(x;\kappa_3) = \frac{\kappa_3}{6}(1-x^2)\varphi(x))$ and $\kappa_{k,\epsilon}$ is the kth cumulant of $X_\epsilon(1)/\sigma(\epsilon)$. For example, if $Q(dx) = |x|^{-1-\alpha}L(x)\,dx$ where L is slowly varying at 0 and $0<\alpha<2$, then

$$\kappa_{k,\epsilon} \;\sim\; L_k(\epsilon)\epsilon^{\alpha(k-2)/2} \quad \text{where} \quad L_k(\epsilon) \;=\; (L(\epsilon)+(-1)^k L(-\epsilon))^{1-k/2}\frac{2-\alpha}{k-\alpha}\,. \tag{3.8}$$

The implication of (3.7) is that the error $\mathbb{P}(X_\epsilon/\sigma(\epsilon)\le x)-\Phi(x)$ in general is of order $|\kappa_{3,\epsilon}|$, which is the same as the order $\delta(\epsilon)$ is as in the Berry–Esseen bound (3.1) (unless for the special case $(L(\epsilon)-L(-\epsilon))/(L(\epsilon)+L(-\epsilon))\to 0$ as $\epsilon\to 0$; then up to the slowly varying function $L_3(\epsilon)$, the order is $\epsilon^{\alpha/2}$). However, in the symmetric case, (3.7) shows that the error rate $\kappa_{4,\epsilon}$ (of order roughly ϵ^α) is effectively smaller than the bound (3.1).

The expansion (3.7), which we do not verify rigorously here (the key step is to bound tail integrals of the characteristic function of $X_\epsilon(1)/\sigma(\epsilon)$, see e.g. Theorem 3.1 of [16] and the following discussion), has an interesting interpretation in terms of the computational effort required to reach a given accuracy η by the simulation. Say we want the deviations of F_ϵ from Φ to be of order η and identify the computational effort to generate $X_\epsilon(1)$ with the number $M=M_\epsilon$ of Poisson events. Then in the asymmetric case ϵ is determined by

$$\eta = |\kappa_{3,\epsilon}| \tag{3.9}$$

whereas

$$M_\epsilon \;\approx\; \int_{|x|\ge\epsilon} Q(dx). \tag{3.10}$$

Consider for simplicity the stable case as in Example 2.5. In the skewed case $a\ne b$ (say $a=1$, $b=0$), we get $\kappa_{3,\epsilon}=\epsilon^{\alpha/2}(2-\alpha)^{3/2}/(3-\alpha)$ which using (3.10) yields

$$M \;\approx\; \frac{\epsilon^{-\alpha}}{\alpha} \;=\; \frac{(2-\alpha)^3}{\alpha(3-\alpha)^2}\cdot\frac{1}{\eta^2}.$$

In the symmetric case $a=b$ (say $a=b=1/2$), we get $\eta=\kappa_{4,\epsilon}=\epsilon^\alpha(2-\alpha)^2/(4-\alpha)$,

$$M \;\approx\; \frac{\epsilon^{-\alpha}}{\alpha} \;=\; \frac{(2-\alpha)^2}{\alpha(4-\alpha)^2}\cdot\frac{1}{\eta}.$$

Thus in both cases, there is again a pay–off of having α close to 2, whereas the better convergence rate in the symmetric case is reflected in the computational error going slower to infinity as $\eta\downarrow 0$.

4 Possible limits — how bad it can be

For a fixed Lévy process X with an infinite Lévy measure Q and $b = 0$, consider the family $\{\mathcal{L}(\sigma(\epsilon)^{-1}X_\epsilon(1)) : 0 < \epsilon \le 1\}$. Since $\mathbf{Var}(X_\epsilon(1)) = \sigma^2(\epsilon)$, this family is tight. Therefore, one can always find a sequence of truncations $\epsilon_n \searrow 0$ and a Lévy process Z such that $\sigma(\epsilon_n)^{-1}X_{\epsilon_n} \xrightarrow{\mathcal{D}} Z$ in the Skorokhod topology. If (2.2) holds, then the only possible limit Z is the standard Brownian motion. But if (2.2) fails, then there are other limits as well. We examine the question how possibly large can be the class of such limits of normalized small jump parts of a one Lévy process. Since the convergence of Lévy processes is determined by the convergence at $t = 1$, it is enough to examine the set

$$\mathcal{I}(Q) \;=\; \bigcap_{\delta>0} \overline{\{\mathcal{L}(\sigma(\epsilon)^{-1}X_\epsilon(1)) : 0 < \epsilon < \delta\}}$$

of (weak) limit points. Necessarily, $\mathcal{I}(Q)$ is a subset of $\mathcal{I}_0$, the set of infinitely divisible distributions with mean zero and variance at most one. Note that any $F \in \mathcal{I}_0$ can be written as $F = F_1 * F_2$ where F_2 is $\mathcal{N}(0, \sigma_2^2)$ and F_1 has characteristic function of the form

$$\hat{F}_1(u) \;=\; \exp\left[\int_{-\infty}^{\infty}(e^{iux} - 1 - iux)Q_1(dx)\right], \tag{4.1}$$

where $0 \le \sigma_1^2 + \sigma_2^2 \le 1$, $\sigma_1^2 = \int_{-\infty}^{\infty} x^2 Q_1(dx)$.

Theorem 4.1 *There exists a Lévy measure Q such that $\mathcal{I}(Q) = \mathcal{I}_0$. In other words, there exists a Lévy process X such that for any Lévy process Z with $\mathcal{L}(Z(1)) \in \mathcal{I}_0$, there exists a sequence $\epsilon_n \searrow 0$ such that $\sigma(\epsilon_n)^{-1}X_{\epsilon_n} \xrightarrow{\mathcal{D}} Z$ in the Skorochod topology.*

Proof Let $\mathcal{I}_{0,1}$ be the subset of $F_1 \in \mathcal{I}_0$ such that $\hat{F}_1$ has the form (4.1) with $\sigma_1^2 = 1$. We first show how to construct Q such that $F_1 \in \mathcal{I}(Q)$ for a given $F_1 \in \mathcal{I}_{0,1}$. Write

$$a_k \;:=\; \frac{1}{(k-1)!^2}, \quad J_k \;:=\; (-a_k, -a_{k+1}) \cup (a_{k+1}, a_k),\; k = 2, 3, \ldots,$$

and let $Q_1^{(k)}$ be a finite measure concentrated on J_k given by

$$Q_1^{(k)}(B) = Q_1\left(k a_k^{-1} B \cap \{k^{-1} < |x| < k\}\right).$$

Define

$$Q(B) \;:=\; \sum_{k=2}^{\infty} Q_1^{(k)}(B), \quad B \in \mathcal{B}(\mathbb{R}).$$

Let

$$\omega_k^2 \;:=\; \int_{\mathbb{R}} x^2 Q_1^{(k)}(dx) \;=\; \frac{a_k^2}{k^2}\int_{\{1/k<x<k\}} x^2 Q_1(dx), \quad \varpi_k^2 \;:=\; \omega_k^2 + \omega_{k+1}^2 + \cdots.$$

Then $\omega_k^2 \sim a_k^2/k^2$ and $\overline{\omega}_k \sim \omega_k$. In particular, $\overline{\omega}_2^2 < \infty$ which shows that Q is a legitimate Lévy measure. The Lévy measure Q_{1,a_k} of $\sigma(a_k)^{-1}X_{a_k}(1)$ is given by

$$Q_{1,a_k}(B) = \sum_{n=k}^{\infty} Q_1^{(n)}(\overline{\omega}_k B),$$

see (2.3). Therefore, $\sigma(a_k)^{-1}X_{a_k}(1)$ can be represented as a series of independent compensated compound Poisson random variables $V_{k,n}$, $n \geq k$ with Lévy measures $Q_1^{(n)}(\overline{\omega}_k(\cdot))$. It is easy to check that the characteristic function of $V_{k,k}$ converges to $\hat{F}_1$ as $k \to \infty$, whereas

$$\sum_{n=k+1}^{\infty} \mathbf{Var}(V_{k,n}) = \sum_{n=k+1}^{\infty} \frac{a_n^2}{n^2 \overline{\omega}_k^2} \int_{n^{-1}<|x|<n} x^2\, Q(dx) \ \to \ 0$$

as $k \to \infty$. Hence $\mathcal{L}(\sigma(a_k)^{-1}X_{a_k}(1)) \xrightarrow{w} F_1$.

Let next $\{F_j\}$ be dense in $\mathcal{I}_{0,1}$ and let $\{F_j\}$ be the set of corresponding Lévy measures. Modifying the above construction of Q by setting

$$Q(B) \ := \ \sum_{k=2}^{\infty} Q_{m_k}^{(k)}(B),$$

where $\{m_k\}$ is the sequence $1, 2, 1, 2, 3, 1, 2, 3, 4, 1 \ldots$, it then follows in the same way that $F_j \in \mathcal{I}(Q)$ for all j. Hence $\mathcal{I}_{0,1} \subseteq \mathcal{I}(Q)$.

Finally note that $\mathcal{I}_{0,1}$ is dense in $\mathcal{I}_0$. Indeed, if $F \in \mathcal{I}_0$ is written as $F = F_1 * F_2$ as above, then $F_1 = \lim G_n$, $F_2 = \lim H_n$ where $\hat{G}_n, \hat{H}_n$ are of the form (4.1) and the variances of G_n, H_n are $1 - \sigma_2^2$ $(\geq \sigma_1^2)$, σ_2^2, respectively. Since $G_n * H_n \xrightarrow{w} F$ and $G_n * H_n \in \mathcal{I}_{0,1}$, we have $F \in \overline{\mathcal{I}_{0,1}}$. □

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