A unified approach to resolvent expansions at thresholds

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Abstract

Results are obtained on resolvent expansions around zero energy for Schrödinger operators $H = -\Delta + V(\mathbf{x})$ on $L^2(\mathbf{R}^m)$, where $V(\mathbf{x})$ is a sufficiently rapidly decaying real potential. The emphasis is on a unified approach, valid in all dimensions, which does not require one to distinguish between $\int V(\mathbf{x})d\mathbf{x} = 0$ and $\int V(\mathbf{x})d\mathbf{x} \neq 0$ in dimensions $m = 1, 2$. It is based on a factorization technique and repeated decomposition of the Lippmann-Schwinger operator. Complete results are given in dimensions $m = 1$ and $m = 2$.

1 Introduction

In this paper we revisit some results on resolvent expansions for Schrödinger operators. We consider Schrödinger operators

$$H = H_0 + V, \quad H_0 = -\Delta,$$

on $L^2(\mathbf{R}^m)$, where V is multiplication by a real-valued function with decay at least $V(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-2-\delta})$ as $|\mathbf{x}| \to \infty$. The free resolvent $R_0(\zeta) = (H_0 - \zeta)^{-1}$ has an explicit integral kernel, which can be used to give asymptotic

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expansions around zero in $\zeta^{1/2}$ for m odd, and in ζ and $\ln\zeta$ for m even. We give the form of the leading terms in dimensions $m = 1, 2, 3$ here.

$$m = 1 \qquad R_0(\zeta) = \zeta^{-1/2}G_{-1} + G_0 + \zeta^{1/2}G_1 + \zeta G_2 + \dots\,, \tag{1.1}$$

$$m = 2 \qquad R_0(\zeta) = \ln\zeta G_{0,-1} + G_{0,0} + \zeta\ln\zeta G_{2,-1} + \zeta G_{2,0} + \dots\,, \tag{1.2}$$

$$m = 3 \qquad R_0(\zeta) = G_0 + \zeta^{1/2}G_1 + \zeta G_2 + \dots\,. \tag{1.3}$$

These expansions are valid in operator norm on $L^2(\mathbf{R}^m)$, if we put weight functions on either side of the resolvent. One possible choice is $\rho(\mathbf{x}) = \langle\mathbf{x}\rangle^{-s}$, and then expansion up to a given order $\mathcal{O}(|\zeta|^k)$ is valid for a sufficiently large s. Another possibility is to use $\rho(\mathbf{x}) = |V(\mathbf{x})|^{1/2}$ as the weight function, which is what we choose to do in this paper. Expansion to higher order then requires faster decay at infinity of the potential. The two approaches lead to different, but equivalent, formulations of the main results.

The decay imposed on $V(\mathbf{x})$ implies that we can obtain expansions for the resolvent $R(\zeta) = (H-\zeta)^{-1}$, using a perturbation procedure. The case $m = 3$ was treated using this approach in [9]. The form of the expansion is

$$m = 3 \qquad R(\zeta) = -\zeta^{-1}P_0 + \zeta^{-1/2}C_{-1} + C_0 + \zeta^{1/2}C_1 + \mathcal{O}(\zeta), \tag{1.4}$$

where generically we have $P_0 = 0$ and $C_{-1} = 0$. Three kinds of exceptional cases occur. (i) The point zero is an L^2-eigenvalue of H. In this case P_0 is the projection onto the eigenspace, and C_{-1} is an operator of rank at most three. (ii) The equation $H\Psi = 0$ has a non-zero solution in a space slightly larger than $L^2(\mathbf{R}^3)$. In this case $P_0 = 0$ and $C_{-1} = i\langle\Psi,\cdot\rangle\Psi$ is a rank one operator (here Ψ should be suitably normalized). In this case we say that H has a zero-resonance. (iii) The combination of the previous two cases.

The purpose of this paper is to give a unified approach to such resolvent expansions, and in particular to give complete and unified results in the two cases $m = 1$ and $m = 2$. These cases are difficult to handle, due to the singularity in the free resolvent, see (1.1) and (1.2). We use a repeated decomposition technique, where we localize the singularity in subspaces of decreasing dimension. Each reduction step increases the singularity. Due to the estimate $|\zeta|\|\rho(\mathbf{x})R(\zeta)\rho(\mathbf{x})\| \le C$ this reduction process must stop after a few steps, leading to invertibility of a key reduced operator.

Our approach is unified in the sense that this reduction procedure applies in all dimensions, without separating out various special cases. Another key idea is the use of the factorization technique in the following form. We factor $V(\mathbf{x}) = v(\mathbf{x})w(\mathbf{x})$, where $v(\mathbf{x}) = |V(\mathbf{x})|^{1/2}$, $U(\mathbf{x}) = 1$ for $V(\mathbf{x}) \ge 0$ and $U(\mathbf{x}) = -1$ for $V(\mathbf{x}) < 0$, and $w(\mathbf{x}) = U(\mathbf{x})v(\mathbf{x})$. Then the crucial term to invert is

$$M(\zeta) = U + vR_0(\zeta)v, \tag{1.5}$$

see (4.3). Now an important point is that this operator is self-adjoint for $\operatorname{Re}\zeta < 0$, $\operatorname{Im}\zeta = 0$, which eliminates the need to distinguish between geometric and algebraic eigenspaces, and gives a canonical choice for the projection onto the eigenspace.

Let us briefly state the form of the expansions in the two cases considered in detail. We state the results in the same form as in (1.4).

$$m = 1 \qquad\qquad R(\zeta) = \zeta^{-1/2}C_{-1} + C_0 + \zeta^{1/2}C_1 + \mathcal{O}(\zeta). \tag{1.6}$$

In the case $m = 1$, and under the assumption $V(x) = \mathcal{O}(|x|^{-2-\delta})$ as $|x| \to \infty$, zero cannot be an L^2-eigenvalue. But there may exist a non-zero solution to $H\Psi = 0$, which satisfies $\Psi \in L^\infty(\mathbf{R})$. In this case $C_{-1} = ic_0\langle\Psi, \cdot\rangle\Psi$, where c_0 is a constant which is computed explicitly. This is the exceptional case, and we say that H has a zero-resonance. Generically with respect to a coupling constant we have $C_{-1} = 0$.

The case $m = 2$ is considerably more complicated. We start by explaining our terminology. Recall from [6] that in order to get an asymptotic expansion we need to have an asymptotic sequence of functions, which is a sequence of functions $\{\phi_j(\zeta)\}_{j\in\mathbf{N}}$, indexed by the non-negative integers, such that for all j we have

$$\phi_{j+1}(\zeta) = o(\phi_j(\zeta)) \quad \text{for } \zeta \to 0. \tag{1.7}$$

Formal computations lead in the case $m = 2$ to expansions of the form

$$\sum_{k=-1}^{\infty} \sum_{\ell=-\infty}^{\infty} \zeta^k (\ln\zeta)^\ell c_{k\ell}. \tag{1.8}$$

Such an expansion cannot be transformed into an asymptotic expansion, since the doubly indexed family of functions $\{\zeta^k(\ln\zeta)^\ell\}_{-1\le k<\infty, -\infty<\ell<\infty}$ cannot be re-indexed by the integers in such a manner that we get an asymptotic sequence. The problem is that a given entry may not have a finite number of predecessors according to the ordering implied by (1.7). In our case it turns out that the problem can be solved by using different functions in the asymptotic expansions. In one of the cases we replace the function $1/\ln\zeta$ and its nonnegative powers by the function $(a - \ln\zeta)^{-1}$, where a is a certain nonzero number. In the other case we introduce a rank two operator for a similar purpose. Note that an asymptotic sequence of functions cannot contain both $(\ln\zeta)^{-1}$ and $(a - \ln\zeta)^{-1}$, since $|(\ln\zeta)^{-1}/(a - \ln\zeta)^{-1}| \to 1$ as $\zeta \to 0$.

We use the terminology "bad" expansions for (formal) expansions that cannot be re-indexed to give asymptotic expansions.

The main results in the case $m = 2$ are too complicated to state in detail here. See the statement of Theorem 6.2. We note that as in the case $m = 3$

we have to distinguish between the regular (generic) case, where there is no singularity in the expansion, and three exceptional cases. (i) Zero is an L_2-eigenvalue of H. (ii) There exist non-zero solutions to $H\Psi = 0$ in $L^\infty(\mathbf{R}^2)$, which do not belong to L^2. There can be up to three linearly independent solutions. (iii) Combinations of the cases (i) and (ii).

In the exceptional case (i) the expansion can be rewritten in the form

$$m = 2 \qquad R(\zeta) = -\zeta^{-1}P_0 + (\ln\zeta)^{-1}C_{0,-1} + C_{0,0} + o(1), \tag{1.9}$$

where we have extracted the leading term in the complicated second term in the full expansion. Here P_0 is the eigenprojection for eigenvalue zero of H, and $C_{0,-1}$ is an operator of rank at most 3.

We have decided not to state any results on resolvent expansions in the cases $m \geq 3$, since our approach leads to results identical to those obtained in [9, 7, 8]. However, we do give the necessary formulae for the free resolvent expansion in Section 3. In Proposition 7.1 we then give a general result on the expansion coefficients, which in odd dimensions shows that the coefficients to odd powers of $\zeta^{1/2}$ are finite rank operators. A similar statement holds for the even dimensional cases.

Let us now give some comments on the literature. The first results on asymptotic expansions of resolvents of the type considered here were obtained in [15], in a very general (and not very explicit) framework, using properties of Fredholm operators. A different approach for the Schrödinger operator was introduced in [9] in the $m = 3$ case. This approach allows one to compute the coefficients explicitly. Using the same approach the cases $m \geq 5$ were treated in [7]. In [8] a good expansion was obtained for the case $m = 4$, by using a function $(a - \ln\zeta)^{-1}$ in the asymptotic expansion.

In [13] a general class of elliptic operators was considered, and resolvent expansions were obtained, using a Fredholm operator technique in combination with a truncated Lippmann-Schwinger operator. The methods allow for explicit computation of expansion coefficients. Our method is quite close to the one used in [13], in the sense that both rely on the fact that for any compact operator A one can find a finite rank operator F such that $(1+A+F)^{-1}$ exists. The key point of our approach is a canonical choice of F in terms of projections onto the subspaces of zero energy bound states and/or resonances. It is this choice which allows us to compute explicitly the expansion coefficients, without relying on operators given only implicitly as solutions of some operator equations. Actually, our choice can be viewed as a method for solving the equations for J, K, and Q in [13].

The case $m = 1$ has been treated in [5], in the case $\int V(x)dx \neq 0$, and in [3, 4] in the case $\int V(x)dx = 0$, with an exponential decay condition on the potential. This strong decay condition allows one to obtain convergent expansions in $\zeta^{1/2}$. In these papers the authors use the standard factorization, leading to the study of the operator $I + vR_0(\zeta)w$ and the consequent need to distinguish between the two cases. This should be compared with our unified approach.

More recently, in [11, 12] a study has been initiated of the case $m = 1$ for general non-local V with polynomial decay. The methods used are a combination of those in [9] and [5, 3, 4].

The case $m = 2$ has been studied in [2], under the additional condition $\int V(\mathbf{x})d\mathbf{x} \neq 0$, and with exponential decay of the potential, which leads to convergent expansions. The case $\int V(\mathbf{x})d\mathbf{x} = 0$ has not previously been treated explicitly in the literature, as far as we know. Note again that our unified approach makes it unnecessary to distinguish between the two cases.

Resolvent expansions of the type obtained here have many applications. The papers [15, 9, 7, 13, 8] all contain applications to the time decay of the corresponding non-stationary equations. Applications to scattering theory are also given in many of the papers previously cited. A survey of such results is given in [1]. The results have also been of importance in the study of mapping properties of the propagator, and of the wave operators, see for example [10, 16], and references therein.

Finally, let us briefly describe the contents of this paper. In Section 2 we state our essential lemmas from operator theory. In Section 3 we give the explicit expansions for free resolvents, in all dimensions, for reference. In Section 4 we explain our choice of factorization technique. Then Section 5 contains the results in the case $m = 1$ and Section 6 the results in the case $m = 2$. Finally, in Section 7 we give a result on the properties of expansion coefficients, valid in all dimensions, and collect some remarks about possible generalizations.

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2 Preliminaries: inversion formulae

We give here some elementary inversion formulae for operator matrices of a special form, which we need in the following sections. The proofs are omitted.

Lemma 2.1. *Let A be a closed operator and S a projection. Suppose $A+S$ has a bounded inverse. Then A has a bounded inverse if and only if*

$$a \equiv S - S(A+S)^{-1}S \tag{2.1}$$

has a bounded inverse in $S\mathcal{H}$, and in this case

$$A^{-1} = (A+S)^{-1} + (A+S)^{-1}Sa^{-1}S(A+S)^{-1}. \tag{2.2}$$

Corollary 2.2. *Let $F \subset \mathbf{C}$ have zero as an accumulation point. Let $A(z)$, $z \in F$, be a family of bounded operators of the form*

$$A(z) = A_0 + zA_1(z) \tag{2.3}$$

with $A_1(z)$ uniformly bounded as $z \to 0$. Suppose 0 is an isolated point of the spectrum of A_0, and let S be the corresponding Riesz projection. Then for sufficiently small $z \in F$ the operator $B(z) : S\mathcal{H} \to S\mathcal{H}$ defined by

$$\begin{aligned} B(z) &= \frac{1}{z}\big(S - S(A(z)+S)^{-1}S\big) \\ &= \sum_{j=0}^{\infty}(-1)^j z^j S[A_1(z)(A_0+S)^{-1}]^{j+1}S \end{aligned} \tag{2.4}$$

is uniformly bounded as $z \to 0$. The operator $A(z)$ has a bounded inverse in $\mathcal{H}$ if and only if $B(z)$ has a bounded inverse in $S\mathcal{H}$, and in this case

$$A(z)^{-1} = (A(z)+S)^{-1} + \frac{1}{z}(A(z)+S)^{-1}SB(z)^{-1}S(A(z)+S)^{-1}. \tag{2.5}$$

The next lemma contains the Feshbach formula in a somewhat abstract form.

Lemma 2.3. *Let A be an operator matrix on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$:*

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad a_{ij} : \mathcal{H}_j \to \mathcal{H}_i, \tag{2.6}$$

where a_{11}, a_{22} are closed and a_{12}, a_{21} are bounded. Suppose a_{22} has a bounded inverse. Then A has a bounded inverse if and only if

$$a \equiv (a_{11} - a_{12}a_{22}^{-1}a_{21})^{-1} \tag{2.7}$$

exists and is bounded. Furthermore, we have

$$A^{-1} = \begin{pmatrix} a & -aa_{12}a_{22}^{-1} \\ -a_{22}^{-1}a_{21}a & a_{22}^{-1}a_{21}aa_{12}a_{22}^{-1} + a_{22}^{-1} \end{pmatrix}. \tag{2.8}$$

Remark 2.4. We shall use the Feshbach formula in the following particular case. Suppose a_{11} is of the form

$$a_{11} = \frac{1}{z}k + b(z), \tag{2.9}$$

where k has a bounded inverse and $b(z)$ is uniformly bounded as $z \to 0$. In this case a_{11} has a bounded inverse for z sufficiently small, viz.

$$a_{11}^{-1} = z(k + zb(z))^{-1}. \tag{2.10}$$

Notice that

$$\lim_{z \to 0} \|a_{11}^{-1}\| = 0. \tag{2.11}$$

It follows that for sufficiently small z the inverse $\left(1 - a_{12}a_{22}^{-1}a_{21}a_{11}^{-1}\right)^{-1}$ exists, and then the inverse

$$a = a_{11}^{-1}\left(1 - a_{12}a_{22}^{-1}a_{21}a_{11}^{-1}\right)^{-1} \tag{2.12}$$

also exists.

3 The free resolvents

In this section we collect the formulae for the low energy expansions of the integral kernels of $R_0(\lambda) = (H_0 - \lambda)^{-1}$, $H_0 = -\Delta$, in $L^2(\mathbf{R}^m)$. We state the results for arbitrary dimensions.

It is well known that the kernel is given by

$$(H_0 - \lambda)^{-1}(|\mathbf{x} - \mathbf{y}|) = \frac{i}{4}\left(\frac{\lambda^{1/2}}{2\pi|\mathbf{x} - \mathbf{y}|}\right)^{\frac{m}{2}-1} H^{(1)}_{\frac{m}{2}-1}(\lambda^{1/2}|\mathbf{x} - \mathbf{y}|), \tag{3.1}$$

where $H^{(1)}_\nu$ are the modified Hankel functions and $\lambda \in \mathbf{C}\backslash[0,\infty)$; the determination for $\lambda^{1/2}$ is such that $\operatorname{Im}\lambda^{1/2} > 0$.

We shall use the variable

$$\kappa = -i\lambda^{1/2}; \quad \lambda = -\kappa^2. \tag{3.2}$$

Notice that for $\lambda < 0$ one has $\kappa > 0$. Thus the relevant domain for the parameter κ is $|\kappa| < \delta$ and $\operatorname{Re}\kappa > 0$ for a sufficiently small $\delta > 0$. Using the identity

$$H^{(1)}_\nu(i\zeta) = \frac{2}{i\pi}e^{-i\pi\nu/2}K_\nu(\zeta), \tag{3.3}$$

where $K_\nu(\zeta)$ are the Macdonald's functions [14, §17], one obtains

$$\begin{aligned}R_0(\kappa;\mathbf{x}-\mathbf{y}) &\equiv (H_0+\kappa^2)^{-1}(|\mathbf{x}-\mathbf{y}|)\\ &= \frac{1}{2\pi}\left(\frac{\kappa}{2\pi|\mathbf{x}-\mathbf{y}|}\right)^{\frac{m}{2}-1} K_{\frac{m}{2}-1}(\kappa|\mathbf{x}-\mathbf{y}|).\end{aligned} \tag{3.4}$$

For convenience we give the formulae for K_ν for ν integer or half integer; notice that they are real for

$$\zeta \equiv \kappa|\mathbf{x}-\mathbf{y}| > 0. \tag{3.5}$$

i. $\nu = n$**,** $n \geq 0$ **integer:**

$$\begin{aligned}K_n(\zeta) =& (-1)^{n-1} I_n(\zeta)\ln(\zeta/2) + \frac{1}{2}\sum_{k=0}^{n-1}(-1)^k \frac{(n-k-1)!}{k!}(\zeta/2)^{2k-n}\\ &+ \frac{(-1)^n}{2}\sum_{p=0}^{\infty}\frac{(\zeta/2)^{2p+n}}{p!(p+n)!}(\psi(p+n+1)+\psi(p+1)),\end{aligned} \tag{3.6}$$

where

$$I_n(\zeta) = \sum_{p=0}^{\infty}\frac{(\zeta/2)^{2p+n}}{p!(n+p)!}, \quad \text{and} \quad \psi(k) = \sum_{j=1}^{k-1}\frac{1}{j} - \gamma. \tag{3.7}$$

Here γ is the Euler constant and the sum is taken to be zero for $k = 1$. In particular, $\psi(1) = -\gamma$, $\psi(2) = 1-\gamma$.

ii. $\nu = n - 1/2$**,** $n \geq 0$ **integer:**

$$K_{n-1/2}(\zeta) = \left(\frac{\pi}{2\zeta}\right)^{1/2} \zeta^n \left(-\frac{1}{\zeta}\frac{d}{d\zeta}\right)^n e^{-\zeta}. \tag{3.8}$$

Using (3.6) and (3.8) one can write down the needed expansions for arbitrary m, up to arbitrary order. Consider first m even. In this case from (3.6) one obtains, using a convenient mixed notation (see (3.5)),

$$R_0(\kappa;|\mathbf{x}-\mathbf{y}|) = \kappa^{2n}\ln\zeta \sum_{p=0}^{\infty} c_{m,p}\zeta^{2p} + \frac{1}{|\mathbf{x}-\mathbf{y}|^{2n}}\sum_{p=0}^{\infty} d_{m,p}\zeta^{2p}, \tag{3.9}$$

where

$$n = \frac{m}{2} - 1 \tag{3.10}$$

and $c_{m,p}$, $d_{m,p}$ are numerical coefficients.

In the odd case one has from (3.8)

$$\begin{aligned} R_0(\kappa; |\mathbf{x}-\mathbf{y}|) &= \frac{1}{2(2\pi)^{\frac{m-1}{2}}}\kappa^{m-2}\left(-\frac{1}{\zeta}\frac{d}{d\zeta}\right)^n e^{-\zeta} \\ &= \begin{cases} \dfrac{1}{2\kappa}\displaystyle\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\zeta^p & \text{if } m=1, \\ \dfrac{1}{|\mathbf{x}-\mathbf{y}|^{m-2}}\displaystyle\sum_{p=0}^{\infty} f_{m,p}\zeta^p & \text{if } m\geq 1, \end{cases} \end{aligned} \tag{3.11}$$

where $f_{m,p}$ are numerical coefficients. From (3.8) one can see that for $m \geq 5$ one has $f_{m,1} = 0$. Actually one has $f_{m,p} = 0$ for $p = 1, 3, \dots, m-4$, see [7, Lemma 3.3].

From (3.9) and (3.11) one concludes that for all $m \geq 5$:

$$R_0(\kappa; |\mathbf{x}-\mathbf{y}|) = \frac{a_m}{|\mathbf{x}-\mathbf{y}|^{m-2}} + \kappa^2\frac{b_m}{|\mathbf{x}-\mathbf{y}|^{m-4}} + \dots \tag{3.12}$$

which implies that there are no threshold resonances [7].

We list below, for future reference, the first terms for $m = 1$ and $m = 2$.

$m = 1$

$$R_0(\kappa; |x-y|) = \frac{1}{2\kappa}e^{-\kappa|x-y|} = \frac{1}{2\kappa} - \frac{|x-y|}{2} + \kappa\frac{|x-y|^2}{4} + \mathcal{O}(\kappa^2), \tag{3.13}$$

$m = 2$

$$\begin{aligned} R_0(\kappa; |\mathbf{x}-\mathbf{y}|) =& \frac{1}{2\pi}\Bigg(-\ln\kappa - (\gamma + \ln(|\mathbf{x}-\mathbf{y}|/2)) - \frac{|\mathbf{x}-\mathbf{y}|^2}{4}\kappa^2\ln\kappa \\ & +\kappa^2\frac{|\mathbf{x}-\mathbf{y}|^2}{4}\left(1-\gamma-\ln(|\mathbf{x}-\mathbf{y}|/2)\right)\Bigg) + \mathcal{O}(\kappa^4\ln\kappa). \end{aligned} \tag{3.14}$$

4 Low energy expansions: generalities

We consider $H = H_0 + V$ looking for the low energy behavior of $(H+\kappa^2)^{-1}$. We suppose V to be sufficiently short range. More precisely, we assume

$$\langle\cdot\rangle^{\beta+\frac{m}{p}}V \in L^p(\mathbf{R}^m), \tag{4.1}$$

where

$$p = \begin{cases} 2 & \text{if } m \leq 4, \\ \frac{m}{2} & \text{if } m \geq 5, \end{cases} \tag{4.2}$$

with β sufficiently large. There is a relation between the value of β and the order up to which one can write the expansion of $(H + \kappa^2)^{-1}$. At the expense of some technicalities stronger local singularities of the potential can be handled. It is also possible to include a class of non-local potentials, see the remarks in Section 7. Under the stated conditions V is H_0-bounded with relative bound zero, hence H is self-adjoint on $\mathcal{D}(H_0)$.

We start from the resolvent formula written in the symmetrized form

$$\begin{aligned} (H + \kappa^2)^{-1} &= (H_0 + \kappa^2)^{-1} \\ &\quad - (H_0 + \kappa^2)^{-1} v (U + v(H_0 + \kappa^2)^{-1} v)^{-1} v (H_0 + \kappa^2)^{-1} \\ &= (H_0 + \kappa^2)^{-1} - (H_0 + \kappa^2)^{-1} v M(\kappa)^{-1} v (H_0 + \kappa^2)^{-1}, \end{aligned} \tag{4.3}$$

where

$$v(\mathbf{x}) = |V(\mathbf{x})|^{1/2}, \qquad U(\mathbf{x}) = \begin{cases} 1 & \text{if } V(\mathbf{x}) \geq 0, \\ -1 & \text{if } V(\mathbf{x}) < 0, \end{cases} \tag{4.4}$$

and

$$M(\kappa) \equiv U + v(H_0 + \kappa^2)^{-1} v. \tag{4.5}$$

We also define

$$w(\mathbf{x}) = U(\mathbf{x}) v(\mathbf{x}). \tag{4.6}$$

From the identity

$$(1 - w(H + \kappa^2)^{-1} v)(1 + w(H_0 + \kappa^2)^{-1} v) = 1 \tag{4.7}$$

one obtains

$$w(H + \kappa^2)^{-1} w = U - M(\kappa)^{-1}. \tag{4.8}$$

From (4.8) and (4.3) one can see that it suffices to obtain the expansion of $M(\kappa)^{-1}$. Notice also that the scattering (or transfer) operator has a simple expression in terms of $M(\kappa)^{-1}$, viz.

$$T(\lambda) \equiv v(U + v(H_0 + \kappa^2)^{-1} v)^{-1} v = v M(\kappa)^{-1} v. \tag{4.9}$$

Since we suppose at least $V(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-2-\delta})$ as $|\mathbf{x}| \to \infty$, there exists a $\kappa_0 > 0$ such that for $\kappa \in (0, \kappa_0)$ we have $\lambda = -\kappa^2 \in \rho(H)$.

Since H is self-adjoint, we have

$$\limsup_{\kappa \searrow 0} \|\kappa^2 (H + \kappa^2)^{-1}\| \leq 1$$

and then from (4.8)

$$\limsup_{\kappa \searrow 0} \|\kappa^2 M(\kappa)^{-1}\| < \infty. \tag{4.10}$$

From the results in Section 3, $M(\kappa)$ has known expansions in powers of κ (and $1/\ln \kappa$ for even dimensions) up to an order depending upon β. More precisely, the problem is to prove that $M(\kappa)^{-1}$ also has expansions in powers of κ (and $1/\ln \kappa$ for even dimensions) up to some order and to compute the coefficients. If the leading term in the expansion of $M(\kappa)$ is invertible, the problem is solved by the Neumann expansion. The obstruction comes from the existence of a nontrivial null subspace of the leading term. The whole idea of this paper is that by using the inversion formulae in Section 2 one can reduce the initial inversion problem to an inversion problem in the null subspace of the leading term and then iterate the procedure. Since each iteration adds to the singularity of $M(\kappa)^{-1}$, after a few iterations the leading term must be invertible and the process stops, due to (4.10). As expected, these null subspaces are directly connected to the threshold eigenvalues and resonances of H. The rest of this paper consists of some concrete realizations of this procedure. As noted in the introduction, we limit ourselves to considering the cases $m = 1$ and $m = 2$.

5 The one dimensional case

The following elementary lemma gives the expansion of $M(\kappa)$, defined in (4.5). We suppose that $v(x)$ is not identically zero.

Lemma 5.1. (i) *Assume*

$$\langle \cdot \rangle^{\beta + \frac{1}{2}} V \in L^2(\mathbf{R}) \tag{5.1}$$

for some $\beta > 7$, and let p be the largest integer satisfying

$$\beta > 2p + 3. \tag{5.2}$$

Then $M(\kappa)-(\alpha/2\pi)\kappa^{-1}P-U$ is a uniformly bounded compact operator valued function in

$$F = \{\kappa \mid \operatorname{Re}\kappa \geq 0, |\kappa| \leq 1\} \tag{5.3}$$

and has the following asymptotic expansion for small $\kappa \in F$:

$$M(\kappa) = \frac{\alpha P}{2}\kappa^{-1} + \sum_{j=0}^{p-1} M_j \kappa^j + \kappa^p \mathcal{R}_0(\kappa), \tag{5.4}$$

where

$$P = \alpha^{-1}\langle v, \cdot\rangle v, \quad \alpha = \|v\|^2, \tag{5.5}$$

and $M_0 - U$, M_j, $j = 1, 2, \ldots, p-1$, are integral operators given by the kernels

$$(M_0 - U)(x,y) = -\frac{1}{2}v(x)|x-y|v(y), \tag{5.6}$$

$$M_j(x,y) = \frac{(-1)^{j+1}v(x)|x-y|^{j+1}v(y)}{2(j+1)!}, \tag{5.7}$$

and $\mathcal{R}_0(\kappa)$ is uniformly bounded in norm.

The operators $M_0 - U$, M_j, $j = 1, 2, \ldots$, are compact and self-adjoint, and for j odd the operators M_j are of finite rank.
(ii) *If $e^{\beta|x|}V(x) \in L^2(\mathbf{R})$ for some $\beta > 0$, then $M(\kappa)$ has a convergent expansion in κ, $0 < |\kappa| < \beta$.*

Proof. Use the Taylor expansion (with remainder) of the kernel of the free resolvent, cf. (3.11), in the definition of $M(\kappa)$, and then use the fact that

$$\iint_{\mathbf{R}^2} |M_j(x,y)|^2 dxdy \leq c \iint_{\mathbf{R}^2} v(x)^2(|x|^{2j+2} + |y|^{2j+2})v(y)^2 dxdy < \infty,$$

i.e. the M_j are actually Hilbert-Schmidt operators. In the same way one sees that $\mathcal{R}_0(\kappa)$ is also Hilbert-Schmidt. Part (ii) is obvious. □

Our main result in the one dimensional case is summarized as follows.

Theorem 5.2. *Assume*

$$\langle\,\cdot\,\rangle^{\beta+\frac{1}{2}}V \in L^2(\mathbf{R}) \tag{5.8}$$

for some $\beta > 7$, and let p be the largest integer satisfying

$$\beta > 2p + 3. \tag{5.9}$$

Then the following results hold.
(i) *Let* $Q = 1 - P$*, with* P *given by* (5.5)*, and let* $S : QL^2(\mathbf{R}) \to QL^2(\mathbf{R})$ *be the orthogonal projection on* $\ker QM_0Q$*. Then* $\dim S \le 1$*.*
(ii) *Suppose* $S \neq 0$ *and let* $\Phi \in SL^2(\mathbf{R})$, $\|\Phi\| = 1$*. If* Ψ *is defined by*

$$\Psi(x) = \frac{1}{\alpha}\langle v, M_0\Phi\rangle + \frac{1}{2}\int_{\mathbf{R}} |x-y|v(y)\Phi(y)dy, \tag{5.10}$$

then

$$w\Psi = \Phi, \tag{5.11}$$

$\Psi \notin L^2(\mathbf{R})$, $\Psi \in L^\infty(\mathbf{R})$*, and in the distribution sense*

$$H\Psi = 0. \tag{5.12}$$

Conversely, if there exists $\Psi \in L^\infty(\mathbf{R})$ *satisfying* (5.12) *in the distribution sense, then*

$$\Phi = w\Psi \in SL^2(\mathbf{R}). \tag{5.13}$$

(iii) *There exists* $\kappa_0 > 0$ *such that for* $|\kappa| \le \kappa_0$, $\operatorname{Re}\kappa \ge 0$*, and* $\kappa \neq 0$, $M(\kappa)^{-1}$ *has the expansion*

$$M(\kappa)^{-1} = \sum_{j=-1}^{q-1} \mathcal{M}_j\kappa^j + \kappa^q\mathcal{R}(\kappa), \tag{5.14}$$

where

$$q = \begin{cases} p & \text{if } S = 0, \\ p-2 & \text{if } S \neq 0. \end{cases} \tag{5.15}$$

Here $\mathcal{R}(\kappa)$ *is uniformly bounded and the coefficients* $\mathcal{M}_j$ *can be computed explicitly* (*see formula* (5.18) *below*)*. In particular*

$$\mathcal{M}_{-1} = -\frac{S}{\tilde{c}^2} \tag{5.16}$$

with (*for* $\dim S = 1$)

$$\tilde{c}^2 = \frac{2}{\alpha^2}|\langle v, M_0\Phi\rangle|^2 + \frac{1}{2}|\langle v, X\Phi\rangle|^2 > 0, \tag{5.17}$$

where X *is the operator of multiplication with* x*.*
(iv) *If* $e^{\beta|x|}V(x) \in L^2(\mathbf{R})$ *for some* $\beta > 0$*, then* $q = \infty$ *and the expansion* (5.14) *is convergent for* $0 < |\kappa| < \beta$*.*

Remark 5.3. Before giving the proof we state the formula obtained below for $M(\kappa)^{-1}$.

$$\begin{aligned}M(\kappa)^{-1} = &\frac{2\kappa}{\alpha}(1+\kappa\widetilde{M}(\kappa))^{-1}\\ &+\frac{2}{\alpha}(1+\kappa\widetilde{M}(\kappa))^{-1}Q(m_0+S+\kappa m_1(\kappa))^{-1}Q(1+\kappa\widetilde{M}(\kappa))^{-1}\\ &+\kappa^{-1}\frac{2}{\alpha}(1+\kappa\widetilde{M}(\kappa))^{-1}Q(m_0+S+\kappa m_1(\kappa))^{-1}Sq(\kappa)^{-1}S\times\\ &\quad\times(m_0+S+\kappa m_1(\kappa))^{-1}Q(1+\kappa\widetilde{M}(\kappa))^{-1},\end{aligned}\tag{5.18}$$

where

$$\widetilde{M}(\kappa)=\frac{2}{\alpha}\sum_{j=0}^{p-1}M_j\kappa^j+\kappa^p\mathcal{R}_0(\kappa)\equiv\frac{2}{\alpha}(M_0+\kappa M_1+\kappa^2M_2(\kappa)),\tag{5.19}$$

$$\begin{aligned}m(\kappa)&=\sum_{j=0}^{\infty}\kappa^j(-1)^jQ\left(\frac{2}{\alpha}M_0+\frac{2}{\alpha}\kappa M_1+\frac{2}{\alpha}\kappa^2M_2(\kappa)\right)^{j+1}Q\\ &\equiv\frac{2}{\alpha}QM_0Q-\frac{2}{\alpha}\kappa Q(\frac{2}{\alpha}M_0^2-M_1)Q+\kappa^2m_2(\kappa)\\ &\equiv m_0+\kappa(m_1+\kappa m_2(\kappa))\\ &\equiv m_0+\kappa m_1(\kappa),\end{aligned}\tag{5.20}$$

and

$$q(\kappa)=\sum_{j=0}^{\infty}\kappa^j(-1)^jS\left(m_1(\kappa)(m_0+S)^{-1}\right)^{j+1}S\tag{5.21}$$

as an operator in $SL^2(\mathbf{R})$ with

$$q(0)\equiv q_0=Sm_1S=-\frac{2}{\alpha}\tilde{c}^2S.\tag{5.22}$$

The formula (5.18) is our main formula for the one dimensional case; it contains all the cases. In particular the generic case, i.e. the case when there is no threshold resonance, is obtained by taking $S=0$ in (5.18). Expanding everything in powers of κ one obtains the expansion of $M(\kappa)^{-1}$. The order up to which one can expand $M(\kappa)^{-1}$ depends on whether S vanishes or not. Namely, if $S=0$, then the order of expansion for $M(\kappa)^{-1}$ equals p, i.e. is the same as for $M(\kappa)$, while if $S\neq 0$ it equals $p-2$. Indeed, $m(\kappa)$ has expansion up to order p (see (5.20)) so when m_0^{-1} exists, $m(\kappa)^{-1}$ has expansion up to order p and this gives the result for the generic case. If $S \neq 0$ then since (see again (5.20)) $m_1(\kappa)$ has expansion up to order $p-1$, $q(\kappa)$ and then (remember that q_0 is invertible) $q(\kappa)^{-1}$ has expansion to order $p-1$. This together with (5.18) gives the result for the singular case, since the last term contains a factor κ^{-1} leading to order $p-2$.

Formula (5.18) can be used to obtain the coefficients in the expansion of

$$M(\kappa)^{-1} = \sum_{j=-1}^{q-1} \mathcal{M}_j \kappa^l + \kappa^q \mathcal{R}(\kappa) \tag{5.23}$$

where $q = p$ in the generic case and $q = p-2$ in the singular one, up to the desired order, provided one assumes sufficient decay of V, see (5.9).

5.1 Proof of Theorem 5.2

The rest of this section is devoted to the proof of Theorem 5.2. Writing

$$M(\kappa) \equiv \frac{\alpha}{2\kappa}(P + \kappa \widetilde{M}(\kappa)) \tag{5.24}$$

and applying Corollary 2.2 to $P + \kappa\widetilde{M}(\kappa)$ (see (2.4) and (2.5)) one obtains that for sufficiently small κ (this is a shorthand for "there exists $\kappa_1 > 0$ such that for $\kappa \in F$, $|\kappa| \leq \kappa_1, \ldots$ "):

$$\begin{aligned} M(\kappa)^{-1} = \frac{2\kappa}{\alpha}\{&(1+\kappa\widetilde{M}(\kappa))^{-1} \\ &+ \kappa^{-1}(1+\kappa\widetilde{M}(\kappa))^{-1} Q m(\kappa)^{-1} Q (1+\kappa\widetilde{M}(\kappa))^{-1}\}, \end{aligned} \tag{5.25}$$

where

$$Q = 1 - P, \tag{5.26}$$

and

$$\begin{aligned} m(\kappa) &= \sum_{j=0}^{\infty} \kappa^j (-1)^j Q \left(\frac{2}{\alpha}M_0 + \frac{2}{\alpha}\kappa M_1 + \frac{2}{\alpha}\kappa^2 M_2(\kappa)\right)^{j+1} Q \\ &= \frac{2}{\alpha} Q M_0 Q - \frac{2}{\alpha}\kappa Q(\frac{2}{\alpha}M_0^2 - M_1)Q + \kappa^2 m_2(\kappa) \\ &= m_0 + \kappa(m_1 + \kappa m_2(\kappa)) \\ &= m_0 + \kappa m_1(\kappa). \end{aligned} \tag{5.27}$$

In the last chain of equalities we defined the following operators on $QL^2(\mathbf{R})$:

$$m_0 \equiv \frac{2}{\alpha} Q M_0 Q, \tag{5.28}$$

$$m_1 \equiv -\frac{2}{\alpha}Q(\frac{2}{\alpha}M_0^2 - M_1)Q, \tag{5.29}$$

$$\begin{aligned} m_2(\kappa) \equiv &\frac{4}{\alpha^2}QM_0^2Q \\ &+\sum_{j=1}^{\infty}\kappa^{j-1}(-1)^jQ\left(\frac{2}{\alpha}M_0+\frac{2}{\alpha}\kappa M_1+\frac{2}{\alpha}\kappa^2M_2(\kappa)\right)^{j+1}Q, \end{aligned} \tag{5.30}$$

$$m_1(\kappa) = m_1 + \kappa m_2(\kappa). \tag{5.31}$$

We continue now by applying Corollary 2.2 to $m(\kappa)$. Note that the spectrum of m_0 in $QL^2(\mathbf{R})$ outside $\{-\frac{2}{\alpha}, \frac{2}{\alpha}\}$ is discrete. This follows from the fact that

$$\begin{aligned} QM_0Q &= (1-P)M_0(1-P) \\ &= U + (M_0 - U) - PM_0 - M_0P + PM_0P = U + K, \end{aligned}$$

where K is compact, which together with the fact that $\sigma(U) \subset \{-1, 1\}$ implies that as a self-adjoint operator in $L^2(\mathbf{R})$, QM_0Q has discrete spectrum outside $\{-1, 1\}$. Accordingly, if S is the orthogonal projection on $\operatorname{Ker} m_0$ (in $QL^2(\mathbf{R})$) then since m_0 is self-adjoint, we have $\dim S < \infty$, $(m_0+S)^{-1}$ exists and is bounded, and

$$S = (m_0 + S)^{-1}S = S(m_0 + S)^{-1}. \tag{5.32}$$

Applying now Corollary 2.2 to $m(\kappa)$ (see (2.4) and (2.5)) one obtains that for sufficiently small κ:

$$\begin{aligned} m(\kappa)^{-1} = &(m_0 + S + \kappa m_1(\kappa))^{-1} + \kappa^{-1}(m_0 + S + \kappa m_1(\kappa))^{-1} \times \\ &\times Sq(\kappa)^{-1}S(m_0 + S + \kappa m_1(\kappa))^{-1}, \end{aligned} \tag{5.33}$$

where

$$q(\kappa) = \sum_{j=0}^{\infty}\kappa^j(-1)^jS(m_1(\kappa)(m_0+S)^{-1})^{j+1}S \tag{5.34}$$

as an operator on $SL^2(\mathbf{R})$.

Taking into account (5.32) one has

$$q(\kappa) = q_0 + \kappa q_1(\kappa), \quad \text{where} \quad q_0 = Sm_1S, \tag{5.35}$$

and

$$q_1(\kappa) = Sm_2(\kappa)S + \sum_{j=1}^{\infty} \kappa^{j-1}(-1)^j S(m_1(\kappa)(m_0 + S)^{-1})^{j+1}S. \tag{5.36}$$

The following lemma shows that the "obstruction" subspace is related to the zero energy resonances of H and that there is no need for further iterations of the procedure.

Lemma 5.4. (i) *Suppose $S \neq 0$ and let $\Phi \in SL^2(\mathbf{R})$, $\|\Phi\| = 1$. If Ψ is defined by*

$$\Psi(x) = c_1 + \frac{1}{2}\int_{\mathbf{R}} |x-y|v(y)\Phi(y)dy \tag{5.37}$$

with

$$c_1 = \frac{1}{\alpha}\langle v, M_0\Phi\rangle, \tag{5.38}$$

then

$$w\Psi = \Phi, \tag{5.39}$$

$\Psi \notin L^2(\mathbf{R})$, $\Psi \in L^\infty(\mathbf{R})$, and in the distribution sense

$$H\Psi = 0. \tag{5.40}$$

(ii) *Suppose there exists $\Psi \in L^\infty(\mathbf{R})$ satisfying* (5.40) *in the distribution sense. Then*

$$\Phi = w\Psi \in SL^2(\mathbf{R}). \tag{5.41}$$

(iii) *We have* $\dim S \leq 1$, *and if* $\dim S = 1$, *then*

$$Sm_1S = -\frac{2}{\alpha}\tilde{c}^2 S \tag{5.42}$$

with $\tilde{c}^2 > 0$, where $\tilde{c}^2$ is given by (5.17).

Proof. The proof of (5.39) is a direct computation using (5.6):

$$\begin{aligned} w\Psi &= c_1w + U(U - M_0)\Phi = c_1w + \Phi - UM_0\Phi \\ &= c_1w + \Phi - UPM_0\Phi = c_1w + \Phi - \frac{1}{\alpha}U\langle v, M_0\Phi\rangle v = \Phi, \end{aligned}$$

and (5.40) follows from (5.37) and (5.39) by differentiation in the distribution sense.

With the notation

$$c_2 = \frac{1}{2} \int_{\mathbf{R}} y v(y) \Phi(y) dy \tag{5.43}$$

and taking into account that $P\Phi = 0$, i.e. $\int_{\mathbf{R}} v(y)\Phi(y)dy = 0$, one obtains from (5.37)

$$\begin{aligned}
\Psi(x) &= c_1 + \begin{cases} -c_2 + \displaystyle\int_x^{\infty} (y-x)v(y)\Phi(y)dy \\ c_2 + \displaystyle\int_{-\infty}^{x} (x-y)v(y)\Phi(y)dy \end{cases} \\
&= c_1 - c_2 \operatorname{sign} x + \begin{cases} \displaystyle\int_x^{\infty} (y-x)v(y)\Phi(y)dy & \text{for } x \geq 0, \\ \displaystyle\int_{-\infty}^{x} (x-y)v(y)\Phi(y)dy & \text{for } x \leq 0. \end{cases}
\end{aligned} \tag{5.44}$$

Suppose now that $c_1 = c_2 = 0$. Then from (5.39) and (5.44) one has

$$\Psi(x) = \int_x^{\infty} (y-x)V(y)\Psi(y)dy.$$

This is a homogeneous Volterra equation which gives $\Psi(x) = 0$ for x sufficiently large (provided $V(x) = \mathcal{O}(|x|^{-2-\varepsilon})$ as $|x| \to \infty$), and then by uniqueness of solutions to the differential equation, $\Psi \equiv 0$. Then (5.39) implies $\Phi = 0$ which in turn implies that c_1 and c_2 cannot be zero simultaneously, since we have assumed $\|\Phi\| = 1$.

From (5.44) it follows in particular that $\Psi \in L^{\infty}(\mathbf{R})$ and also

$$\lim_{x\to\infty} \Psi(x) = c_1 - c_2, \quad \lim_{x\to-\infty} \Psi(x) = c_1 + c_2,$$

which implies that $\Psi \notin L^2(\mathbf{R})$, and the first point of the lemma is proved.

To prove (ii), suppose there exists $\Psi \in L^{\infty}(\mathbf{R})$ satisfying $H\Psi = 0$ in the distribution sense. Define $\Phi = w\Psi$. Then again in the distribution sense

$$\frac{d^2}{dx^2}\Psi(x) = V(x)\Psi(x) = v(x)\Phi(x).$$

Let $\phi \in C_0^{\infty}(\mathbf{R})$ such that $\phi(x) = 1$ for $|x| \leq 1$ and $\phi(x) = 0$ for $|x| > 2$.

Then for any $\delta > 0$ we have

$$\begin{aligned}\left|\int_{\mathbf{R}} v(x)\Phi(x)\phi(\delta x)dx\right| &= \left|\int_{\mathbf{R}} (\frac{d^2}{dx^2}\Psi(x))\phi(\delta x)dx\right| \\ &= \left|\int_{\mathbf{R}} \Psi(x)(\frac{d^2}{dx^2}\phi(\delta x))dx\right| \\ &= \left|\int_{\mathbf{R}} \Psi(x)\delta^2\phi^{''}(\delta x)dx\right| \\ &\leq \delta\|\Psi\|_\infty \int_{\mathbf{R}} |\phi^{''}(x)|dx.\end{aligned}$$

Taking the limit $\delta \to 0$ and using the Lebesgue dominated converge theorem, one obtains that $\int_{\mathbf{R}} v(x)\Phi(x)dx = 0$, i.e.

$$\Phi \in QL^2(\mathbf{R}). \tag{5.45}$$

Consider now

$$\Xi(x) = \frac{1}{2}\int_{\mathbf{R}} |x-y|v(y)\Phi(y)dy = \frac{1}{2}\int_{\mathbf{R}} |x-y|V(y)\Psi(y)dy. \tag{5.46}$$

By differentiation in the distribution sense we find

$$\frac{d^2}{dx^2}\Xi(x) = V(x)\Psi(x) = \frac{d^2}{dx^2}\Psi(x),$$

so that

$$\Xi(x) = \Psi(x) + a + bx$$

for some $a, b \in \mathbf{C}$. Notice now that $\Xi \in L^\infty(\mathbf{R})$ by a computation analogous to the one leading to (5.44), so that $b = 0$. By multiplying (5.46) with $v(x)$ and using (5.6) one obtains $(U - M_0)\Phi = U\Phi + av$, i.e. $M_0\Phi = -av$ so that $QM_0\Phi = 0$, which together with (5.45) finishes the proof of (ii).

To prove (iii), suppose that there are two linearly independent Φ, $\tilde{\Phi} \in SL^2(\mathbf{R})$ and correspondingly for $x \geq 0$

$$\Psi(x) = c_1 - c_2 + \int_x^\infty (y-x)v(y)\Phi(y)dy$$

and

$$\tilde{\Psi}(x) = \tilde{c_1} - \tilde{c_2} + \int_x^\infty (y-x)v(y)\tilde{\Phi}(y)dy.$$

There exists $a \in \mathbf{C}$ such that

$$c_1 - c_2 = -a(\tilde{c_1} - \tilde{c_2}),$$

which gives

$$(\Psi + a\tilde{\Psi})(x) = \int_x^\infty (y - x)V(y)(\Psi + a\tilde{\Psi})(y)dy$$

and then by the Volterra equation argument used above we get $\Psi + a\tilde{\Psi} = 0$. Hence $\Phi + a\tilde{\Phi} = 0$, which proves that $\dim S = 1$.

We are left with the computation of $\tilde{c}^2$ in (5.42). Suppose $\dim S = 1$ and let $\Phi \in SL^2(\mathbf{R})$, $\|\Phi\| = 1$. Then (see (5.29))

$$-\langle \Phi, m_1\Phi \rangle = \frac{4}{\alpha^2}\langle \Phi, M_0^2\Phi \rangle - \frac{2}{\alpha}\langle \Phi, M_1\Phi \rangle \tag{5.47}$$

Using $QM_0\Phi = 0$ and (5.38) we get

$$\langle \Phi, M_0^2\Phi \rangle = \langle \Phi, M_0PM_0\Phi \rangle = \frac{1}{\alpha}|\langle \Phi, M_0v \rangle|^2 = \alpha|c_1|^2. \tag{5.48}$$

On the other hand (see (5.7) and (5.43), and remember that $P\Phi = 0$)

$$\begin{aligned} \langle \Phi, M_1\Phi \rangle &= \frac{1}{4}\iint_{\mathbf{R}^2} \overline{\Phi(x)}v(x)(x^2 - 2xy + y^2)v(y)\Phi(y)dxdy \\ &= -\frac{1}{2}\left|\int_{\mathbf{R}} xv(x)\Phi(x)dx\right|^2 = -2|c_2|^2 \end{aligned} \tag{5.49}$$

Combining (5.47) with (5.48) and (5.49) one obtains

$$\tilde{c}^2 = 2(|c_1|^2 + |c_2|^2). \tag{5.50}$$

Since $c_1 = c_2 = 0$ implies $\Phi = 0$, the proof of lemma is finished. □

Coming back to the expansion $M(\kappa)^{-1}$ the above procedure gives (5.18) (see (5.27) and (5.34)) and the proof of the theorem is finished. □

Remark 5.5. Let us note that results similar to those in Lemma 5.4 have been obtained in [5, 3, 4, 11, 12].

Remark 5.6. In order to compare our results with the results in [5, 3, 4, 11, 12] we can use the result in Theorem 5.2 also to give the leading term in the expansion of $(H+\kappa^2)^{-1}$ as a map between weighted spaces. In the case where we have a zero resonance, the leading term is

$$-\frac{1}{\kappa}\frac{1}{\tilde{c}^2}\langle\Psi,\cdot\rangle\Psi+\mathcal{O}(1). \tag{5.51}$$

Here Ψ is the solution to $H\Psi=0$ in $L^\infty(\mathbf{R})$, normalized by $\|w\Psi\|=1$, and the constant $\tilde{c}^2$ is given by (5.17). With appropriate identifications our results agree with the results in the papers cited.

Let us finish the results on the one-dimensional case with an example showing that the result on absence of zero-eigenvalue in Theorem 5.2 is optimal with respect to decay rate. Note that the proof given requires a decay rate $\mathcal{O}(|x|^{-2-\delta})$ as $|x|\to\infty$, for some $\delta>0$.

Example 5.7. For $x\in\mathbf{R}$ we write $\langle x\rangle=(1+x^2)^{1/2}$ as usual, and define

$$\Psi_\beta(c)=e^{-\langle x\rangle^\beta},$$
$$V_\beta(x)=\beta^2x^2\langle x\rangle^{2\beta-4}-\beta\langle x\rangle^{\beta-2}-\beta(\beta-2)x^2\langle x\rangle^{\beta-4}.$$

Let

$$H_\beta=-\frac{d^2}{dx^2}+V_\beta(x).$$

Then a simple computation shows that $H_\beta\Psi_\beta=0$. Thus for $\beta<0$ the potential satisfies $V_\beta(x)=\mathcal{O}(|x|^{\beta-2})$ as $|x|\to\infty$, and zero is a resonance with resonance function Ψ_β. For $0<\beta<1$ we have $V_\beta(x)=\mathcal{O}(|x|^{2\beta-2})$ as $|x|\to\infty$, and zero is an L^2-eigenvalue with eigenfunction Ψ_β.

6 The two dimensional case

With the notation

$$\eta=1/\ln\kappa \tag{6.1}$$

the expansion of $M(\kappa)$, defined in (4.5) and (3.9), takes the form:

Lemma 6.1. (i) *Let*

$$\langle\,\cdot\,\rangle^{\beta+1}V\in L^2(\mathbf{R}^2). \tag{6.2}$$

Suppose $\beta>9$ and let p be the largest integer satisfying

$$\beta>4p+2. \tag{6.3}$$

Then $M(\kappa) - \eta^{-1}M_{0,-1} - U$ is a uniformly bounded, compact operator valued function in

$$F = \{\kappa | \operatorname{Re}\kappa \geq 0, |\kappa| \leq 1\}, \tag{6.4}$$

and $M(\kappa)$ has the following asymptotic expansion for small κ:

$$M(\kappa) = \sum_{j=0}^{p-1} \kappa^{2j}(M_{2j,0} + \eta^{-1}M_{2j,-1}) + \kappa^{2p}\eta^{-1}\mathcal{R}_0(\kappa), \tag{6.5}$$

where

$$M_{0,-1} = -\frac{\alpha}{2\pi}P, \quad \textit{with} \quad P = \alpha^{-1}\langle v, \cdot\rangle v, \quad \alpha = \|v\|^2, \tag{6.6}$$

$M_{0,0} - U$, $M_{2j,0}$, $M_{2j,-1}$, $j = 1, 2, \ldots, p-1$, are integral operators. In particular,

$$(M_{0,0} - U)(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi}v(\mathbf{x})\ln(\frac{e^{\gamma}|\mathbf{x}-\mathbf{y}|}{2})v(\mathbf{y}), \tag{6.7}$$

$$M_{2,-1}(\mathbf{x}, \mathbf{y}) = -\frac{1}{8\pi}v(\mathbf{x})|\mathbf{x}-\mathbf{y}|^2 v(\mathbf{y}), \tag{6.8}$$

$$M_{2,0}(\mathbf{x}, \mathbf{y}) = \frac{1}{8\pi}v(\mathbf{x})|\mathbf{x}-\mathbf{y}|^2(1 - \gamma - \ln(\frac{|\mathbf{x}-\mathbf{y}|}{2}))v(\mathbf{y}). \tag{6.9}$$

The operators $M_{0,0} - U$, $M_{2j,0}$, and $M_{2j,-1}$, $j = 1, 2, \ldots, p-1$, are compact and self-adjoint, the $M_{2j,-1}$ are of finite rank, and $\mathcal{R}_0(\kappa)$ is uniformly bounded.
(ii) *If $e^{\beta|\mathbf{x}|}V(\mathbf{x}) \in L^\infty(\mathbf{R}^2)$ for some $\beta > 0$, then the series $\sum_{j=0}^{\infty}\kappa^{2j}M_{2j,0}$ and $\sum_{j=0}^{\infty}\kappa^{2j}M_{2j,-1}$ are norm convergent for $|\kappa| < \beta$.*

Proof. Similar to the one dimensional case. Details are omitted. □

The main result concerning the expansion of $M(\kappa)^{-1}$ for the two dimensional case is contained in the following Theorem. In the statements obvious changes have to be made, if any of the three projections S_j, $j = 1, 2, 3$, equal zero. See also Remark 6.6.

Theorem 6.2. *Let*

$$\langle\,\cdot\,\rangle^{\beta+1}V \in L^2(\mathbf{R}^2). \tag{6.10}$$

Suppose $\beta > 9$ and let p be the largest integer satisfying

$$\beta > 4p + 2 \tag{6.11}$$

Then we have the following results.
(i) *Let* $Q = 1 - P$, *with* P *given by* (6.6), *and let* $Q \geq S_1 \geq S_2 \geq S_3$ *be the orthogonal projections on* $\operatorname{Ker} QM_{0,0}Q$, $\operatorname{Ker} S_1M_{0,0}PM_{0,0}S_1$, *and* $\operatorname{Ker} S_2M_{2,-1}S_2$, *respectively. Let*

$$T_2 = S_1 - S_2, \tag{6.12}$$

$$T_3 = S_2 - S_3. \tag{6.13}$$

Then $\operatorname{Ran} T_2$ *has dimension at most* 1 *and is spanned by the function*

$$\Theta_0 = S_1 M_{0,0} v \tag{6.14}$$

($\dim T_2 = 0$ *is equivalent with* $\Theta_0 = 0$), *and* $\operatorname{Ran} T_3$ *has dimension at most* 2 *and is spanned by the functions*

$$\Theta_j = S_2 X_j v, \tag{6.15}$$

where X_j *are the operators of multiplication with* x_j ($\mathbf{x} = (x_1, x_2)$). *In the cases where* $\dim T_3 < 2$, *one or both* Θ_j *vanish or are linearly dependent.*

Any $\Phi \in S_1 L^2(\mathbf{R}^2)$ *has the (orthogonal) decomposition*

$$\begin{aligned} &\Phi = \Phi_s + \Phi_p + \Phi_b, \\ &\Phi_s = b_s \Theta_0, \quad b_s \in \mathbf{C}, \\ &\Phi_p = \sum_{j=1}^{2} b_{p,j} \Theta_j, \quad b_{p,j} \in \mathbf{C}, \\ &\Phi_b \in L^2(\mathbf{R}^2). \end{aligned} \tag{6.16}$$

If Ψ *is defined by*

$$\Psi(\mathbf{x}) = c_0 + \frac{1}{2\pi} \int_{\mathbf{R}^2} \ln(|\mathbf{x} - \mathbf{y}|) v(\mathbf{y}) \Phi(\mathbf{y}) d\mathbf{y} \tag{6.17}$$

with

$$c_0 = \frac{1}{\alpha} \langle v, M_0 \Phi \rangle, \tag{6.18}$$

then

$$w\Psi = \Phi, \tag{6.19}$$

and in the sense of distributions

$$H\Psi = 0. \tag{6.20}$$

Furthermore, $\Psi \in L^\infty(\mathbf{R}^2)$ *and has the decomposition, cf.* (6.16),

$$\Psi = b_s \Psi_s + \sum_{j=1}^{2} b_{p,j} \Psi_{p,j} + \Psi_b, \tag{6.21}$$

where either $\Psi_s = 0$ *or*

$$\Psi_s \in L^\infty(\mathbf{R}^2); \quad \Psi_s \notin L^q(\mathbf{R}^2) \quad \text{for all } q < \infty, \tag{6.22}$$

$$\Psi_{p,j} \in L^q(\mathbf{R}^2) \quad \text{for all } q > 2, \tag{6.23}$$

and if $\Psi_{p,j} \neq 0$*, then* $\Psi_{p,j} \notin L^2(\mathbf{R}^2)$*,*

$$\Psi_b \in L^2(\mathbf{R}^2). \tag{6.24}$$

Suppose $\Psi(\mathbf{x}) = c + \Lambda(\mathbf{x})$ *with* $c \in \mathbf{C}$ *and* $\Lambda = \Lambda_1 + \Lambda_2$*, where* $\Lambda_1 \in L^q(\mathbf{R}^2)$ *for some* $2 < q < \infty$*, and* $\Lambda_2 \in L^2(\mathbf{R}^2)$*. If* Ψ *satisfies* (6.20) *in the distribution sense, then*

$$\Phi = w\Psi \in S_1 L^2(\mathbf{R}^2). \tag{6.25}$$

Furthermore, $\Phi_1, \Phi_2 \in S_1 L^2(\mathbf{R}^2)$ *are linear independent if and only if the corresponding* Ψ_1*,* Ψ_2 *are linear independent. In particular,* $\dim \operatorname{Ran} S_3$ *equals the dimension of the spectral subspace of* H *corresponding to zero energy.*
(ii) *There exists* $\kappa_0 > 0$ *such that for* $0 < |\kappa| \le \kappa_0$*, and* $\operatorname{Re}\kappa \ge 0$*, the inverse* $M(\kappa)^{-1}$ *can be computed by the formula*

$$\begin{aligned} M(\kappa)^{-1} &= (M(\kappa) + S_1)^{-1} \\ &\quad - g(\kappa)(M(\kappa) + S_1)^{-1} S_1 (M_1(\kappa) + S_2)^{-1} S_1 (M(\kappa) + S_1)^{-1} \\ &\quad - \frac{g(\kappa)}{\kappa^2 \eta^{-1}} (M(\kappa) + S_1)^{-1} S_1 (M_1(\kappa) + S_2)^{-1} S_2 \times \\ &\quad \times \{T_3 m(\kappa)^{-1} T_3 - T_3 m(\kappa)^{-1} b(\kappa) d(\kappa)^{-1} S_3 - S_3 d(\kappa)^{-1} c(\kappa) m(\kappa)^{-1} T_3 \\ &\qquad + S_3 d(\kappa)^{-1} c(\kappa) m(\kappa)^{-1} b(\kappa) d(\kappa)^{-1} S_3 + S_3 d(\kappa)^{-1} S_3\} \times \\ &\quad \times S_2 (M_1(\kappa) + S_2)^{-1} S_1 (M(\kappa) + S_1)^{-1}, \end{aligned} \tag{6.26}$$

where

$$\begin{aligned} (M(\kappa) + S_1)^{-1} = g(\kappa)^{-1}\{&P - P M_0(\kappa) Q D_0(\kappa) Q - Q D_0(\kappa) Q M_0(\kappa) P \\ &+ Q D_0(\kappa) Q M_0(\kappa) P M_0(\kappa) Q D_0(\kappa) Q\} + Q D_0(\kappa) Q \end{aligned} \tag{6.27}$$

with

$$M_0(\kappa) \equiv M(\kappa) + \frac{\alpha}{2\pi\eta} P = M_{0,0} + \kappa^2 \eta^{-1} M_{2,-1} + \kappa^2 M_{2,0} + \dots\,, \tag{6.28}$$

$$D_0(\kappa) \equiv (Q(M_0(\kappa) + S_1)Q)^{-1} : QL^2(\mathbf{R}^2) \to QL^2(\mathbf{R}^2), \tag{6.29}$$

$$\begin{aligned} g(\kappa) &= -\frac{\alpha}{2\pi}\eta^{-1} + \mathrm{Tr}\{PM_0(\kappa)P - PM_0(\kappa)QD_0(\kappa)QM_0(\kappa)P\} \\ &\equiv \eta^{-1}(-\frac{\alpha}{2\pi} + \eta h(\kappa)). \end{aligned} \tag{6.30}$$

As an operator in $S_1L^2(\mathbf{R}^2)$

$$\begin{aligned} M_1(\kappa) &= -g(\kappa)(S_1 - g(\kappa)S_1D_0(\kappa)QM_0(\kappa)PM_0(\kappa)QD_0(\kappa)S_1 \\ &\quad - S_1D_0(\kappa)S_1) \\ &= M_{1;0,0} + \kappa^2\eta^{-2}M_{1;2,-2} + \kappa^2\eta^{-1}M_{1;2,-1} + \dots , \end{aligned} \tag{6.31}$$

with

$$M_{1;0,0} = S_1M_{0,0}PM_{0,0}S_1, \tag{6.32}$$

$$M_{1;2,-2} = \frac{\alpha}{2\pi}S_1M_{2,-1}S_1. \tag{6.33}$$

As an operator in $S_2L^2(\mathbf{R}^2)$

$$M_2(\kappa) = -\eta^{-1}\sum_{j=0}^{\infty}(-1)^j(\kappa\eta^{-1})^{2j} \times \tag{6.34}$$

$$\times\, S_2[\kappa^{-2}\eta^2(M_1(\kappa) - M_{1;0,0})(M_{1;0,0} + S_2)^{-1}]^{j+1}S_2 \tag{6.35}$$

$$= \eta^{-1}M_{2;0,-1} + M_{2;0,0} + \dots \tag{6.36}$$

with

$$M_{2;0,-1} = \frac{\alpha}{2\pi}S_2M_{2,-1}S_2, \tag{6.37}$$

$$M_{2;0,0} = \frac{\alpha}{2\pi}S_3M_{2,0}S_3. \tag{6.38}$$

Finally, $a(\kappa)$*,* $b(\kappa)$*,* $c(\kappa)$*, and* $d(\kappa)$ *are the matrix elements of* $M_2(\kappa)$ *according to the decomposition* $S_2 = T_3 + S_3$

$$a(\kappa) = T_3M_2(\kappa)T_3, \qquad b(\kappa) = T_3M_2(\kappa)S_3, \tag{6.39}$$

$$c(\kappa) = S_3M_2(\kappa)T_3, \qquad d(\kappa) = S_3M_2(\kappa)S_3. \tag{6.40}$$

As an operator in $S_3L^2(\mathbf{R}^2)$*,*

$$d(0) = \frac{\alpha}{2\pi}S_3M_{2,0}S_3 \tag{6.41}$$

has a bounded inverse.

As an operator in $T_3 L^2(\mathbf{R}^2)$

$$m(\kappa) = \eta^{-1}\frac{\alpha}{8\pi^2}\sum_{j=1}^{r\leq 2}\langle\Theta_j, \cdot\rangle\Theta_j + f(\kappa) \tag{6.42}$$

with bounded $f(\kappa)$.
(iii) *All the inverses appearing in* (6.26) *have invertible leading terms so they can be computed using Neumann series. Only the expansions of the numerical factor,* $g(\kappa)^{-1}$, *and of* $m(\kappa)^{-1}$ (*as an operator in* $T_3 L^2(\mathbf{R}^2)$) *can lead to "bad" expansions.*

Remark 6.3. (i) Writing (see (6.30))

$$h(\kappa) \equiv h(0) + \kappa^2\eta^{-1}h_1(\kappa) \tag{6.43}$$

(notice that $h_1(\kappa)$ has a good expansion) and defining

$$\delta_0(\kappa) \equiv 1 + \eta d_0 = 1 + \eta\frac{2\pi}{\alpha}h(0), \tag{6.44}$$

$g(\kappa)^{-1}$ takes the form

$$g(\kappa)^{-1} = -\eta\frac{2\pi}{\alpha}\delta_0(\kappa)^{-1}\sum_{j=0}^{\infty}\left(\frac{2\pi}{\alpha}\delta_0(\kappa)^{-1}\kappa^2 h_1(\kappa)\right)^j. \tag{6.45}$$

Analogously with

$$f(\kappa) = f(0) + \kappa^2\eta^{-1}f_1(\kappa), \tag{6.46}$$

$m(\kappa)^{-1}$ takes the form:

$$m(\kappa)^{-1} = \eta\Delta(\kappa)^{-1}\sum_{j=0}^{\infty}(-1)^j\left(\Delta(\kappa)^{-1}\kappa^2 d_1(\kappa)\right)^j \tag{6.47}$$

where

$$\Delta(\kappa) = k + \eta f(0). \tag{6.48}$$

Now since k is strictly positive we can write

$$\begin{aligned} k + \eta f(0) &= k^{1/2}(1 + \eta k^{-1/2}f(0)k^{-1/2})k^{1/2} \\ &= k^{1/2}(1 + \eta\sum_{j=1}^{2}f_j P_j)k^{1/2} \end{aligned} \tag{6.49}$$

where $\sum_{j=1}^{2} f_j P_j$ is the spectral decomposition of $k^{-1/2} f(0) k^{-1/2}$. Accordingly

$$\Delta(\kappa)^{-1} = \sum_{j=1}^{2} \delta_j(\kappa)^{-1} k^{-1/2} P_j k^{-1/2} \tag{6.50}$$

where

$$\delta_j(\kappa) = 1 + \eta f_j. \tag{6.51}$$

Summing up, we see that all the "bad" expansions are confined in the inverses of at most three numerical factors, $\delta_j(\kappa)$, $j = 0, 1, 2$.

(ii) The asymptotic expansion of $M(\kappa)^{-1}$ can be obtained from (6.26) by straightforward (though lengthy for higher terms) computations. In particular, the leading terms in various cases can be directly "read" from (6.26):

(a) $S_3 \neq 0$ (there are zero energy bound states). In this case, taking into account that $g(\kappa)^{-1} \sim \eta$, $m(\kappa)^{-1} \sim \eta$, $\frac{g(\kappa)}{\kappa^2 \eta^{-1}} = -\frac{\alpha}{2\pi\kappa^2} + \mathcal{O}(\kappa^{-2}\eta)$, $S_2 S_3 = S_3$, and

$$(M_{0,0} + S_1)^{-1} S_1 (M_{0,0} + S_2)^{-1} S_2 = S_2,$$

one obtains from (6.26) (remark that only the last term in (6.26) gives contribution to the most singular term)

$$M(\kappa)^{-1} = \frac{1}{\kappa^2} (S_3 M_{2,0} S_3)^{-1} + \mathcal{O}(\kappa^{-2}\eta). \tag{6.52}$$

Notice that (6.52) holds true irrespective of the existence of zero energy resonances.

(b) $S_3 = 0$, $T_3 = S_2 \neq 0$ (no zero energy bound states but there are "p-wave" resonances). Again only the last term in (6.26) contributes to the most singular term; more exactly we have to extract the most singular contribution from

$$\frac{g(\kappa)}{\kappa^2 \eta^{-1}} T_3 m(\kappa)^{-1} T_3.$$

Taking into account (6.42) one obtains

$$M(\kappa)^{-1} = 4\pi \frac{\eta}{\kappa^2} T_3 \left(\sum_{j=1}^{r \leq 2} \langle \Theta_j, \cdot \rangle \Theta_j \right)^{-1} T_3 + \mathcal{O}(\kappa^{-2}\eta^2). \tag{6.53}$$

(c) $S_3 = T_3 = 0$, $\dim S_1 = 1$, $\Theta_0 = S_1 M_{0,0} v \neq 0$ (neither zero energy bound states nor "p-wave" resonances but there is an "s-wave" zero energy resonance) In this case $S_2 = 0$ so the last term in (6.26) vanishes and the most singular term is to be extracted from the second term in the r.h.s. of (6.26). Due to the fact that $(M(0) + S_1)^{-1} S_1 = S_1$ one obtains

$$M(\kappa)^{-1} = \frac{\alpha}{2\pi\eta\|\Theta_0\|^2} S_1 + \mathcal{O}(1). \tag{6.54}$$

(d) Finally in the generic case, i.e. $S_1 = 0$,

$$M(\kappa)^{-1} = (QM_{0,0}Q)^{-1} + \mathcal{O}(\eta). \tag{6.55}$$

(iii) The next remark concerns with the order of expansion of $M(\kappa)^{-1}$ as a function of β. As in the one-dimensional case, in general the order of expansion of $M(\kappa)^{-1}$ is lower than the order of expansion of $M(\kappa)$; the rule is that the loss in the order of expansion equals the square of the most singular term.

6.1 Proof of Theorem 6.2

Before starting the somewhat complicated procedure of expanding $M(\kappa)^{-1}$ a few guiding remarks might be useful. Suppose in (6.5) we factor out η^{-1} and then apply Corollary 2.2. The starting expansion parameters are η, $\kappa^2\eta$ and κ^2. By making the Neumann expansions in (2.5), the result will contain a series of the form $\sum_{l=0}^{l=\infty} d_l \eta^l$ which is obviously "bad" in view of its slow convergence, so if we are looking for a power like error one needs to sum it. A way out is not to expand the terms giving "bad" series. Let us recall that for the 4-dimensional case this has been achieved by Jensen [8] who proved that all "bad" expansions can be confined in a single numerical factor. As stated in the theorem above a similar result (albeit a bit more complicated one) holds true here: all "bad" expansions can be confined in a numerical factor (i.e. a rank one operator) and in a rank two operator. The way of achieving that is as follows: if one has to invert an expression like

$$A + \eta(B + \text{ good expansion}),$$

then rewrite it as

$$\eta(A\eta^{-1} + B + \text{ good expansion})$$

and apply Lemma 2.3 to

$$A\eta^{-1} + B + \text{ good expansion} + S_B,$$

where S_B is the orthogonal projection on $\operatorname{Ker} B$. Then it turns out that the "bad" expansion is confined to $\operatorname{Ran} A$ which in our case will be one or two dimensional subspaces. It turns out that all the "bad" expansions are contained in the inverses of at most three numerical factors of the form $1+\eta d_j$, $j=0,1,2$.

We use a notation similar to the one used in the proof of Theorem 5.2. As in the one dimensional case we set $Q=1-P$ and let S_1 be the orthogonal projection on $\mathrm{Ker}\, QM_{0,0}Q$ as an operator in $QL^2(\mathbf{R}^2)$. By the same argument as in the one dimensional case, $QM_{0,0}Q$ is self-adjoint and has discrete spectrum outside $\{-1,1\}$. It follows that, as an operator in $QL^2(\mathbf{R}^2)$, $Q(M_{0,0}+S_1)Q = QM_{0,0}Q+S_1$ has a bounded inverse $(Q(M_{0,0}+S_1)Q)^{-1}$ and

$$(Q(M_{0,0}+S_1)Q)^{-1}S_1 = S_1 \tag{6.56}$$

$$\dim \mathrm{Ran}\, S_1 = N < \infty \tag{6.57}$$

It follows that for sufficiently small κ, the operator $(Q(M_0(\kappa)+S_1)Q)$, where

$$M_0(\kappa) \equiv M_{0,0} + \kappa^2\eta^{-1}M_{2,-1} + \kappa^2 M_{2,0} + \dots\,, \tag{6.58}$$

has a bounded inverse in $QL^2(\mathbf{R}^2)$:

$$(Q(M_0(\kappa)+S_1)Q)^{-1} \equiv D_0(\kappa). \tag{6.59}$$

Then by Lemma 2.3 (see also Remark 2.4)

$$M(\kappa)+S_1 = -\frac{\alpha}{2\pi}\eta^{-1}P + M_0(\kappa) + S_1 \tag{6.60}$$

has a bounded inverse given by the formula

$$\begin{aligned}(M(\kappa)+S_1)^{-1} =& g(\kappa)^{-1}\{P - PM_0(\kappa)QD_0(\kappa)Q - QD_0(\kappa)QM_0(\kappa)P + \\ & QD_0(\kappa)QM_0(\kappa)PM_0(\kappa)QD_0(\kappa)Q\} + QD_0(\kappa)Q,\end{aligned} \tag{6.61}$$

where

$$\begin{aligned} g(\kappa) &= -\frac{\alpha}{2\pi}\eta^{-1} + \mathrm{Tr}\{PM_0(\kappa)P - PM_0(\kappa)QD_0(\kappa)QM_0(\kappa)P\} \\ &\equiv \eta^{-1}(-\frac{\alpha}{2\pi} + \eta h(\kappa)). \end{aligned} \tag{6.62}$$

Remark that $h(\kappa)$ has a "good" expansion and that the same is true for $g(\kappa)$.

We claim now that the application of Lemma 2.1 gives:

$$\begin{aligned} M(\kappa)^{-1} =& (M(\kappa)+S_1)^{-1} \\ & - g(\kappa)(M(\kappa)+S_1)^{-1}S_1M_1(\kappa)^{-1}S_1(M(\kappa)+S_1)^{-1}\end{aligned} \tag{6.63}$$

where

$$M_1(\kappa) = M_{1;0,0} + \kappa^2\eta^{-2}M_{1;2,-2} + \kappa^2\eta^{-1}M_{1;2,-1} + \dots \tag{6.64}$$

with

$$M_{1;0,0} = S_1 M_{0,0} P M_{0,0} S_1, \tag{6.65}$$

$$M_{1;2,-2} = \frac{\alpha}{2\pi} S_1 M_{2,-1} S_1. \tag{6.66}$$

Indeed, the use of Lemma 2.1 gives

$$\begin{aligned} M(\kappa)^{-1} = {} & (M(\kappa) + S_1)^{-1} \\ & + (M(\kappa) + S_1)^{-1} S_1 \tilde{M}_1(\kappa)^{-1} S_1 (M(\kappa) + S_1)^{-1}, \end{aligned} \tag{6.67}$$

where (see (6.61))

$$\begin{aligned} \tilde{M}_1(\kappa) = {} & S_1 \\ & - g(\kappa)^{-1} S_1 D_0(\kappa) Q M_0(\kappa) P M_0(\kappa) Q D_0(\kappa) S_1 - S_1 D_0(\kappa) S_1. \end{aligned} \tag{6.68}$$

Now

$$\begin{aligned} D_0(\kappa) &= (QM_{0,0}Q + S_1 + \kappa^2\eta^{-1}(Q(M_{2,-1} + \eta M_{2,0} + \dots)Q)^{-1} \\ &= (QM_{0,0}Q + S_1)^{-1} \\ & \quad [1 + \kappa^2\eta^{-1} Q(M_{2,-1} + \eta M_{2,0} + \dots)Q(QM_{0,0}Q + S_1)^{-1}]^{-1}. \end{aligned} \tag{6.69}$$

Taking into account (6.56) one has from (6.69) (remember that $QS_1 = S_1$)

$$S_1 - S_1 D_0(\kappa) S_1 = \kappa^2\eta^{-1} S_1 M_{2,-1} S_1 + \kappa^2 S_1 M_{2,0} S_1 + \dots \tag{6.70}$$

On the other hand

$$\begin{aligned} & S_1 D_0(\kappa) Q M_0(\kappa) P M_0(\kappa) Q D_0(\kappa) S_1 \\ & \qquad = S_1 M_{0,0} P M_{0,0} S_1 + \dots \mathcal{O}(\kappa^2\eta^{-1}) \end{aligned} \tag{6.71}$$

which together with (6.70) and (6.62) gives:

$$\begin{aligned} \tilde{M}_1(\kappa) &= -g(\kappa)^{-1}[S_1 M_{0,0} P M_{0,0} S_1 - g(\kappa)\kappa^2\eta^{-1} S_1 M_{2,-1} S_1 + \dots] \\ &= -g(\kappa)^{-1}[S_1 M_{0,0} P M_{0,0} S_1 + \kappa^2\eta^{-2} \frac{\alpha}{2\pi} S_1 M_{2,-1} S_1 + \dots] \\ &\equiv -g(\kappa)^{-1} M_1(\kappa) \end{aligned} \tag{6.72}$$

which proves (6.63)–(6.66).

We are left with the computation of $M_1(\kappa)^{-1}$. We shall use Corollary 2.2; it gives a "good" expansion and also a good start for the next iteration. Notice first that (as an operator in $S_1 L^2(\mathbf{R}^2)$) $M_{1;0,0} = S_1 M_{0,0} P M_{0,0} S_1$ is of rank at most one, so

$$\dim \operatorname{Ker} M_{1;0,0} \geq N - 1. \tag{6.73}$$

Let S_2 be the orthogonal projection on $\operatorname{Ker} M_{1;0,0} \subset S_1 L^2(\mathbf{R}^2)$. If

$$M_{1;0,0} = S_1 M_{0,0} P M_{0,0} S_1 = 0, \tag{6.74}$$

then

$$S_2 = S_1. \tag{6.75}$$

Coming back to $M_1(\kappa)^{-1}$, by Corollary 2.2,

$$\begin{aligned} M_1(\kappa)^{-1} &= (M_1(\kappa) + S_2)^{-1} \\ &\quad + \frac{\eta^2}{\kappa^2}(M_1(\kappa) + S_2)^{-1} S_2 \tilde{M}_2(\kappa)^{-1} S_2 (M_1(\kappa) + S_2)^{-1}, \end{aligned} \tag{6.76}$$

where

$$\begin{aligned} \tilde{M}_2(\kappa) &= \sum_{j=0}^{\infty} (-1)^j (\kappa\eta^{-1})^{2j} \times \\ &\quad \times S_2[(M_{1;2,-2} + \eta M_{1;2,-1} + \dots)(M_{1;0,0} + S_2)^{-1}]^{j+1} S_2. \end{aligned} \tag{6.77}$$

Expanding (6.77) one obtains:

$$\begin{aligned} \tilde{M}_2(\kappa) &= S_2 M_{1;2,-2} S_2 - \eta S_2 M_{1;2,-1} S_2 + \dots \\ &= \eta[\eta^{-1} S_2 M_{1;2,-2} S_2 + S_2 M_{1;2,-1} S_2 + \dots] \\ &\equiv \eta M_2(\kappa). \end{aligned} \tag{6.78}$$

Taking this into account (6.76) becomes

$$\begin{aligned} M_1(\kappa)^{-1} &= (M_1(\kappa) + S_2)^{-1} \\ &\quad + \frac{\eta}{\kappa^2}(M_1(\kappa) + S_2)^{-1} S_2 M_2(\kappa)^{-1} S_2 (M_1(\kappa) + S_2)^{-1} \end{aligned} \tag{6.79}$$

with

$$M_2(\kappa) = \eta^{-1} S_2 M_{1;2,-2} S_2 + S_2 M_{1;2,-1} S_2 + \dots \tag{6.80}$$

Computing $M_{1;2,-1}$ in (6.64) and observing that all contributions coming from the development of $S_1 D_0(\kappa) Q M_0(\kappa) Q D_0(\kappa) S_1$ vanish due to the fact that $P M_{0,0} S_2 = 0$, one obtains (6.38).

Notice that $M_2(\kappa)$ has the right structure to apply Lemma 2.3. Consider first $S_2 M_{1;2,-2} S_2$. By (6.66) (remember that $S_2 \leq S_1$)

$$S_2 M_{1;2,-2} S_2 = \frac{\alpha}{2\pi} S_2 M_{2,-1} S_2. \tag{6.81}$$

Since $S_2 \leq S_1$ and $P S_1 = 0$, it follows that $P S_2 = 0$ and then (see (6.8))

$$S_2 M_{1;2,-2} S_2 = -\frac{\alpha}{16\pi^2} S_2 T S_2 = \frac{\alpha}{8\pi^2} S_2 W S_2, \tag{6.82}$$

where T and W are integral operators with integral kernels $v(\mathbf{x})(\mathbf{x}^2 - 2\mathbf{x} \cdot \mathbf{y} + \mathbf{y}^2) v(\mathbf{y})$ and $v(\mathbf{x}) \mathbf{x} \cdot \mathbf{y} v(\mathbf{y})$, respectively. Let X_j be the operator of multiplication with x_j ($\mathbf{x} = (x_1, x_2)$), $j = 1, 2$, and

$$\Theta_j = S_2 X_j v \in L^2(\mathbf{R}^2). \tag{6.83}$$

Then from (6.82) and (6.83):

$$S_2 M_{1;2,-2} S_2 = \frac{\alpha}{8\pi^2} \sum_{j=1}^{j=2} \langle \Theta_j, \cdot \rangle \Theta_j. \tag{6.84}$$

It follows that $S_2 M_{1;2,-2} S_2$ is positive and of rank at most 2 (one or both Θ_j can be zero or they can be linearly dependent). So if T_3 is the orthogonal projection on $\operatorname{Ran} S_2 M_{1;2,-2} S_2$, then

$$\dim \operatorname{Ran} T_3 \leq 2. \tag{6.85}$$

Let S_3 be the orthogonal projection on $\operatorname{Ker} S_2 M_{1;2,-2} S_2$, i.e.

$$S_2 = T_3 + S_3. \tag{6.86}$$

Writing $M_2(\kappa)$ as a 2×2 matrix according to the decomposition (6.86)

$$M_2(\kappa) = \begin{pmatrix} a(\kappa) & b(\kappa) \\ c(\kappa) & d(\kappa) \end{pmatrix}, \tag{6.87}$$

where

$$\begin{aligned} a(\kappa) &= T_3 M_2(\kappa) T_3, & b(\kappa) &= T_3 M_2(\kappa) S_3, \\ c(\kappa) &= S_3 M_2(\kappa) T_3, & d(\kappa) &= S_3 M_2(\kappa) S_3. \end{aligned}$$

We compute now $M_2(\kappa)^{-1}$ by using Lemma 2.3. For, observe that since $T_3M_{1;2,-2}T_3$ is strictly positive on $T_3L^2(\mathbf{R}^2)$ and $T_3M_{1;2,-2}S_3 = 0$, $a(\kappa)$ has the form (2.9) and $b(\kappa)$, $c(\kappa)$, $d(\kappa)$ are uniformly bounded as $\kappa \to 0$. We shall argue now that $d(0)$ must be invertible and then $d(\kappa)$ is invertible for small enough κ. Indeed, since (see Remark 2.4) for κ small, $a(\kappa)$ has a bounded inverse by reversing the roles of a_{11} and a_{22} in Lemma 2.3, one obtains that $d(\kappa)^{-1}$ remains bounded in the limit $\kappa \to 0$ if and only if $M_2(\kappa)^{-1}$ does. But $M_2(\kappa)^{-1}$ must remain bounded as $\kappa \to 0$ since otherwise (see (6.62), (6.63), and (6.79)) the inequality (4.10) will be violated. Then by Lemma 2.3

$$\begin{aligned}M_2(\kappa)^{-1} =& T_3m(\kappa)^{-1}T_3 - T_3m(\kappa)^{-1}b(\kappa)d(\kappa)^{-1}S_3 - S_3d(\kappa)^{-1}c(\kappa)m(\kappa)^{-1}T_3 \\ &+ S_3d(\kappa)^{-1}c(\kappa)m(\kappa)^{-1}b(\kappa)d(\kappa)^{-1}S_3 + S_3d(\kappa)^{-1}S_3, \end{aligned} \tag{6.88}$$

where

$$m(\kappa)^{-1} = \big(a(\kappa) - b(\kappa)d(\kappa)^{-1}c(\kappa)\big)^{-1} : T_3L^2(\mathbf{R}^2) \to T_3L^2(\mathbf{R}^2). \tag{6.89}$$

Summing up (6.63), (6.79), and (6.88), one arrives at the final formula for $M(\kappa)^{-1}$ (see (6.26)).

As in the one dimensional case the "obstruction" subspaces $\operatorname{Ker} S_j$, $j = 1, 2, 3$, are related to zero energy resonances and bound states of H. We restate some of the results as a Lemma and prove it before we continue with the proof of the Theorem.

Lemma 6.4. (i) *Suppose $S_1 \neq 0$ and let $\Phi \in S_1L^2(\mathbf{R}^2)$, $\|\Phi\| = 1$. If Ψ is defined by*

$$\Psi(\mathbf{x}) = c_0 + \frac{1}{2\pi}\int_{\mathbf{R}^2} \ln(|\mathbf{x}-\mathbf{y}|)v(\mathbf{y})\Phi(\mathbf{y})d\mathbf{y} \tag{6.90}$$

with

$$c_0 = \frac{1}{\alpha}\langle v, M_0\Phi\rangle, \tag{6.91}$$

then

$$w\Psi = \Phi, \tag{6.92}$$

and in the sense of distributions

$$H\Psi = 0. \tag{6.93}$$

Furthermore, $\Psi \in L^\infty(\mathbf{R}^2)$, and

$$\Psi(\mathbf{x}) = c_0 + \sum_{j=1}^{2} c_j \frac{x_j}{\langle \mathbf{x}\rangle^2} + \tilde{\Psi}(\mathbf{x}), \tag{6.94}$$

where $\tilde{\Psi} \in L^2(\mathbf{R}^2)$ *and*

$$c_j = -\frac{1}{2\pi}\langle v, X_j\Phi\rangle = -\frac{1}{2\pi}\int_{\mathbf{R}^2} x_j v(\mathbf{x})\Phi(\mathbf{x})d\mathbf{x}. \tag{6.95}$$

(ii) *Suppose* $\Psi(\mathbf{x}) = c + \Lambda(\mathbf{x})$ *with* $c \in \mathbf{C}$ *and* $\Lambda = \Lambda_1 + \Lambda_2$*, where* $\Lambda_1 \in L^p(\mathbf{R}^2)$ *for some* p*,* $2 < p < \infty$*, and* $\Lambda_2 \in L^2(\mathbf{R}^2)$*. If* Ψ *satisfies* $H\Psi = 0$ *in the distribution sense, then*

$$\Phi = w\Psi \in S_1 L^2(\mathbf{R}^2). \tag{6.96}$$

Proof. We give a detailed proof of the results. Assume $\Phi \in \operatorname{Ran} S_1$, and $\Phi \neq 0$. Notice that due to $P\Phi = 0$ we have

$$\int_{\mathbf{R}^2} \ln(|\mathbf{x}-\mathbf{y}|)v(\mathbf{y})\Phi(\mathbf{y})d\mathbf{y} = \int_{\mathbf{R}^2} \ln(e^{\gamma}|\mathbf{x}-\mathbf{y}|/2)v(\mathbf{y})\Phi(\mathbf{y})d\mathbf{y}.$$

Let Ψ be given by (6.90) and (6.91). Then using (6.7) and $QM_{0,0}\Phi = 0$ we get

$$\begin{aligned} w\Psi &= c_0 w + U(U - M_{0,0})\Phi \\ &= c_0 w + \Phi - UM_{0,0}\Phi \\ &= c_0 w + \Phi - UPM_{0,0}\Phi \\ &= c_0 w + \Phi - \frac{1}{\alpha}\langle v, M_{0,0}\Phi\rangle w \\ &= \Phi, \end{aligned}$$

which proves (6.92). Differentiation in the sense of distributions yields (6.93).

We now establish the results in (6.94). It suffices to consider $|\mathbf{x}| \geq 4$. We use the following $\mathbf{x}$-dependent decomposition of $\mathbf{R}^2$.

$$\begin{aligned} R_0 &= \{\mathbf{y} \in \mathbf{R}^2 \,|\, |\mathbf{x}-\mathbf{y}| \leq 2\}, \\ R_1 &= \{\mathbf{y} \in \mathbf{R}^2 \backslash R_0 \,|\, |\mathbf{y}| \geq |\mathbf{x}|/8, |\mathbf{x}| \leq |\mathbf{x}-\mathbf{y}|\}, \\ R_2 &= \{\mathbf{y} \in \mathbf{R}^2 \backslash R_0 \,|\, |\mathbf{y}| \geq |\mathbf{x}|/8, |\mathbf{x}| > |\mathbf{x}-\mathbf{y}|\}, \\ R_3 &= \{by \in \mathbf{R}^2 \backslash R_0 \,|\, |\mathbf{y}| < |\mathbf{x}|/8\}. \end{aligned}$$

Using $P\Phi = 0$ once more we have

$$\Psi(\mathbf{x}) = c_0 + \frac{1}{4\pi}\sum_{j=0}^{3}\int_{R_j} \ln(\frac{|\mathbf{x}-\mathbf{y}|^2}{|\mathbf{x}|^2})v(\mathbf{y})\Phi(\mathbf{y})d\mathbf{y}.$$

Each term is now estimated. For $\mathbf{y} \in R_0$ we have for any $\varepsilon > 0$ the estimate $|\ln|\mathbf{x}-\mathbf{y}|| \le c_\varepsilon |\mathbf{x}-\mathbf{y}|^{-\varepsilon}$. We also note that $\langle \mathbf{x}\rangle^s \langle \mathbf{y}\rangle^{-s}$ is bounded on R_0, since $2 \le |\mathbf{x}| - 2 \le |\mathbf{y}|$ there. Thus we have

$$\begin{aligned}\left|\int_{R_0} \ln(\frac{|\mathbf{x}-\mathbf{y}|^2}{|\mathbf{x}|^2}) v(\mathbf{y})\Phi(\mathbf{y})d\mathbf{y}\right| &\le C\int_{R_0} |\mathbf{x}-\mathbf{y}|^{-\varepsilon}|\mathbf{y}|^{\varepsilon}|v(\mathbf{y})||\Phi(\mathbf{y})|d\mathbf{y} \\ &\quad + C\int_{R_0} |\ln|\mathbf{x}||\, \langle \mathbf{x}\rangle^s \langle \mathbf{y}\rangle^{-s} |v(\mathbf{y})||\Phi(\mathbf{y})|d\mathbf{y} \\ &\le \langle \mathbf{x}\rangle^{-\mu}\end{aligned}$$

for some $\mu > 1$, due to the assumption on V.

For $\mathbf{y} \in R_1$ we have $|\mathbf{x}-\mathbf{y}|/|\mathbf{x}| \ge 1$, and $2 \le |\mathbf{x}-\mathbf{y}| \le 9|\mathbf{y}|$. Thus

$$\begin{aligned}\left|\int_{R_1} \ln(\frac{|\mathbf{x}-\mathbf{y}|^2}{|\mathbf{x}|^2}) v(\mathbf{y})\Phi(\mathbf{y})d\mathbf{y}\right| &\le C\int_{R_1} (|\ln|\mathbf{y}|| + |\ln|\mathbf{x}||)|v(\mathbf{y})||\Phi(\mathbf{y})|d\mathbf{y} \\ &\le \langle \mathbf{x}\rangle^{-\mu}\end{aligned}$$

for some $\mu > 1$.

For $\mathbf{y} \in R_2$ we use an estimate $|\ln(|\mathbf{x}-\mathbf{y}|/|\mathbf{x}|)| \le C|\mathbf{x}|^{\varepsilon}|\mathbf{x}-\mathbf{y}|^{-\varepsilon} \le |\mathbf{y}|^{\varepsilon}$ and again get that the contribution from R_2 is estimated by $C\langle \mathbf{x}\rangle^{-\mu}$ for some $\mu > 1$.

Finally we consider the region R_3. We write

$$\ln(\frac{|\mathbf{x}-\mathbf{y}|^2}{|\mathbf{x}|^2}) = \ln(1 + \frac{|\mathbf{y}|^2}{|\mathbf{x}|^2} - 2\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|^2}).$$

Now for $\mathbf{y} \in R_3$ we have

$$\left|\frac{|\mathbf{y}|^2}{|\mathbf{x}|^2} - 2\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|^2}\right| \le \frac{|\mathbf{y}|^2}{|\mathbf{x}|^2} + 2\frac{|\mathbf{y}|}{|\mathbf{x}|} \le \frac{1}{64} + \frac{1}{4} < \frac{1}{2}.$$

Taylor's formula with remainder yields

$$\ln(1+h) = h + h^2\rho(h), \quad |h| \le \tfrac{1}{2},$$

where $|\rho(h)| \le C$ for $|h| \le \frac{1}{2}$. Thus we have

$$\begin{aligned}\int_{R_3} \ln(\frac{|\mathbf{x}-\mathbf{y}|^2}{|x|^2}) v(\mathbf{y})\Phi(\mathbf{y})d\mathbf{y} &= -2\int_{R_3} \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|^2} v(\mathbf{y})\Phi(\mathbf{y})d\mathbf{y} \\ &\quad + \int_{R_3} \frac{|\mathbf{y}|^2}{|\mathbf{x}|^2} v(\mathbf{y})\Phi(\mathbf{y})d\mathbf{y} \\ &\quad + \int_{R_3} \left(\frac{|\mathbf{y}|^2}{|\mathbf{x}|^2} - 2\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|^2}\right)^2 \rho(\cdot) v(\mathbf{y})\Phi(\mathbf{y})d\mathbf{y}.\end{aligned}$$

The second and third terms can be estimated by $\langle x\rangle^{-2}$. The first term is rewritten

$$-2\int_{R_3}\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|^2}v(\mathbf{y})\Phi(\mathbf{y})d\mathbf{y} = -2\int_{\mathbf{R}^2}\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|^2}v(\mathbf{y})\Phi(\mathbf{y})d\mathbf{y} + 2\int_{\mathbf{R}^2\backslash R_3}\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|^2}v(\mathbf{y})\Phi(\mathbf{y})d\mathbf{y}.$$

On $\mathbf{R}^2\backslash R_3$ we have $|\mathbf{y}|\geq|\mathbf{x}|/8$, hence we get a decay estimate of order $\langle x\rangle^{-\mu}$ for some $\mu>1$, as above. This completes the proof of (6.94) and (6.95). Note that we have also established that $\Psi\in L^\infty(\mathbf{R}^2)$.

We now continue to prove part (ii) of the Lemma. Assume that $\Psi(\mathbf{x}) = c+\Lambda(x)$, $\Lambda=\Lambda_1+\Lambda_2$, where $\Lambda_1\in L^p(\mathbf{R}^2)$ for some p, $2<p<\infty$, and $\Lambda_2\in L^2(\mathbf{R}^2)$. Assume furthermore that $H\Psi=0$ in the sense of distributions. Define $\Phi=w\Psi$. Then we have

$$\Delta\Psi = V\Psi = v\Phi.$$

Now choose a nonnegative function $\phi\in C_0^\infty(\mathbf{R}^2)$ with support in $|\mathbf{x}|\leq 2$ and with $\phi(\mathbf{x})=1$ for $|\mathbf{x}|\leq 1$. Then we compute as follows.

$$\begin{aligned}\int_{\mathbf{R}^2}v(\mathbf{x})\Phi(\mathbf{x})\phi(\delta\mathbf{x})d\mathbf{x} &= \int_{\mathbf{R}^2}(\Delta\Psi)(\mathbf{x})\phi(\delta\mathbf{x})d\mathbf{x}\\ &= \int_{\mathbf{R}^2}(\Delta\Lambda)(\mathbf{x})\phi(\delta\mathbf{x})d\mathbf{x}\\ &= \delta^2\int_{\mathbf{R}^2}\Lambda(\mathbf{x})(\Delta\phi)(\delta\mathbf{x})d\mathbf{x}.\end{aligned}$$

Using the assumptions on Λ this leads to an estimate of the absolute value by

$$\delta^{2/p}\|\Lambda_1\|_p\|\Delta\phi\|_{p'}+\delta\|\Lambda_2\|_2\|\Delta\phi\|_2,$$

which tends to zero as $\delta\to 0$. Using Lebesgue's dominated convergence theorem we conclude

$$\int_{\mathbf{R}^2}v(\mathbf{x})\Phi(\mathbf{x})d\mathbf{x}=0,\quad\text{or}\quad P\Phi=0.$$

Thus $\Phi\in QL^2(\mathbf{R}^2)$. Define

$$\Xi(\mathbf{x})=\frac{1}{2\pi}\int\ln(|\mathbf{x}-\mathbf{y}|)v(\mathbf{y})\Phi(\mathbf{y})d\mathbf{y}.$$

Then in the sense of distributions we have

$$\Delta\Xi = V\Psi = \Delta\Psi,$$

which means that $\Psi - \Xi$ is harmonic on $\mathbf{R}^2$. The assumptions on Ψ and the proof of part (i) together show that $\Psi - \Xi \in L^\infty(\mathbf{R}^2) + L^2(\mathbf{R}^2)$. But then by well-known properties of harmonic functions in the plane we have $\Psi - \Xi = c$ for some constant. Thus we have proved that

$$\Psi(\mathbf{x}) = c + \frac{1}{2\pi}\int \ln(|\mathbf{x}-\mathbf{y}|)v(\mathbf{y})\Phi(\mathbf{y})d\mathbf{y}.$$

Hence

$$\Phi = w\Psi = cw + U(U - M_{0,0})\Phi = cw + \Phi - UM_{0,0}\Phi,$$

or

$$cv = Ucw = M_{0,0}\Phi. \tag{6.97}$$

Apply P on both sides of (6.97) to get

$$cv = \frac{1}{\alpha}\langle v, M_{0,0}\Phi\rangle v.$$

Since by assumption V is not identically zero, we conclude

$$c = \frac{1}{\alpha}\langle v, M_{0,0}\Phi\rangle.$$

We now use $Q\Phi = \Phi$ and apply Q to both sides of (6.97) to get

$$QM_{0,0}Q\Phi = 0, \quad \text{or} \quad \Phi \in S_1L^2.$$

This proves part (ii) of the Lemma. $\square$

Remark 6.5. Let us note that most of the results in Lemma 6.4 have been obtained in [2] in the $\int V(\mathbf{x})d\mathbf{x} \neq 0$ case, with different proofs.

We now proceed with the proof of the first part of the Theorem. Recalling the definitions of the various projections S_j, $j = 1, 2, 3$, and T_j, $j = 2, 3$, and the self-adjointness of the operators defining the kernels, we immediately get that

$$\operatorname{Ran} T_2 = S_1L^2 \cap \operatorname{Ran} S_1M_{0,0}PM_{0,0}S_1,$$

which by the definitions of the operators is spanned by the vector

$$\Theta_0 = S_1M_{0,0}v. \tag{6.98}$$

We also get that

$$\operatorname{Ran} T_3 = S_2L^2 \cap \operatorname{Ran} S_2M_{2,-1}S_2.$$

Again using the definitions we get that this space is spanned by

$$\Theta_j = S_2 X_j v, \quad j = 1, 2, \tag{6.99}$$

where X_j denotes multiplication by the coordinate x_j. This proves the first half of part (i) of the Theorem.

Let us now establish the connection between the eigenspace

$$\mathcal{N} = \{\Psi \in L^2 \mid H\Psi = 0\}$$

and $\operatorname{Ran} S_3$. Suppose first that $\Phi \in \operatorname{Ran} S_3$. Then Φ is orthogonal to both $\operatorname{Ran} T_2$ and $\operatorname{Ran} T_3$, which implies

$$\langle v, M_{0,0}\Phi\rangle = 0, \quad \langle v, X_j\Phi\rangle = 0, \quad j = 1, 2. \tag{6.100}$$

Now define Ψ by (6.90). Then (6.93), (6.94), and (6.100) imply that $\psi \in \mathcal{N}$.

Conversely, assume $\Psi \in \mathcal{N}$, and define $\Phi = w\Psi$. Since $\Psi \in L^2$, we can use part (ii) of the Lemma to conclude via (6.94) that $\Phi \in S_1 L^2$, and furthermore that (6.100) hold for this particular Φ. Thus $\Phi \in \operatorname{Ran} S_3$. The correspondence is clearly one-to-one and onto, thus $\dim \operatorname{Ran} S_3 = \dim \mathcal{N}$.

Finally, let us establish the decomposition results. Define Ψ_s by using (6.90) with $\Phi = \Theta_0$ from (6.98). It follows that (6.94) holds for Ψ_s with $c_1 = c_2 = 0$. We conclude that if $\Psi_s \neq 0$, then $c_0 \neq 0$, $\Psi_s \in L^\infty$ and $\Psi \notin L^q$, for any $q < \infty$.

For $j = 1, 2$ define $\Psi_{p,j}$ by taking $\Phi = \Theta_j$ from (6.99) in (6.90). It follows from the above results that $\langle v, M_{0,0}\Theta_j\rangle = 0$. Then we get from (6.94) that $\Psi_{p,j} \in L^q$ for all $q > 2$. If $\Theta_j \neq 0$, and consequently $\Psi_{p,j} \neq 0$, then $c_j \neq 0$, and (6.94) shows that $\Psi_{p,j} \notin L^2$.

This concludes the proof of the Theorem. □

Remark 6.6. Of course some or all of T_2, T_3, S_3 can be zero and in this case the formula (6.26) takes a simpler form. One can obtain the formula of $M(\kappa)^{-1}$ in these cases either from specialising (6.26) or by repeating the procedure which led to (6.26). Let us mention that the two ways can lead to formulae which looks different but they are the same due to various identities. Consider, for example, that $T_2 = T_3 = 0$ i.e. $S_1 = S_3$ (no zero energy resonances). Then formula (6.26) gives (remember that in this case $S_1 = S_2 = S_3$)

$$\begin{aligned} M(\kappa)^{-1} = {} & (M(\kappa) + S_1)^{-1} \\ & - g(\kappa)(M(\kappa) + S_1)^{-1} S_1 (M_1(\kappa) + S_1)^{-1} S_1 (M(\kappa) + S_1)^{-1} \\ & - \frac{\eta g(\kappa)}{\kappa^2}(M(\kappa) + S_1)^{-1} S_1 (M_1(\kappa) + S_1)^{-1} S_1 \times \\ & \times S_1 d(\kappa)^{-1} S_1 (M_1(\kappa) + S_1)^{-1} S_1 (M(\kappa) + S_1)^{-1}, \end{aligned} \tag{6.101}$$

while the procedure stops after the first application of Corollary 2.2, which gives

$$
\begin{aligned}
M(\kappa)^{-1} &= (M(\kappa)+S_1)^{-1} \\
&\quad - g(\kappa)(M(\kappa)+S_1)^{-1}S_1M_1(\kappa)^{-1}S_1(M(\kappa)+S_1)^{-1}.
\end{aligned}
\tag{6.102}
$$

Now (see the definitions of T_2 and T_3) $M_{1;0,0} = M_{1;2,-2} = 0$, i.e.

$$
M_1(\kappa) = \kappa^2\eta^{-1}(M_{1;2,-1} + \dots) \equiv \kappa^2\eta^{-1}M_{1;2,-1}(\kappa). \tag{6.103}
$$

Since $g(\kappa)\kappa^2\eta_{-1} = \mathcal{O}(\kappa^2)$, $M_{1;2,-1}$ and then $M_{1;2,-1}(\kappa)$ must be invertible, so that one obtains finally

$$
\begin{aligned}
M(\kappa)^{-1} = (M(\kappa)+S_1)^{-1} - \frac{\eta}{\kappa^2 g(\kappa)}(M(\kappa)+S_1)^{-1} \times \\
\times S_1M_{1;2,-1}(\kappa)^{-1}S_1(M(\kappa)+S_1)^{-1}.
\end{aligned}
\tag{6.104}
$$

Still (6.101) and (6.104) are identical, since by Corollary 2.2

$$
\begin{aligned}
M_1(\kappa)^{-1} &= (M_1(\kappa)+S_1)^{-1} \\
&\quad + \frac{\eta}{\kappa^2}(M_1(\kappa)+S_1)^{-1}S_1d(\kappa)^{-1}S_1(M_1(\kappa)+S_1)^{-1}.
\end{aligned}
\tag{6.105}
$$

One particular case of the above results is of separate interest. It is a computation of the singular part of $M(\kappa)^{-1}$ in the case when there are no zero energy resonances. In this case (see Theorem 6.2) $T_2 = T_3 = 0$ or in other words $S_1 = S_2 = S_3 \equiv S$, where S is the orthogonal projection onto $\operatorname{Ran} S_3$, which is isomorphic to the subspace of zero energy bound states.

We compute just the leading term, expanding the good expressions obtained in the theorem.

Proposition 6.7. *Assume $T_2 = T_3 = 0$ and let $S = S_3$. Let*

$$
\tilde{m}_{2,0} = SM_{2,0}S, \tag{6.106}
$$

$$
\tilde{m}_{4,-1} = SM_{4,-1}S + \frac{2\pi}{\alpha}SM_{2,-1}PM_{2,-1}S. \tag{6.107}
$$

Then (as an operator in $SL^2(\mathbf{R}^2)$) $\tilde{m}_{2,0}$ is invertible and

$$
M(\kappa)^{-1} = \kappa^{-2}\tilde{m}_{2,0}^{-1} - \eta^{-1}\tilde{m}_{2,0}^{-1}\tilde{m}_{4,-1}\tilde{m}_{2,0}^{-1} + \mathcal{O}(1). \tag{6.108}
$$

The range $\operatorname{Ran}\tilde{m}_{4,-1}$ *is spanned by the functions Sx_1^2v, Sx_2^2v, and Sx_1x_2v.*

Proof. The reason for the simple form of (6.108) is that many terms in the expansion vanish. We have to use that

$$PS = SP = 0, \tag{6.109}$$

and that in the given case

$$\Theta_0 = SM_{0,0}v = 0, \tag{6.110}$$

$$\Theta_j = SX_j v = 0, \quad j = 1, 2. \tag{6.111}$$

Now (6.109)–(6.111) imply that the following operators are zero:

$$SM_{0,0}P = PM_{0,0}S = SM_{2,-1}S = SM_{2,-1}Q = QM_{2,-1}S = 0. \tag{6.112}$$

We compute $M(\kappa)^{-1}$ using (6.67), (6.68), (6.61), and (6.62). We start by computing $D_0(\kappa)$ up to $\mathcal{O}(\kappa^4)$. With the notation

$$QAQ \equiv A^Q, \quad D_0(0) \equiv D_{0,0} = (M^Q_{0,0} + S)^{-1}, \tag{6.113}$$

one has

$$\begin{aligned} D_0(\kappa) &= (M^Q(\kappa) + S)^{-1} \\ &= D_{0,0} - \kappa^2 \eta^{-1} D_{0,0} M^Q_{2,-1} D_{0,0} - \kappa^2 D_{0,0} M^Q_{2,0} D_{0,0} \\ &\quad + \kappa^4 \eta^{-2} D_{0,0} M^Q_{2,-1} D_{0,0} M^Q_{2,-1} D_{0,0} \\ &\quad + \kappa^4 \eta^{-1} [D_{0,0} M^Q_{2,-1} D_{0,0} M^Q_{2,0} D_{0,0} \\ &\qquad + D_{0,0} M^Q_{2,0} D_{0,0} M^Q_{2,-1} D_{0,0} - D_{0,0} M^Q_{4,-1} D_{0,0}] + \mathcal{O}(\kappa^4). \end{aligned} \tag{6.114}$$

From (6.114), (6.112), and the fact that

$$QS = SQ = S, \quad D_{0,0}S = SD_{0,0} = S, \tag{6.115}$$

one obtains

$$SD_0(\kappa)S = S - \kappa^2 SM_{2,0}S - \kappa^4 \eta^{-1} SM_{4,-1}S + \mathcal{O}(\kappa^4), \tag{6.116}$$

$$SD_0(\kappa)Q = S + \mathcal{O}(\kappa^2), \tag{6.117}$$

$$QD_0(\kappa)S = S + \mathcal{O}(\kappa^2). \tag{6.118}$$

We compute now $\tilde{M}_1(\kappa)$ from (6.68):

$$\tilde{M}(\kappa) = S - SD_0(\kappa)S - g(\kappa)^{-1} SD_0(\kappa)QM_0(\kappa)PM_0(\kappa)QD_0(\kappa)S.$$

Taking into account that

$$g(\kappa)^{-1} = -\frac{2\pi}{\alpha}\eta + \mathcal{O}(\eta^2),$$

and also (6.112) and (6.115), one gets

$$-g(\kappa)^{-1}SD_0(\kappa)QM_0(\kappa)PM_0(\kappa)QD_0(\kappa)S$$
$$= \kappa^4\eta^{-1}\frac{2\pi}{\alpha}SM_{2,-1}PM_{2,-1}S + \mathcal{O}(\kappa^4),$$

which together with (6.116) gives

$$\begin{aligned}\tilde{M}_1(\kappa) &= \kappa^2 SM_{2,0}S + \kappa^4\eta^{-1}(SM_{4,-1}S + \frac{2\pi}{\alpha}SM_{2,-1}PM_{2,-1}S) + \mathcal{O}(\kappa^4) \\ &= \tilde{m}_{2,0}\kappa^2 + \tilde{m}_{4,-1}\kappa^4\eta^{-1} + \mathcal{O}(\kappa^4). \qquad (6.119)\end{aligned}$$

From $g(\kappa)^{-1} \sim \eta$ and (6.117)–(6.118) one has (see (6.61))

$$(M(\kappa) + S)^{-1}S = S + \mathcal{O}(\kappa^2), \qquad (6.120)$$
$$S(M(\kappa) + S)^{-1} = S + \mathcal{O}(\kappa^2). \qquad (6.121)$$

Since $(M(\kappa)+S)^{-1} = \mathcal{O}(1)$, the proposition follows from (6.67), and (6.119)-(6.121).

The last result follows from the definitions of the operators and the assumption that $T_2 = T_3 = 0$. $\square$

Remark 6.8. The invertibility of the operator $\tilde{m}_{2,0}$ was obtained from the general singularity argument in the proof of Theorem 6.2, see the discussion before (6.88) concerning the invertibility of $d(0)$.

Let us briefly indicate how this result can be proved directly. We give the discussion in the context of Proposition 6.7. Let P_0 denote the orthogonal projection onto the eigenspace of eigenvalue zero of H. Then we claim that

$$P_0 wM_{2,0}wP_0 = P_0. \qquad (6.122)$$

This is seen as follows. Let $\Psi_1, \Psi_2 \in L^2(\mathbf{R})$. Using the definitions we get

$$\langle\Psi_1, P_0wM_{2,0}wP_0\Psi_2\rangle$$
$$= \lim_{\kappa\to 0}\frac{1}{\kappa^2}\langle\Psi_1, P_0wv(H_0 + \kappa^2)^{-1}vwP_0\Psi_2 + P_0w(U - M_{0,0})wP_0\Psi_2\rangle.$$

Computing the right hand side in Fourier space (see similar computation in [8, Lemma 2.6]) and using the results from Lemma 6.4, one finds that the limit equals $\langle\Psi_1, P_0\Psi_2\rangle$. The rather lengthy computations are omitted.

Using this result one then finds that

$$\tilde{m}_{2,0}^{-1} = wP_0w. \qquad (6.123)$$

Example 6.9. Let us give a few examples. First we note that Example 5.7 has an immediate generalization to dimension two. Let $\beta < 0$. Then $\Psi = e^{-\langle \mathbf{x} \rangle^\beta}$ is a zero resonance eigenfunction of s-wave type, and the potential is in this case

$$V_\beta(\mathbf{x}) = \beta^2 \mathbf{x}^2 \langle \mathbf{x} \rangle^{2\beta-4} - 2\beta \langle \mathbf{x} \rangle^{\beta-2} - \beta(\beta-2)\mathbf{x}^2 \langle \mathbf{x} \rangle^{\beta-4}.$$

It is also easy to give examples where we have resonances of p-wave type. Let

$$V(\mathbf{x}) = \frac{-8}{(1+x_1^2+x_2^2)^2}.$$

Then $-\Delta + V$ has a zero resonance of p-wave type with resonance functions

$$\Psi_1(\mathbf{x}) = \frac{x_1}{1+x_1^2+x_2^2}, \quad \Psi_2(\mathbf{x}) = \frac{x_2}{1+x_1^2+x_2^2}.$$

Concerning zero eigenvalues, then taking V to depend only on $r = |\mathbf{x}|$ it is easy to construct potentials, for example a well, where we have zero eigenvalues. In particular, Proposition 6.7 shows that only solutions with angular momentum 2 (d-wave type) will have a nonzero second term in the expansion (6.108).

7 Further results and generalizations

In this section we give some further results and then discuss some possible generalizations of the results obtained above.

Let us first note the following result on the expansion coefficients. The result applies to all coefficients that can be obtained for a given V. We also note that the proof applies to all dimensions.

Proposition 7.1. (i) *The coefficients in the asymptotic expansion of* $M(\kappa)^{-1}$ *are bounded self-adjoint operators.*
(ii) *The coefficients in the asymptotic expansion of* $\operatorname{Im} M(\kappa)^{-1}$, *for* $\operatorname{Re}\kappa = 0$, *are finite rank operators.*

Remark 7.2. Let $E(\lambda)$ denote the spectral family of H. The second result and (4.8) then show that the coefficients in the asymptotic expansion of $wE'(\lambda)w$, $\lambda \downarrow 0$, are finite rank operators.

Proof. For a given V with a specified decay rate we have expansions up to an order p. The results hold for the coefficients in this expansion. For

$\kappa \in (0, \kappa_0)$ the operator $M(\kappa)$ is self-adjoint and therefore $M(\kappa)^{-1}$ is also self-adjoint. But then uniqueness of the expansion coefficients in an asymptotic expansion gives the result. Note that uniqueness holds, once we have fixed the asymptotic family of functions to be used in the expansion.

To prove (ii) we first note that (4.5) implies

$$\begin{aligned} 2i \operatorname{Im} wM(\kappa)^{-1}w &= wM(\kappa)^{-1}w - w(M(\kappa)^{-1})^*w \\ &= wM(\kappa)^{-1}v \left(R_0(-\kappa^2) - R_0(-\kappa^2)^* \right) v(M(\kappa)^{-1})^*w \end{aligned} \tag{7.1}$$

It follows from the formulae in Section 3 for the kernel of the free resolvent in various dimensions that the terms in the expansion of $v(R_0(-\kappa^2) - R_0(-\kappa^2)^*)v$, for κ purely imaginary, all are finite rank operators, since for dimensions $m \geq 5$, m odd, the expansions do not contain terms $|\mathbf{x} - \mathbf{y}|^{2p}$ for $p < 0$, due to [7, Lemma 3.3], and the similar result for even dimensions, $m \geq 6$, given in [7, (3.10)]. The result (ii) then follows from (7.1) and the existence of the asymptotic expansion of $M(\kappa)^{-1}$. □

Let us now consider the question of extending the class of potentials V. As mentioned previously, it is just a matter of technicalities to extend the results to $V(\mathbf{x})$ such that V is a quadratic form perturbation of H_0, and with sufficient decay in $\mathbf{x}$.

It is also possible to include certain classes of non-local potentials. For example, one can assume that the operator V has a factorization $V = vUv$ with v self-adjoint, and with suitable mapping properties, and with U satisfying $U^2 = I$. But here the analysis of the possible null spaces arising in the reduction process is different and requires a different approach. For example, in the one-dimensional case with a local potential there can be at most one zero resonance function, and no L^2-eigenvalue, as proved in Theorem 5.2. But with a non-local potential one can have two linearly independent zero resonance functions, and simultaneously an L^2-eigenvalue of arbitrarily large (finite) multiplicity. A study of this case has been initiated in [11, 12].

More general operators can also be treated by the approach used here, including non-self-adjoint perturbations, as in [13]. The analysis of the kernels and their relation to the original operator may be complicated in this case.

A class of two-channel Hamiltonians can easily be analyzed with the technique developed in Section 2. Details will be given elsewhere.

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