NO-ARBITRAGE AND COMPLETENESS FOR THE LINEAR AND EXPONENTIAL MODELS BASED ON LÉVY PROCESSES

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Abstract. Let $(X_t)_{t\ge 0}$ be a Lévy process. We consider the linear model

$$S_t = S_0 + X_t, \quad t \in [0,T]$$

and the exponential model

$$S_t = S_0 e^{X_t}, \quad t \in [0,T]$$

for an asset price. We also consider the models

$$S_t = S_0 + X_{\tau_t}, \quad t \in [0,T]$$

and

$$S_t = S_0 e^{X_{\tau_t}}, \quad t \in [0,T],$$

where $(\tau_t)_{t\in[0,T]}$ is a time-change that is independent of $(X_t)_{t\ge 0}$.

We present the necessary and sufficient conditions for the absence of arbitrage and for the completeness of these 4 models. It turns out that they are arbitrage-free except for some trivial cases. Furthermore, they are not complete except for some special cases.

Key words and phrases. Lévy processes, time-changed Lévy processes, no-arbitrage, completeness, fundamental theorems of asset pricing, predictable representation property, martingale transforms.

1 Introduction

1. No-arbitrage. First, we will cite the definition of some basic notions of the mathematical finance.

Let $T \ge 0$. Let $(S_t)_{t\in[0,T]}$ be a one-dimensional semimartingale on a filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P}\big)$. The financial interpretation is as follows: S may represent the (discounted) price of a stock, an exchange rate or a financial index. We will call the collection $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P}; S\big)$ a *model* of a financial market.

Definition 1.1. A (self-financing) *strategy* π is a pair (x, H), where $x \in \mathbb{R}$ and $H = (H_t)_{t\in[0,T]}$ is a $(\mathcal{F}_t)$-predictable S-integrable process, i.e. there exists the stochastic integral $\int_0^t H_u dS_u$, $t \in [0,T]$. The *capital process* of a strategy $\pi = (x, H)$ is given by

$$V_t^\pi := x + \int_0^t H_u dS_u, \quad t \in [0,T].$$

We do not specify here the class of S-integrable processes. The precise definition can be found, for example, in [23], [22; Ch. VII, §1a], [9].

We now cite the definition of the *free lunch with vanishing risk*. This notion, introduced by F. Delbaen and W. Schachermayer in [10], is a relevant continuous-time analogue of the no-arbitrage property.

Definition 1.2. A sequence of strategies $\pi_k = (x_k, H_k)$, $k = 1, 2, \dots$ realizes *free lunch with vanishing risk* if

i) for each k, $x_k = 0$;

ii) for each k, there exists a constant a_k such that

$$\mathsf{P}\big(\forall t \in [0,T],\, V_t^{\pi_k} \geq a_k\big) = 1;$$

iii) for each k,

$$V_T^{\pi_k} \geq -\frac{1}{k} \quad \mathsf{P}\text{-a.s.};$$

iv) there exist constants $\delta_1 > 0$, $\delta_2 > 0$ such that, for each k,

$$\mathsf{P}\big(V_T^{\pi_k} > \delta_1\big) > \delta_2.$$

A model satisfies the *no free lunch with vanishing risk* condition if such a sequence of strategies does not exist. Notation: (*NFLVR*).

`Remark`. There exist several other analogues of the no-arbitrage property in the continuous time: (*NFLBR*), (*NFL*). However, they are equivalent to (*NFLVR*) (see [10]). □

Definition 1.3. A process S is called a $(\mathcal{F}_t, \mathsf{P})$*-martingale transform* if there exist a $(\mathcal{F}_t, \mathsf{P})$-local martingale M and a $(\mathcal{F}_t)$-predictable M-integrable process H such that

$$S_t = S_0 + \int_0^t H_s dM_s, \quad t \in [0,T].$$

`Remarks`. (i) Any local martingale is a martingale transform. The reverse is not true. Indeed, if a process H is locally bounded, then the stochastic integral $\int_0^t H_s dM_s$ is again a local martingale. However, the class of M-integrable processes is much larger than the class of locally bounded processes, so that $\int_0^t H_s dM_s$ may not be a local martingale. The corresponding example was given by M. Émery [14] (it is also cited in [11], [22; Ch. VII, §1a], [23]).

(ii) The processes that are called here martingale transforms were introduced by C.S. Chou [7] and M. Émery [14] under the name "semimartingales de la classe Σ_m". F. Delbaen and W. Schachermayer [11] called these processes "sigma-martingales". We prefer the term "martingale transforms". □

Proposition 1.4. (The First fundamental theorem of asset pricing). *A model satisfies the condition (NFLVR) if and only if there exists an equivalent martingale transform measure, i.e. a measure* $\widetilde{\mathsf{P}} \sim \mathsf{P}$ *such that* S *is a* $(\mathcal{F}_t, \widetilde{\mathsf{P}})$*-martingale transform.*

For the proof, see [11].

2. Completeness. In all the considerations concerning the completeness we assume that $\mathcal{F}_0$ is P-trivial and $\mathcal{F} = \bigvee_{t\in[0,T]} \mathcal{F}_t$.

Definition 1.5. A model is *complete* if for each bounded $\mathcal{F}$-measurable function f, there exists a strategy π such that

i) there exist constants a and b such that

$$\mathsf{P}\big(\forall t \in [0,T],\ a \le V_t^\pi \le b\big) = 1;$$

ii) $f = V_T^\pi$.

Definition 1.6. A semimartingale S on $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P}\big)$ has the *predictable representation property* if for any $(\mathcal{F}_t, \mathsf{P})$-local martingale M there exists a $(\mathcal{F}_t)$-predictable S-integrable process H such that

$$M_t = M_0 + \int_0^t H_u dS_u, \quad t \in [0,T].$$

Proposition 1.7. (The Second fundamental theorem of asset pricing). *Suppose that a model satisfies the property (NFLVR). Then the following conditions are equivalent:*

(i) *the model is complete;*

(ii) *there exists a unique equivalent martingale transform measure;*

(iii) *there exists an equivalent measure such that* S *has the predictable representation property with respect to this measure.*

For the proof, see [23].

`Remark`. If the process S is continuous or has (locally) bounded jumps, then the First and the Second fundamental theorems of asset pricing admit simpler formulations with a "local martingale measure" instead of a "martingale transform measure". However, if S has unbounded jumps (and this is the case for models (1.1)–(1.4) considered below), the use of the martingale transforms is essential. The corresponding (counter-)examples are given in [11] and [23]. □

3. Linear and exponential Lévy models. We will say that X is a $(\mathcal{F}_t)$*-Lévy process* if X is a Lévy process, X is $(\mathcal{F}_t)$-adapted and, for any $s \le t$, the increment $X_t - X_s$ is independent of $\mathcal{F}_s$.

The linear Lévy model has the form

$$S_t = S_0 + X_t, \quad t \in [0,T], \tag{1.1}$$

where $S_0 \in \mathbb{R}$ and X is a $(\mathcal{F}_t)$-Lévy process. However, it has the disadvantage that S can take negative values.

A more realistic model is the exponential model:

$$S_t = S_0 e^{X_t}, \quad t \in [0, T], \tag{1.2}$$

where $S_0 \in (0, \infty)$ and X is a $(\mathcal{F}_t)$-Lévy process.

Models of type (1.2) have been investigated in many papers (see, for example, [12], [13], [15], [19]). Note that the Black-Sholes model is a particular case of (1.2).

There are two main advantages of model (1.2):

- By varying the Lévy process X one can achieve a large variety of the marginal distributions for the increments of the logarithmic price process $\ln S$.
- The increments of $\ln S$ in this model are stationary that is in accordance with the real data.

In Section 3 we prove that models (1.1), (1.2) do not satisfy the condition (*NFLVR*) if and only if S is increasing or S is decreasing (Theorem 3.1). In other words, in "almost all" the cases these models have the (*NFLVR*) property.

Moreover, we prove that if model (1.1) (model (1.2)) satisfies the condition (*NFLVR*), then there exists a measure $\widetilde{\mathsf{P}} \sim \mathsf{P}$ such that X is a $(\mathcal{F}_t, \widetilde{\mathsf{P}})$-Lévy process and S is a $(\mathcal{F}_t, \widetilde{\mathsf{P}})$-martingale (Theorem 3.2).

We also prove that if S has the form (1.1) or (1.2) and S is a martingale transform, then S is a martingale (Theorem 3.3).

Furthermore, we prove that models (1.1), (1.2) are complete if and only if X is a Brownian motion with a drift or a Poisson process with a drift (Theorem 3.4). In other words, in "almost all" the cases these models are not complete.

It is well known that if X is a Brownian motion, then it has the predictable representation property (assuming that $\mathcal{F}_t = \mathcal{F}_t^X$). The same result is known for the compensated Poisson process. We prove that these are the only Lévy processes that have the predictable representation property (Corollary 3.5). The similar result is also established for the exponential Lévy processes.

4. Linear and exponential time-changed Lévy models. The main disadvantage of model (1.2) is that in this model the increments of $\ln S$ are independent that is not in accordance with the real financial data. In order to imitate the long memory of the prices, P. Carr, H. Geman, D. Madan and M. Yor [6] proposed to consider the time-changed Lévy processes.

We first describe the linear time-changed Lévy model. Let $\big(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \geq 0}, \mathsf{P}\big)$ be a filtered probability space. Let $(X_t)_{t \geq 0}$ be a $(\mathcal{G}_t)$-Lévy process. Let $(\tau_t)_{t \in [0,T]}$ be an increasing càdlàg process with $\tau_0 = 0$ and such that τ is $(\mathcal{G}_0)$-adapted (in particular, this means that $(\tau_t)_{t \in [0,T]}$ and $(X_t)_{t \geq 0}$ are independent). Set

$$S_t = S_0 + X_{\tau_t}, \quad t \in [0, T], \tag{1.3}$$

where $S_0 \in \mathbb{R}$. The filtration $(\mathcal{F}_t)$ is an arbitrary filtration such that $\mathcal{F}_t^S \subseteq \mathcal{F}_t \subseteq \mathcal{G}_{\tau_t}$. Here, $\mathcal{F}_t^S$ denotes the natural filtration of S: $\mathcal{F}_t^S = \sigma(S_u;\, u \leq t)$.

In the exponential time-changed Lévy model the processes X, τ and the filtration $(\mathcal{F}_t)$ are the same, while S has the form

$$S_t = S_0 e^{X_{\tau_t}}, \quad t \in [0, T], \tag{1.4}$$

where $S_0 \in (0, \infty)$.

Models of the form (1.4) are also considered in the paper [4] by O.E. Barndorff-Nielsen, E. Nicolato and N. Shephard.

The advantages of model (1.4) are:

- By varying the Lévy process X one can get a large variety of marginal distributions for the increments of $\ln S$. In particular, it is possible to imitate the following features of marginals observed in the real prices: skewness and heavy tails (for more information on the statistics of one-dimensional distributions, see [1], [2], [22; Ch. IV]).
- If the process τ has stationary increments, then $\ln S$ has stationary increments. The property that $\ln S$ should have stationary increments is supported by the real data.
- If $\mathsf{E}X_t^2 < \infty$, $\mathsf{E}X_t = 0$ for any $t \geq 0$ and $\mathsf{E}\tau_T < \infty$, then the increments of $\ln S$ over disjoint intervals are uncorrelated. The motivation of this feature from the real financial data is described in [3], [22; Ch. IV].
- By the appropriate choice of the process τ one can achieve a strong correlation of the squared increments of $\ln S$ over disjoint intervals. This effect is observed with the real prices and is termed long-range dependence, clustering, persistence of volatility, etc; see [3], [22; Ch. IV].

In Section 4 we prove that models (1.3), (1.4) do not satisfy the condition (*NFLVR*) if and only S is increasing or S is decreasing (Theorem 4.1).

Moreover, we prove that if model (1.3) (model (1.4)) satisfies the condition (*NFLVR*), then there exists a measure $\widetilde{\mathsf{P}} \sim \mathsf{P}$ such that X is an independently time-changed Lévy process with respect to $\widetilde{\mathsf{P}}$ and S is a $(\mathcal{F}_t, \widetilde{\mathsf{P}})$-martingale (Theorem 4.3).

Furthermore, we prove that models (1.3), (1.4) are complete if and only if τ is deterministic and continuous, while X is a Brownian motion with a drift or a Poisson process with a drift (Theorem 4.4).

2 Known Facts and Preliminary Results

1. Lévy processes. The basic references on Lévy processes are [5] and [21].

Proposition 2.1. *Let $(X_t)_{t \geq 0}$ be a Lévy process with*

$$\mathsf{E}e^{i\lambda X_t} = \exp\Big\{t\Big[i\lambda b - \frac{c}{2}\lambda^2 + \int_{\mathbb{R}} \big(e^{i\lambda x} - 1 - i\lambda x I(|x| \leq a)\big)\nu(dx)\Big]\Big\}.$$

Here, $a \geq 0$. In the case $a = 0$ we assume that

$$\int_{\mathbb{R}} (|x| \wedge 1)\nu(dx) < \infty.$$

(a) *Suppose that*

$$\int_{\{|x|>1\}} e^x \nu(dx) < \infty.$$

Then, for any $t \geq 0$, $\mathsf{E}e^{X_t} < \infty$ *and*

$$\mathsf{E}e^{X_t} = \exp\Big\{t\Big[b + \frac{c}{2} + \int_{\mathbb{R}} \big(e^x - 1 - xI(|x| \leq a)\big)\nu(dx)\Big]\Big\}.$$

(b) *Suppose that*

$$\int_{\{|x|>1\}} |x|\nu(dx) < \infty.$$

Then, for any $t \geq 0$, $\mathsf{E}|X_t| < \infty$ *and*

$$\mathsf{E}X_t = t\Big[b + \int_{\{|x|>a\}} x\nu(dx)\Big].$$

For the proof, see [21; §25].

`Notational remark`. In what follows, the expectation sign E with no subscript will always stand for the expectation with respect to the original measure P. □

Lemma 2.2. (Change of measure for compound Poisson processes). *Let* ν *and* $\widetilde{\nu}$ *be two finite positive measures on* $\mathbb{R}$ *such that* $\widetilde{\nu} \sim \nu$. *Let* $X = (X_t)_{t\in[0,T]}$ *be a* $(\mathcal{F}_t)$*-Lévy process with*

$$\mathsf{E}e^{i\lambda X_t} = \exp\Big\{t\int_{\mathbb{R}} \big(e^{i\lambda x} - 1\big)\nu(dx)\Big\}.$$

Set

$$M_t = \exp\Big\{t\nu(\mathbb{R}) - t\widetilde{\nu}(\mathbb{R}) + \sum_{s\leq t} \ln \rho(\Delta X_s)\Big\},$$

where $\rho = \frac{d\widetilde{\nu}}{d\nu}$. *Then* M *is a* $(\mathcal{F}_t, \mathsf{P})$*-martingale. If we set* $\widetilde{\mathsf{P}}^u = M_u\mathsf{P}$, *where* $u \geq 0$, *then the process* $(X_t)_{t\in[0,u]}$ *is a* $(\mathcal{F}_t, \widetilde{\mathsf{P}}^u)$*-Lévy process with*

$$\mathsf{E}_{\widetilde{\mathsf{P}}^u} e^{i\lambda X_t} = \exp\Big\{t\int_{\mathbb{R}} \big(e^{i\lambda x} - 1\big)\widetilde{\nu}(dx)\Big\}, \quad t \in [0, u]. \tag{2.1}$$

`Proof`. As $\nu(\mathbb{R}) < \infty$, the process X has a.s. only a finite number of jumps (it is a compound Poisson process) and thus, M is defined correctly. For any $0 \leq s \leq t \leq u$, we have

$$\begin{aligned}\mathsf{E}[M_t \mid \mathcal{F}_s] &= M_s \exp\{(t-s)(\nu(\mathbb{R}) - \widetilde{\nu}(\mathbb{R}))\}\ \mathsf{E} \prod_{s<r\leq t} \rho(\Delta X_r) \\ &= M_s \exp\{-(t-s)\widetilde{\nu}(\mathbb{R})\} \sum_{k=0}^{\infty} \frac{\big((t-s)\nu(\mathbb{R})\big)^k}{k!} \Big(\int_{\mathbb{R}} \frac{\rho(x)}{\nu(\mathbb{R})}\nu(dx)\Big)^k = M_s.\end{aligned}$$

In order to prove that $(X_t)_{t\in[0,u]}$ is a $(\mathcal{F}_t,\widetilde{\mathsf{P}}^u)$-Lévy process and it has the characteristic function given by (2.1), it is sufficient to note that, for any $0\le s\le t\le u$ and $\lambda\in\mathbb{R}$,

$$\begin{aligned}
&\mathsf{E}_{\widetilde{\mathsf{P}}^u}\left[e^{i\lambda(X_t-X_s)}\,\middle|\,\mathcal{F}_s\right]=\mathsf{E}\left[e^{i\lambda(X_t-X_s)}\frac{M_t}{M_s}\,\middle|\,\mathcal{F}_s\right]\\
&=\mathsf{E}\exp\Big\{\sum_{s<r\le t}i\lambda\Delta X_r+(t-s)(\nu(\mathbb{R})-\widetilde{\nu}(\mathbb{R}))+\sum_{s<r\le t}\ln\rho(\Delta X_r)\Big\}\\
&=\exp\{-(t-s)\widetilde{\nu}(\mathbb{R})\}\sum_{k=0}^{\infty}\frac{\big((t-s)\nu(\mathbb{R})\big)^k}{k!}\Big(\int_{\mathbb{R}}\frac{e^{i\lambda x+\ln\rho(x)}}{\nu(\mathbb{R})}\nu(dx)\Big)^k\\
&=\exp\Big\{-(t-s)\widetilde{\nu}(\mathbb{R})+(t-s)\int_{\mathbb{R}}e^{i\lambda x+\ln\rho(x)}\nu(dx)\Big\}\\
&=\exp\Big\{(t-s)\int_{\mathbb{R}}\big(e^{i\lambda x}-1\big)\widetilde{\nu}(dx)\Big\}.
\end{aligned}$$

□

Lemma 2.3. *Let $X=(X_t)_{t\in[0,T]}$ be a $(\mathcal{F}_t)$-Lévy process with*

$$\mathsf{E}e^{i\lambda X_t}=\exp\Big\{t\Big[i\lambda b-\frac{c}{2}\lambda^2+\int_{\mathbb{R}}\big(e^{i\lambda x}-1-i\lambda xI(|x|\le a)\big)\nu(dx)\Big]\Big\},$$

where $a>0$. Suppose that $\widetilde{\nu}$ is a positive measure such that
(i) $\widetilde{\nu}=\nu$ *on* $\{|x|\le a\}$;
(ii) $\widetilde{\nu}\sim\nu$ *on* $\{|x|>a\}$;
(iii) $\widetilde{\nu}(\{|x|>a\})<\infty$.
Then there exists a process $(M_t)_{t\ge 0}$ such that
(i) *M is a strictly positive $(\mathcal{F}_t,\mathsf{P})$-martingale;*
(ii) *M is $(\mathcal{F}_t^X)$-adapted;*
(iii) *if we set $\widetilde{\mathsf{P}}^u=M_u\mathsf{P}$, then, for any $u\ge 0$, the process $(X_t)_{t\in[0,u]}$ is a $(\mathcal{F}_t,\widetilde{\mathsf{P}}^u)$-Lévy process with*

$$\mathsf{E}_{\widetilde{\mathsf{P}}^u}e^{i\lambda X_t}=\exp\Big\{t\Big[i\lambda b-\frac{c}{2}\lambda^2+\int_{\mathbb{R}}\big(e^{i\lambda x}-1-i\lambda xI(|x|\le a)\big)\widetilde{\nu}(dx)\Big]\Big\}.\qquad(2.2)$$

`Proof`. Set

$$\begin{aligned}
X_t^1&=\sum_{s\le t}\Delta X_s\,I(|\Delta X_s|>a),\\
X_t^2&=X_t-X_t^1.
\end{aligned}$$

It follows from the Lévy-Itô decomposition (see [21; §19]) that X^1 and X^2 are independent Lévy processes with

$$\begin{aligned}
\mathsf{E}e^{i\lambda X_t^1}&=\exp\Big\{t\int_{\{|x|>a\}}\big(e^{i\lambda x}-1\big)\nu(dx)\Big\},\\
\mathsf{E}e^{i\lambda X_t^2}&=\exp\Big\{t\Big[i\lambda b-\frac{c}{2}\lambda^2+\int_{\{|x|\le a\}}\big(e^{i\lambda x}-1-i\lambda x\big)\nu(dx)\Big]\Big\}.
\end{aligned}$$

Moreover, it is seen from the explicit form of X^1 and X^2 that the two-dimensional process (X^1, X^2) is a $(\mathcal{F}_t)$-Lévy process.

Set

$$M_t = \exp\Big\{t\nu(\{|x| > a\}) - t\widetilde{\nu}(\{|x| > a\}) + \sum_{s\le t} \ln \rho(\Delta X^1_s)\Big\},$$

where $\rho = \frac{d\widetilde{\nu}}{d\nu}$. Similarly as in the previous lemma we verify that M is a $(\mathcal{F}_t, \mathsf{P})$-martingale. According to Lemma 2.2, for any $u \ge 0$, the process $(X^1_t)_{t\in[0,u]}$ is a $(\mathcal{F}_t, \widetilde{\mathsf{P}})$-Lévy process with

$$\mathsf{E}_{\widetilde{\mathsf{P}}^u} e^{i\lambda X^1_t} = \exp\Big\{t \int\limits_{\{|x|>a\}} \big(e^{i\lambda x} - 1\big)\widetilde{\nu}(dx)\Big\}, \quad t \in [0, u].$$

For any $0 \le s \le t \le u$ and $\lambda_1, \lambda_2 \in \mathbb{R}$, we have

$$\begin{aligned}\mathsf{E}_{\widetilde{\mathsf{P}}^u}\big[e^{i\lambda_1(X^1_t - X^1_s) + i\lambda_2(X^2_t - X^2_s)} \,\big|\, \mathcal{F}_s\big] &= \mathsf{E}\bigg[e^{i\lambda_1(X^1_t - X^1_s) + i\lambda_2(X^2_t - X^2_s)} \frac{M_t}{M_s} \,\bigg|\, \mathcal{F}_s\bigg]\\ &= \mathsf{E}\bigg[e^{i\lambda_1(X^1_t - X^1_s) + i\lambda_2(X^2_t - X^2_s)} \frac{M_t}{M_s}\bigg]\\ &= \mathsf{E}\bigg[e^{i\lambda_1(X^1_t - X^1_s)} \frac{M_t}{M_s}\bigg] \mathsf{E} e^{i\lambda_2(X^2_t - X^2_s)}.\end{aligned}$$

We used here the independence of $e^{i\lambda_1(X^1_t - X^1_s) + i\lambda_2(X^2_t - X^2_s)} \frac{M_t}{M_s}$ and $\mathcal{F}_s$ (it follows from the fact that (X^1, X^2) is a $(\mathcal{F}_t)$-Lévy process). Hence, the two-dimensional increment $(X^1_t - X^1_s, X^2_t - X^2_s)$ is $\widetilde{\mathsf{P}}^u$-independent of $\mathcal{F}_s$. Taking $s = 0$, $\lambda_1 = 0$, we conclude that $\mathsf{E}_{\widetilde{\mathsf{P}}^u} e^{i\lambda X^2_t} = \mathsf{E} e^{i\lambda X^2_t}$. Furthermore, X^1 and X^2 are $\widetilde{\mathsf{P}}^u$-independent since X^1 and X^2 are independent and the density $\frac{d\widetilde{\mathsf{P}}^u}{d\mathsf{P}}$ is a functional of X^1. Thus, $(X_t)_{t\in[0,u]}$ is a $(\mathcal{F}_t, \widetilde{\mathsf{P}}^u)$-Lévy process, and its characteristic function is given by (2.2). □

2. Jump measures and compensators. If $(X_t)_{t\in[0,T]}$ is a semimartingale, then its *jump measure* is a random measure μ (i.e. a family $\{\mu(\omega, dt, dx);\, \omega \in \Omega\}$ of measures on $\mathcal{B}([0, T] \times \mathbb{R})$) defined by

$$\mu(\omega, A) = \sum_{s\le T} I\big(\Delta X_s(\omega) \ne 0,\ (s, \Delta X_s(\omega)) \in A\big), \quad A \in \mathcal{B}([0, T] \times \mathbb{R}).$$

The *compensator of the jump measure* of X is a predictable (for the definition, see [16; Ch. II, (1.6)]) random measure $\eta = \{\eta(\omega, dt, dx);\, \omega \in \Omega\}$ such that, for any nonnegative $\mathcal{P} \times \mathcal{B}(\mathbb{R})$-measurable function $W = W(\omega, t, x)$ (here, $\mathcal{P}$ denotes the predictable σ-field on $\Omega \times [0, T]$), one has

$$\mathsf{E} \int\limits_{[0,T]\times\mathbb{R}} W(\omega, t, x)\mu(\omega, dt, dx) = \mathsf{E} \int\limits_{[0,T]\times\mathbb{R}} W(\omega, t, x)\eta(\omega, dt, dx). \tag{2.3}$$

If X is a $(\mathcal{F}_t)$-Lévy process, then it is a $(\mathcal{F}_t)$-semimartingale and its compensator of the jump measure has the form:

$$\eta(\omega, dt, dx) = dt \times \nu(dx),$$

where ν is the Lévy measure of X (in particular, $\eta(\omega, dt, dx)$ is the same for all ω's).

For more information on compensators, see [16; Ch. II, §1a].

3. Martingale transforms. We will need the following property of the stochastic integrals.

Proposition 2.4. (Associativity). *Let Z be a semimartingale and H be a predictable Z-integrable process. Set $Y_t = \int_0^t H_s dZ_s$. Then a predictable process K is Y-integrable if and only if KH is Z-integrable. In this case*

$$\int_0^t K_s dY_s = \int_0^t (K_s H_s) dZ_s.$$

For the proof, see [23].

Lemma 2.5. *Let X and Y be two martingale transforms on the same filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P})$. Then their sum is again a martingale transform.*

`Proof`. By the definition, there exist local martingales M, N and processes H, K such that

$$X_t = X_0 + \int_0^t H_s dM_s, \quad Y_t = Y_0 + \int_0^t K_s dN_s.$$

Set

$$H_t^1 = H_t I(|H_t| \le 1) + I(|H_t| > 1),$$
$$H_t^2 = I(|H_t| \le 1) + H_t I(|H_t| > 1).$$

Then $H_t = H_t^1 H_t^2$ and, in view of Proposition 2.4, we may write

$$X_t = X_0 + \int_0^t H_s^2 d\overline{M}_s,$$

where $\overline{M}_t = \int_0^t H_s^1 dM_s$. Note that $|H_t^1| \le 1$, and therefore, $\overline{M}$ is a local martingale.

In a similar way we define K^1, K^2 and $\overline{N}$. Set

$$\widetilde{M}_t = \int_0^t \frac{1}{K_s^2} d\overline{M}_s, \quad \widetilde{N}_t = \int_0^t \frac{1}{H_s^2} d\overline{N}_s.$$

In view of the inequalities $|H^2| \ge 1$, $|K^2| \ge 1$, the processes $\widetilde{M}$, $\widetilde{N}$ are local martingales. By Proposition 2.4,

$$X_t = X_0 + \int_0^t H_s^2 K_s^2 d\widetilde{M}_s,$$
$$Y_t = Y_0 + \int_0^t H_s^2 K_s^2 d\widetilde{N}_s.$$

Hence,

$$X_t + Y_t = X_0 + Y_0 + \int_0^t H_s^2 K_s^2 d(\widetilde{M} + \widetilde{N})_s.$$

As a sum of two local martingales is again a local martingale, we conclude that $X + Y$ is a martingale transform. □

Lemma 2.6. *Suppose that Z is a martingale transform and is an increasing process. Then, for each t, $Z_t = Z_0$ a.s.*

`Proof`. By the definition, there exist a local martingale M and a M-integrable process H such that

$$Z_t = Z_0 + \int_0^t H_s dM_s.$$

In view of Proposition 2.4, we can write

$$Z_t = Z_0 + \int_0^t \widetilde{H}_s d\widetilde{M}_s,$$

where

$$\widetilde{H}_t = |H_t| + I(H_t = 0),$$
$$\widetilde{M}_t = \int_0^t \operatorname{sgn} H_s dM_s$$

(we put $\operatorname{sgn} 0 = 0$). The process $\widetilde{M}$ is a local martingale since $\operatorname{sgn} H$ is bounded. Furthermore, by Proposition 2.4,

$$\widetilde{M}_t = \widetilde{M}_0 + \int_0^t \frac{1}{\widetilde{H}_s} dZ_s.$$

As $\widetilde{H} > 0$ and Z is an increasing process, the integral in the last equality is also an increasing process. As $\widetilde{M}$ is a local martingale, we deduce that, for each t, $\widetilde{M}_t = \widetilde{M}_0$ a.s. This leads to the desired statement. □

Proposition 2.7. *Let $(Z_t)_{t\in[0,T]}$ be a semimartingale on $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P}\big)$, whose compensator of the jump measure is given by*

$$\eta(\omega, dt, dx) = K(\omega, t, dx) dA_t(\omega),$$

where A is a predictable increasing process and K is a transition kernel from $(\Omega\times[0,T], \mathcal{P})$ (here, $\mathcal{P}$ denotes the predictable σ-field) to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Suppose that Z is a martingale transform. Then

$$\int_{\mathbb{R}} |x| \wedge x^2 K(\omega, t, dx) < \infty$$

for $\mathsf{P} \times dA$-almost all (ω, t).

For the proof, see [18; Lemma 3].

3 Linear and Exponential Lévy Models

1. The results. In Theorem 3.1 we exclude the trivial case $X \equiv 0$.

Theorem 3.1. (No-arbitrage). *Model* (1.1) (*model* (1.2)) *does not satisfy the condition* (*NFLVR*) *in the following cases only:*
(i) *S is increasing;*
(ii) *S is decreasing.*

Theorem 3.2. *Suppose that in model* (1.1) (*model* (1.2)) S *is neither increasing nor decreasing. Then there exists a measure* $\widetilde{\mathsf{P}} \sim \mathsf{P}$ *such that* X *is a* $(\mathcal{F}_t, \widetilde{\mathsf{P}})$*-Lévy process and* S *is a* $(\mathcal{F}_t, \widetilde{\mathsf{P}})$*-martingale.*

`Remark`. The statement of Theorem 3.2 for model (1.2) was proved in the paper [17] by P. Jakubėnas. However, we will present an alternative proof here. □

Theorem 3.3. *Suppose that in model* (1.1) (*model* (1.2)) S *is a* $(\mathcal{F}_t, \mathsf{P})$*-martingale transform. Then* S *is a* $(\mathcal{F}_t, \mathsf{P})$*-martingale.*

`Remark`. It follows from Theorem 3.3 that if X is a $(\mathcal{F}_t, \mathsf{P})$-Lévy process and is a $(\mathcal{F}_t, \mathsf{P})$-local martingale, then X is a $(\mathcal{F}_t, \mathsf{P})$-martingale. This result can be found in the paper [24] by R. Sidibe. □

In the statements below B denotes a standard Brownian motion and N denotes a standard Poisson process.

Theorem 3.4. (Completeness). *Suppose that* S *is neither increasing nor decreasing and* $\mathcal{F}_t = \mathcal{F}_t^S$. *Model* (1.1) (*model* (1.2)) *is complete in the following cases only:*
(i) $X_t = \alpha B_t + \beta T$, *where* $\alpha, \beta \in \mathbb{R}$ (*the case* $\alpha = 0$, $\beta \neq 0$ *is excluded*);
(ii) $X_t = \alpha N_{\gamma t} + \beta t$, *where* $\alpha, \beta \in \mathbb{R}$, $\gamma > 0$ *and* $\alpha\beta < 0$.

2. An application to the predictable representation property. The above results lead to

Corollary 3.5. *Suppose that in model* (1.1) (*model* (1.2)) $\mathcal{F}_t = \mathcal{F}_t^S$.
(a) *In model* (1.1) S *has the predictable representation property* (*with respect to the original measure* P) *in the following cases only:*
(i) $X_t = \alpha B_t$, *where* $\alpha \in \mathbb{R}$;
(ii) $X_t = \alpha N_{\gamma t} - \alpha\gamma t$, *where* $\alpha \in \mathbb{R}$, $\gamma > 0$.
(b) *In model* (1.2) S *has the predictable representation property* (*with respect to the original measure* P) *in the following cases only:*
(i) $X_t = \alpha B_t - \frac{\alpha^2}{2} t$, *where* $\alpha \in \mathbb{R}$;
(ii) $X_t = \alpha N_{\gamma t} - (e^\alpha - 1)\gamma t$, *where* $\alpha \in \mathbb{R}$, $\gamma > 0$.

`Proof`. We will give the proof only for model (1.1).

First, it is well known that in both cases (i) and (ii) X has the predictable representation property (see [20; Ch. V, (3.4)] for (i) and [16; Ch. III, (4.37)] for (ii)).

Now, suppose that X has the predictable representation property. By the Second fundamental theorem of asset pricing, the model is complete. By Theorem 3.4, X is either a Brownian motion with a drift or a Poisson process with a drift. Suppose first that $X_t = \alpha B_t + \beta t$ and $\beta \neq 0$. Take a non-degenerate martingale M and consider its representation:

$$M_t = M_0 + \int_0^t H_s dX_s = M_0 + \alpha \int_0^t H_s dB_s + \beta \int_0^t H_s ds, \quad t \in [0, T].$$

The process $\int_0^t H_s dB_s$ is a local martingale. Hence, $\int_0^t H_s ds$ is a local martingale. Since this process is continuous and has finite variation, it should be equal to zero. This means that $\int_0^T |H_s|ds = 0$ a.s. Hence, $\int_0^t H_s dB_s = 0$ which means that M is degenerate. The contradiction shows that $\beta = 0$ and hence X has the form (i).

In a similar way we consider the case, where X is a Poisson process with a drift. □

Remarks. (i) If X has the form (i) or (ii) of Corollary 3.5 (a) and is a $(\mathcal{F}_t)$-Lévy process, then X may not have the predictable representation property on $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P}\big)$. Consider, for example, the process $X_t = B_t^1$, where $B = (B^1, B^2)$ is a two-dimensional Brownian motion, and take $\mathcal{F}_t = \mathcal{F}_t^B$. Then the $(\mathcal{F}_t)$-martingale $M_t = B_t^2$ cannot be represented as a stochastic integral $\int_0^t K_s dX_s$ since, for any predictable X-integrable process K, $\langle M, \int_0^\cdot K_s dX_s\rangle_t = 0$, while $\langle M, M\rangle_t = t$.

(ii) However, in some cases X may have the predictable representation property even if $(\mathcal{F}_t)$ is strictly larger than $(\mathcal{F}_t^X)$. Consider, for example, the process

$$X_t = \int_0^t \operatorname{sgn} B_s dB_s,$$

where B is a standard linear Brownian motion. Take $\mathcal{F}_t = \mathcal{F}_t^B$. Then $\mathcal{F}_t^X = \mathcal{F}_t^{|B|} \subset \mathcal{F}_t^B$ (see [20; Ch. VI, (2.2)]). On the other hand, any $(\mathcal{F}_t)$-local martingale M can be represented as

$$M_t = M_0 + \int_0^t K_s dB_s = M_0 + \int_0^t K_s \operatorname{sgn} B_s dX_s.$$

(iii) The predictable representation property means that each $(\mathcal{F}_t^X)$-local martingale is representable as a stochastic integral with respect to X. However, for any Lévy process X, any $(\mathcal{F}_t^X)$-local martingale can be represented as a sum of a stochastic integral with respect to the continuous martingale part of X and a stochastic integral with respect to the compensated jump measure of X (see [16; Ch. III, (4.34)]).

(iv) C.S. Chou and P.-A. Meyer [8] proved that if $(X_t)_{t\ge 0}$ is a Lévy process that is not a Brownian motion with a drift or a compensated Poisson process with a drift, then there exists no $(\mathcal{F}_t^X)$-martingale Y such that all the $(\mathcal{F}_t^X)$-local martingales are stochastic integrals with respect to Y.

(v) M. Yor and J. de Sam Lazaro [25; Appendix] proved the following result. Suppose that $(X_t)_{t\ge 0}$ is a martingale such that, for any $s \ge 0$, $\text{Law}(X_t;\, t \ge 0) = \text{Law}(X_{t+s} - X_s;\, t \ge 0)$. Set $\mathcal{F}_t = \mathcal{F}_t^X$. Then X has the predictable representation property if and only if X is a Brownian motion or a compensated Poisson process. □

3. The proofs. We will prove only the statements related to model (1.2). The statements related to model (1.1) are verified in a similar (and even simpler) way.

Lemma 3.6. *Suppose that X is neither increasing nor decreasing. Then there exists a process $(M_t)_{t\ge 0}$ such that*

(i) *M is a strictly positive $(\mathcal{F}_t, \mathsf{P})$-martingale;*

(ii) *M is $(\mathcal{F}_t^X)$-adapted;*

(iii) *if we set $\widetilde{\mathsf{P}}^u = M_u \mathsf{P}$, then, for any $u \ge 0$, the process $(X_t)_{t\in[0,u]}$ is a $(\mathcal{F}_t, \widetilde{\mathsf{P}}^u)$-Lévy process and the process $(e^{X_t})_{t\in[0,u]}$ is a $(\mathcal{F}_t, \widetilde{\mathsf{P}}^u)$-martingale.*

Proof. Let ν denote the Lévy measure of X.

Case I. Suppose that there exists $a > 0$ such that $\nu((-\infty, -a)) > 0$ and $\nu((a, \infty)) > 0$. The characteristic function of X can be written as follows:

$$\mathsf{E}e^{i\lambda X_t} = \exp\Big\{t\Big[i\lambda b - \frac{c}{2}\lambda^2 + \int_{\mathbb{R}}\big(e^{i\lambda x} - 1 - i\lambda x I(|x| \le a)\big)\nu(dx)\Big]\Big\}.$$

There exists a positive measure $\widetilde{\nu}$ such that

$$\widetilde{\nu} = \nu \text{ on } \{|x| \le a\}, \tag{3.1}$$
$$\widetilde{\nu} \sim \nu \text{ on } \{|x| > a\}, \tag{3.2}$$
$$\widetilde{\nu}(\{|x| > a\}) < \infty, \tag{3.3}$$
$$\int_{\{|x|>a\}} e^x \widetilde{\nu}(dx) < \infty, \tag{3.4}$$
$$b + \frac{c}{2} + \int_{\mathbb{R}}\big(e^x - 1 - xI(|x| \le a)\big)\widetilde{\nu}(dx) = 0. \tag{3.5}$$

In order to construct such a measure, it is sufficient to take first a rapidly decreasing at infinity function $\overline{\rho}$ such that $\overline{\rho} > 0$, $\overline{\rho} = 1$ on $[-a, a]$ and the measure $\overline{\nu} = \overline{\rho}\nu$ satisfies conditions (3.1)–(3.4). Then, using the density of the form $\widetilde{\rho} = \alpha I(x \le a) + \beta I(x > a)$ with $\alpha > 0$, one can construct a measure $\widetilde{\nu} = \widetilde{\rho}\overline{\nu}$ that satisfies conditions (3.1)–(3.5).

According to Lemma 2.3, there exists a process $(M_t)_{t\ge 0}$ satisfying conditions (i), (ii) and such that, for any $u \ge 0$, $(X_t)_{t\in[0,u]}$ is a $(\mathcal{F}_t, \widetilde{\mathsf{P}}^u)$-Lévy process with

$$\mathsf{E}_{\widetilde{\mathsf{P}}^u} e^{i\lambda X_t} = \exp\Big\{t\Big[i\lambda b - \frac{c}{2}\lambda^2 + \int_{\mathbb{R}}\big(e^{i\lambda x} - 1 - i\lambda x I(|x| \le a)\big)\widetilde{\nu}(dx)\Big]\Big\}, \quad t \in [0, u].$$

By Proposition 2.1 (a) combined with equality (3.5), $\mathsf{E}_{\widetilde{\mathsf{P}}^u} e^{X_t} = 1$ for any $t \in [0, u]$. Consequently, for any $0 \le w \le v \le u$,

$$\mathsf{E}_{\widetilde{\mathsf{P}}^u}[S_v \mid \mathcal{F}_w] = S_w \mathsf{E}_{\widetilde{\mathsf{P}}^u}\big[e^{X_v - X_w} \bigm| \mathcal{F}_w\big] = S_w \mathsf{E}_{\widetilde{\mathsf{P}}^u} e^{X_v - X_w} = S_w.$$

Thus, $(S_t)_{t\in[0,u]}$ is a $(\mathcal{F}_t, \widetilde{\mathsf{P}}^u)$-martingale.

Case II. Suppose that $\nu \ne 0$, ν is concentrated on $(0, \infty)$ and $\int_0^1 x\nu(dx) = \infty$. The characteristic function of X can be written as

$$\mathsf{E}e^{i\lambda X_t} = \exp\Big\{t\Big[i\lambda b(a) - \frac{c}{2}\lambda^2 + \int_0^\infty \big(e^{i\lambda x} - 1 - i\lambda x I(|x| \le a)\big)\nu(dx)\Big]\Big\},$$

where $a \in (0, 1]$ and

$$b(a) = b(1) - \int_{\{a<x\le 1\}} x\nu(dx).$$

Due to the condition $\int_0^1 x\nu(dx) = \infty$, we can take $a \in (0, 1]$ such that $\nu((a, \infty)) > 0$ and

$$b(a) + \frac{c}{2} + \int_{\{x\le a\}} \big(e^x - 1 - x\big)\nu(dx) < 0.$$

Obviously, there exists a positive measure $\widetilde{\nu}$ that satisfies properties (3.1)–(3.4) and the following one:

$$b(a)+\frac{c}{2}+\int_0^\infty \bigl(e^x-1-xI(x\le a)\bigr)\nu(dx)=0. \tag{3.6}$$

By Lemma 2.3, there exists a process $(M_t)_{t\ge 0}$ satisfying conditions (i), (ii) and such that, for any $u\ge 0$, $(X_t)_{t\in[0,u]}$ is a $(\mathcal{F}_t,\widetilde{\mathsf{P}}^u)$-Lévy process with

$$\mathsf{E}_{\widetilde{\mathsf{P}}^u}e^{i\lambda X_t}=\exp\Bigl\{t\Bigl[i\lambda b-\frac{c}{2}\lambda^2+\int_{\mathbb{R}}\bigl(e^{i\lambda x}-1-i\lambda xI(|x|\le a)\bigr)\widetilde{\nu}(dx)\Bigr]\Bigr\},\quad t\in[0,u].$$

Proposition 2.1 (a) combined with equality (3.6) shows that $(S_t)_{t\in[0,u]}$ is a $(\mathcal{F}_t,\widetilde{\mathsf{P}}^u)$-martingale.

Case III. Suppose that $\nu\ne 0$, ν is concentrated on $(0,\infty)$, $\int_0^1 x\nu(dx)<\infty$ and $c\ne 0$, where c stands for the diffusion coefficient of X. Then the characteristic function of X can be written as

$$\mathsf{E}e^{i\lambda X_t}=\exp\Bigl\{t\Bigl[i\lambda b-\frac{c}{2}\lambda^2+\int_{\mathbb{R}}\bigl(e^{i\lambda x}-1\bigr)\nu(dx)\Bigr]\Bigr\}.$$

There exists a positive measure $\widetilde{\nu}$ that satisfies properties (3.1)–(3.4) with $a=1$. Take $\widetilde{b}$ such that

$$\widetilde{b}+\frac{c}{2}+\int_{\mathbb{R}}(e^x-1)\widetilde{\nu}(dx)=0. \tag{3.7}$$

By the Lévy-Itô decomposition, the process X can be represented as a sum $X=X^1+X^2$, where X^1, X^2 are independent Lévy processes,

$$\mathsf{E}e^{i\lambda X_t^1}=\exp\Bigl\{t\Bigl[i\lambda b+\int_{\mathbb{R}}\bigl(e^{i\lambda x}-1\bigr)\nu(dx)\Bigr]\Bigr\}$$

and $X_t^2=\sqrt{c}B_t$. Moreover, the two-dimensional process (X^1,X^2) is a $(\mathcal{F}_t,\mathsf{P})$-Lévy process (this follows from the explicit form of the Lévy-Itô decomposition; see [21; Theorem 19.2]).

Set

$$\begin{aligned}M_t&=\exp\Bigl\{t\nu(\{|x|>1\})-t\widetilde{\nu}(\{|x|>1\})+\sum_{s\le t}\ln\rho(\Delta X_s^1)I(|\Delta X_s^1|>1)\Bigr\}\\&\quad\times\exp\Bigl\{\frac{\widetilde{b}-b}{\sqrt{c}}B_t-\frac{(\widetilde{b}-b)^2}{2c}t\Bigr\}=M_t^1M_t^2,\end{aligned}$$

where $\rho=\frac{d\widetilde{\nu}}{d\nu}$.

Since (X^1,X^2) is a $(\mathcal{F}_t,\mathsf{P})$-Lévy process, we can write, for $s\le t$,

$$\mathsf{E}[M_t\mid\mathcal{F}_s]=M_s\mathsf{E}\biggl[\frac{M_t}{M_s}\biggm|\mathcal{F}_s\biggr]=M_s\mathsf{E}\frac{M_t}{M_s}=M_s.$$

Thus, M is a $(\mathcal{F}_t,\mathsf{P})$-martingale. Condition (ii) is trivially satisfied. Furthermore, for any $0 \le s \le t \le u$ and $\lambda \in \mathbb{R}$, we have

$$\begin{aligned}
&\mathsf{E}_{\widetilde{\mathsf{P}}^u}\big[e^{i\lambda(X_t-X_s)}\,\big|\,\mathcal{F}_s\big]\\
&= \mathsf{E}\bigg[e^{i\lambda(X_t^1-X_s^1)}\frac{M_t^1}{M_s^1}e^{i\lambda(X_t^2-X_s^2)}\frac{M_t^2}{M_s^2}\,\bigg|\,\mathcal{F}_s\bigg]\\
&= \mathsf{E}\bigg[e^{i\lambda(X_t^1-X_s^1)}\frac{M_t^1}{M_s^1}e^{i\lambda(X_t^2-X_s^2)}\frac{M_t^2}{M_s^2}\bigg]\\
&= \mathsf{E}\bigg[e^{i\lambda(X_t^1-X_s^1)}\frac{M_t^1}{M_s^1}\bigg]\mathsf{E}\bigg[e^{i\lambda(X_t^2-X_s^2)}\frac{M_t^2}{M_s^2}\bigg]\\
&= \exp\Big\{(t-s)\Big[i\lambda b+\int_{\mathbb{R}}\big(e^{i\lambda x}-1\big)\widetilde{\nu}(dx)\Big]\Big\}\exp\Big\{(t-s)\Big[i\lambda\widetilde{b}-i\lambda b-\frac{c}{2}\lambda^2\Big]\Big\}.
\end{aligned}$$

Thus, $(X_t)_{t\in[0,u]}$ is a $(\mathcal{F}_t,\widetilde{\mathsf{P}}^u)$-Lévy process. By letting $s=0$, we see that

$$\mathsf{E}_{\widetilde{\mathsf{P}}^u}e^{i\lambda X_t} = \exp\Big\{t\Big[i\lambda\widetilde{b}-\frac{c}{2}\lambda^2+\int_{\mathbb{R}}\big(e^{i\lambda x}-1\big)\widetilde{\nu}(dx)\Big]\Big\}.$$

Proposition 2.1 (a) combined with equality (3.7) shows that $(S_t)_{t\in[0,u]}$ is a $(\mathcal{F}_t,\widetilde{\mathsf{P}}^u)$-martingale.

Case IV. Suppose that $\nu \neq 0$, ν is concentrated on $(0,\infty)$, $\int_0^1 x\nu(dx) < \infty$, $c=0$ and $b<0$, where b is given by (3.7). Take $a>0$ such that $\nu((a,\infty))>0$ and

$$b+\int\limits_{\{x\le a\}}\big(e^x-1\big)\nu(dx)<0.$$

Obviously, there exists a positive measure $\widetilde{\nu}$ that satisfies properties (3.1)–(3.4) and the following one:

$$b+\int_0^\infty\big(e^x-1\big)\widetilde{\nu}(dx)=0. \tag{3.8}$$

By Lemma 2.3, there exists a process $(M_t)_{t\ge0}$ satisfying conditions (i), (ii) and such that, for any $u\ge0$, $(X_t)_{t\in[0,u]}$ is a $(\mathcal{F}_t,\widetilde{\mathsf{P}}^u)$-Lévy process with

$$\mathsf{E}_{\widetilde{\mathsf{P}}^u}e^{i\lambda X_t} = \exp\Big\{t\Big[i\lambda b+\int_0^\infty\big(e^{i\lambda x}-1\big)\widetilde{\nu}(dx)\Big]\Big\},\quad t\in[0,u].$$

Proposition 2.1 (a) combined with equality (3.8) shows that $(S_t)_{t\in[0,u]}$ is a $(\mathcal{F}_t,\widetilde{\mathsf{P}}^u)$-martingale.

Case V. Suppose that $\nu\neq0$, ν is concentrated on $(0,\infty)$, $\int_0^1 x\nu(dx)<\infty$, $c=0$ and $b\ge0$. In this case X (and consequently, S) is an increasing process. So, the conditions of the lemma are not satisfied.

Case VI. Suppose that $\nu=0$. In this case $X_t=\alpha B_t+\beta t$, where $\alpha,\beta\in\mathbb{R}$ and B is a standard Brownian motion. For such a process X, the desired statement is an easy consequence of Girsanov's theorem.

In a similar way as above we consider the cases where $\nu\neq0$ and ν is concentrated on $(-\infty,0)$. □

`Proof of Theorem 3.1 (for model (1.2)).` Suppose that S is increasing and the condition (*NFLVR*) is satisfied. By the First fundamental theorem of asset pricing, there exists a measure $\widetilde{\mathsf{P}} \sim \mathsf{P}$ such that S is a $(\mathcal{F}_t, \widetilde{\mathsf{P}})$-martingale transform. The process S is increasing also under the measure $\widetilde{\mathsf{P}}$, so, by Lemma 2.6, $S \equiv S_0$. But this case is excluded. As a result, the condition (*NFLVR*) is not satisfied.

Suppose that S is neither increasing nor decreasing. By Lemma 3.6, there exists a measure $\widetilde{\mathsf{P}} \sim \mathsf{P}$ such that $(S_t)_{t\in[0,T]}$ is a $(\mathcal{F}_t, \widetilde{\mathsf{P}})$-martingale. Hence, S is also a $(\mathcal{F}_t, \widetilde{\mathsf{P}})$-martingale transform, and thus, the condition (*NFLVR*) is satisfied. □

`Proof of Theorem 3.2 (for model (1.2)).` This statement follows immediately from Lemma 3.6.

`Proof of Theorem 3.3 (for model (1.2)).` Let η denote the compensator of the jump measure of the process S and let ν denote the Lévy measure of X. Using the definition of the compensator, one can verify that

$$\eta(\omega, dt, dx) = K(\omega, t, dx)dt,$$

where $K(\omega, t, dx)$ is the image of the measure $\nu(dx)$ under the map

$$\mathbb{R} \ni x \longmapsto e^{X_{t-}(\omega)}(e^x - 1) \in \mathbb{R}.$$

Suppose that S is a martingale transform. By Proposition 2.7, we have

$$\int_{\mathbb{R}} |x| \wedge x^2 K(\omega, t, dx) < \infty$$

for $\mathsf{P} \times dt$-almost all (ω, t). Using the explicit form of K described above, we get

$$\int_{\mathbb{R}} e^{X_{t-}(\omega)}|e^x - 1| \wedge e^{2X_{t-}(\omega)}(e^x - 1)^2 \nu(dx) < \infty$$

for $\mathsf{P} \times dt$-almost every (ω, t). Hence,

$$\int_{\{|x|>1\}} |e^x - 1|\nu(dx) < \infty$$

that leads to

$$\int_{\{|x|>1\}} e^x \nu(dx) < \infty. \tag{3.9}$$

The characteristic function of X can be written as

$$\mathsf{E}e^{i\lambda X_t} = \exp\Big\{t\Big[i\lambda b - \frac{c}{2}\lambda^2 + \int_{\mathbb{R}} \big(e^{i\lambda x} - 1 - i\lambda x I(|x| \le 1)\big)\nu(dx)\Big]\Big\}.$$

By Proposition 2.1 (a) combined with equality (3.9), $\mathsf{E}e^{X_t} = e^{\alpha t}$, where

$$\alpha = b + \frac{c}{2} + \int_{\mathbb{R}} \big(e^x - 1 - xI(|x| \le 1)\big)\nu(dx).$$

Then the process $M_t = e^{X_t - \alpha t}$ is a $(\mathcal{F}_t, \mathsf{P})$-martingale.

Suppose that $\alpha \neq 0$. We may write $S_t = S_0 e^{\alpha t} M_t$. By Itô's formula,

$$S_t = S_0 + \int_0^t \alpha e^{\alpha s} M_{s-} ds + \int_0^t e^{\alpha s} dM_s.$$

The process $\int_0^t e^{\alpha s} dM_s$ is a martingale transform. By Lemma 2.5, the process

$$N_t = \int_0^t \alpha e^{\alpha s} M_{s-} ds$$

is also a martingale transform. Consequently, the process

$$t = \int_0^t \frac{1}{\alpha e^{\alpha s} M_{s-}} dN_s$$

is a martingale transform (we use here Proposition 2.4). But this contradicts the statement of Lemma 2.6. As a result, $\alpha = 0$. Hence, S is a martingale. □

`Proof of Theorem 3.4 (for model (1.2)).` *Step 1.* Let us first prove that in cases (i), (ii) the model is complete.

In case (i) this is just the Black-Scholes model, and its completeness is widely known.

In case (ii) we have, by Itô's formula (see [16; Ch.I, (4.57)]),

$$\begin{aligned}
S_t &= S_0 + \int_0^t e^{X_{s-}} dX_s + \sum_{s \le t} \left(e^{X_s} - e^{X_{s-}} - e^{X_{s-}} \Delta X_s\right) \\
&= S_0 + \alpha \int_0^t e^{X_{s-}} dN_{\gamma s} + \beta \int_0^t e^{X_{s-}} ds + \sum_{s \le t} e^{X_{s-}} \left(e^{\alpha \Delta N_{\gamma s}} - 1 - \alpha \Delta N_{\gamma s}\right) \\
&= S_0 + \sum_{s \le t} e^{X_{s-}} \left(e^{\alpha \Delta N_{\gamma s}} - 1\right) + \beta \int_0^t e^{X_{s-}} ds \\
&= S_0 + (e^{\alpha} - 1) \int_0^t e^{X_{s-}} dN_{\gamma s} + \beta \int_0^t e^{X_{s-}} ds \\
&= S_0 + (e^{\alpha} - 1) \int_0^t e^{X_{s-}} \, d\left(N_{\gamma s} - \frac{\beta}{1 - e^{\alpha}} s\right).
\end{aligned}$$

By Lemma 2.2, there exists a measure $\widetilde{\mathsf{P}} \sim \mathsf{P}$ such that with respect to $\widetilde{\mathsf{P}}$ the process $(N_{\gamma t})_{t \in [0,T]}$ is a Poisson process with intensity $\frac{\beta}{1 - e^{\alpha}}$ (note that $\frac{\beta}{1 - e^{\alpha}} < 0$ since $\alpha\beta < 0$). Then $\frac{\beta}{1 - e^{\alpha}} t$ is the $\widetilde{\mathsf{P}}$-compensator of $N_{\gamma t}$, and it is known that any $(\mathcal{F}_t, \widetilde{\mathsf{P}})$-local martingale (recall that $\mathcal{F}_t = \mathcal{F}_t^S = \mathcal{F}_t^N$) can be represented as

$$M_t = M_0 + \int_0^t K_s \, d\left(N_{\gamma s} - \frac{\beta}{1 - e^{\alpha}} s\right)$$

(see [16; Ch. III, (4.37)]). Hence, M can also be represented as a stochastic integral with respect to S. Now, it follows from the Second fundamental theorem of asset pricing that the model is complete.

Step 2. Let us now prove that model (1.2) is complete only in cases (i) and (ii). Let ν denote the Lévy measure of the process X. Suppose that the support of ν contains more than one point. The analysis of the proof of Theorem 3.1 shows that in this case one can construct two different measures $\widetilde{\mathsf{P}}$ and $\widetilde{\mathsf{P}}'$ that are equivalent to P and such that S is a martingale with respect to both of them. So, in this case the model is not complete.

Suppose now that the Lévy measure of X is concentrated at one point. Then X can be represented as

$$X_t = \alpha B_t + \alpha' N_{\gamma t} + \beta t, \quad t \in [0,T], \tag{3.10}$$

where $\alpha, \alpha', \beta \in \mathbb{R}$, $\gamma > 0$, B is a Brownian motion, N is a Poisson process and B, N are independent. By Itô's formula,

$$S_t = S_0 + \int_0^t e^{X_{s-}} d\Big(\alpha B_s + \big(e^{\alpha'} - 1\big) N_{\gamma s} + \frac{\alpha^2}{2} s + \beta s\Big). \tag{3.11}$$

Using Girsanov's theorem and Lemma 2.2, we can, for each $b \in \mathbb{R}$, $\lambda > 0$, construct a measure $\widetilde{\mathsf{P}}_{b\lambda} \sim \mathsf{P}$ such that with respect to this measure B is a Brownian motion with drift b and N is a Poisson process with intensity λ. If

$$\alpha b + \big(e^{\alpha'} - 1\big)\gamma\lambda + \frac{\alpha^2}{2} + \beta = 0, \tag{3.12}$$

then, by (3.11), the process S is also a local martingale (note that the process $e^{X_{t-}}$ is locally bounded).

Suppose now that $\alpha, \alpha \neq 0$. Then there exists infinitely many pairs (b, λ) satisfying (3.12) and hence, infinitely many equivalent local martingale measures for S. As a result, model (1.2) based on process (3.10) can be complete only if $\alpha = 0$ or $\alpha' = 0$. But these are exactly the cases (i) and (ii). □

4 Linear and Exponential Time-Changed Lévy Models

1. The results. In Theorem 4.1 we exclude the trivial cases, where $X \equiv 0$ or $\tau \equiv 0$ a.s.

Theorem 4.1. (No-arbitrage). *Model* (1.3) (*model* (1.4)) *does not satisfy the condition* (*NFLVR*) *in the following cases only:*
(i) S *is increasing;*
(ii) S *is decreasing.*

Definition 4.2. A process $(Y_t)_{t\in[0,T]}$ is an *independently time-changed Lévy process* if there exists a Lévy process $(Z_t)_{t\geq 0}$ and an increasing càdlàg process $(\tau_t)_{t\in[0,T]}$ with $\tau_0 = 0$ such that Z, τ are independent and

$$\text{Law}(Y_t;\, t \in [0,T]) = \text{Law}(Z_{\tau_t};\, t \in [0,T]).$$

Theorem 4.3. *Suppose that in model* (1.3) (*model* (1.4)) S *is neither increasing nor decreasing. Then there exists a measure* $\widetilde{\mathsf{P}} \sim \mathsf{P}$ *such that* $(X_{\tau_t})_{t\in[0,T]}$ *is an independently time-changed Lévy process with respect to* $\widetilde{\mathsf{P}}$ *and* S *is a* $(\mathcal{F}_t, \widetilde{\mathsf{P}})$*-martingale.*

Theorem 4.4. (Completeness). *Suppose that S is neither increasing nor decreasing and $\mathcal{F}_t = \mathcal{F}_t^S$. Model* (1.3) (*model* (1.4)) *is complete in the following cases only:*

(i) $X_t = \alpha B_t + \beta t$, *where* $\alpha, \beta \in \mathbb{R}$ (*the case* $\alpha = 0$, $\beta \neq 0$ *is excluded*);

(ii) $X_t = \alpha N_{\gamma t} + \beta t$, *where* $\alpha, \beta \in \mathbb{R}$, $\gamma > 0$, $\alpha\beta < 0$ *and* τ *is a deterministic continuous function.*

2. The proofs. We will prove only the statements related to model (1.4).

`Proof of Theorem 4.3 (for model (1.4)).` The conditions of the theorem imply that X is neither increasing nor decreasing. By Lemma 3.6, there exists a strictly positive $(\mathcal{G}_t, \mathsf{P})$-martingale $M = M(X)$ such that, for any $u \geq 0$, the process $(X_t)_{t\in[0,u]}$ is a $(\mathcal{G}_t, \widetilde{\mathsf{P}}^u)$-Lévy process and $(e^{X_t})_{t\in[0,u]}$ is a $(\mathcal{G}_t, \widetilde{\mathsf{P}}^u)$-martingale (here, $\widetilde{\mathsf{P}}^u = M_u\mathsf{P}$).

Denote

$$\begin{aligned} \mathsf{Q} &= \mathrm{Law}(X_t;\, t \geq 0), \\ \widetilde{\mathsf{Q}}^u &= \mathrm{Law}(X_t;\, t \geq 0 \mid \widetilde{\mathsf{P}}^u), \quad u \geq 0, \\ \mathsf{R} &= \mathrm{Law}(\tau_t;\, t \in [0,T]), \end{aligned}$$

so that Q, $\widetilde{\mathsf{Q}}^u$ are measures on $D(\mathbb{R}_+)$ and R is a measure on $D([0,T])$. Since $(X_t)_{t\in[0,u]}$ is a $\widetilde{\mathsf{P}}^u$-Lévy process and $\mathsf{E}_{\widetilde{\mathsf{P}}^u} e^{X_t} = 1$, $t \in [0,u]$, we conclude by [21; Theorem 25.18] that

$$\mathsf{E}_{\widetilde{\mathsf{P}}^u} \sup_{t\leq u} e^{X_t} < \infty, \quad u \geq 0.$$

Obviously, there exists a measure $\widetilde{\mathsf{R}} \sim \mathsf{R}$ such that the density $\rho = \frac{d\widetilde{\mathsf{R}}}{d\mathsf{R}}$ is bounded and

$$\int\limits_{D([0,T])} \int\limits_{D(\mathbb{R}_+)} \sup_{t\leq\tau_T} e^{X_t}\, \widetilde{\mathsf{Q}}^{\tau_T}(dX)\, \widetilde{\mathsf{R}}(d\tau) < \infty. \tag{4.1}$$

(In order to construct such a measure, it is sufficient to consider the density of the form $\rho(\tau) = \varphi(\tau_T)$, where φ is a bounded rapidly decreasing at infinity function).

Set $\widetilde{\mathsf{P}} = \rho(\tau) M_{\tau_T}(X)\mathsf{P}$. It follows from the equalities

$$\mathsf{E}\rho(\tau)M_{\tau_T}(X) = \int\limits_{D([0,T])} \widetilde{\rho}(\tau) \int\limits_{D(\mathbb{R}_+)} M_{\tau_T}(X)\, \mathsf{Q}(dX)\, \mathsf{R}(d\tau) = \int\limits_{D([0,T])} \rho(\tau)\, \mathsf{R}(d\tau) = 1$$

and

$$\begin{aligned} \mathsf{P}\big(\rho(\tau)M_{\tau_T}(X) > 0\big) &= \int\limits_{D([0,T])} I(\rho(\tau) > 0) \int\limits_{D(\mathbb{R}_+)} I(M_{\tau_T}(X) > 0)\, \mathsf{Q}(dX)\, \mathsf{R}(d\tau) \\ &= \int\limits_{D([0,T])} I(\rho(\tau) > 0)\, \mathsf{R}(d\tau) = 1 \end{aligned}$$

that $\widetilde{\mathsf{P}}$ is a probability measure and $\widetilde{\mathsf{P}} \sim \mathsf{P}$.

Denote $Y_t = X_{t\wedge\tau_T}$. Then

$$\begin{aligned}\mathsf{E}_{\widetilde{\mathsf{P}}} \sup_{t\ge 0} e^{Y_t} = \mathsf{E}_{\widetilde{\mathsf{P}}} \sup_{t\le\tau_T} e^{X_t} &= \int_{D([0,T])} \rho(\tau) \int_{D(\mathbb{R}_+)} \sup_{t\le\tau_T} e^{X_t} M_{\tau_T}(X)\,\mathsf{Q}(dX)\,\mathsf{R}(d\tau) \\ &= \int_{D([0,T])} \int_{D(\mathbb{R}_+)} \sup_{t\le\tau_T} e^{X_t}\,\widetilde{\mathsf{Q}}^{\tau_T}(dX)\,\widetilde{\mathsf{R}}(d\tau) < \infty\end{aligned} \tag{4.2}$$

(see (4.1)). Hence, the process $(e^{Y_t})_{t\ge 0}$ is uniformly integrable with respect to $\widetilde{\mathsf{P}}$.

For any $u\ge 0$, the process $(M_t e^{X_t})_{t\in[0,u]}$ is a $(\mathcal{G}_t,\mathsf{P})$-martingale (see [16; Ch. II, (3.8)]). Hence, $(M_t e^{X_t})_{t\ge 0}$ is a $(\mathcal{G}_t,\mathsf{P})$-martingale. Consequently, $(M_{t\wedge\tau_T} e^{Y_t})_{t\ge 0}$ is a $(\mathcal{G}_t,\mathsf{P})$-martingale. Since $\rho(\tau)$ is bounded and $\mathcal{G}_0$-measurable, the process $(\rho(\tau) M_{t\wedge\tau_T} e^{Y_t})_{t\ge 0}$ is a $(\mathcal{G}_t,\mathsf{P})$-martingale. Notice that $(\rho(\tau)M_{t\wedge\tau_T})_{t\ge 0}$ is the density process of $\widetilde{\mathsf{P}}$ with respect to P. Thus, $(e^{Y_t})_{t\ge 0}$ is a $(\mathcal{G}_t,\widetilde{\mathsf{P}})$-martingale (see [16; Ch. II, (3.8)]). Combining this with (4.2), we conclude that $(e^{Y_t})_{t\ge 0}$ is a uniformly integrable $(\mathcal{G}_t,\widetilde{\mathsf{P}})$-martingale.

The theory of martingales ensures that there exists a random variable ξ such that, for any $t\in[0,T]$, $e^{Y_{\tau_t}} = \mathsf{E}_{\widetilde{\mathsf{P}}}\big[\xi \mid \mathcal{G}_{\tau_t}\big]$. This implies that the process $(e^{Y_{\tau_t}})_{t\in[0,T]}$ is a $(\mathcal{G}_{\tau_t},\widetilde{\mathsf{P}})$-martingale.

Finally, there exists a Lévy process $(Z_t)_{t\ge 0}$ such that, for any $u\ge 0$,

$$\operatorname{Law}(X_t;\, t\in[0,u] \mid \widetilde{\mathsf{P}}^u) = \operatorname{Law}(Z_t;\, t\in[0,u]).$$

Obviously,

$$\operatorname{Law}(X_{\tau_t};\, t\in[0,T] \mid \widetilde{\mathsf{P}}) = \operatorname{Law}(Z_{\sigma_t};\, t\in[0,T]),$$

where Z, σ are independent and $\operatorname{Law}(\sigma_t;\, t\in[0,T]) = \widetilde{\mathsf{R}}$. Thus, $(X_{\tau_t})_{t\in[0,T]}$ is an independently time-changed Lévy process. □

`Proof of Theorem 4.1 (for model (1.4)).` If S is increasing, then the reasoning is the same as in the proof of Theorem 3.1.

If S is neither increasing nor decreasing, then the desired statement follows from Theorem 4.3 and the First fundamental theorem of asset pricing. □

`Proof of Theorem 4.4 (for model (1.4)).` *Step 1.* Let us prove that in case (i) the model is complete. There exists a measure $\widetilde{\mathsf{P}}\sim\mathsf{P}$ such that $X_t = \alpha\widetilde{B}_t - \frac{\alpha^2}{2}t$, where $(\widetilde{B}_t)_{t\in[0,\tau_T]}$ is a Brownian motion with respect to $\widetilde{\mathsf{P}}$.

Let $(M_t)_{t\in[0,T]}$ be a $(\mathcal{F}_t,\widetilde{\mathsf{P}})$-martingale. Set

$$\sigma_t = \inf\{s\ge 0 : \tau_s\ge t\},\quad t\in[0,\tau_T].$$

Then σ is an increasing right-continuous function. The process $(M_{\sigma_t})_{t\in[0,\tau_T]}$ is a $(\mathcal{F}_{\sigma_t},\widetilde{\mathsf{P}})$-martingale. Note that $\mathcal{F}_t = \mathcal{F}_t^S = \mathcal{F}_{\tau_t}^X = \mathcal{F}_{\tau_t}^{\widetilde{B}}$, $t\in[0,T]$. In view of the continuity of τ, we have $\mathcal{F}_{\sigma_t} = \mathcal{F}_{\tau_{\sigma_t}}^{\widetilde{B}} = \mathcal{F}_t^{\widetilde{B}}$, $t\in[0,\tau_T]$. Hence, there exists a $(\mathcal{F}_t^{\widetilde{B}})$-predictable $\widetilde{B}$-integrable process $(H_t)_{t\in[0,\tau_T]}$ such that

$$M_{\sigma_t} = M_0 + \int_0^t H_s\,d\widetilde{B}_s,\quad t\in[0,\tau_T].$$

In view of the equality

$$S_0 e^{X_t} = S_0 + \alpha S_0 \int_0^t e^{X_s} d\widetilde{B}_s,$$

we have

$$M_{\sigma_t} = M_0 + \int_0^t \alpha^{-1} S_0^{-1} e^{-X_s} d(S_0 e^{X_s}).$$

Using the time-change formula for the stochastic integrals (see [20; Ch, V, §1]), we deduce that

$$M_{\sigma_{\tau_t}} = M_0 + \int_0^t H_{\tau_u} dS_u, \quad t \in [0,T].$$

Let $[a,b]$ be an interval of constancy of τ, i.e. $\tau_a = \tau_b$. Then $\mathcal{F}_a = \mathcal{F}_b$ up to P-null sets and hence, $M_a = M_b$ a.s. Since M is càdlàg, this means that almost all the paths of M are constant over all the intervals of constancy of τ. Hence, $M_{\sigma_{\tau_t}} = M_t$, $t \in [0,T]$.

Thus, any $(\mathcal{F}_t, \widetilde{\mathsf{P}})$-martingale can be represented as a stochastic integral with respect to S. Then this is also true for all the $(\mathcal{F}_t, \widetilde{\mathsf{P}})$-local martingales. By the Second fundamental theorem of asset pricing, the model is complete.

The proof of the completeness for case (ii) is similar.

Step 2. Now, suppose that the model is complete. Suppose first that τ is not deterministic. Then there exists $r \in [0,T]$ such that the support of $\mathrm{Law}(\tau_r)$ contains at least two points a, b. We can choose a sequence of bounded densities $\rho_n(\tau)$ that satisfy (4.1) and such that

$$\mathrm{Law}(\tau_r \mid \rho_n \mathsf{R}) \xrightarrow[n\to\infty]{w} \delta_a.$$

Here $\mathsf{R} = \mathrm{Law}(\tau_t;\, t \in [0,T])$ and δ_a is the Dirac mass at point a. There also exists a sequence $\rho'_n(\tau)$ satisfying the same conditions and such that

$$\mathrm{Law}(\tau_r \mid \rho'_n \mathsf{R}) \xrightarrow[n\to\infty]{w} \delta_b.$$

If we set $\widetilde{\mathsf{P}}_n = \rho_n(\tau) M_{\tau_T}(X)\mathsf{P}$, $\widetilde{\mathsf{P}}'_n = \rho'_n(\tau) M_{\tau_T}(X)\mathsf{P}$, where M is given by Lemma 3.6, then $(S_t)_{t\in[0,T]}$ is a martingale with respect to all the measures $\widetilde{\mathsf{P}}_n$, $\widetilde{\mathsf{P}}'_n$ (see the proof of Theorem 4.3).

There exists a Lévy process $(Z_t)_{t\geq 0}$ such that, for any $u \geq 0$,

$$\mathrm{Law}(X_t;\, t \in [0,u] \mid \widetilde{\mathsf{P}}^u) = \mathrm{Law}(Z_t;\, t \in [0,u]),$$

where $\widetilde{\mathsf{P}}^u = M_u \mathsf{P}$. Then

$$\mathrm{Law}(X_{\tau_r} \mid \widetilde{\mathsf{P}}_n) \xrightarrow[n\to\infty]{w} \mathrm{Law}(Z_a),$$
$$\mathrm{Law}(X_{\tau_r} \mid \widetilde{\mathsf{P}}'_n) \xrightarrow[n\to\infty]{w} \mathrm{Law}(Z_b),$$

which shows that there exists n such that

$$\mathrm{Law}(X_{\tau_r} \mid \widetilde{\mathsf{P}}_n) \neq \mathrm{Law}(X_{\tau_r} \mid \widetilde{\mathsf{P}}'_n).$$

Hence, there exist different equivalent martingale measures for S. By the Second fundamental theorem of asset pricing, the model is not complete. Thus, τ is a deterministic function.

Now, the arguments used in the proof of Lemma 3.6 show that the model can be complete only if X is a Brownian motion with a drift or a Poisson process with a drift.

Finally, let us prove that τ is continuous. Suppose that there exists $r \in [0,T]$ such that $\tau_{r-} \neq \tau_r$. Then, by changing the distribution of the process X on $[\tau_{r-}, \tau_r]$, we can construct different equivalent martingale measures for S. This completes the proof. □

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