The recovery problem for time-changed Lévy processes*

Matthias Winkel†
MaPhySto‡

October 5, 2001

Summary. Given a Lévy process Y and an independent time change A, we identify the cases when A and Y are completely determined by $X = Y \circ A$. We calculate the nontrivial conditional law of A and Y given X when A is a pure jump subordinator. For Y Brownian motion this study was initiated by Geman et al. [10]. We deduce from our higher generality some of their main results using new methods. In their setting, we go a step further and allow A to be the sum of a continuous process and a pure jump process with independent increments.

Keywords. Lévy processes, time-change, subordination, recovery problem, financial modelling, stochastic volatility

A.M.S. Classification. Primary 60 G 51; secondary 91 B 28

Email. winkel@imf.au.dk

1 Introduction

For a stochastic process Y we consider the process

$$X = Y \circ A, \qquad \text{i.e.} \qquad X_t = Y_{A_t}, \quad t \geq 0, \tag{1}$$

obtained from Y by an independent random change of time A, increasing. In mathematical finance, several models of the stock price X have been based on this concept to address the inadequacy of normally distributed short time returns and the constancy of the volatility parameter in the Black-Scholes model based on Brownian motion. We

*This research was supported by a PhD scholarship of the DAAD (German Academic Exchange Service) within the scope of the common programme HSP III of the German Federal and Länder Governments

†Department of Mathematical Sciences, University of Aarhus, Ny Munkegade, DK-8000 Aarhus C, Denmark; and Laboratoire de Probabilités et Modèles Aléatoires, Université Pierre et Marie Curie, 175 rue du Chevaleret, F-75013 Paris, France

‡Centre for Mathematical Physics and Stochastics, funded by the Danish National Research Foundation

mention here only the variance gamma model (Madan and Seneta [13]), the generalized hyperbolic model (Eberlein [7]), and approaches to stochastic volatility, cf. Barndorff-Nielsen [1], Barndorff-Nielsen and Shephard [3], also Eberlein et al. [8], Geman et al. [11] and Carr et al. [6]. The procedure (1) is known as the subordination of Bochner [5] when Y is a Lévy process and A a subordinator, cf. Bertoin [4]. On the other hand, when A is right-continuously differentiable, its derivative can be viewed as stochastic volatility of the price process X, cf. Barndorff-Nielsen and Shephard [3].

We assume here that Y is a Lévy process. We interpret X as the observed stock price and suppose that Y and A cannot be observed. Since A bears the information on the volatility in financial models, it is natural to ask what we can say about A. This problem has been treated by Geman et al. [10] when Y is more particularly Brownian motion, and the interesting case then is when A is not continuous, since A can be fully recovered from X if A is continuous. We make the same observation for all Lévy processes Y that are not compound Poisson. Like [10] we then focus on the case when A is a subordinator and obtain results concerning the law of A given X. Our results are more general and more explicit than Geman et al. and allow to reprove some of their main results by different methods. Next, we return to the Brownian motion setting and let A be the sum of a continuous and a discontinuous part. We can then combine the two preceding results to recover the continuous part and give the conditional law of the discontinuous part, when it has independent increments. Once the conditional law of A given X is known, the conditional law of Y given X and A can be constructed as a concatenation of Lévy (or Brownian) bridges.

2 Characterisation of full recovery

Let Y be a Lévy process whose marginal laws are given by the Lévy-Khintchine representation of their characteristic exponent

$$\psi_Y(\lambda) = -\log E\left(\exp\{i\lambda Y_1\}\right) = -i\alpha\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{\mathbb{R}^*}\left(1 - e^{i\lambda z} - i\lambda z 1_{\{|z|<1\}}\right)\nu(dz) \quad (2)$$

where $\alpha \in \mathbb{R}$, $\sigma^2 \geq 0$, $\int_{\mathbb{R}^*}(1 \wedge z^2)\nu(dz) < \infty$ and $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$. Y is then Brownian motion with drift α and variance σ^2 with (for $|z| < 1$ compensated!) independent jumps added following a Poisson point process $(\Delta Y_a)_{a\geq 0}$ with intensity measure ν. This motivates to refer to σ^2 as the Gaussian coefficient and to ν as the jump measure. It also allows to decompose $Y = Y^c + Y^j$ into a Brownian motion Y^c with drift and an independent Lévy process Y^j with zero Gaussian coefficient. We refer to the two books on Lévy processes by Bertoin [4] and Sato [16].

The following theorem shows the most important situations in which A can be recovered completely from $X = Y \circ A$ and how to do the recovery. After the proof of the theorem we give a discussion why these are essentially the only situations where full recovery is possible.

Theorem 1 *Let $(Y_a)_{a\geq 0}$ be a Lévy process but not compound Poisson, $(A_t)_{t\geq 0}$ independent continuous increasing and $X = Y \circ A$.*

a) If Y has an infinite jump measure $\nu(\mathbb{R}^*) = \infty$, then

$$A_t = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} N_k,$$

where $N_k = \#\{s \leq t : \Delta X_s \in [z_{k+1}, z_k)\}$ and $(z_k)_{k\geq 0}$ such that $\nu(|z| \geq z_k) = k$. Existence problems of $(z_k)_{k\geq 0}$ due to atoms of ν may be circumvented by randomising accordingly whether a jump of size z_k is counted by N_k or N_{k+1} etc.

b) If Y has bounded variation, i.e. $Y_t = \beta t + \sum_{s\leq t} \Delta Y_s$, and a nontrivial drift coefficient $\beta \neq 0$, then

$$A_t = \frac{1}{\beta}\left(X_t - \sum_{s\leq t} \Delta X_s\right).$$

c) If Y has a positive Gaussian coefficient $\sigma^2 > 0$, then

$$A_t = \frac{1}{\sigma^2}\left([X]_t - \sum_{s\leq t} (\Delta X_s)^2\right), \quad t \geq 0,$$

where the quadratic variation process $[X]$ of X is adapted to the natural filtration of X.

Proof: a) Conditionally on A, the N_k are independent and identically Poisson distributed with parameter A_t since the jump heights of Y on $[0, A_t]$ are the same as the jumps of X on $[0, t]$ and they form a homogeneous Poisson point process with intensity measure ν. The result now follows from the strong law of large numbers.

b) This is an elementary observation.

c) The jumps of the quadratic variation process $[X]$ of a semi-martingale X are the squares $\Delta[X]_s = (\Delta X_s)^2$ of the jump sizes of X. Removing these jumps, we retain the quadratic variation of the continuous martingale part. Cf. Protter [14] Section II.6.

Now the result follows as an application of the well-known Dubins-Schwarz theorem, cf. Revuz and Yor [15] Section V.1. □

Furthermore, it is clear that when Y is compound Poisson, A is only determined by X in trivial cases. More precisely, we study here the determination of $(A_s)_{0\leq s\leq t}$ by $(X_s)_{0\leq s\leq t}$ for all $t \geq 0$. Roughly, any randomness of A before it passes the level given by the first jump time of Y, is not reflected by X. Since Y and A are independent, there is positive probability that $Y_{A_t} = 0$ for any $t \geq 0$, thus A must be deterministic, regardless whether A was assumed continuous or not.

On the other hand, if A has jumps, the homogeneity of Lévy processes only allows us to recover the jump height if either the jump height itself or the Lévy process are deterministic. Deterministic Lévy processes are multiples of the identity process in which case A coincides with X up to this constant multiple. If the jump heights of A are deterministic, the situation is more complicated since one has to make sure that jump times of A are recognised from observing $X = Y \circ A$ which jumps both when A jumps and when Y jumps on the closed range of A which may have positive Lebesgue measure. Still, we consider this a degenerate situation and stop its discussion here.

3 Conditional laws for Bochner's subordination of Lévy processes

In Theorem 1 the continuity of A is essential to ensure that all jumps of X are jumps of Y. In fact, if Y is not compound Poisson, X jumps whenever A jumps and also inherits those jumps of Y that take place on the range of A. If A is continuous, all jumps of Y are in the range of A. The other extreme is when the closed range of A has zero Lebesgue measure, because then the independence of Y and A ensures that a.s. no jump time of Y lies in the closed range of A. Having a range of zero Lebesgue measure is in fact equivalent to moving only by jumps - we then call A purely discontinuous. For our second result we shall furthermore assume that A is a subordinator, i.e. an increasing Lévy process.

Theorem 2 *Let $(Y_a)_{a\geq 0}$ be a Lévy process with $P(Y_a \in dy) = p_a(y)dy$, $(A_t)_{t\geq 0}$ a purely discontinuous subordinator with jump measure ν_A and $X = Y \circ A$.*

Then A has the same jump times as X. The heights of the jumps of A are conditionally independent given X and their conditional laws are

$$P(\Delta A_t \in dz | T = t, \Delta X_t = y) = \frac{1}{c(y)} p_z(y) \nu_A(dz) \qquad \text{for a.e. } y \in \mathbb{R}$$

where $c(y) < \infty$ is a normalisation constant and T any (stopping) time for X with $y \in supp(\Delta X_T)$ and $t \in supp(T)$. As the right hand side is independent of T, we may introduce the convention to drop the condition $T = t$.

Proof: A and X have countable numbers of jumps. During any jump $(T, \Delta A_T) = (t, z)$ of A (where T is any stopping time for A), the independent process Y moves from a starting point $Y(A_{t-}) = x$ to a terminal point $Y(A_t) = Y(A_{t-} + z)$ distributed according to $p_z(y-x)dy$. In particular, Y is not at x at the terminal point a.s. and $(t, Y(A_t) - Y(A_{t-}))$ is a jump of X. Vice versa, take any jump $(T, \Delta X_T) = (t, y)$ of X (where T is now any stopping time for X), i.e.

$$y = X_t - X_{t-} = Y_{A_t} - \lim_{s\uparrow t} Y_{A_s} > 0.$$

Assume that t is not a jump time of A, then $A_t = A_{t-}$ must be a jump time of Y. But, since A is purely discontinuous, its range has zero Lebesgue measure whereas any of the countable number of jump times of Y does a.s. not lie in this set of zero Lebesgue measure. Thus, contrary to our assumption, t must be a jump time of A. We have established that A and X have the same jump times.

Also, we deduce from the independence of increments of Y that the jump heights of X are conditionally independent given A. The height distribution only depends on the size of the jump of A and follows the transition kernel of Y. Therefore $(\Delta X_t, \Delta A_t)_{t\geq 0}$ is a Poisson point process with intensity measure $p_z(y)dy\nu_A(dz)$. In particular, $(\Delta X_t)_{t\geq 0}$ has an absolutely continuous intensity measure $c(y)dy$, and $c(y) = \int_{(0,\infty)} p_z(y)\nu_A(dz)$ is a version of the density, hence finite for a.e. $y \in \mathbb{R}$.

Note that X has no Brownian component, since the Gaussian coefficient of X is

$$2\lim_{|\lambda|\to\infty}\frac{\psi_X(\lambda)}{\lambda^2}=2\lim_{|\lambda|\to\infty}\frac{\Phi_A(\psi_Y(\lambda))}{\lambda^2}=d\sigma^2=0$$

due to the zero drift coefficient $d=0$ of A, cf. Proposition I.2 in [4]; $\Phi_A(q)=-\log E(\exp\{-qA_1\})$ denotes the Laplace exponent of A. Hence, the randomness of X is determined by its jumps, and the Poisson point process property of $(\Delta X_t,\Delta A_t)_{t\geq 0}$ yields the conditional independence of the jump heights of A given X. Looking at $(\Delta X_t,\Delta A_t)_{t\geq 0}$ as a marked Poisson point process, cf. Kingman [12] Section 5.2, the Marking Theorem identifies the marking kernel

$$q(y,dz)=\frac{1}{c(y)}p_z(y)\nu_A(dz).$$

For any (random) time T which is measurable in the σ-algebra generated by X such that $\Delta X_T\neq 0$ a.s., we obtain furthermore

$$P(\Delta A_t\in dz|\Delta X_t=y,T=t)=\frac{1}{c(y)}p_z(y)\nu_A(dz)$$

for a.e. $y\in\mathbb{R}$ and all $t\in supp(T)$ since

$$\begin{aligned}
&E\left(f(\Delta A_T)g(\Delta X_T,T)\right)\\
&\quad=E\left(E\left(f(\Delta A_T)g(\Delta X_T,T)|\,X\right)\right)\\
&\quad=E\left(g(\Delta X_T,T)\int_{(0,\infty)}f(z)q(\Delta X_T,dz)\right)\\
&\quad=\int_{(0,\infty)\times\mathbb{R}}\int_{(0,\infty)}f(z)g(y,t)\frac{1}{c(y)}p_z(y)\nu_A(dz)P(T\in dt,\Delta X_T\in dy).
\end{aligned}$$

□

One can generalise Theorem 2 by replacing the subordinator A by any purely discontinuous increasing additive process (processes with independent but inhomogeneous increments), cf. Sato [16] for background on additive processes, in particular their time-dependent Lévy-Khintchine characteristics in analogy with (2). The conditional laws then depend on t. We do not spell this out here, but refer to Theorem 3 where a similar result is established in a slightly different setting.

Example 1 Let Y be Brownian motion, A an independent standard gamma subordinator, then

$$P(\Delta A_t\in dz|\Delta X_t=y)=\frac{|y|}{\sqrt{2\pi}}e^{\sqrt{2}|y|}z^{-3/2}\exp\left\{-\frac{1}{2}\left(y^2z^{-1}+2z\right)\right\}dz$$

and this is the inverse Gaussian law of parameters $\sqrt{2}$ and $|y|$. Using the additivity property of independent inverse Gaussian random variables (associated with a countable family of stopping times), we deduce that the conditional law of A_t given X is inverse Gaussian with parameters $\sqrt{2}$ and the total variation of X on $[0,t]$ which is the sum of the moduli of its jumps. This is Theorem 1 of Geman et al. [10].

Example 2 Let Y be Brownian motion, A an independent α-stable subordinator, then X is symmetric 2α-stable and

$$P(\Delta A_t \in dz | \Delta X_t = y) = \frac{1}{\Gamma(\alpha+1/2)} \left(\frac{y^2}{2}\right)^{\alpha+1/2} z^{-\alpha-3/2} \exp\left\{-\frac{y^2}{2} z^{-1}\right\} dz$$

which is a reciprocal gamma distribution with parameters $\alpha + 1/2$ and $y^2/2$.

Examples 1 and 2 can be seen as limit cases of a more general setting where A is an $\gamma^2/2$-exponentially tilted α-stable subordinator. The conditional law is then generalised inverse Gaussian with parameters $-\alpha - 1/2$, y and γ. The densities of these laws can be expressed in terms of the Bessel function of the third kind, cf. [2]. Example 1 corresponds to $\alpha = 0$, $\gamma = \sqrt{2}$, and in Example 2, $\gamma = 0$.

Theorem 2 is at the same time more general and more explicit than Theorem 3 in Geman et al. [10] where they give transform identities in terms of some stochastic exponential (for the special case where Y is Brownian motion):

Corollary 1 ([10] for Y Brownian motion) *In the setting of Theorem 2, we have*

$$E\left(e^{-\lambda A_t}\middle| (X_s)_{s\le t}\right) = e^{-t\psi(\lambda)} \mathcal{E}_t\left(\frac{\phi_\lambda - \phi_0}{\phi_0}\right),$$

where $\psi(\lambda) = -\log E(e^{-\lambda A_1})$ is the Laplace exponent of A, $\mathcal{E}(f)$ is the stochastic exponential of

$$f * (\mu_X|_{[0,t]} - t\nu_X) = \int_{\mathbb{R}} f(x) \left(\sum_{0\le s\le t} \delta_{\Delta X_s} - t\nu_X\right)(dx), \qquad t \ge 0,$$

ν_X is the jump measure of X and

$$\phi_\lambda(x) = \int_{(0,\infty)} e^{-\lambda z} p_z(x) \nu_A(dz).$$

Proof: We calculate both sides of the asserted equality, first by Theorem 2

$$\begin{aligned} E\left(e^{-\lambda A_t}\middle| (X_s)_{s\le t}\right) &= \prod_{s\le t} E\left(e^{-\lambda \Delta A_s}\middle| \Delta X_s\right) \\ &= \prod_{s\le t} \frac{1}{c(\Delta X_s)} \phi_\lambda(\Delta X_s) = \prod_{s\le t} \frac{\phi_\lambda(\Delta X_s)}{\phi_0(\Delta X_s)} \end{aligned}$$

and then the definition of the closed form representation of the stochastic exponential for the compensated pure jump process

$$Z_t = f * (\mu_X|_{[0,t]} - t\nu_X) = \sum_{s\le t} f(\Delta X_s) - t \int_{\mathbb{R}} f(x)\phi_0(x) dx$$

yields for $f = (\phi_\lambda - \phi_0)/\phi_0$

$$\begin{aligned} e^{-t\psi(\lambda)}\mathcal{E}_t\left(\frac{\phi_\lambda - \phi_0}{\phi_0}\right) &= e^{-t\psi(\lambda)}e^{Z_t}\prod_{s\le t}(1+\Delta Z_s)e^{-\Delta Z_s} \\ &= \exp\left\{-t\left(\psi(\lambda) + \int_{\mathbb{R}}(\phi_\lambda(x)-\phi_0(x))dx\right)\right\}\prod_{s\le t}\frac{\phi_\lambda(\Delta X_s)}{\phi_0(\Delta X_s)} \end{aligned}$$

cf. Theorem II.36 in Protter [14]. Now the equality is established by noting

$$\begin{aligned} -\int_{\mathbb{R}}(\phi_\lambda(x)-\phi_0(x))dx &= \int_{\mathbb{R}}\int_{(0,\infty)}(1-e^{-\lambda z})p_z(x)\nu_A(dz)dx \\ &= \int_{(0,\infty)}(1-e^{-\lambda z})\nu_A(dz) = \psi(\lambda). \end{aligned}$$

□

4 Combination of the preceding in the Brownian case

In the following $A = \alpha + \gamma$ is an increasing process that has both a continuous component α and a purely discontinuous component γ. We shall not need any restrictions upon α. As for γ, a subordinator is a convenient choice, also from the point of view of applications but as we remarked after Theorem 2 our method only requires the independent increment property, i.e. we suppose γ is an additive process. Furthermore, the result can be best presented when the Lévy measures ν_t of γ_t, $t \ge 0$, are such that there are density measures n_s in the sense that

$$\nu_t(C) = \int_0^t n_s(C)ds, \qquad t \ge 0,\ C \in \mathcal{B}. \tag{3}$$

Such a process is a subordinator if and only if n_s does not depend on $s \in [0,\infty)$. n_s is then the jump measure of the subordinator.

Theorem 3 *Let B be Brownian motion and $A = \alpha+\gamma$ an independent increasing process with continuous part α and purely discontinuous part γ such that (3) holds. Observing $X = B \circ A$, we recover*

$$\alpha_t = [X]_t - \sum_{s\le t}(\Delta X_s)^2,$$

and obtain

$$P(\Delta\gamma_t \in dz|\Delta X_t = y) = \frac{1}{c_t(y)}p_z(y)n_t(dz), \qquad t \ge 0,\ z \ge 0, y \in \mathbb{R},$$

where $p_z(y)$ is the Brownian transition density and $c_t(y)$ a normalisation constant.

Proof: The essential idea of the proof is to write $X_t = B(\alpha_t + \gamma_t) \sim B(\alpha_t) + \tilde{B}(\gamma_t)$ as an identity in law of processes where $\tilde{B}$ is an independent Brownian motion. To check this we calculate

$$\begin{aligned} E\left(\exp\left\{i\int_{(0,t]} f(s)dB_{\alpha_s+\gamma_s}\right\}\right) &= E\left(\exp\left\{-\frac{1}{2}\int_{(0,t]} f^2(s)d(\alpha_s+\gamma_s)\right\}\right) \\ &= E\left(\exp\left\{i\left(\int_{(0,t]} f(s)dB_{\alpha_s} + \int_{(0,t]} f(s)d\tilde{B}_{\gamma_s}\right)\right\}\right) \end{aligned}$$

where the equalities follow from a conditioning w.r.t. α and γ.

Since $A = \alpha + \gamma$ is the decomposition in continuous and purely discontinuous component, the same is true for the decomposition $X \sim \tilde{X} = B \circ \alpha + \tilde{B} \circ \gamma$. Specifically, $B \circ \alpha$ is obviously continuous, and $\tilde{B} \circ \gamma$ is purely discontinuous since this process moves when and only when γ jumps. In particular

$$\sum_{s \le t} \Delta \tilde{X}_s = \tilde{B}_{\gamma_t} \tag{4}$$

converges (but not necessarily absolutely) for all $t \ge 0$.

The result now follows from our Theorems 1 and 2. □

In our presentation we implicitly assumed that α and γ are independent. This assumption may be dropped provided that γ is an additive process given α. Its Lévy measures n_t may then be allowed to depend on the current value of α.

Example 3 (Ornstein-Uhlenbeck type stochastic volatility with jumps) Let Y be Brownian motion and $(U_t)_{t\ge 0}$ a process of Ornstein-Uhlenbeck type (OU process) associated with the so-called background driving subordinator Z by

$$U_t = e^{-ct}U_0 + e^{-ct}\int_{(0,t]} e^{cs}dZ_s.$$

Under a weak logarithmic moment condition on Z we may choose U_0 in such a way that $(U_t)_{t\ge 0}$ is stationary. The stochastic volatility model

$$dX_t = U_t dY_t$$

is then equivalent to subordinating a Brownian motion by the integrated OU process $\alpha_t = \int_0^t U_s ds$, cf. Barndorff-Nielsen and Shephard [3]. By our Theorem 1, α and hence U can be completely recovered from observing the whole path of X. When fitting data, the continuity of α is not always adequate. A mathematically tractable way to remedy this is by adding independent jumps to α, following a subordinator γ. One can e.g. choose the same law for the stationary law of U and the increments of γ, inverse Gaussian, say. According to Barndorff-Nielsen [1] the law of short time returns is then approximately normal inverse Gaussian. This composite model can be seen as a mixture of two types of stochastic volatility models via the decomposition in the proof of Theorem 3, which allows to recover α and obtain the conditional law of γ given the observation X.

5 Recovering Y from observing X

Though not so interesting from an applied point of view, mathematically, Y has the same right to be recovered from X as A. In the setting of a continuous time change A to a non-compound-Poisson Lévy process Y, Theorem 1 gives full recovery of A. Since A is continuous increasing, Y is just X composed with the inverse of A - hence itself completely determined by X.

When A is a purely discontinuous subordinator, Theorem 2 shows that the jump times of A can be recovered and gives the conditional law of the jump heights. The following theorem says that this determines the conditional law of Y on the range of A given X and A, and that Y behaves like a concatenation of independent Lévy bridges. For details on bridges we refer to Fitzsimmons et al. [9].

Theorem 4 *In the situation of Theorem 2 denote the closed range of A by $\mathcal{R} = \{A_t : t \geq 0\}^{cl}$, its subsets of points isolated to the left or right by $\mathcal{G}$ and $\mathcal{D}$ respectively. Then*

$$P(Y_a = X_{L_a -}, a \in \mathcal{R}\backslash\mathcal{D}, Y_a = X_{L_a}, a \in \mathcal{R}\backslash\mathcal{G} | X, A) = 1 \qquad a.s.$$

where $L_a = \inf\{t \geq 0 : A_t > a\}$. Furthermore, the subpaths of Y between each two successive points in the regenerative set $\mathcal{R}$ are conditionally independent given X and A and we have a.s. for all $t \geq 0$

$$P\left((Y_{A_{t-}+u})_{0\leq u \leq \Delta A_t} \in d\omega \middle| X, A\right) = Q^{\Delta A_t}_{X_{t-}\to X_t}(d\omega)$$

where $Q^z_{x\to y}$ denotes the law of the Lévy bridge of length z spanning from x to y associated with Y.

Proof: The first statement is immediate by the definition of $X = Y \circ A$ which yields precisely the assertion when composed with L.

For the second statement, note that there is a.s. only a countable number of jumps, and the assertion is trivial when A or (and) X do not jump. As for the jump times, we associate a Poisson point process in path space $\Gamma_t = (Y_{A_{t-}+u} - Y_{A_{t-}})_{0\leq u\leq \Delta A_t}$ if $\Delta A_t > 0$, $t \geq 0$, where the Poisson property follows from the independence of increments of Y, the independence of Y and A and the Poisson property of $(\Delta A_t)_{t\geq 0}$. We proceed now as in the proof of Theorem 2: the intensity measure of $(\Gamma_t)_{t\geq 0}$ is given by

$$P((Y_u)_{0\leq u\leq z} \in d\omega)\nu_A(dz)$$

and this can be interpreted as a Poisson point process $(\Delta A_t)_{t\geq 0}$ with marks Γ_t associated with $\Delta A_t > 0$. Even more, it can be interpreted as a Poisson point process $(\Delta A_t, \Delta X_t)_{t\geq 0}$ where the mark distributions depend on $\Delta X_t = Y(A_{t-} + \Delta A_t) - Y(A_{t-})$ by the conditioning to reach a fixed terminal value - this is the definition of a bridge, cf. [9]. The bridge exists since Y has absolutely continuous transition densities. The mark kernel is now given by $Q^z_{0\to y}(d\omega)$ since

$$P((Y_u)_{0\leq u\leq z} \in d\omega, Y_z \in dy)\nu_A(dz) = Q^z_{0\to y}(d\omega)p_z(y)dy\nu_A(dz).$$

Now it suffices to calculate

$$
\begin{aligned}
&E\left(f(A_{T-},A_T,X_{T-},X_T,T)g(Y_{A_{T-}+u}-Y_{A_{T-}},0\le u\le \Delta A_T)\right)\\
&\quad = E\left(E\left(f(A_{T-},A_T,X_{T-},X_T,T)g(Y_{A_{T-}+u}-Y_{A_{T-}},0\le u\le \Delta A_T)\middle|\, A,X\right)\right)\\
&\quad = E\left(f(A_{T-},A_T,X_{T-},X_T,T)\int g(\omega)Q^{\Delta A_T}_{0\to\Delta X_T}(d\omega)\right)\\
&\quad = \int_{(0,\infty)^2\times\mathbb{R}^2\times(0,\infty)}\int f(a,a+z,x,y,t)g(\omega)Q^z_{0\to y-x}(d\omega)\\
&\qquad\qquad\qquad\qquad P(A_{T-}\in da,\Delta A_T\in dz,X_{T-}\in dx,X_T\in dy,T\in dt).
\end{aligned}
$$

to conclude by a linear transformation

$$P\left((Y_{a+u})_{0\le u\le z}\in d\omega \middle|\, A_{t-}=a,A_t=a+z,X_{t-}=x,X_t=y\right)=Q^z_{x\to y}(d\omega)$$

in the sense introduced in the statement of Theorem 2. □

The same description is also valid in the setting of Theorem 3. A convenient way to present this is via the decomposition $X\sim B\circ\alpha+\tilde{B}\circ\gamma$ into independent continuous and discontinuous time-change given in the proof of Theorem 3. Then (B,α) is deterministic given X and the remainder of the conditional law of (B,γ), trivially conditionally independent of (B,α), can be given in terms of (Brownian) bridges as above.

Acknowledgements

Ole E. Barndorff-Nielsen suggested to me to study the recovery problem for Lévy processes. I also thank him for valuable comments on an early draft of the paper. Theorem 3 was motivated by a discussion with Neil Shephard. Marc Yor suggested some improvements of the presentation.

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