Chapter 1

Introduction and preliminaries

1.1 Introduction

J. Tits' theory of buildings for groups of Lie type (see e.g. [30] or Chapter 11 of [3] for a survey) and their more general companions, the diagram geometries mainly developed by F. Buekenhout (see e.g. various Chapters of [3] or [25] for a survey) provide a major tool for understanding the interplay between groups and geometries. Since then, various traces have been pursued. For example, given a particular group G, classify all geometries (under certain assumptions) for G (see e.g. [7], also their list of references), or classify all geometries and their automorphism groups having a diagram of a certain type (see e.g. [14]), or use a particular geometry to characterize its automorphism group. In this last branch various (computer-free) existence and uniqueness proofs for sporadic groups have been completed in the past few years (see [1], [16], [28] and [29] for computer-free proofs or [19]). Such proofs follow a two step program:

- Prove that the geometry is simply connected (giving the uniqueness) and
- construct a suitable and faithful representation of the amalgam of the geometry in some $GL(V)$ (giving the existence).

The origin of this thesis was to give such an existence and uniqueness proof for the sporadic O'Nan group ($O'N$) - discovered by M. O'Nan in [24] - using the two known flag-transitive geometries for that group. The importance of such a construction is simply given by the fact that the group $O'N$ is the only sporadic group which has not been contructed computer-freely up to now.

Unfortunately, this original attempt failed. The main reason for this lies in the subgroup structure of $O'N$. Its maximal subgroups are themselves quite small or their maximal subgroups are small. This leads to e.g. to the fact that, if one tries to get the point-line graph of a geometry under control, one may have good control over its points (in the sense that we have a small permutation degree) but for the lines there is complete wilderness. Also the representation which was chosen suits the geometry but the module carries no further structure like a form respected by $O'N$.

Nevertheless, this thesis presents some new results. The following will be proved:

- The amalgams of the geometries of Buekenhout and of Ivanov and Shpectorov for the group $O'N$ are uniquely determined by their diagram and residues of rank $n-1$.
- The Buekenhout geometry for $O'N$ is simply connected.
- The 3-fold cover for $3O'N$ of the Ivanov-Shpectorov geometry for $O'N$ is its universal cover.
- Every completion of the amalgam of the Buekenhout geometry has an irreducible 154-dimensional $GF(3)$-module.
- Every completion of that amalgam is also a completion of the amalgam related to the Ivanov-Shpectorov geometry.

The proofs of the first result are computer-free. The proofs for the simply connectedness of the geometries for $O'N$, resp. $3O'N$ involve computer use for coset enumeration. The computer is also involved (but just in a small way) in the construction of the representation. Furthermore this construction does not make use of the fact that the universal completion of the amalgam related the Buekenhout geometry is $O'N$. This is also not used in the last chapter of the thesis proving that any completion of this amalgam also acts on the Ivanov-Shpectorov geometry.

1.2 Preliminaries

For the basic definitions for (coset) geometries, diagrams and (universal) coverings we refer to [25], [5] or [3]. We will briefly state the most important group theoretic tools for this thesis.

Definition 1.2.1 [10] Let I be a finite set. An *amalgam* $\mathcal{A}$ consists of a family $(G_J)_{J\subset I}$ of groups and a family of group homomorphism $\delta_{JK} : G_J \to G_K$ for every pair $J, K \subset I$ with $K \subset J$ satisfying the following conditions:

1. For all $J, K, L \subset I$ with $L \subset K \subset J$ the composite $\delta_{JK}\delta_{KL}$ equals δ_{JL}.
2. We have $\delta_{JJ} = id$ for every $J \subset I$.

We shall give an easy example for this definition. Let Γ be some geometry of rank n and $C = \{x_1, x_2, \ldots x_n\}$ be a chamber in Γ. Let G be a flag-transitive group of type preserving automorphisms of Γ and set for $J \subset \{1, 2, \ldots, n\} =: I$ the group $G_J := G_{\{x_i : i\in J\}}$. Then $(G_J)_{J\subset I}$ forms an amalgam with δ_{JK} being the inclusion mapping for all $J, K \subset I$ with $K \subset J$.

Definition 1.2.2 [10] A *completion* of an amalgam $\mathcal{A}$ is a group G and a familiy of homorphisms $\eta_J : G_J \to G$ for all $J \subset I$ such that:

1. $\eta_J = \delta_{JK}\eta_K$ for all $K \subset J$ and
2. $G :=< G_J\eta_J : J \subset I >$.

For two completions G and $\hat{G}$ with homomorphisms η_J and $\hat{\eta}_J$ of $\mathcal{A}$ a *morphism of completions* is a homomorphism $\psi : G \to \hat{G}$ such that $\hat{\eta}_J = \eta_J\psi$ for all $J \subset I$. A completion of $\mathcal{A}$ is called *universal* if and only if there is a unique morphism of completions from it to any given completion.

Returning to our above example of the amalgam $(G_J)_{J\subset I}$ of a flag-transitive geometry Γ, we see that G is a completion of $(G_J)_{J\subset I}$ if and only if $G =< G_1, G_2, \ldots, G_n >$ which holds if Γ is connected.

The existence of the universal completion of an amalgam is ensured by the following (see e.g. [10]):

Proposition 1.2.3 *Let $\mathcal{A}$ be an amlgam. Then $\mathcal{A}$ has a universal completion (possibly infinite), unique up to isomorphism of completions.*

□

The next proposition establishes a connection between universal covers of flag-transitive geometries and the universal completions of the related amalgams:

Proposition 1.2.4 *[26], [31] Let Γ be a geometry and let $G \leq Aut\Gamma$ be flag-transitive. Denote by $\mathcal{A}$ the amalgam of maximal parabolic subgroups associated with the action of G on Γ and by $U(\mathcal{A})$ the universal completion of $\mathcal{A}$. Then $\Gamma(U(\mathcal{A}), \mathcal{A})$ is the universal cover of $\Gamma(G, \mathcal{A})$.*

□

In the following, we describe a technique to determine the universal completion $U(\mathcal{A})$ of some amalgam $\mathcal{A} = (G_J)_{J\subset I}$ (see [27], also [14] or [25]) in terms of generators and relations.

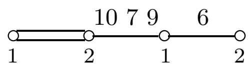
For every $i \in I$ denote by $\mathcal{X}_i$ a set of generators and by $\mathcal{R}^i_{xy}$ a set of relations between the elements of $\mathcal{X}_i$ such that $G_i \simeq< \mathcal{X}_i : \mathcal{R}^i_{xy} >$. Put moreover $\mathcal{X}_U := \bigcup_{i\in I} \mathcal{X}_i$ and $\mathcal{R}^U_{xy} := \bigcup_{i\in I} \mathcal{R}^i_{xy}$. Then we find

$$U(\mathcal{A}) \simeq< \mathcal{X}_U : \mathcal{R}^U_{xy} >.$$

Note that every relation in $\mathcal{R}^U_{xy}$ holds in at least one of the groups G_i.

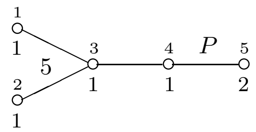
1.2.1 The known geometries for the O'Nan sporadic group

In his 1985-paper [4], Buekenhout gives the diagram of a rank four geometry Γ admitting $O'N$ as a flag-transitive automorphism group. This geometry, no. (102) in the notation of [4], can be constructed from two geometries of rank three for the groups $L_3(7) : \mathbb{Z}_2$ (no. (100) of the list in [4]) and for $\mathbb{Z}_4 L_3(4) : \mathbb{Z}_2$ (no. (101) in [4]). Note that the latter group is the centralizer of an involution in $O'N$. The Buekenhout diagram for the $O'N$-geometry is the following:



If we denote the maximal parabolic subgroups of this geometry by G_1, G_2, G_3 and G_4, from the left to the right of the diagram nodes, we have $G_1 \simeq L_3(7) : \mathbb{Z}_2$ and $G_4 \simeq \mathbb{Z}_4 L_3(4) : \mathbb{Z}_2$. Note that the involutions in $G_4 - G_4'$ are unitary. This will be sometimes indicated by writing $G_4 \simeq \mathbb{Z}_4 L_3(4) : 2_1$ following the notation of [8] or [22].

In 1986, a second geometry, now of rank five, was found by A. Ivanov and S. Shpectorov [17] admitting the group $O'N$ as a flag-transitive automorphism group. This geometry involves the Petersen graph as a residue of rank two. Its diagram is the following:



The maximal parabolics are $G_1 \simeq J_1$, $G_2 \simeq M_{11}$ and $G_5 \simeq (\mathbb{Z}_4 * Q_8 * Q_8) : A_5$ the latter group being a maximal parabolic of $\mathbb{Z}_4 L_3(4)$. This geometry also admits a 3-fold cover with automorphism group $3O'N$ such that its center acts as a deck transformation group (see e.g. [18]).

For both geometries it is not known whether they are simply connected, resp. the 3-fold cover is universal. This will be shown in this thesis using the technique to determine the universal completions of amalgams which was described above.