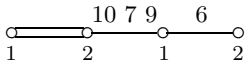
Chapter 2

Generators and relations for the Buekenhout geometry

2.1 The geometry of Buekenhout for the O'Nan group

In this section we give the corresponding amalgam for Buekenhout's geometry [4]. To recall, this is a geometry of rank four having the following Buekenhout diagram.



We denote the maximal parabolics of this geometry by G_1, G_2, G_3 and G_4, read from the left to the right of the diagram nodes. In [4], we see that $G_1 \simeq L_3(7) : \mathbb{Z}_2$ and $G_4 \simeq \mathbb{Z}_4 L_3(4) : \mathbb{Z}_2$. The complete amalgam of the residue Γ_4 for $G_4/K_4 \simeq L_3(4) : \mathbb{Z}_2$ (where $K_4 = Z(G_4')$ denotes the kernel of the action of G_4) is given in [11]. Since $G_{14} = G_1 \cap G_4$ is the centralizer of an inner involution of G_1, we get $G_{14} \simeq (\mathbb{Z}_4 * SL_2(7)) : \mathbb{Z}_2$ (see e.g. [8]). Futhermore, since any outer automorphism of G_4' in G_4 inverts the center of G_4', we obtain $B = G_{1234} \simeq \mathbb{Z}_2 \times D_8$. Then [11] implies $G_{34} \simeq (\mathbb{Z}_3^2 : \mathbb{Z}_4) \times D_8$, $G_{134} \simeq S_3 \times D_8$ and $G_{234} \simeq \mathbb{Z}_4 \times D_8$. In all these groups the direct factor D_8 is generated by K_4 and an automorphism of G_4. From [4] we draw that G_{12} must be the centralizer in G_1 of an outer involution, thus, $G_{12} \simeq \mathbb{Z}_2 \times PGL_2(7)$ (see e.g. [8]). Since we have $|G_{124}| = 2^5$, we have therefore $G_{124} \in Syl_2(G_{12})$, hence $G_{124} \simeq \mathbb{Z}_2 \times D_{16}$. This implies that the structure of G_{24} can be described as $G_{24} \simeq (\mathbb{Z}_4 \times D_8) : \mathbb{Z}_2$. By [4], we hold that $G_{13} \simeq S_3 \times S_4$ where the direct factor S_4 is not contained in G_1'. Now G_2 is described in [4] as a group of shape $([2] \times PGL_2(7))2$. Since $G_{12} \leq G_2 \geq G_{24}$, we can describe the structure of $G_2 \simeq (\mathbb{Z}_4 \times L_2(7)) : \mathbb{Z}_2$. Then G_2 acts on a geometry which is the direct sum of a folded projective plane of order two with an isolated node. Thus $|G_2 : G_{23}| = 14$ and $G_{23} \simeq \mathbb{Z}_4 \times S_4$. Now the geometry for G_3 is also some direct sum of a 3×3-grid and some isolated node implying $G_3 \simeq (\mathbb{Z}_3^2 : \mathbb{Z}_4) \times S_4$ because of its subgroups.

We will construct a presentation of the universal completion of the amalgam $\mathcal{A}$ of the Buekenhout geometry and establish that this completion is the group $O'N$. In order to achieve this, we derive a representation for G_1, G_2, G_3 and G_4. Since the residues for G_2 and G_3 are direct sums of geometries, we get $G_2 =< G_{12}, G_{23}, G_{24} >$ and $G_3 =< G_{13}, G_{23}, G_{34} >$. By a result of Ronan [27], the residues for G_1 and G_4 have an infinite universal cover. Therefore we have to identify the groups $L_3(7) : \mathbb{Z}_2 \simeq G_1 \leq < G_{12}, G_{13}, G_{14} >$ and $\mathbb{Z}_4 L_3(4) : \mathbb{Z}_2 \simeq G_4 \leq < G_{14}, G_{24}, G_{34} >$. This will be done by giving a canonical presentation for these groups and indentifications of the canonical generators as words in the geometric generators for G_1 and, respectively, the geometric generators as words in the canonical ones for G_4.

2.2 Generators and relations for Buekenhout's geometry

In this section we draw generators and relations from the amalgam of Buekenhout's geometry for the O'Nan group.

2.2.1 A presentation for the Borel subgroup

The Borel subgroup B of the Buekenhout geometry is isomorphic to $\mathbb{Z}_2 \times D_8$. We give a presentation of $B =< z, X, x >$ such that the generator z will become the center of the group $G_4' \simeq \mathbb{Z}_4 L_3(4)$ and X become an automorphism of that group. This implies the following relations:

$$z^4 = X^2 = 1, z^X = z^{-1}.$$

We construct a second subgroup D_8 of B and the amalgamation via

$$x^4 = 1, x^X = x^{-1}, x^2 = z^2, [x, z] = 1.$$

Then we get

$$\mathbb{Z}_2 \times D_8 \simeq B =< z, X, x \mid z^4 = X^2 = 1, x^2 = z^2, [z, x] = 1, z^X = z^{-1}, x^X = x^{-1} >,$$
$$Z(B) =< z^2, z^{-1}x >.$$

2.2.2 Presentations for the minimal parabolic subgroups

2.2.2.1 $G_{123} \simeq \mathbb{Z}_2 \times S_4$

We add a generator Z of order four. In order to produce a group of shape S_4, we set

$$Z^4 = 1, Z^{x^2} = Z^{-1}, x^{Z^2} = x^{-1}, (xZ)^3 = 1.$$

We see that $< x^2, Z^2 > \trianglelefteq < x, Z >$, $< x^2, Z^2, xZ > \simeq A_4$ and $(xZ)^{x^2 Z^{-1}} = ZxZ^2 = Z^{-1}x^{-1}$. To generate $\mathbb{Z}_2 \times S_4$ with center $< z^{-1}x >$, we generate the second S_4 via

$$(z^2)^{xZ} = z^{-1}X, [Z, z^{-1}x] = 1.$$

Then the first relation implies $z^2Z^2 = z^{-1}X$, hence an indentification $Z^2 = zX$. The second relation implies $Z^{z^{-1}} = Z^{x^{-1}}$ and from both relations we hold $(xZ)^X = (xZ)^{z^{-1}Z^2} = x^2Zx = Z^{-1}x^{-1}$. Therefore we get a presentation of G_{123} as follows:

$$\mathbb{Z}_2 \times S_4 \simeq G_{123} =< z, X, x, Z \mid z^4 = X^2 = Z^4 = 1, x^2 = z^2, [z, x] = 1, z^X = z^{-1}, x^X = x^{-1}, Z^{x^2} = Z^{-1}, x^{Z^2} = x^{-1}, (xZ)^3 = 1, (z^2)^{xZ} = z^{-1}X, [Z, z^{-1}x] = 1 >$$

2.2.2.2 $G_{124} \simeq \mathbb{Z}_2 \times D_{16}$

We add a generator Y which also will become an automorphism of G_4' such that $Z(G_{124}) = Z(B)$. Therefore we generate a subgroup D_{16} as follows:

$$(XY)^8 = Y^2 = 1.$$

Then we identify B inside G_{124} and generate a second $\mathbb{Z}_2 \times D_8$ by

$$(XY)^2 = x, z^Y = z^{-1}.$$

Thereby we hold a presentation of G_{124} via

$$\mathbb{Z}_2 \times D_{16} \simeq G_{124} =< z, X, Y, x \mid z^4 = X^2 = Y^2 = 1, x^2 = z^2, [z, x] = 1, z^X = z^{-1}, x^X = x^{-1}, (XY)^2 = x, z^Y = z^{-1} >, Z(G_{124}) = Z(B).$$

2.2.2.3 $G_{134} \simeq S_3 \times D_8$

We add a generator ρ of order three. Furthermore the amalgam implies that $< X, z > \simeq D_8$ is the (unique) direct factor of G_{134} of shape D_8, thus

$$\rho^3 = 1, [\rho, z] = [\rho, X] = 1.$$

Therefore we set

$$\rho^x = \rho^{-1}$$

and get a presentation of G_{134}:

$$S_3 \times D_8 \simeq G_{134} =< z, X, x, \rho \mid z^4 = X^2 = \rho^3 = 1, x^2 = z^2, [z, x] = 1, z^X = z^{-1}, x^X = x^{-1}, [\rho, z] = [\rho, X] = 1, \rho^x = \rho^{-1} >.$$

2.2.2.4 $G_{234} \simeq \mathbb{Z}_4 \times D_8$

Again, $< X, z >$ is a direct factor D_8 of G_{234} and the square of the new generator a is in $Z(B) - \{z^2\}$. Thus

$$a^2 = z^{-1}x, [a, z] = [a, X] = [a, x] = 1$$

which gives the following presentation for G_{234}:

$$\mathbb{Z}_4 \times D_8 \simeq G_{234} =< z, X, x, a \mid z^4 = X^2 = 1, x^2 = z^2, [z, x] = 1, z^X = z^{-1}, x^X = x^{-1}, a^2 = z^{-1}x, [a, z] = [a, X] = [a, x] = 1 >.$$

2.2.3 Presentations for the groups generated by two minimal parabolics

2.2.3.1 $< G_{123}, G_{124} > \simeq \mathbb{Z}_2 \times PGL_2(7)$

The group $< G_{123}, G_{124} >$ acts on the folded projective plane of order two. Thus we add the Weyl relation as

$$(ZZ^Y)^3 = 1$$

and get a presentation as:

$$\mathbb{Z}_2 \times PGL_2(7) \simeq < G_{123}, G_{124} >=< z, X, x, Z, Y \mid z^4 = X^2 = Z^4 = Y^2 = 1, x^2 = z^2, [z, x] = 1, z^X = z^{-1}, x^X = x^{-1}, Z^{x^2} = Z^{-1}, x^{Z^2} = x^{-1}, (xZ)^3 = 1, (z^2)^{xZ} = z^{-1}X, [Z, z^{-1}x] = 1, (XY)^2 = x, z^Y = z^{-1}, (ZZ^Y)^3 = 1 >.$$

2.2.3.2 $< G_{123}, G_{134} > \simeq S_3 \times S_4$

The direct factor of a group $S_3 \times S_4$ of shape S_4 is uniquely determined. We proceed as follows. The group $< G_{123}, G_{134} >$ contains $G_{134} \simeq S_3 \times D_8$ with the uniquely determined direct factor $< z, X > \simeq D_8$. Thus we choose $< z, X, xZ > \simeq S_4$ as direct factor. This implies

$$[\rho, xZ] = 1$$

and yields the following presentation:

$$S_3 \times S_4 \simeq < G_{123}, G_{134} >=< z, X, x, Z, \rho \mid z^4 = X^2 = Z^4 = \rho^3 = 1, x^2 = z^2, [z, x] = 1, z^X = z^{-1}, x^X = x^{-1}, Z^{x^2} = Z^{-1}, x^{Z^2} = x^{-1}, (xZ)^3 = 1, (z^2)^{xZ} = z^{-1}X, [Z, z^{-1}x] = 1, [\rho, z] = [\rho, X] = 1, \rho^x = \rho^{-1}, [\rho, xZ] = 1 >.$$

Therefore the involution X is not contained in G_1' because it is contained in the direct factor S_4 but not in its commutator subgroup.

2.2.3.3 $< G_{123}, G_{234} > \simeq \mathbb{Z}_4 \times S_4$

We construct $< G_{123}, G_{234} >$ such that a is contained in its center. Therefore we add

$$[a, Z] = 1$$

and hold:

$$\mathbb{Z}_4 \times S_4 \simeq < G_{123}, G_{234} > = < z, X, x, Z, a \mid z^4 = X^2 = Z^4 = a^4 = 1, x^2 = z^2, [z, x] = 1, z^X = z^{-1}, x^X = x^{-1}, Z^{x^2} = Z^{-1}, x^{Z^2} = x^{-1}, (xZ)^3 = 1, (z^2)^{xZ} = z^{-1}X, [Z, z^{-1}x] = 1, a^2 = z^{-1}x, [a, z] = [a, X] = [a, x] = 1, [a, Z] = 1 >.$$

2.2.3.4 $< G_{124}, G_{134} > \simeq (\mathbb{Z}_4 * SL_2(7)) : \mathbb{Z}_2$

The group $< G_{124}, G_{134} >$ acts on the Coxeter graph for $PGL_2(7)$. This graph is defined as follows: The vertices are the anti-flags of the projective plane $PG(2, 2)$ and two anti-flags are called adjacent if the union of their point sets covers the plane. Therefore it is clear that the stabilizer of an anti-flag and the stabilizer of an edge generate a Singer cycle of the plane. Thus we add

$$(\rho z^{-1} XY)^7 = 1$$

yielding the presentation

$$(\mathbb{Z}_4 * SL_2(7)) : \mathbb{Z}_2 \simeq < G_{124}, G_{134} > = < z, X, x, Y, \rho \mid z^4 = X^2 = Y^2 = \rho^3 = 1, x^2 = z^2, [z, x] = 1, z^X = z^{-1}, x^X = x^{-1}, (XY)^2 = x, z^Y = z^{-1}, \rho^3 = 1, [\rho, z] = [\rho, X] = 1, (\rho z^{-1} XY)^7 = 1 >.$$

2.2.3.5 $< G_{124}, G_{234} > \simeq (\mathbb{Z}_4 \times D_8) : \mathbb{Z}_2$

We have to add a relation such that Y also acts on $Z(G_{234})$, hence

$$a^Y = a^{-1}.$$

This yields a presentation

$$(\mathbb{Z}_4 \times D_8) : \mathbb{Z}_2 \simeq < G_{124}, G_{234} > = < z, X, Y, x, a \mid z^4 = X^2 = 1, x^2 = z^2, [z, x] = 1, z^X = z^{-1}, x^X = x^{-1}, (XY)^2 = x, z^Y = z^{-1}, a^2 = z^{-1}x, [a, z] = [a, X] = [a, x] = 1, a^Y = a^{-1} >.$$

2.2.3.6 $< G_{134}, G_{234} > \simeq (\mathbb{Z}_3^2 : \mathbb{Z}_4) \times D_8$

For this group it is easy to see that we have to add

$$[\rho, \rho^a] = 1$$

to get a presentaion as

$$(\mathbb{Z}_3^2 : \mathbb{Z}_4) \times D_8 \simeq < G_{134}, G_{234} > = < z, X, x, \rho \mid z^4 = X^2 = \rho^3 = 1, x^2 = z^2, [z, x] = 1, z^X = z^{-1}, x^X = x^{-1}, [\rho, z] = [\rho, X] = 1, \rho^x = \rho^{-1}, a^2 = z^{-1}x, [a, z] = [a, X] = [a, x] = 1, [\rho, \rho^a] = 1 >.$$

2.3 Canonical generators and relations for $L_3(7) : 2$

In this section we give a presentation of $L_3(7) : \mathbb{Z}_2$ using the projective plane $PG(2, 7)$. We will call these generators and relations *canonical*.

2.3.1 A presentation for a maximal parabolic subgroup for $PG(2, 7)$

It is well known that the stabilizer of a point or a line of $PG(2, 7)$ in $L_3(7)$ is a maximal subgroup of shape $\mathbb{Z}_7^2 : (SL_2(7) : \mathbb{Z}_2)$ (see e.g. [8]). We will construct a presentation of such a group.

We set $\nu := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $\rho := \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$ and $x := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ as matrices over $GF(7)$. Then we have $< \nu, \rho, x > \simeq SL_2(7)$ with $x^2 \in Z(< \nu, \rho, x >)$.

Lemma 2.3.1 $G := < \nu, \rho, x \mid \nu^7 = \rho^3 = x^4 = 1, \nu^\rho = \nu^4, \rho^x = \rho^{-1}, (\nu x)^3 = x^2, [\nu, x^2] = 1 > \simeq SL_2(7)$.

Proof. We set $B := < \nu, \rho, x^2 > \simeq \mathbb{Z}_2 \times (\mathbb{Z}_7 : \mathbb{Z}_3)$ and $N := < x > \simeq \mathbb{Z}_4$ and show that these groups provide a rank one BN-pair for G. Therefore it remains to show that $G = BNB$. Since $x^2 \in B$, we need to prove that $G = B \cup BxB$. The claim follows if $B \cup BxB$ is a subgroup of G because $x, \nu, \rho \in B \cup BxB$. Thus we need to show that for any $a, b \in BxB$ we have $ab^{-1} \in BxB$. Let $a = b_1 x b_2$, $b = b_3 x b_4$. Then $ab^{-1} = b_1 x b_2 b_4^{-1} x^{-1} b_3^{-1} = b_1 x b_2' x b_3'$ with $b_2' = b_2 b_4^{-1}$ and $b_3' = x^2 b_3^{-1}$. Now x normalizes $< \rho, x^2 >$, thus we only need to prove $x\nu^i x \in BxB \cup B$ for any power of ν. Since $\nu^\rho = \nu^4$ and $x\nu^{-1}x = (x\nu x)^{-1}$, we only have to verify $x\nu x \in BxB$. But we have $(\nu x)^3 = x^2$, hence $x\nu x = \nu^{-1} x \nu^{-1}$ and the lemma is proved. □

Clearly, an involutory automorphism of our (matrix) group $SL_2(7)$ is $i := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This provides a presentation of $SL_2(7) : \mathbb{Z}_2$ as

$$SL_2(7) : \mathbb{Z}_2 \simeq < \nu, \rho, x, i \mid \nu^7 = \rho^3 = x^4 = i^2 = 1, \nu^\rho = \nu^4, \rho^x = \rho^{-1}, (\nu x)^3 = x^2, [\nu, x^2] = 1, [\rho, i] = 1, x^i = x^{-1}, \nu^i = \nu^{-1} >.$$

We add a natural module $\mathbb{Z}_7^2 \simeq < v_1, v_2 >$ to this group via the identification $v_1 \sim (1,0)$ and $v_2 \sim (0,1)$. Thus

$$\mathbb{Z}_7^2 : (SL_2(7) : \mathbb{Z}_2) \simeq < \nu, \rho, x, i, v_1, v_2 \mid \nu^7 = \rho^3 = x^4 = i^2 = v_1^7 = v_2^7 = 1, \nu^\rho = \nu^4, \rho^x = \rho^{-1}, (\nu x)^3 = x^2, [\nu, x^2] = 1, [\rho, i] = 1, x^i = x^{-1}, \nu^i = \nu^{-1}, [v_1, v_2] = [v_1, \nu] = 1, v_2^\nu = v_1 v_2, v_1^\rho = v_1^2, v_2^\rho = v_2^4, v_1^x = v_2, v_2^x = v_1^{-1}, [v_1, i] = 1, v_2^i = v_2^{-1} >.$$

2.3.2 Adding the graph automorphism

We add a graph automorphism u to the presentation of $\mathbb{Z}_7^2 : (SL_2(7) : \mathbb{Z}_2)$ obtained in the previous subsection such that $< \nu, \rho, x, i, v_1, v_2 > \cap < \nu, \rho, x, i, v_1, v_2 >^u$ is a Borel group of $L_3(7)$. Thus we construct u such that u normalizes $\mathbb{Z}_7^{1+2} \simeq < v_1, v_2, \nu >$ and maps $< v_1, v_2 >$ to $< v_1, \nu >$. Moreover we can choose u to fulfill $[u, \rho] = 1$ and $(x^2)^u = x^2 i$. Then it remains to add the Weyl relation, e. g. $(xx^u)^3 = 1$. Thus we get

$$L_3(7) : \mathbb{Z}_2 \simeq < \nu, \rho, x, i, v_1, v_2, u \mid \nu^7 = \rho^3 = x^4 = i^2 = v_1^7 = v_2^7 = u^2 = 1, \nu^\rho = \nu^4, \rho^x = \rho^{-1}, (\nu x)^3 = x^2, [\nu, x^2] = 1, [\rho, i] = 1, x^i = x^{-1}, \nu^i = \nu^{-1}, [v_1, v_2] = [v_1, \nu] = 1, v_2^\nu = v_1 v_2, v_1^\rho = v_1^2, v_2^\rho = v_2^4, v_1^x = v_2, v_2^x = v_1^{-1}, [v_1, i] = 1, v_2^i = v_2^{-1}, v_1^u = v_1^{-1}, v_2^u = \nu, \nu^u = v_2, [\rho, u] = 1, (x^2)^u = x^2 i, (xx^u)^3 = 1 >.$$

2.3.3 3×3-matrices for the canonical generators

In this subsection we identify our canonical generators with the corresponding 3×3-matrices in $L_3(7)$. These matrices will be used to identify the geometric generators as words in the canonical ones.

Our construction from the above subsections suggests the following identification: $\nu = \begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&1&1 \end{pmatrix}$, $\rho = \begin{pmatrix} 1&0&0 \\ 0&2&0 \\ 0&0&-3 \end{pmatrix}$, $x = \begin{pmatrix} 1&0&0 \\ 0&0&1 \\ 0&-1&0 \end{pmatrix}$ and $i = \begin{pmatrix} -1&0&0 \\ 0&1&0 \\ 0&0&-1 \end{pmatrix}$.

Set $v_1 := (a_{ij})$. We construct v_1 and v_2 such that $< v_1, v_2, \nu, \rho, x, i >$ fixes the vector $(1,0,0)$. Thus $a_{12} = a_{13} = 0$. Using the relations $[\nu, v_1] = [i, v_1] = 1$, we hold $v_1 = \begin{pmatrix} a_{11}&0&0 \\ 0&a_{22}&0 \\ a_{31}&0&a_{33} \end{pmatrix}$. Proceeding with $v_1\rho = \rho v_1^2$, we get $a_{ii} = a_{ii}^2$, hence $a_{ii} = 1$ (for $i = 1, 2, 3$), and a_{31} to be some non-zero element in $GF(7)$. Therefore we set $v_1 = \begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 1&0&1 \end{pmatrix}$. By $v_1^x = v_2$, this implies $v_2 = \begin{pmatrix} 1&0&0 \\ -1&1&0 \\ 0&0&1 \end{pmatrix}$.

2.4 Identification of the geometric generators

In this section we identify the geometric generators for $L_3(7) : \mathbb{Z}_2$ as words in the canonical generators. For this purpose we use the relations as given in 2.3.2 and Section 2.2 as well as the matrices obtained in the previous subsection. Since u cannot be expressed as a 3×3-matrix we also use $(x^2)^u = x^2 i = \begin{pmatrix} 0\,1\,0 \\ -1\,0\,0 \\ 0\,0\,1 \end{pmatrix}$ and $\rho^u = 2\rho = \begin{pmatrix} 2 & 0\,0 \\ 0\,-3\,0 \\ 0 & 0\,1 \end{pmatrix}$ since we compute inside $SL_3(7)$ using the matrices.

Set $w_1 := v_1^{-1}$, $w_2 := \nu$ and $\omega := v_2$. Then x^u acts in the same way on $< w_1, w_2, \omega >$ as x on $< v_1, v_2, \nu >$. Therefore x^u has to fulfill $w_1^{x^u} = w_2$, $w_2^{x^u} = w_1^{-1}$, $(\rho^u)^{x^u} = (\rho^u)^{-1}$ and $(x^2)^u = x^2 i$. Set $x^u = (a_{ij})$. Then we use these relations and compute $x^u = \begin{pmatrix} 0\,-1\,0 \\ 1 & 0\,0 \\ 0 & 0\,1 \end{pmatrix}$.

2.4.1 The generators z, x and ρ

We construct z such that $z^2 = t := x^2$ because $t \in Z(< \nu, \rho, x, i >)$ and the group $M :=< z, X, Y, \rho >$ will be the centralizer of z^2 in $L_3(7) : \mathbb{Z}_2$. Set $z' := ut$. Then $(z')^2 = utut = i$ and $o(z') = 4$. Moreover $x^i = x^{-1}$ leads to $i^x = ti$ and therefore we set $z := (z')^{xu} = (ut)^{xu}$. This implies $M =< \nu, \rho, x, i, z >$. We set the *geometrical* generator ρ to be the *canonical* ρ because, using $[u, \rho] = 1$ and $\rho^x = \rho^{-1}$, we find $[z, \rho] = 1$, also we use the canonical x as the geometric generator x .

2.4.2 The generator X

Set $L :=< v_1, v_2, \nu, \rho, x, i, u >' \simeq L_3(7)$. Then we get $M \cap L =< \nu, \rho, x, i >$ and $C_{M \cap L}(\rho) =< \rho, t, i >$. Therefore we have $C_M(\rho) =< \rho, z, i > \simeq \mathbb{Z}_3 \times D_8$. Furthermore we must find $X \in < \rho, z, i >$ since $[\rho, X] = 1$. By 2.2.3.2, $X \notin L$. Thus we set $X := zi$ since $z^i = z^{-1}$ and $z^X = z^{-1}$ has to hold.

2.4.3 The generator Y

We construct $Y \in M \cap L$ such that $x = (XY)^2$. Since $X \notin L$, we must find $XY \notin L$. We proceed as follows. From the relations between the geometric generators we draw $[XY, z] = 1$. Thus we construct some element $e \in M' \simeq SL_2(7)$ such that $e^2 = x^{-1}$. Then $ez \notin L$ and $(ez)^2 = x^{-1}x^2 = x$. We compute e inside $L_2(7) \simeq < \nu, \rho, x \mid \nu^7 = \rho^3 = x^2 = 1, \rho^x = \rho^{-1}, \nu^\rho = \nu^4, (\nu x)^3 = 1 >$. Hence we can identify $\nu = (1234567)$, $\rho = (253)(467)$ and $x = (23)(47)$. Then $\nu^3 = (1473625)$, $x^{\nu^3} = (37)(56)$ and $xx^{\nu^3} =$

$(2743)(56)$. Furthermore $(xx^{\nu^3})^2 = (24)(37)$, $x^\nu = (34)(51)$ and $((xx^{\nu^3})^2)^{x^\nu} = x$. Therefore we set $e := (xx^{\nu^3})^{x^\nu}$. Using our 3×3-matrices, we get $e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & -2 & -2 \end{pmatrix}$ with $e^2 = x^{-1}$. Thus our 'candidates' for Y are $Y_j = zi(ez)^j = tie^j$, $j = 1, 5$ and $t = z^2$. For $j = 5$ we get $Y_5 = ie$. Using our matrices, we get $Y_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & -2 \end{pmatrix}$ and $Y_5 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & 2 \end{pmatrix}$.

Now Y must fulfill the relation $(\rho z^{-1}XY)^7 = 1$. With $z^{-1}X = i$ we compute $\rho i Y_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 3 \\ 0 & 1 & 1 \end{pmatrix}$ and $(\rho i Y_1)^7 = t$ since $\begin{pmatrix} -3 & 3 \\ 1 & 1 \end{pmatrix}^7 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Thus for Y_5 we get $\rho i Y_5 = t\rho i Y_1$. Therefore we set $Y = ie = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & 2 \end{pmatrix}$.

2.4.4 The generator Z

We set $H :=< z, X, Y, Z >$. Then $H \cap L$ must be isomorphic to $PGL_2(7)$. Since $Z(H) =< z^{-1}x >$ has to hold, we compute the centralizer of $z^{-1}x$ in L. Since $x^Y = x^{-1}$, we have $D_{16} \simeq < Y, i, x > \leq H \cap L$. Using the relations between the canonical generators, we find $z^{-1}x = (ux)^3$. Then we get $[x^u, (ux)^3] = (xu)^2 t (ux)^2 = xtuiiux = 1$. Thus $H \cap L =< Y, i, x, x^u >$. Furthermore the relations imply $[u, (ux)^3] = (xu)^2 t (ux)^2 = 1$ so $Y^u \in H \cap L$. We set $Z' := (iY^u)^2 = (e^u)^2$. Again, we use our relations between the canonical generators and get $x^\nu u = uv_2^{-1}x^u v_2$ and $\nu^3 u = uv_2^3$, thus $e^u = (xx^{\nu^3})^{x^{\nu} u} = (x^u x^{uv_2^3})^{v_2^{-1}x^u v_2}$. Using the matrices, we get $e^u = \begin{pmatrix} -1 & -1 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $Z' = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (x^{-1})^u$.

Then we compute (using the matrices) $o(Z'(Z')^Y) = 3$. Since $x^i = x^{-1}$, we obtain $(Z')^{x^2} = ((x^{-1})^u)^t = x^u = (Z')^{-1}$. Furthermore $xZ' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, thus $o(xZ') = 3$. Moreover $x^{(Z')^2} = x^{ti} = x^i = x^{-1}$ and $(Z')^2 = (x^2)^u = ti = z^2 i = zX$ since $X = zi$. Then we compute, using the matrices, $[\rho, xZ] = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$, so $[\rho, xZ] = 1$ in L. Therefore we set $Z := Z'$.

2.5 Identification of the canonical generators

In this section we identify our canonical generators for $L_3(7) : \mathbb{Z}_2$ as words in the geometric ones. For this purpose, we use the matrices obtained in the previous section.

2.5.1 The generators ρ, x and i

Since the geometric ρ and the canonical ρ coincide, we keep the geometric generator ρ. The previous section implies to set $x = (XY)^2$ and $i = z^{-1}X$.

2.5.2 The generator ν

Using the matrices of the previous section, we obtain $((\rho iY)^2)^{i(\rho^{-1})^Y} = \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,1\,1 \end{pmatrix}$. Thus we get $\nu = ((\rho z^{-1}XY)^2)^{z^{-1}X(\rho^{-1})^Y}$ since $i = z^{-1}X$.

2.5.3 The generator u

We have the relation $z = (ut)^{xu}$. Thus we get $z = ux^{-1}utxu = uxuiux = x^uixu = x^ux^{-1}ui$. Therefore we obtain $X = zi = x^ux^{-1}u$. Since $x^u = Z^{-1}$, we find $X = Z^{-1}x^{-1}u = (zZ)^2u$ or $u = xZX = (XY)^2ZX$.

2.5.4 The generators v_1 and v_2

Because of our canonical relations, we set $v_2 = \nu^u$ and $v_1 = v_2^{-1}v_2^{\nu}$.

2.5.5 The presentation for $L_3(7) : \mathbb{Z}_2$

Putting together the geometric relations, the identification of the canonical generators and the canonical relations, we get the following presentation of $G_1 \simeq L_3(7) : \mathbb{Z}_2$ of the amalgam of the Buekenhout geometry:

$$L_3(7) : \mathbb{Z}_2 \simeq G_1 =< z, X, Y, Z, \rho, x, v_1, v_2, \nu, u, i \mid z^4 = X^2 = Y^2 = Z^4 = \rho^3 = 1, x^2 = z^2, z^X = z^{-1}, x^X = x^{-1}, [x, z] = [z^-1x, Z] = 1, x^{Z^2} = x^{-1}, Z^{x^2} = Z^{-1}, (xZ)^3 = 1, (z^2)^{xZ} = z^{-1}X, x = (XY)^2, z^Y = z^{-1}, [z, \rho] = [X, \rho] = 1, \rho^x = \rho^{-1}, (ZZ^Y)^3 = 1, [\rho, xZ] = 1, (\rho z^{-1}XY)^7 = 1, i = z^{-1}X, \nu = ((\rho z^{-1}XY)^2)^{z^{-1}X(\rho^{-1})^Y}, u = xZX, v_2 = \nu^u, v_1 = v_2^-1v_2^{\nu}, \nu^\rho = \nu^4, [\rho, i] = 1, (\nu x)^3 = x^2, \nu^i = nu^{-1}, [\nu, x^2] = 1, x^i = x^{-1}, v_1^u = v_1^{-1}, v_2^u = \nu, \nu^u = v_2, [\rho, u] = 1, (x^2)^u = x^2i, (xx^u)^3 = 1 >.$$

2.6 Canonical generators and relations for $L_3(4) : 2_1$

In this section we construct a presentation of $L_3(4)$ extended by a unitary polarity using the projective plane of order four. Again, we will call these generators and relations the *canonical* ones.

2.6.1 A presentation for a maximal parabolic subgroup for $PG(2,4)$

We give a presentation of a point or line stabilizer of $PG(2,4)$ in $L_3(4)$. This is a group of shape $\mathbb{Z}_2^4 : A_5$ where its O_2 is the natural $L_2(4)$-module.

We set $\tilde{a} := \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix}$, $\rho_n := \begin{pmatrix} \xi^2 & 0 \\ 0 & \xi \end{pmatrix}$ and $\tau := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ as matrices over $GF(4)$ where $< \xi >:= GF(4)^*$. Then clearly, $A_5 \simeq L_2(4) \simeq < \tilde{a}, \rho_n, \tau >$. This leads to the following presentation:

$$A_5 \simeq < \tilde{a}, \rho_n, \tau \mid \tilde{a}^2 = \rho_n^3 = \tau^2 = 1, (\rho_n \tilde{a})^3 = 1, \rho_n^\tau = \rho_n^{-1}, (\tau\rho_n\tilde{a})^5 = 1 >$$

where $A_4 \simeq < \tilde{a}, \rho_n >$ (see e.g. [9]), $S_3 \simeq < \rho_n, \tau >$ and $(\tau\rho_n\tilde{a})^5 = 1$ is a $(2,3,5)$-relation.

We add a natural module $\mathbb{Z}_2^4$ by the following. We identify involutions f_1, f_2, f_3 and f_5 via $f_1 \sim (1,0)$, $f_2 \sim (0,1)$, $f_3 \sim (\xi,0)$ and $f_5 \sim (0,\xi)$. Using this identification, we get a presentation of a maximal parabolic subgroup of $L_3(4)$ as

$$\mathbb{Z}_2^4 : A_5 \simeq < \tilde{a}, \rho_n, \tau, f_1, f_2, f_3, f_5 \mid \tilde{a}^2 = \rho_n^3 = \tau^2 = f_1^2 = f_2^2 = f_3^2 = f_5^2 = 1 (\rho_n\tilde{a})^3 = 1, \rho_n^\tau = \rho_n^{-1}, (\tau\rho_n\tilde{a})^5 = 1 [f_1, f_2] = [f_1, f_3] = [f_1, f_5] = 1, [f_2, f_3] = [f_2, f_5] = 1, [f_3, f_5] = 1, f_1^{\rho_n} = f_1 f_3, f_2^{\rho_n} = f_5, f_3^{\rho_n} = f_1, f_5^{\rho_n} = f_5 f_2, f_1^\tau = f_2, f_3^\tau = f_5, [f_1, \tilde{a}] = [f_3, \tilde{a}] = 1, f_2^{\tilde{a}} = f_2 f_3, f_5^{\tilde{a}} = f_1 f_3 f_5 >.$$

2.6.2 Adding the unitary polarity

We have to add a unitary polarity β. This is a polarity of $PG(2,4)$ such that every point is incident with its polar. Group theoretically β is the product of the field automorphism of $GF(4)$ with a graph automorphism of $PG(2,4)$. Then we have $C_{L_3(4)}(\beta) \simeq \mathbb{Z}_3^2 : Q \simeq U_3(2)$. Thus, if $S \in Syl_2(L_3(4))$ is normalized by β, then β acts non-trivially on $Z(S)$ and therefore β does not centralize an element of order three in $N_{L_3(4)}(S)$. Setting $N :=< N_{L_3(4)}(S), \beta >$, thus $N/S \simeq S_3$ and S has a complement in N. Thus there exists an element d of order three in N with $d^\beta = d^{-1}$.

We construct β such that β nomalizes $B :=< f_1, f_2, f_3, f_5, \tilde{a}, \rho_n >$. We set $f_4 := \tilde{a}\tilde{a}^{\rho_n}$. Then $S =< f_1, f_2, f_3, f_4, f_5, \tilde{a} >= O_2(B) \in Syl_2(L_3(4))$ with $Z(S) =< f_1, f_3 >$. We set $\rho_n^\beta := \rho_n^{-1}$. Furthermore we define the remaining β-relations such that they respect the other relations, i.e., if $\mathcal{R}(x,y)$ holds for $x, y \in \{f_1, f_2, f_3, f_5, \tilde{a}, \rho_n, \tau\}$, then also $\mathcal{R}(x^\beta, y^\beta)$ is true. We identify our generators in the following way with 3×3-matrices:

$\tilde{a} = \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,\xi\,1 \end{pmatrix}$, $\rho_n = \begin{pmatrix} 1 & 0\,0 \\ 0\,\xi^2\,0 \\ 0 & 0\,\xi \end{pmatrix}$, $\tau = \begin{pmatrix} 1\,0\,0 \\ 0\,0\,1 \\ 0\,1\,0 \end{pmatrix}$. Now $< B, \tau >$ should fix the vector $(1,0,0)$. Then our relations suggest the following: $f_1 = \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 1\,0\,1 \end{pmatrix}$, $f_3 = \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ \xi\,0\,1 \end{pmatrix}$, $f_2 = \begin{pmatrix} 1\,0\,0 \\ 1\,1\,0 \\ 0\,0\,1 \end{pmatrix}$ and $f_5 = \begin{pmatrix} 1\,0\,0 \\ \xi\,1\,0 \\ 0\,0\,1 \end{pmatrix}$.

Since β centralizes an element in $< f_1, f_3 >$, we set w.l.o.g. $[f_1, \beta] = 1$ and therefore $f_3^\beta = f_1^{\rho_n^{-1}\beta} = f_1^{\beta\rho_n} = f_1 f_3$. Now $f_2^\beta \in < f_1, f_3, f_4, \tilde{a} > - < f_1, f_3 >$. Furthermore we want $(f_2^\beta)^{\tilde{a}^\beta} = f_2^\beta f_3^\beta = f_2^\beta f_1 f_3$ to hold. Since $Z(< f_1, f_2, f_3, f_4, f_5, \tilde{a} >) =< f_1, f_3 >$, we only need to check this relation for $f_2^\beta \in < f_4, \tilde{a} >$. Assume that $f_2^\beta = \tilde{a}$. Then $\tilde{a}^\beta = f_2$ and $f_2^\beta (f_2^\beta)^{\tilde{a}^\beta} = \tilde{a}\tilde{a}^{f_2} = f_3 \neq f_1 f_3$. Assume $f_2^\beta = f_4\tilde{a}$. Then $f_5^\beta = f_2^{\beta\rho_n^{-1}} = \tilde{a}$, thus $\tilde{a}^\beta = f_5$. This yields $f_2^\beta (f_2^\beta)^{\tilde{a}^\beta} = f_4\tilde{a}(f_4\tilde{a})^{f_5} = f_1 \neq f_1 f_3$. Hence we get $f_2^\beta \in f_4 < f_1, f_3 >$ and we set w.l.o.g. $f_2^\beta := f_3 f_4$. Thereby $f_4^\beta = f_1 f_2 f_3$, $f_5^\beta = f_2^{\beta\rho_n^{-1}} = (f_3 f_4)^{\rho_n^{-1}} = f_1 f_3 f_4 \tilde{a}$. This gives $\tilde{a}^\beta = f_1 f_1 f_3 f_1 f_2 f_3 f_5 = f_1 f_2 f_5$. Adding the Weyl relation $(\tau\tau^\beta)^3 = 1$ leads to the following presentation:

$$L_3(4) : 2_1 \simeq < \tilde{a}, \rho_n, \tau, f_1, f_2, f_3, f_4, f_5, \beta \mid \tilde{a}^2 = \rho_n^3 = \tau^2 = f_1^2 = f_2^2 = f_3^2 = f_5^2 = \beta^2 = 1, (\rho_n\tilde{a})^3 = 1, \rho_n^\tau = \rho_n^{-1}, (\tau\rho_n\tilde{a})^5 = 1, [f_1, f_2] = [f_1, f_3] = [f_1, f_5] = 1, [f_2, f_3] = [f_2, f_5] = 1, [f_3, f_5] = 1, f_1^{\rho_n} = f_1 f_3, f_2^{\rho_n} = f_5, f_3^{\rho_n} = f_1, f_5^{\rho_n} = f_5 f_2, f_1^\tau = f_2, f_3^\tau = f_5, [f_1, \tilde{a}] = [f_3, \tilde{a}] = 1, f_2^{\tilde{a}} = f_2 f_3, f_5^{\tilde{a}} = f_1 f_3 f_5, f_4 = \tilde{a}\tilde{a}^{\rho_n}, [f_1, \beta] = 1, f_3^\beta = f_1 f_3, \rho_n^\beta = \rho_n^{-1}, \tilde{a}^\beta = f_5 f_2 f_1, f_2^\beta = f_3 f_4, f_5^\beta = \tilde{a} f_1 f_3 f_4, (\tau\tau^\beta)^3 = 1 >,$$

where $P_1 :=< f_1, f_2, f_3, f_5, \tilde{a}, \rho_n, \tau > \simeq \mathbb{Z}_2^4 : A_5$, $P_2 := P_1^\beta$ with $B := P_1 \cap P_2 \simeq \mathbb{Z}_2^{2+4} : \mathbb{Z}_3$, generate the commutator subgroup $L_3(4)$ because of the Weyl relation and β is a unitary polarity by construction, so $< P_1, \beta > \simeq L_3(4) : 2_1$.

2.7 Generators and relations for $\mathbb{Z}_4 L_3(4) : 2_1$

In this section we add a central generator z of order four to obtain a presentation of $\mathbb{Z}_4 L_3(4) : \mathbb{Z}_2$ such that its factor group modulo $< z >$ satisfies the relations of the previous section and we keep $< \tau, \rho_n, \tilde{a} > \simeq A_5$ as a complement of $O_2(P_1)$ in the maximal parabolic P_1. First we prove the following lemma:

Lemma 2.7.1 *Let $G \simeq \mathbb{Z}_4 L_3(4)$, $S \in Syl_2(G)$, $\bar{S} := S/Z(G)$. Then the preimage of $Z(\bar{S})$ is abelian.*

Proof. If P is a maximal parabolic of G, then $O_2(P) \simeq \mathbb{Z}_4 * Q_8 * Q_8$. Hence, if we assume that the preimage C of $Z(\bar{S})$ is non-abelian, we have that $C \simeq \mathbb{Z}_4 * Q_8$.

Let $< z >:= Z(G)$, $C =:< z, a, b >$ with $< a, b > \simeq Q_8$. Then we can choose involutions $i, j, k, l \in C_S(< a, b >)$ with $< i, j > \simeq D_8 \simeq < k, l >$, i.e. $< z, a, b, i, j >$ and $< z, a, b, k, l >$ are the preimages of the two elementary abelian subgroups of order 16 in $\bar{S}$. W.l.o.g., we have $i^k = iaz$ and $i^l = ibz^\nu$ where $\nu \in \{-1, 1\}$. Thus $i^{kl} = (iaz)^l = i^l az = ibaz^{\nu+1}$. In both cases we hold $(ibaz^{\nu+1})^2 = (ba)^2 \neq 1$, a contradiction. □

To construct a presentation of $\mathbb{Z}_4 L_3(4) : 2_1$, we start by setting $z^4 = 1$, $[z, r] = 1$ for $r \in \{f_1, f_2, f_3, f_5, \tilde{a}, \tau, \rho_n\}$, $z^\beta = z^{-1}$ and $f_i^2 = z^2$ for $i \in \{1, 2, 3, 5\}$. The above lemma yields $[f_1, f_3] = 1$ and, since $< z, f_1, f_3 >^\tau / < z > =< z, f_2, f_5 > / < z >$, $[f_2, f_5] = 1$. Now we must find $f_1^\beta = f_1 z^\epsilon$, $\epsilon \in \{0, 2\}$ and $f_3^\beta = f_1 f_3 z^\delta$, $\delta \in \{-1, 1\}$ because $o(f_1 f_3) = 2$. Then $f_3 = f_3^{\beta^2} = (f_1 f_3 z^\delta)^\beta = f_1 z^\epsilon f_1 f_3 z^\delta z^{-\delta} = f_1 z^\epsilon f_1 f_3$, hence $f_1^\beta = f_1^{-1}$. Furthermore $f_1^{\rho_n} = f_1 f_3 z^\gamma$, $\gamma \in \{-1, 1\}$ and $f_3^{\rho_n} = f_1 z^\epsilon$, $\epsilon \in \{0, 2\}$. Using the relation $\rho_n^\beta = \rho_n^{-1}$, we hold $f_1^{-1} z^\epsilon = f_3^{\rho_n \beta} = f_3^{\beta \rho_n^{-1}} = (f_1 f_3 z^\delta)^{\rho_n^{-1}} = f_1^{-1} z^{\gamma+\delta}$, thus $\epsilon = \gamma + \delta$. It is easy to see that all relations are consistent for any choice of γ and δ, so we set w.l.o.g. $f_3^\beta = f_1 f_3 z^{-1}$, $f_1^{\rho_n} = f_1 f_3 z^{-1}$ and thereby $f_3^{\rho_n} = f_1^{-1}$.

For the τ-relations, we get $f_1^\tau = f_2 z^\epsilon$ and $f_3^\tau = f_5 z^\gamma$, where $\epsilon, \gamma \in \{0, 2\}$. Moreover we have $f_2^{\rho_n} = f_5 z^\delta$, $\delta \in \{0, 2\}$, and $f_5^{\rho_n} = f_2 f_5 z^\nu$, $\nu \in \{-1, 1\}$. Using $\rho_n^\tau = \rho_n^{-1}$, we find $f_2 z^\epsilon f_5 z^\gamma z^{-1} = (f_1 f_3 z^{-1})^\tau = f_1^{\rho_n \tau} = f_1^{\tau \rho_n^{-1}} = (f_2 z^\epsilon)^{\rho_n^{-1}} = f_2 f_5 z^\nu z^\delta z^\epsilon$. Thus we hold $\gamma - 1 = \nu + \delta$. Also $f_2^{-1} z^\epsilon = (f_1^{-1})^\tau = f_3^{\rho_n \tau} = f_3^{\tau \rho_n^{-1}} = (f_5 z^\gamma)^{\rho_n^{-1}} = (f_2 f_5 z^\nu)^{\rho_n} z^\gamma = f_5 z^\delta f_2 f_5 z^{2\nu} z^\gamma$. This yields $\epsilon + 2 = \delta + \gamma$. Furthermore $f_2 f_5 z^\nu z^\delta = (f_5 z^\delta)^{\rho_n} = f_3^{\tau \rho_n} = f_3^{\rho_n^{-1} \tau} = (f_1^{-1})^{\rho_n \tau} = (f_1 f_3 z)^\tau = f_2 z^\epsilon f_5 z^\gamma z$. Hence $\nu + \delta = \epsilon + \gamma + 1$. Using the above equation, we get $\gamma - 1 = \epsilon + \gamma + 1 \Leftrightarrow 2 = \epsilon$. This gives $\delta + \gamma = 0 \Leftrightarrow \delta = \gamma$ and $\gamma - 1 = \delta + \nu \Leftrightarrow \nu = -1$. Thus $f_1^\tau = f_2^{-1}$ and $f_5^{\rho_n} = f_2 f_5 z^{-1}$. We can also see that the relations are consistent for any choice of γ, so we set w.l.o.g. $f_3^\tau = f_5$ which implies $f_2^{\rho_n} = f_5$.

Now we settle the relations $[f_i, f_j]$. Since the O_2 of a maximal parabolic in $\mathbb{Z}_2 L_3(4)$ is elementary abelian, we hold that $[f_i, f_j] \in \{1, z^2\}$. Moreover $\mathbb{Z}_4 * Q_8 * Q_8$ has no abelian subgroup of order 2^5. Therefore we obtain that at least one of $[f_1, f_2]$ and $[f_2, f_3]$, one of $[f_1, f_2]$ and $[f_1, f_5]$, one of $[f_1, f_5]$ and $[f_3, f_5]$, and one of $[f_3, f_5]$ and $[f_2, f_3]$ equals z^2. Furthermore we get $[f_2, f_3] = [f_2^{\rho_n}, f_3^{\rho_n}] = [f_5, f_1^{-1}] = [f_1, f_5]$ and $[f_3, f_5] = [f_3^{\rho_n}, f_5^{\rho_n}] = [f_1^{-1}, f_2 f_5 z^{-1}] = [f_1, f_2 f_5] = [f_1, f_5][f_1, f_2]^{f_5} = [f_1, f_5][f_1, f_2]$. This yields three possibilities: If $[f_3, f_5] = 1$, then $[f_1, f_5] = [f_1, f_2] = [f_2, f_3] = z^2$. If $[f_1, f_2] = 1$, then $[f_1, f_5] = [f_3, f_5] = [f_2, f_3] = 1$. And, if $[f_3, f_5] = [f_1, f_2] = z^2$, then $[f_1, f_5] = [f_2, f_3] = z^2$. It is easy to see that all three possibilities are consistent with the relations obtained so far and in all cases we find $< z, f_1, f_3, f_2, f_5 > \simeq \mathbb{Z}_4 * Q_8 * Q_8$. Therefore we choose w.l.o.g. $[f_3, f_5] = 1$ yielding $[f_1, f_5] = [f_1, f_2] = [f_2, f_3] = z^2$.

Adding the $\tilde{a}$-relations, we see that $f_1^{\tilde{a}} = f_1 z^\epsilon$, $f_3^{\tilde{a}} = f_3 z^\gamma$, $f_2^{\tilde{a}} = f_2 f_3 z^\delta$, where $\epsilon, \gamma, \delta \in \{0, 2\}$, and $f_5^{\tilde{a}} = f_1 f_3 f_5 z^\nu$, with $\nu \in \{-1, 1\}$ since $(f_1 f_3 f_5)^2 = 1$. Now $f_2 = f_2^{\tilde{a}^2} = (f_2 f_3 z^\delta)^{\tilde{a}} = f_2 f_3 f_3 z^\gamma z^{2\delta} = f_1 f_3 f_3 z^\gamma$, hence $f_3^{\tilde{a}} = f_3^{-1}$. This implies

$f_5 = f_5^{\tilde{a}^2} = (f_1f_3f_5z^\nu)^{\tilde{a}} = f_1^{\tilde{a}}f_3^{-1}f_1f_3f_5z^{2\nu}$, hence $[f_1, \tilde{a}] = 1$ since $z^{2\nu} = z^2$. Again, we can see that the relations are consistent for each choice of δ and ν, so w.l.o.g. we set $f_2^{\tilde{a}} = f_2f_3$ and $f_5^{\tilde{a}} = f_1f_3f_5z^{-1}$.

For the remaining relations we get $f_4 = \tilde{a}\tilde{a}^{\rho_n}z^\delta$, with $\delta \in \{-1, 1\}$, and $f_2^\beta = f_3f_4z^\epsilon$, where $\epsilon \in \{-1, 1\}$ since $(f_3\tilde{a}\tilde{a}^{\rho_n})^2 = z^2$. Then we hold $f_4^\beta = (f_3^{-1})^\beta f_2z^\epsilon = f_1f_3f_2z^{1+\epsilon}$. Moreover $f_5^\beta = f_2^{\rho_n\beta} = f_2^{\beta\rho_n^{-1}} = (f_3f_4)^{\rho_n^{-1}}z^\epsilon = f_1f_3z\tilde{a}^{\rho_n^{-1}}\tilde{a}z^{\delta+\epsilon} = f_1f_3\tilde{a}\tilde{a}^{\rho_n}z^\delta\tilde{a}z^{1+\epsilon} = \tilde{a}f_1f_3f_4z^{\epsilon-1}$. Thus for $\epsilon = 1$, we hold $f_5^\beta = \tilde{a}f_1f_3f_4$, and $f_5^\beta = \tilde{a}f_1f_3f_4z^2$ if $\epsilon = -1$. Let us abreviate this as $f_5^\beta = \tilde{a}f_1f_3f_4z^\gamma$. Then $\tilde{a}^\beta = f_5(f_4^{-1})^\beta(f_3^{-1})^\beta f_1z^\gamma = f_5z^{-\epsilon}f_2^{-1}f_3^\beta(f_3^{-1})^\beta f_1z^\gamma = f_5f_2f_1z^{\gamma-\epsilon-2}$. Thus if $\epsilon = 1$, we get $\gamma - \epsilon - 2 = -3$, and, if $\epsilon = -1$, we get $\gamma - \epsilon - 2 = 1$. Therefore we obtain $\tilde{a}^\beta = f_5f_2f_1z$ in both cases. Now $f_1f_3f_2zz^\epsilon = f_4^\beta = (\tilde{a}\tilde{a}^{\rho_n}z^\delta)^\beta = \tilde{a}^\beta\tilde{a}^{\beta\rho_n^{-1}}z^{-\delta} = f_1f_3f_2zz^{-\delta}$. Thereby we hold $\delta = -\epsilon$. Moreover one can check that both choices for ϵ are consistent with all relations obtained so far. Therefore we set $f_2^\beta = f_3f_4z^{-1}$ implying $f_4 = \tilde{a}\tilde{a}^{\rho_n}z$ and $f_5^\beta = \tilde{a}f_1f_3f_4z^2$.

As a summary, we obtain the following presentation of $\mathbb{Z}_4L_3(4) : 2_1$:

$$\mathbb{Z}_4L_3(4) : 2_1 \simeq < z, \tilde{a}, \rho_n, \tau, f_1, f_2, f_3, f_4, f_5, \beta \mid \tilde{a}^2 = \rho_n^3 = \tau^2 = \beta^2 = z^4 = 1, f_1^2 = f_2^2 = f_3^2 = f_5^2 = z^2, (\rho_n\tilde{a})^3 = 1, \rho_n^\tau = \rho_n^{-1}, (\tau\rho_n\tilde{a})^5 = 1, [f_1, f_3] = [f_2, f_5] = [f_3, f_5] = 1, [f_1, f_2] = [f_1, f_5] = [f_2, f_3] = z^2, [f_1, z] = [f_2, z] = [f_3, z] = [f_5, z] = 1, [\rho_n, z] = [\tau, z] = [\tilde{a}, z] = 1, z^\beta = z^{-1}, f_1^{\rho_n} = f_1f_3z^{-1}, f_2^{\rho_n} = f_5, f_3^{\rho_n} = f_1^{-1}, f_5^{\rho_n} = f_2f_5z^{-1}, f_1^\tau = f_2^{-1}, f_3^\tau = f_5, [f_1, \tilde{a}] = 1, f_3^{\tilde{a}} = f_3^{-1}, f_2^{\tilde{a}} = f_2f_3, f_5^{\tilde{a}} = f_1f_3f_5z^{-1}, f_4 = \tilde{a}\tilde{a}^{\rho_n}z, f_1^\beta = f_1^{-1}, f_3^\beta = f_1f_3z^{-1}, \rho_n^\beta = \rho_n^{-1}, \tilde{a}^\beta = f_5f_2f_1z, f_2^\beta = f_3f_4z^{-1}, f_5^\beta = \tilde{a}f_1f_3f_4z^2, (\tau\tau^\beta)^3 = 1 >.$$

2.8 Identification of the geometric generators

In this section we identify the geometric generators for $\mathbb{Z}_4L_3(4) : \mathbb{Z}_2$ as words in the canonical generators. For this purpose we use the relations given above. Again, we have that $M :=< z, \rho, X, Y > \simeq (\mathbb{Z}_4 * SL_2(7)) : \mathbb{Z}_2$. We start with a maximal parabolic $P =< z, f_1, f_3, f_2, f_5, \tau, \rho_n, \tilde{a} >$ of $\mathbb{Z}_4L_3(4)$. We set $B :=< z, f_1, f_2, f_3, f_5, \rho_n, \tilde{a} >$ and $B_\beta :=< B, \beta >$. W.l.o.g., we can assume that $\bar{P} := P \cap M$ is the preimage of a maximal parabolic of $< M', z > / < z > \simeq L_3(2)$, hence $\bar{P} \simeq \mathbb{Z}_4 * GL_2(3)$.

2.8.1 The generators z, $(XY)^2$ and a

Of course, we identify the *geometric* generator z with the *canonical* z. Also we set w.l.o.g. $(XY)^2 := f_1$. Since we must have $[a, (XY)^2] = [a, f_1] = 1$, we can assume that $a \in C_B(f_1)$. Therefore we can set $a := \tilde{a}f_2f_5$, because $(\tilde{a}f_2f_5)^2 = f_2f_3\tilde{a}f_5\tilde{a}f_2f_5 = f_2f_3f_1f_3f_5z^{-1}\tilde{a}^2f_2f_5 = f_2f_1f_5^{-1}f_2f_5z^{-1} = f_2f_1f_2z^{-1} = z^{-1}f_1 = z^{-1}(XY)^2$. Since $z \in Z(B)$, we also have $[z, a] = 1$.

2.8.2 The generators X and Y

Set $\bar{X} := \beta^{f_3 f_5}$. Then $z^{\bar{X}} = z^{-1}$ and $f_1^{\bar{X}} = f_1^{f_5\beta f_3 f_5} = (f_1^{-1})^{\beta f_3 f_5} = f_1^{f_5} = f_1^{-1}$. Using our relations, we hold $a^{\bar{X}} = (f_1^{-1} f_3 \tilde{a} z f_2^{-1} f_5)^{\beta f_3 f_5} = (f_3^{-1} f_5 f_2 f_3 f_1 \tilde{a} f_1 f_3 z^2)^{f_3 f_5}$ since $f_1^{f_4} = f_1^{-1}$. Now $\tilde{a} f_1 f_3 = f_1 f_3 \tilde{a} z^2$, hence we obtain $a^{\bar{X}} = (f_3^{-1} f_5 f_2 \tilde{a})^{f_3 f_5} = f_3^{-1} f_5 f_2^{-1} f_1^{-1} f_3 \tilde{a} z = f_5 f_2 f_1^{-1} \tilde{a} z = \tilde{a} f_1 f_3 f_5 f_2 f_3 f_1^{-1} = \tilde{a} f_2 f_5 = a$. Thus we can set $X := \bar{X}$.

Setting $\bar{Y} := a\beta$, we get $a^{\bar{Y}} = a^{\beta} = f_5 f_2 f_1 f_3 \tilde{a} f_1 f_3 z^2 = f_5 f_2 \tilde{a} = a^{-1}$ and $z^{\bar{Y}} = z^{-1}$. Also we get $X\bar{Y} = f_3 f_5 \beta f_3 f_5 a\beta = f_3 f_5 f_4 z^{-1} (f_2 f_5)^{\tilde{a}} = f_3 f_5 f_4 f_2 f_1 f_5 = f_3 f_4^{f_5} f_2 f_1$. Using $[f_4, f_5] = f_3 z$, we get $X\bar{Y} = f_3 f_4 f_3 f_2 f_1 z = f_4 f_2 f_1 z^{-1}$ since $[f_3, f_4] = 1$. This implies $(X\bar{Y})^2 = f_4 f_2 f_1 f_4 f_2 f_1^{-1} = f_4 f_2 f_1 f_2 f_4$ because we have $[f_2, f_4] = f_1^{-1}$. Thus we find $(X\bar{Y})^2 = f_4 f_2 f_2^{-1} f_1 f_4 = f_4 f_1 f_4 = f_1$ since $[f_1, f_4] = z^2$. Therefore we set w.l.o.g. $Y := \bar{Y}$. Using the results obtained so far, we also see that $f_3 = [a, f_2]$.

2.8.3 The generator ρ

In order to identify our generator ρ, we construct the subgroup M explicitly using the canonical and geometric generators obtained so far.

We start to construct $\bar{P}$. Since $< f_1, f_2 > \simeq Q_8$, we can assume that $< z, f_1, f_2 > = O_2(\bar{P})$. Furthermore $f_1^{f_4} = f_1^{-1}$ and $[f_2, f_4] = f_1^{-1}$, thus $f_2^{f_4} = f_2 f_1^{-1}$. Also, τ normalizes $< z, f_1, f_2 >$. Therefore we can assume that $< z, f_1, f_2, f_4 > \in Syl_2(\bar{P})$ and $< z, f_1, f_2, \tau > \in Syl_2(\bar{P})$.

We see that $(f_4\tau)^3 = (\tilde{a}\tilde{a}^{\rho_n}\tau)^3 z^{-1}$. Using $(\tilde{a}\rho_n\tau)^3 = 1$, we hold $(\tilde{a}\tilde{a}^{\rho_n}\tau)^3 = (\tilde{a}\tau)^2 \rho_n \tilde{a} \rho_n \tau \tilde{a} \rho_n^{-1} \tilde{a} \rho_n \tau = (\tilde{a}\tau)^2 \tau \rho_n \tilde{a} \rho_n \tilde{a} \tau \rho_n^{-1} \tilde{a} = \tilde{a}\tau\tilde{a}\rho_n\tilde{a}\rho_n\tilde{a}\tau\rho_n^{-1}\tilde{a} = \tilde{a}\tau(\tilde{a}\rho_n)^3\tau\tilde{a} = 1$ because $(\tilde{a}\rho_n)^3 = 1$. Thus $o(f_4\tau) = 12$ and $(f_4\tau)^3 = z^{-1}$. So we set $\rho_1 = (f_4\tau)^4 = f_4\tau z^{-1}$ and get $f_1^{\rho_1} = f_2$, $f_2^{\rho_1} = f_1^{-1} f_2$ and $f_4 \in \bar{P} = < z, f_1, f_2, \tau, \rho_1 >$.

Moreover $< z, f_1 >^X = < z, f_1 >$ and $f_2^X = f_3 f_4^{f_5} z = f_3 f_4 f_3 z^2 = f_4$. Thus $< z, f_1, f_2, f_4 > \leq \bar{P} \cap \bar{P}^X$. Since $[\tau, f_3 f_5] = 1$, we get $o(\tau\tau^X) = o(\tau\tau^{\beta}) = 1$ and $\tau^X \notin \bar{P}$. This implies that $< z, f_1, f_2, \tau, \rho_1, X > \simeq M$. Using our relations, we get $YX = f_2 f_1 f_5 f_4 f_3 f_5 = f_2 f_1 f_4 f_5 f_3^2 f_5 z = f_2 f_1 f_4 z$. Thereby we see that we have $Y \in < z, f_1, f_2, \tau, \rho_1, X >$, so w.l.o.g. $M = < z, f_1, f_2, \tau, \rho_1, X >$.

Now our generator $\rho \in C_M(X)$. Since $f_1^{\rho_1} = f_2$, we get that f_1 inverts $(\rho_1^{-1})^{X\tau} =: \bar{\rho}$. Then $[z, \bar{\rho}] = 1$ and $\bar{\rho}^{f_1} = \bar{\rho}^{-1}$. Also, $\bar{\rho}^X = (\tau f_4^{-1} z)^{X\tau X} = (\tau f_4^{-1})^{X\tau X} z$. We find $(f_4^{-1})^{X\tau X} = (f_2^{-1})^{\tau X} = f_1^{-1}$ and $\tau^{X\tau X} = (X\tau)^3 X$. So, since $(X\tau)^6 = 1$, $\bar{\rho}^X = (X\tau)^3 X f_1^{-1} z = (\tau X)^3 X f_1^{-1} z = (\tau X)^2 \tau f_1^{-1} z = (\tau X)(\tau X) f_2 \tau z = (\tau X)(\tau f_4 z^{-1})(X\tau) = (\tau X)(\tau f_4^{-1} z)(X\tau) = \bar{\rho}$. Therefore we hold $[\bar{\rho}, X] = 1$.

Now a and $\bar{\rho}$ centralize X, thus $< z, \bar{\rho}, a, X >$ is isomorphic to a subgroup of $(\mathbb{Z}_3^2 : Q_8) \times D_8$ which is the preimage of the centralizer of X in $L_3(4) : 2_1$. This implies $[\bar{\rho}, \bar{\rho}^a] = 1$.

The relation $(\bar{\rho} X Y z^{-1})^7 = 1$ holds if and only if $(\bar{\rho} X Y)^7 = z^{-1}$. Since $XY = f_4 f_2 f_1 z^{-1}$, we get $\bar{\rho} XY = (\tau f_4^{-1} z)^{X\tau} f_4 f_2 f_1 z^{-1} = (\tau X)^2 \tau f_1 f_4 f_2 f_1 z^2 = (\tau X)^2 \tau f_4 f_2$. This implies the equalities $(\bar{\rho} XY)^2 = ((\tau X)^2 \tau f_4 f_2)^2 = (\tau X)^2 \tau f_4 f_2 \tau (X\tau)^2 f_4 f_2 =$

$(\tau X)^2\tau f_4 f_2\tau z^{-1}f_1^{-1}\tau^{X\tau X}f_1^{-1}(X\tau)^2 f_2$. The last equality holds because $(\tau f_4)^3 = z^{-1}$ implies $f_4^\tau = z^{-1}f_4^{-1}\tau f_4^{-1}$ yielding $f_4^{\tau X} = zf_2^{-1}\tau^X f_2^{-1}$ and $f_4^{(\tau X)^2} = z^{-1}f_1^{-1}\tau^{X\tau X}f_1^{-1}$. Using the fact $(X\tau)^2 f_2 = f_4^{-1}(X\tau)^2$, we obtain the following identities: $(\bar\rho XY)^2 = (\tau X)^2(\tau f_4 f_2\tau z^{-1}f_1^{-1}\tau^{X\tau X}f_1^{-1}f_4^{-1})(X\tau)^2 = (\tau X)^2(f_4^{-1}\tau f_4^{-1}\tau^{X\tau X}f_1^{-1}f_4^{-1})(X\tau)^2 = (\tau X)^2(z^{-1}\tau f_4^{-1}(\tau X)^4 f_1^{-1}f_4^{-1})(X\tau)^2 = (\tau X)^2(z^{-1}\tau f_4^{-1}(X\tau)^2 f_1^{-1}f_4^{-1})(X\tau)^2$. This implies that for $(\bar\rho XY)^3$ we hold the following equalities: $(\bar\rho XY)^3 = (\bar\rho XY)^2\bar\rho XY = (\tau X)^2(z^{-1}\tau f_4^{-1}(X\tau)^2 f_1^{-1}f_4^{-1})\tau f_4 f_2 = (\tau X)^2(\tau f_4^{-1}(X\tau)^2 f_1^{-1}f_4^{-1}\tau f_2) = (\tau X)^2(\tau f_4^{-1}(X\tau)f_4 f_2^{-1}f_1)(X\tau) = (\tau X)^2(\tau X f_2^{-1}\tau f_4 f_2^{-1}f_1)(X\tau) = (\tau X)^3\tau(f_4 f_2)(X\tau) = (\tau X)^4 f_2 f_4\tau = (X\tau)^2 f_2 f_4\tau$.

We have $(\bar\rho XY)^7 = z^{-1}$ if and only if we find for $(\bar\rho XY)^6$: $(\bar\rho XY)^6 = z^{-1}YX\bar\rho^{-1} = z^{-1}f_2 f_1 f_4 z(f_4\tau z^{-1})^{X\tau} = f_2 f_4(\tau X)^2\tau z^{-1}$. Conversely, we obtain that $(\bar\rho XY)^6 = ((X\tau)^2 f_2 f_4\tau)^2 = (f_2 f_4)^{(\tau X)^2}(f_2 f_4)^\tau = f_4^{-1}f_1^{-1}\tau^{X\tau X}f_4^{-1}\tau f_4^{-1} = f_4^{-1}f_1^{-1}(\tau X)^2\tau f_4^{-1}\tau f_4^{-1} = f_4^{-1}f_1^{-1}(\tau X)^2 f_4\tau z = f_4 f_2 f_1(\tau X)^2\tau z$. Together with $[f_4, f_2] = f_1$, this gives $(\bar\rho XY)^6 = f_2 f_4(\tau X)^2\tau z^{-1}$. Therefore we can set $\rho = \bar\rho$.

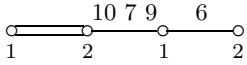
2.8.4 The presentation for $\mathbb{Z}_4 L_3(4) : \mathbb{Z}_2$

Putting together the geometric relations, the identification of the canonical generators and the canonical relations, we get the following presentation of the group $G_4 \simeq \mathbb{Z}_4 L_3(4) : \mathbb{Z}_2$ of the amalgam of the Buekenhout geometry:

$$\mathbb{Z}_4 L_3(4) : \mathbb{Z}_2 \simeq G_1 =< f_1, f_2, f_3, f_4, f_5, \tilde a, \rho_n, \tau, \beta, z, X, Y, \rho, a, x \mid z^4 = X^2 = Y^2 = \rho^3 = 1, x^2 = z^2, z^X = z^{-1}, x^X = x^{-1}, [x, z] = 1, x = (XY)^2, z^Y = z^{-1}, [z, \rho] = [X, \rho] = 1, \rho^x = \rho^{-1}, (\rho z^{-1}XY)^7, [X, a] = 1, a^2 = z^{-1}(XY)^2, [a, z] = 1, a^Y = a^{-1}, [x, a] = 1, [\rho, \rho^a] = 1, f_1 = (XY)^2, a = \tilde a f_2 f_5, X = \beta^{f_3 f_5}, Y = a\beta, \rho = (\tau f_4^{-1}z)^{X\tau}, [f_1, z] = [f_2, z] = [f_3, z] = [f_5, z] = [\tilde a, z] = [\rho_n, z] = [\tau, z] = 1, z^\beta = z^{-1}, f_1^2 = z^2, f_2^2 = z^2, f_3^2 = z^2, f_5^2 = z^2, [f_1, f_2] = z^2, [f_1, f_3] = 1, [f_1, f_5] = z^2, [f_2, f_3] = z^2, [f_2, f_5] = 1, [f_3, f_5] = 1, \tilde a^2 = \rho_n^3 = \tau^2 = 1, (\tilde a\rho_n)^3 = 1, \rho_n^\tau = \rho_n^{-1}, (\tau\rho_n\tilde a)^5 = 1, f_4 = z\tilde a\tilde a^{\rho_n}, f_1^{\rho_n} = f_1 f_3 z^{-1}, f_2^{\rho_n} = f_5, f_3^{\rho_n} = f_1^{-1}, f_5^{\rho_n} = f_5 f_2 z^{-1}, f_1^\tau = f_2^{-1}, f_3^\tau = f_5, [f_1, \tilde a] = 1, f_3^{\tilde a} = f_3^{-1}, f_2^{\tilde a} = f_2 f_3, f_5^{\tilde a} = f_1 f_3 f_5 z^{-1}, \beta^2 = 1, \rho_n^\beta = \rho_n^{-1}, \tilde a^\beta = f_5^{-1}f_2^{-1}f_1 z, f_1^\beta = f_1^{-1}, f_3^\beta = f_1 f_3 z^{-1}, f_2^\beta = f_3 f_4 z^{-1}, f_5^\beta = \tilde a f_1 f_3 f_4 z^2, (\tau\tau^\beta)^3 = 1 >.$$

2.9 A presentation for the amalgam of Buekenhout's geometry

In this section we put together all information obtained so far in order to give a presentation for the universal completion of Buekenhout's geometry Γ for the O'Nan group. We recall that this geometry has the following Buekenhout diagram:



We denote the maximal parabolics of this geometry by G_1, G_2, G_3 and G_4, read from the left to the right of the diagram nodes, with $G_1 \simeq L_3(7) : \mathbb{Z}_2$ and $G_4 \simeq \mathbb{Z}_4 L_3(4) : \mathbb{Z}_2$. Since the residues for G_2 and G_3 are just direct sums of geometries, we do not have to add further relations and obtain that for the universal completion U of the amalgam of this geometry we have $U =< G_1, G_4 \mid [a, Z] = 1 >$. Moreover the following holds:

Theorem 2.9.1 *Let Γ' be a flag-transitive geometry with the same Buekenhout diagram as Γ. Assume furthermore that for a flag-transitive automorphism group H we have $H_1/K_1 \simeq L_3(7) : \mathbb{Z}_2$ and $H_4/K_4 \simeq L_3(4) : 2_1$ where K_i denotes the kernel of the action of H_i on the corresponding residue. Then $K_1 = 1$ and $K_4 \simeq \mathbb{Z}_4$.*

Proof. The hypothesis of the theorem immediately implies that $K_1 \lhd K_4 \leq H_1$ and $K_4/K_1 \simeq \mathbb{Z}_4$ since $H_{14}/K_1 \simeq (\mathbb{Z}_4 * SL_2(7)) : \mathbb{Z}_2$. Therefore we have that K_i must be a 2-group for $i = 1, 4$ since $O^2(K_4) \leq K_1$. The assertion follows if $K_1 = 1$. Let us assume that $K_1 \neq 1$.

We prove that $Z(K_4) \leq Z(K_1)$. Assume $Z(K_4) \not\leq K_1$. Then there exists some $1 \neq x \in Z(K_4) - K_1$ and therefore there exists some $1 \neq \bar{x} \in K_4/K_1$. Since $[x, K_1] = 1$, we get $[< x^{H_1} >, K_1] = 1$. Thus we find at least $[U, K_1] = 1$ where $UK_1/K_1 = (H_1/K_1)' \simeq L_3(7)$. Set $U :=< x^{H_1} >$ and $U_{14} := H_{14} \cap U$. Then $U_{14}K_4/K_4 \simeq L_2(7) : \mathbb{Z}_2$. Since K_4/K_1 is centralized by $(H_{14}/K_1)'$ (recall that $H_{14}/K_1 \simeq (\mathbb{Z}_4 * SL_2(7)) : \mathbb{Z}_2$), we hold $[U'_{14}, K_4] = 1$. Now $C_{H_4}(K_4) \lhd H_4$ and $(U_{14}K_4/K_4)' \leq (H_4/K_4)' \simeq L_3(4)$. Together with the fact that H_{14}/K_4 contains an automorphism of $(H_4/K_4)'$ this implies $K_1 \lhd H_4$, a contradiction.

We use this fact to prove that for every $A \in \mathcal{A}(K_4)$ with $A \not\leq K_1$ we must have $[A, Z(K_1)] \neq 1$. Assume the contrary. Then $[< A^{G_1} >, Z(K_1)] = 1$. Since $K_4/K_1 \simeq \mathbb{Z}_4$, we get that at least $[U, Z(K_1)] = 1$ where U is the preimage of $(H_1/K_1)' \simeq L_3(7)$. Now $H_{14}/K_1 \simeq (\mathbb{Z}_4 * SL_2(7)) : \mathbb{Z}_2$ contains an automorphism α of U normalizing $Z(K_4)$. By $Z(K_4) \leq Z(K_1)$, we hold $Z(K_4) \lhd < U, \alpha >= H_1$, a contradiction. The same argument can be used to prove $[A, \Omega_1(Z(K_1))] \neq 1$ since $Z(K_4) \leq Z(K_1)$ implies $\Omega_1(Z(K_4)) \leq \Omega_1(Z(K_1))$.

Using $K_4/K_1 \simeq \mathbb{Z}_4$, we hold $\mid A : A \cap K_1 \mid = 2$ if $A \not\leq K_1$, $A \in \mathcal{A}(K_4)$. Since A is maximal, we get $\mid (A \cap K_1)\Omega_1(Z(K_1)) \mid \leq \mid A \mid$. Thus we have $\frac{|A \cap K_1||\Omega_1(Z(K_1))|}{|\Omega_1(Z(K_1)) \cap A|} \leq \mid A \mid$ which implies $\frac{|\Omega_1(Z(K_1))|}{|\Omega_1(Z(K_1)) \cap A|} \leq \frac{|A|}{|A \cap K_1|} = 2$. Using $[A, \Omega_1(Z(K_1))] \neq 1$, this leads to $\mid \Omega_1(Z(K_1)) : \Omega_1(Z(K_1)) \cap A \mid = 2$ proving that A induces transvections on $\Omega_1(Z(K_1))$. Let now $a \in A - \Omega_1(Z(K_1))$ and let $d \in G_1$ with $o(d) = p$, $2 \neq p$ a prime and $d^a = d^{-1}$. Since a is a transvection and $< a, d >$ is a dihedral group, we hold that $\mid \Omega_1(Z(K_1)) : C_{\Omega_1(Z(K_1))}(< d >) \mid = 4$ yielding $o(d) = 3$. But this contradicts the fact that, because $(H_1/K_1)' \simeq L_3(7)$, the involution a inverts also elements of order seven. Thereby we have a contradiction to $A \not\leq K_1$, so $A \leq K_1$ implying the final contradiction, namely $J(K_4) = J(K_1)$, proving the theorem. □

Using the MAGMA [2] function for coset enumeration we establish the following theorem:

Theorem 2.9.2 *Let* $G :=< z, X, Y, Z, \rho, x, v_1, v_2, \nu, u, i, f_1, f_2, f_3, f_4, f_5, \tilde{a}, \rho_n, \tau, \beta, \rho_1 \mid z^4 = X^2 = Y^2 = Z^4 = \rho^3 = 1, x^2 = z^2, z^X = z^{-1}, x^X = x^{-1}, [x, z] = [z^{-1}x, Z] = 1, x^{Z^2} = x^{-1}, Z^{x^2} = Z^{-1}, (xZ)^3 = 1, (z^2)^{xZ} = z^{-1}X, x = (XY)^2, z^Y = z^{-1}, [z, \rho] = [X, \rho] = 1, \rho^x = \rho^{-1}, (ZZ^Y)^3 = 1, [\rho, xZ] = 1, (\rho z^{-1}XY)^7 = 1, [X, a] = 1, a^2 = z^{-1}(XY)^2, [a, z] = 1, a^Y = a^{-1}, [x, a] = 1, [\rho, \rho^a] = 1, [Z, a] = 1, i = z^{-1}X, \nu = ((\rho z^{-1}XY)^2)^{z^{-1}X(\rho^{-1})^Y}, u = xZX, v_2 = \nu^u, v_1 = v_2^{-1}v_2^\nu, \nu^\rho = \nu^4, [\rho, i] = 1, (\nu x)^3 = x^2, \nu^i = \nu^{-1}, [\nu, x^2] = 1, x^i = x^{-1}, v_1^u = v_1^{-1}, v_2^u = \nu, \nu^u = v_2, [\rho, u] = 1, (x^2)^u = x^2 i, (xx^u)^3 = 1, f_1 = (XY)^2, a = \tilde{a}f_2f_5, X = \beta^{f_3f_5}, Y = a\beta, \rho = (\tau f_4^{-1}z)^{X\tau}, [f_1, z] = [f_2, z] = [f_3, z] = [f_5, z] = [\tilde{a}, z] = [\rho_n, z] = [\tau, z] = 1, z^\beta = z^{-1}, f_1^2 = z^2, f_2^2 = z^2, f_3^2 = z^2, f_5^2 = z^2, [f_1, f_2] = z^2, [f_1, f_3] = 1, [f_1, f_5] = z^2, [f_2, f_3] = z^2, [f_2, f_5] = 1, [f_3, f_5] = 1, \tilde{a}^2 = \rho_n^3 = \tau^2 = 1, (\tilde{a}\rho_n)^3 = 1, \rho_n^\tau = \rho_n^{-1}, (\tau\rho_n\tilde{a})^5 = 1, f_4 = z\tilde{a}\tilde{a}^{\rho_n}, f_1^{\rho_n} = f_1f_3z^{-1}, f_2^{\rho_n} = f_5, f_3^{\rho_n} = f_1^{-1}, f_5^{\rho_n} = f_5f_2z^{-1}, f_1^\tau = f_2^{-1}, f_3^\tau = f_5, [f_1, \tilde{a}] = 1, f_3^{\tilde{a}} = f_3^{-1}, f_2^{\tilde{a}} = f_2f_3, f_5^{\tilde{a}} = f_1f_3f_5z^{-1}, \beta^2 = 1, \rho_n^\beta = \rho_n^{-1}, \tilde{a}^\beta = f_5^{-1}f_2^{-1}f_1z, f_1^\beta = f_1^{-1}, f_3^\beta = f_1f_3z^{-1}, f_2^\beta = f_3f_4z^{-1}, f_5^\beta = \tilde{a}f_1f_3f_4z^2, (\tau\tau^\beta)^3 = 1 >$. *Then* $G \simeq O'N$.

□

As a corollary we obtain by Proposition 1.2.4:

Corollary 2.9.3 *The Buekenhout geometry for the O'Nan sporadic group is simply connected.*

□