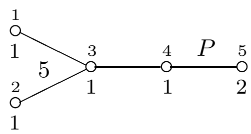
Chapter 3

Generators and relations for the Ivanov-Shpectorov geometry

3.1 The Ivanov-Shpectorov geometry for the O'Nan sporadic group

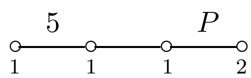
In this chapter we will give generators and relations for the universal completion of the amalgam related to the Ivanov-Shpectorov geometry for the groups $O'N$ and $3O'N$. To recall, this is a geometry of rank five with the following with the following Buekenhout diagram:



Also recall that the geometry of $3O'N$ is a triple cover of the $O'N$-geometry where $Z(3O'N)$ acts as a group of deck transformations. The maximal parabolics are $G_1 \simeq J_1$, $G_2 \simeq M_{11}$ and $G_5 \simeq (\mathbb{Z}_4 * Q_8 * Q_8) : A_5$ (G_3 and G_4 will be constructed later on). In difference to the Buekenhout geometry, we can construct a presentation for this geometry successively using simply connected geometries for $L_2(11)$, M_{11}, J_1 and $\mathbb{Z}_2^5 : A_5$.

3.2 A presentation of J_1

It is known by [15], that the group $G \simeq J_1$ acts flag-transitively on a rank four geometry Γ with the following Buekenhout diagram:



The maximal parabolic subgroups are $G_1 \simeq L_2(11)$, $G_2 \simeq \mathbb{Z}_2 \times A_5$, $G_3 \simeq S_3 \times D_{10}$ and $G_4 \simeq \mathbb{Z}_2 \times A_5$, read from the left to the right in the diagram. Furthermore we have $B = G_{1234} \simeq \mathbb{Z}_2$ and $B = Z(G_4)$ (see e.g. [12]).

It is known from [15], that Γ is 3-simply connected. We use Γ to construct a presentation for J_1. The following amalgam corresponds to Γ (see e.g. [12]): $G_{12} \simeq A_5$, $G_{13} \simeq D_{12}$, $G_{14} \simeq D_{12}$, $G_{23} \simeq D_{12}$, $G_{24} \simeq \mathbb{Z}_2^3$, $G_{34} \simeq D_{20}$, $G_{123} \simeq S_3$, $G_{124} \simeq G_{134} \simeq G_{234} \simeq \mathbb{Z}_2^2$ and $B \simeq \mathbb{Z}_2$.

We set $B =:< z >$, $G_{123} =:< z, t >$, $G_{124} =:< z, b >$, $G_{134} =:< z, a_n >$ and $G_{234} =:< z, a >$. Thereby we hold the following relations:

$$\mathcal{R}_o : z^2 = a^2 = a_n^2 = b^2 = t^3 = 1,$$

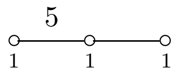
$$\mathcal{R}_m : [z, a] = [z, a_n] = [z, b] = 1, t^z = t^{-1}.$$

The diagram yields the following relations:

$$\mathcal{R}_d : [a, b] = [a, t] = [a_n, t] = (aa_n)^5 = (a_n b)^3 = (tb)^5 = 1.$$

3.2.1 Additional relations for G_4

The residue of G_4 in Γ has the Coxeter diagram:



The corresponding Coxeter group C is isomorphic to $\mathbb{Z}_2 \times A_5$ such that $Z(C)$ acts non-trivially on the geometry. Hence, using the relations obtained so far, we have $< z, a, a_n, b >\simeq \mathbb{Z}_2^2 \times A_5$.

Let $C =< a_1, a_2, a_3 \mid a_1^2 = a_2^2 = a_3^2 = (a_1a_2)^5 = (a_2a_3)^3 = 1 >$. Then $o(a_1a_2a_3) = 10$ and therefore we have $< a_1, a_2, a_3 \mid a_1^2 = a_2^2 = a_3^2 = (a_1a_2)^5 = (a_2a_3)^3 = (a_1a_2a_3)^5 = 1 >\simeq A_5$ by adding a $(2, 3, 5)$-relation to the Coxeter relations.

Since $Z(G_4) =< z >$ acts trivially on the residue of G_4, we have to distinguish two cases, namely, $a_n \in G_4'$ and $a_n \notin G_4'$. If $a_n \notin G_4'$ ($a \in G_4'$), we can assume w.l.o.g. that a and b are also not contained (are contained) in G_4'. Thus we can add either the relation

$\mathcal{R}_{(2,3,5)}.1 : (zaa_nb)^5 = 1$

or

$\mathcal{R}_{(2,3,5)}.2 : (aa_nb)^5 = 1$.

In both cases we hold $< z, a, a_n, b >\simeq \mathbb{Z}_2 \times A_5$. The correct relation will be distinguished by the amalgamation with G_3.

3.2.2 Additional relations for G_1

We have to ensure that $G_{12} =< z, b, t >\simeq A_5$. Using the relations obtained so far we hold $< z, b, t \mid z^2 = b^2 = t^3 = 1, (tb)^5 = 1, [z, b] = 1, t^z = t^{-1} >\simeq \mathbb{Z}_2 \times A_5$ where $< t, b >\simeq A_5$. We identify t with the element (123) of A_5, z with (23)(45) and b with (24)(35). Using this identification we hold $tb = (14253)$ and $[b, t] = (15423)$, thus $z = tb[b, t]^3 = bt[b, t]^2$. Therefore we have to add the relation

$\mathcal{R}_A : z = bt[b, t]^2$.

Using the relations $\mathcal{R}_d$, we find $G_{13} =< z, t, b >\simeq D_{12} \simeq< z, a_n, b >= G_{14}$. By [18], this amalgam determines the group $G_1 \simeq L_2(11)$ because it is the amalgam of a simply connected geometry.

3.2.3 Amalgamation of G_3 and G_4

Using the relations $\mathcal{R}_o$, $\mathcal{R}_m$, $\mathcal{R}_d$ and $\mathcal{R}_A$, we easily see that $a \in Z(< z, b, t, a >)$ and $< z, b, t, a >\simeq \mathbb{Z}_2 \times A_5$. Furthermore we have $G_3 =< z, t, a, a_n >\simeq S_3 \times D_{10}$ where $< z, t >$ and $< a, a_n >$ are the direct factors of G_3. It remains to analyze whether $a_n \in G_4'$ or not.

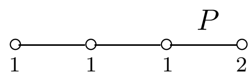
Using the subgroup lattice of J_1 (given in e.g. [12]), we see that J_1 contains a single conjugacy class of subgroups D_{20}. Given such a subgroup U of J_1, its two normal subgroups of shape D_{10} are non-conjugate in J_1. One of these two classes correspond to the direct factors of subgroups of shape $S_3 \times D_{10}$. These subgroups have no supergroup A_5 inside J_1. Thus a and a_n are not contained in G_4'. Together with the fact that Γ is simply connected [15] this yields the following lemma.

Lemma 3.2.1 *Let* $J :=< a, a_n, b, t, z \mid \mathcal{R}_o \cup \mathcal{R}_m \cup \mathcal{R}_d \cup \mathcal{R}_A \cup \mathcal{R}_{(2,3,5)}.1 >$. *Then* $J \simeq J_1$.

□

3.3 A presentation of M_{11}

By [4] it is known, that the Mathieu group M_{11} acts transitively on a geometry Γ, related to its 3-transitive action on 12 points, with the following diagram:



The corresponding maximal subgroups of $G \simeq M_{11}$ are $G_1 \simeq L_2(11)$, $G_2 \simeq S_5$, $G_3 \simeq S_3 \times S_3$ and $G_4 \simeq GL_2(3)$, read from the left to the right in the diagram (see e.g. [6]). It is shown in [18], that Γ is simply connected. Again, we use this geometry to get a presentation for G.

By the previous section, we hold $G_{12} \simeq A_5$, $G_{13} \simeq D_{12}$, $G_{14} \simeq D_{12}$, $G_{123} \simeq S_3$, $G_{124} \simeq \mathbb{Z}_2^2 \simeq G_{234}$ and $B = G_{1234} \simeq \mathbb{Z}_2$.

Since $G_4 \simeq GL_2(3)$, we find $G_{24} \simeq D_8$, $G_{34} \simeq D_{12}$, $G_{234} \simeq \mathbb{Z}_2^2$ and $B = Z(G_4)$ because we have $GL_2(3)$ acting on a geometry with the Coxeter diagram for S_4. This shows that $G_{23} \simeq D_{12}$. As in the last section we set $B =:< z >$, $G_{123} =:< z, t >$, $G_{124} =:< z, b >$ and $G_{134} =:< z, a_n >$. Moreover we set $G_{234} =:< z, v >$. Therefore we find the following relations:

$\mathcal{R}'_o : z^2 = v^2 = a_n^2 = b^2 = t^3 = 1,$

$\mathcal{R}'_m : [z, v] = [z, a_n] = [z, b] = 1, t^z = t^{-1}.$

The diagram yields the following relations:

$\mathcal{R}'_d : [v, t] = [a_n, t] = (a_n b)^3 = (va_n)^3 = (tb)^5 = 1, (vb)^2 = z.$

Since $G_{12} =< z, t, b >\simeq A_5$, we add again the relation

$\mathcal{R}'_A : z = bt[b, t]^2$

in order to get $G_1 =< z, b, t, a_n >\simeq L_2(11)$.

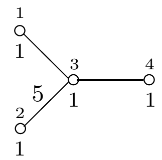
Using the relations $\mathcal{R}'_o$, $\mathcal{R}'_m$ and $\mathcal{R}'_d$, we already see that $G_4 =< z, b, a_n, v >\simeq GL_2(3)$. Furthermore it is easy to see that these relations give $G_2 =< z, t, b, v >\simeq S_5$. Obviously, we have $[t, va_n] = 1$, which is enough to prove $G_3 =< z, t, a_n, v >\simeq S_3 \times S_3$. Then by the results of [18], we have the following lemma:

Lemma 3.3.1 *Let $M :=< z, t, b, a_n, v \mid \mathcal{R}'_o \cup \mathcal{R}'_m \cup \mathcal{R}'_d \cup \mathcal{R}'_A >$. Then $M \simeq M_{11}$.*

□

3.4 Some geometry for $\mathbb{Z}_2^5 : A_5$

It is known by [17], that a maximal parabolic subgroup P of the group $\mathbb{Z}_4 L_3(4)$ acts flag-transitively on a geometry Γ having the following Coxeter diagram:



From the diagram of the rank five geometry of Ivanov and Shpectorov [17] we draw that $P \cap M_{11} \simeq GL_2(3)$, so $\Omega_1(Z(P))$ is contained in the Borel subgroup of this geometry. Therefore we can assume that P is a maximal parabolic in $\mathbb{Z}_2 L_3(4)$. Thus P is isomorphic to $\mathbb{Z}_2^5 : A_5$ such that $Z(P) \simeq \mathbb{Z}_2$, $O_2(P)$ is an indecomposable module for A_5 where $O_2(P)/Z(P)$ is the natural $L_2(4)$-module.

Then the maximal parabolic subgroups of the pair (Γ, P) are: $G_1 \simeq A_5$, $G_2 \simeq S_4$, $G_3 \simeq \mathbb{Z}_2^3$ and $G_4 \simeq A_5$ (the numbering is denoted above the diagram nodes). Furthermore this implies $G_{12} \simeq S_3 \simeq G_{24}$, $G_{13} \simeq \mathbb{Z}_2^2 \simeq G_{23}$ and $G_{14} \simeq D_{10}$. Clearly, all minimal parabolic subgroups are isomorphic to $\mathbb{Z}_2$. We set $< b >:= G_{123}$, $< a_n >:= G_{124}$, $< a >:= G_{134}$ and $< v >:= G_{234}$ and let C be the corresponding (infinite) Coxeter group. Thus we obtain the following relations:

$\tilde{\mathcal{R}}_o : a^2 = a_n^2 = b^2 = v^2 = 1$,
$\tilde{\mathcal{R}'}_c : [a,b] = [a,v] = [b,v] = (aa_n)^5 = (a_nb)^3 = (a_nv)^3 = 1$.

In view of the aim to give generators and relations for the Ivanov-Shpectorov geometry, we denote $\tilde{\mathcal{R}'}_c$ by:

$\tilde{\mathcal{R}}_c : [a,b] = [a,v] = (aa_n)^5 = (a_nb)^3 = (va_n)^3 = 1, (vb)^2 = 1$.

Then $C =< a, a_n, b, v \mid \tilde{\mathcal{R}}_o \cap \tilde{\mathcal{R}}_c >$. Again, we have to ensure that $G_1$and G_4 are both isomorphic to A_5. Thus we have to diminish the order of the two Coxeter elements aa_nb and va_na. Therefore we have to add

$\tilde{\mathcal{R}}_{(2,3,5)} : (aa_nb)^5 = (va_na)^5 = 1$.

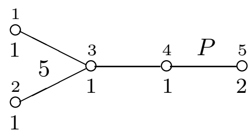
Using the Magma [2] program for coset enumeration, we get the following lemma:

Lemma 3.4.1 *Let $P :=< a, a_n, b, v \mid \tilde{\mathcal{R}}_o \cup \tilde{\mathcal{R}}_c \cup \tilde{\mathcal{R}}_{(2,3,5)} >$. Then P is isomorphic to a maximal parabolic of $\mathbb{Z}_2L_3(4)$.*

□

3.5 Generators and relations for the Ivanov-Shpectorov geometry

We recall that the geometry Γ of Ivanov and Shpectorov for the O'Nan group has the following diagram:



The maximal parabolic subgroups G_1, G_2 and G_5 of the pair (Γ, G) (where $G \in \{O'N, 3O'N\}$) are $G_1 \simeq J_1$, $G_2 \simeq M_{11}$ and G_5 is isomorphic to a maximal parabolic of $\mathbb{Z}_4L_3(4)$.

Let P_1 be a maximal parabolic in $\mathbb{Z}_4L_3(4)$. According to the last section, we get generators and relations for P acting on the geometry Γ_5 simply by adding a center which acts trivially such that $G_{15} \simeq \mathbb{Z}_2 \times A_5 \simeq G_{45}$ and $G_{25} \simeq GL_2(3)$. Thus we add a new generator z and transform the relation $(vb)^2 = 1$ in $\tilde{\mathcal{R}}_c$ in $(vb)^2 = z$. Since P_1 has to contain a subgroup $\mathbb{Z}_2 \times A_5$ of $G_1 \simeq J_1$, we change $\tilde{\mathcal{R}}_{(2,3,5)}$ to

$\bar{\mathcal{R}}_{(2,3,5)} : (zaa_nb)^5 = (va_na)^5 = 1$.

Furthermore we add

$\bar{\mathcal{R}}_z : z^2 = [a,z] = [b,z] = [a_n,z] = [v,z] = 1$.

We set $\bar{\mathcal{R}}_o := \tilde{\mathcal{R}}_o$ and

$\bar{\mathcal{R}}_c : [a,b] = [a,v] = (aa_n)^5 = (a_nb)^3 = (va_n)^3 = 1, (vb)^2 = z$.

Then we have the following:

Lemma 3.5.1 *Let* $P_1 :=< z, a, a_n, b, v \mid \bar{\mathcal{R}}_o \cup \bar{\mathcal{R}}_c \cup \bar{\mathcal{R}}_{(2,3,5)} >$. *Then* P *is isomorphic to a maximal parabolic of* $\mathbb{Z}_4 L_3(4)$.

□

We are now able to give generators and relations for the Ivanov-Shpectorov geometry. First we set $\mathcal{R}_O := \mathcal{R}_o \cup \mathcal{R}'_o \cup \bar{\mathcal{R}}_o$, thus:

$\mathcal{R}_O : z^2 = a^2 = a_n^2 = b^2 = v^2 = t^3 = 1$.

Then we put $\mathcal{R}_M := \mathcal{R}_m \cup \mathcal{R}'_m$, thus:

$\mathcal{R}_M : [z, a] = [z, a_n] = [z, b] = [z, v] = 1, t^z = t^{-1}$.

We set $\mathcal{R}_D := \mathcal{R}_d \cup \mathcal{R}'_d \cup \bar{\mathcal{R}}_c$, thus:

$\mathcal{R}_D : [a, b] = [a, t] = [a_n, t] = [v, t] = [a, v] = (aa_n)^5 = (a_n b)^3 = (va_n)^3 = (tb)^5 = 1, (vb)^2 = z$.

We recall that $\mathcal{R}_A = \mathcal{R}'_A$ and

$\mathcal{R}_A : z = bt[b, t]^2$.

Finally, we set $\mathcal{R}_{(2,3,5)} := \bar{\mathcal{R}}_{(2,3,5)}$, thus:

$\mathcal{R}_{(2,3,5)} : (zaa_n b)^5 = (va_n a)^5 = 1$.

Then we get that

$U :=< z, a, a_n, b, v, t \mid \mathcal{R}_O \cup \mathcal{R}_M \cup \mathcal{R}_D \cup \mathcal{R}_A \cup \mathcal{R}_{(2,3,5)} >$ is the universal completion of the amalgam of G_1, G_2 and G_5. Using MAGMA [2] for coset enumeration, we hold that $G_3 =< z, a, b, v, t > \simeq \mathbb{Z}_2 \times S_5$ and $G_4 =< z, a, a_n, v, t > \simeq S_3 \times A_5$. So U is the universal completion of the amalgam corresponding to the Ivanov-Shpectorov geometry.

Theorem 3.5.2 *Let* Γ' *be a flag-transitive geometry with the same Buekenhout diagram as* Γ. *Assume furthermore that for a flag-transitive automorphism group* H *we have* $H_1/K_1 \simeq J_1$, $H_2/K_2 \simeq M_{11}$ *and* $H_5/K_5 \simeq \mathbb{Z}_2^5 : A_5$ *(a maximal parabolic in* $\mathbb{Z}_2 L_3(4)$*) where* K_i *denotes the kernel of the action of* H_i *on the corresponding residue. Then* $K_1 = K_2 = 1$ *and* $B = K_5 \simeq \mathbb{Z}_2$.

Proof. Clearly, we have that K_i is a subgroup of the Borel group B of Γ' in H for all i. Since the Borel subgroup of the geometry for H_5/K_5 is trivial, we hold $B = K_5$. We have that $H_{12}/K_{12} \simeq L_2(11)$. We have $K_{12}/K_i \lhd H_{12}/K_i$ for $i = 1, 2$, thus $K_{12} = K_i = 1$ since $H_{12}/K_i \simeq L_2(11)$. This implies the assertion. □

Using the *Adaptive Coset Enumerator* ACE, version 3 [13], we establish the following theorem:

Theorem 3.5.3 *Let* $G :=< a, a_n, b, v, t, z \mid a^2 = a_n^2 = b^2 = v^2 = t^3 = z^2 = 1, [z, a] = [z, a_n] = [z, b] = [z, v] = 1, t^z = t^{-1}, [a, b] = [a, t] = [a_n, t] = [v, t] = [a, v] = 1, (a_n a)^5 = (a_n b)^3 = (a_n v)^3 = (tb)^5 = 1, (vb)^2 = z, z = bt[b, t]^2, (zaa_n b)^5 = (aa_n v)^5 = 1 >$. *Then* $G \simeq 3O'N$.

□

As a corollary we hold by Proposition 1.2.4:

Corollary 3.5.4 *The 3-fold cover of the Ivanov-Shpectorov geometry for the sporadic O'Nan group is universal.*

□