Chapter 4

Constructing an irreducible representation for the Buekenhout geometry

4.1 Introduction

In this chapter we construct 154×154-matrices over $GF(3)$ for the generators of the amalgam related to the Buekenhout geometry given in Chapter 2. These matrices provide an irreducible representation of that amalgam. This representation for the O'Nan group has been constructed in [23] using the computer. We are going to give a construction which is largely done by hand. In particular we do not use the fact that the group $O'N$ is the universal completion of the amalgam.

We start with the representation of the group $L_3(7) : \mathbb{Z}_2$, identify the generators z, X, Y, Z and ρ, and then construct a matrix a satisfying all the required relations.

The representation for $L_3(7) : \mathbb{Z}_2$ will split for that group as the direct sum of a 1-dimensional, a 57-dimensional and a 96-dimensional module all being irreducible.

In order to construct the representation for $L_3(7) : \mathbb{Z}_2$, we use the canonical generators and relations given in Chapter 2. To recall:

Set $G_1 :=< v_1, v_2, \nu, \rho, x, u, i | v_1^7 = v_2^7 = \nu^7 = 1, \rho^3 = x^4 = i^2 = 1, [v_1, v_2] = [v1, \nu] = 1, v_2^\nu = v_2 v_1, v_1^\rho = v_1^2, v_2^\rho = v_2^4, v_1^x = v_2, v_2^x = v_1^{-1}, [v_1, i] = 1, v_2^i = v_2^{-1}, \nu^\rho = \nu^4, \rho^x = \rho^{-1}, [\rho, i] = 1, x^i = x^{-1}, (\nu x)^3 = x^2, \nu^i = \nu^{-1}, [\nu, x^2] = 1, v_1^u = v_1^{-1}, v_2^u = \nu, \nu^u = v_2, [\rho, u] = 1, (x^2)^u = x^2 i, (x x^u)^3 = 1 >$.

Then $G_1 \simeq L_3(7) : \mathbb{Z}_2$. More exactly, we have $P_1 :=< v_1, v_2, \nu, \rho, x, i > \simeq \mathbb{Z}_7^2 : SL_2(7) : \mathbb{Z}_2$, where $< v_1, v_2 >= O_7(P_1)$, $< \nu, \rho, x, i > \simeq SL_2(7) : \mathbb{Z}_2$ and $< \nu, \rho, x > \simeq SL_2(7)$. Furthermore the pair (P_1, P_1^u) consists of two incident maximal parabolic subgroups of $L_3(7)$.

4.2 Constructing matrices for $L_3(7) : \mathbb{Z}_2$

4.2.1 The irreducible 57-dimensional $GF(3)$-module

It is known that $L_3(7) : \mathbb{Z}_2$ has an irreducible 57-dimensional $GF(3)$-module V_{57} (see e.g. [22]). We construct a representation simply by calculating matrices satisfying the required relations and do not consider the construction of a specific representation. As above we set $G_1 :=< v_1, v_2, \nu, \rho, x, u, i >$, $P_1 :=< v_1, v_2, \nu, \rho, x, i >$ and $W := O_7(P_1) =< v_1, v_2 >$. Then we have $V_{57} \mid_W = C_{V_{57}}(W) \oplus [V_{57}, W]$ where $[V_{57}, W] = \bigoplus_{j=1}^{8} C_{[V_{57},W]}(H_j)$, H_j the hyperplanes in W.

Since $\frac{x^7-1}{x-1}$ is irreducible over $GF(3)$, the smallest representation of an element of order seven is six dimensional and hence we hold $dim C_{V_{57}}(W) = 9$ and $dim C_{[V_{57},W]}(H_j) = 6$.

We number the hyperplanes in W as follows: $H_1 :=< v_1 >$, $H_j :=< (j-2)v_1 + v_2 >$ for $j = 2, 3, \ldots, 8$. Moreover we set

$$J := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}.$$

This implies the following approach:

$$v_1 = \begin{pmatrix} I_9 &&&&&&&& \\ & I_6 &&&&&&& \\ && J &&&& 0 && \\ &&& J &&&&& \\ &&&& J &&&& \\ &&&&& J &&& \\ && 0 &&&& J && \\ &&&&&&& J & \\ &&&&&&&& J \end{pmatrix}, \quad v_2 = \begin{pmatrix} I_9 &&&&&&&& \\ & J &&&&&&& \\ && I_6 &&&& 0 && \\ &&& J^6 &&&&& \\ &&&& J^5 &&&& \\ &&&&& J^4 &&& \\ && 0 &&&& J^3 && \\ &&&&&&& J^2 & \\ &&&&&&&& J \end{pmatrix}.$$

We set $\mathcal{C} := C_{[V_{57},W]}(H_1)$. Since $[v_1, \nu] = 1$, ν acts on $\mathcal{C}$. Using our generators and relations, we have that $t := x^2 \in Z(< \nu, \rho, x, i >)$, thus t inverts v_1 and v_2 and centralizes ν. Therefore t acts on $\mathcal{C}$ as well as v_2 because $[v_1, v_2] = 1$. Since $< \nu, \rho, x, i >$ acts transitively on the hyperplanes of W, t inverts v_2 on $\mathcal{C}$. Moreover $v_1 = v_2^{\nu} v_2^{-1}$ imlying that $[\nu, v_2] = 1$ on $\mathcal{C}$. Since t inverts v_2 and centralizes ν on $\mathcal{C}$, ν and v_2 cannot induce the same subgroup of order seven on $\mathcal{C}$, thus we can assume that $\nu = I_6$ on $\mathcal{C}$ because $GL_6(3)$ does not contain a subgroup $\mathbb{Z}_7^2$.

We identify v_2 on $\mathcal{C}$ with the permutation (1234567) where the numbers one to six represent the canonical basis vectors and seven their negative sum. Since $v_2^{\rho} = v_2^4$ and

$v_2^t = v_2^{-1}$, we can identify t with (27)(54)(36) and ρ with (253)(674). Thus we have the following on $\mathcal{C}$:

$$t \sim C := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \rho \sim B := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}$$

If we now set $C_{[V_{57},W]}(H_k)^\nu =: C_{[V_{57},W]}(H_{k\nu})$, we hold the identification $\nu \sim$ (2345678) and therefore the following on $[V_{57}, W]$:

$$\nu = \begin{pmatrix} I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 \\ 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad t = x^2 = diag(C)$$

We approach i, ρ and x on $[V_{57}, W]$ by the following:

$i = (A_{ij})_{i,j=1,\dots 8}$, $\rho := (R_{ij})_{i,j=1,\dots 8}$ and $x := (X_{ij})_{i,j=1,\dots 8}$. Where A_{ij}, R_{ij}, X_{ij} are elements of $GF(3)^{6\times 6}$.

Then, using the relation $v_1 i = i v_1$, we hold:

$$v_1 i = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} & A_{17} & A_{18} \\ JA_{21} & JA_{22} & JA_{23} & JA_{24} & JA_{25} & JA_{26} & JA_{27} & JA_{28} \\ \vdots & & & \ddots & & & & \vdots \end{pmatrix} =$$

$$\begin{pmatrix} A_{11} & A_{12}J & A_{13}J & A_{14}J & A_{15}J & A_{16}J & A_{17}J & A_{18}J \\ A_{21} & A_{22}J & A_{23}J & A_{24}J & A_{25}J & A_{26}J & A_{27}J & A_{28}J \\ \vdots & & & \ddots & & & & \vdots \end{pmatrix} = i v_1.$$

Therefore we get $A_{12} = A_{13} = A_{14} = \ldots = A_{18} = 0$, $A_{21} = A_{31} = A_{41} = \ldots = A_{81} = 0$ and $A_{ij}J = JA_{ij}$ for all A_{ij} except A_{11}.

The relation $v_2 i = i v_2^{-1}$ leads to:

$$v_2 i = \begin{pmatrix} JA_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} & A_{27} & A_{28} \\ 0 & J^6A_{32} & J^6A_{33} & J^6A_{34} & J^6A_{35} & J^6A_{36} & J^6A_{37} & J^6A_{38} \\ \vdots & & & \ddots & & & & \vdots \end{pmatrix} =$$

$$\begin{pmatrix} A_{11}J^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{22} & A_{23}J & A_{24}J^2 & A_{25}J^3 & A_{26}J^4 & A_{27}J^5 & A_{28}J^6 \\ 0 & A_{32} & A_{33}J & A_{34}J^2 & A_{35}J^3 & A_{36}J^4 & A_{37}J^5 & A_{38}J^6 \\ \vdots & & & \ddots & & & & \vdots \end{pmatrix} = iv_2.$$

Together with $A_{ij}J = JA_{ij}$ from above, this shows that $A_{ij} = 0$ except for A_{11}, A_{22}, A_{38}, A_{47}, A_{56}, A_{65}, A_{74} and A_{83} which are therefore elements of $GL_6(3)$. Furthermore the relation $\nu^i = \nu^{-1}$ yields $A_{22} = A_{38} = \ldots = A_{83}$. Moreover we have that $J^{A_{11}} = J^{-1}$, $[A_{22}, J] = 1$ and $A_{11}^2 = A_{22}^2 = I_6$. The relation $it = ti$ proves that $[A_{11}, C] = [A_{22}, C] = 1$. Now $\mid C_{GL_6(3)}(< J, C >) \mid = 2 \cdot 13$ and $\mid N_{GL_6(3)}(< J >) \cap C_{GL_6(3)}(C) \mid = 2^2 \cdot 3 \cdot 13$ having a normal 2-Sylow subgroup. Therefore we have $A_{11} = \pm C$ and $A_{22} = \pm I_6$ and choose $A_{11} := -C$ and $A_{22} := -I_6$. Thus we hold:

$$i = \begin{pmatrix} -C & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & -I_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & -I_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & -I_6 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

To compute ρ, we start with the relation $v_1\rho = \rho v_1^2$ leading to

$$v_1\rho = \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} & R_{16} & R_{17} & R_{18} \\ JR_{21} & JR_{22} & JR_{23} & JR_{24} & JR_{25} & JR_{26} & JR_{27} & JR_{28} \\ \vdots & & & \ddots & & & & \vdots \end{pmatrix} =$$

$$\begin{pmatrix} R_{11} & R_{12}J^2 & R_{13}J^2 & R_{14}J^2 & R_{15}J^2 & R_{16}J^2 & R_{17}J^2 & R_{18}J^2 \\ R_{21} & R_{22}J^2 & R_{23}J^2 & R_{24}J^2 & R_{25}J^2 & R_{26}J^2 & R_{27}J^2 & R_{28}J^2 \\ \vdots & & & \ddots & & & & \vdots \end{pmatrix} = \rho v_1^2.$$

Hence $R_{12} = R_{13} = \ldots = R_{18} = 0$ and $R_{21} = R_{31} = \ldots = R_{81} = 0$. Similarily, using $v_2\rho = \rho v_2^4$, we hold $R_{23} = R_{24} = \ldots R_{28} = 0$ and $R_{32} = R_{42} = \ldots = R_{82} = 0$. By the relation $\nu\rho = \rho\nu^4$, we deduce that $R_{ij} = 0$ except for R_{11}, R_{22} and $R_{36} = R_{43} = R_{57} = R_{64} = R_{78} = R_{85} = R_{22}$. Thus $R_{11}, R_{22} \in GL_6(3)$. Moreover $v_1\rho = \rho v_1^2$ yields $J^{R_{22}} = J^2$ and $v_2\rho = \rho v_2^4$ yields $J^{R_{11}} = J^4$. Exploiting $\rho i = i\rho$, we hold $[R_{11}, C] = 1$. Also, since ρ should be of order three, we get $o(R_{11}) = o(R_{22}) = 3$. Thus we can choose $R_{11} := B$ and $R_{22} := B^{-1}$ leading to

$$\rho = \begin{pmatrix} B & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B^{-1} & 0 & 0 \\ 0 & 0 & B^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & B^{-1} & 0 \\ 0 & 0 & 0 & B^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & B^{-1} \\ 0 & 0 & 0 & 0 & B^{-1} & 0 & 0 & 0 \end{pmatrix}.$$

We compute x in a similar way. We start using the reation $v_1 x = x v_2$:

$$v_1 x = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} & X_{17} & X_{18} \\ JX_{21} & JX_{22} & JX_{23} & JX_{24} & JX_{25} & JX_{26} & JX_{27} & JX_{28} \\ JX_{31} & JX_{32} & JX_{33} & JX_{34} & JX_{35} & JX_{36} & JX_{37} & JX_{38} \\ \vdots & & & \ddots & & & & \vdots \end{pmatrix} =$$

$$\begin{pmatrix} X_{11}J & X_{12} & X_{13}J^6 & X_{14}J^5 & X_{15}J^4 & X_{16}J^3 & X_{17}J^2 & X_{18}J \\ X_{21}J & X_{22} & X_{23}J^6 & X_{24}J^5 & X_{25}J^4 & X_{26}J^3 & X_{27}J^2 & X_{28}J \\ X_{31}J & X_{32} & X_{33}J^6 & X_{34}J^5 & X_{35}J^4 & X_{36}J^3 & X_{37}J^2 & X_{38}J \\ \vdots & & & \ddots & & & & \vdots \end{pmatrix} = x v_2.$$

This proves $X_{11} = X_{13} = \ldots = X_{18} = 0$ and $X_{22} = X_{32} = \ldots = X_{82} = 0$ and therefore $X_{12}, X_{21} \in GL_6(3)$. By $v_2 x = x v_1^{-1}$ we hold:

$$v_2 x = \begin{pmatrix} 0 & JX_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ X_{21} & 0 & X_{23} & X_{24} & X_{25} & X_{26} & X_{27} & X_{28} \\ J^{-1}X_{31} & 0 & J^{-1}X_{33} & J^{-1}X_{34} & J^{-1}X_{35} & J^{-1}X_{36} & J^{-1}X_{37} & J^{-1}X_{38} \\ \vdots & & & \ddots & & & & \vdots \end{pmatrix} =$$

$$\begin{pmatrix} 0 & X_{12}J^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ X_{21} & 0 & X_{23}J^{-1} & X_{24}J^{-1} & X_{25}J^{-1} & X_{26}J^{-1} & X_{27}J^{-1} & X_{28}J^{-1} \\ X_{31} & 0 & X_{33}J^{-1} & X_{34}J^{-1} & X_{35}J^{-1} & X_{36}J^{-1} & X_{37}J^{-1} & X_{38}J-1 \\ \vdots & & & \ddots & & & & \vdots \end{pmatrix} = x v_1^{-1}.$$

Thus $X_{21} = X_{31} = \ldots = X_{81} = 0$ from the first column and $X_{23} = X_{24} = \ldots = X_{28} = 0$ from the second row. The third row shows $X_{3j}J^{-1} = J^{-1}X_{3j}$ implying $X_{3j}J = JX_{3j}$. Using this information and the third row in $v_1 x = x v_2$, we get $X_{33} = X_{34} = \ldots = X_{37} = 0$ and $X_{38} \neq 0$.

As a next step we use $\rho x \rho = x$ ($\rho^x = \rho^{-1}$) and find

$$x = \begin{pmatrix} 0 & X_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ X_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & X_{38} \\ 0 & 0 & 0 & 0 & B^{-1}X_{38}B^{-1} & 0 & 0 & 0 \\ 0 & 0 & X_{53} & X_{54} & X_{55} & X_{56} & X_{57} & X_{58} \\ 0 & 0 & 0 & 0 & 0 & 0 & BX_{38}B & 0 \\ 0 & 0 & BX_{56}B & BX_{53}B & BX_{57}B & BX_{54}B & BX_{58}B & BX_{55}B \\ 0 & 0 & B^{-1}X_{54}B^{-1} & B^{-1}X_{56}B^{-1} & B^{-1}X_{58}B^{-1} & B^{-1}X_{53}B^{-1} & B^{-1}X_{55}B^{-1} & B^{-1}X_{57}B^{-1} \end{pmatrix}$$

where $[X_{12}, B] = [X_{21}, B] = 1$ because $X_{12}, X_{21} \in GL_6(3)$. Furthermore since x has to be invertible, $X_{38} \in GL_6(3)$.

Since $x^2 = t = diag(C)$, we hold $X_{12}X_{21} = C = X_{21}X_{12}$, $X_{38}B^{-1}X_{54}B^{-1} = C$, $X_{38}B^{-1}X_{5j}B^{-1} = 0$ for $j \in \{3, 5, 6, 7, 8\}$. Thus $X_{5j} = 0$ except for $X_{54} = BCX_{38}^{-1}B = CBX_{38}^{-1}B$ since $[B, C] = 1$ and

$$x = \begin{pmatrix} 0 & X_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ X_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & X_{38} \\ 0 & 0 & 0 & 0 & B^1X_{38}B^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & CBX_{38}^{-1}B & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & BX_{38}B & 0 \\ 0 & 0 & 0 & 0 & 0 & CB^{-1}X_{38}^{-1}B^{-1} & 0 & 0 \\ 0 & 0 & CX_{38}^{-}1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

furthermore $[X_{38}, C] = 1$. The last relation has to hold since $x^2 = diag(C)$ implies $X_{54} = BCX_{38}B$ as well as $X_{54} = BX_{38}CB$. Now X_{38} also commutes with J as seen above. Thus $X_{38} \in C_{GL_6(3)}(< C, J >)$. By $(\nu x)^3 = t$, we get the relation $X_{12}X_{38}X_{21} = C$, hence $X_{38} = I_6$ since $X_{12}X_{21} = C = X_{21}X_{12}$. The relation $v_2x = xv_1^{-1}$ shows that $J^{X_{12}} = J^{-1}$. Therefore $[X_{12}, X_{21}] = 1$ leads to $X_{21} \in C_{GL_6(3)}(< C, J, B >)$, $X_{12} \in CC_{GL_6(3)}(< C, J, B >)$. Now $C_{GL_6(3)}(< C, J, B >) = \{I_6, -I_6\}$ yields $X_{12} = \pm C$ and $X_{21} = \pm I_6$ and we choose $X_{12} = C$ and therefore $X_{21} = I_6$. Thus

$$x = \begin{pmatrix} 0 & C & 0 & 0 & 0 & 0 & 0 & 0 \\ I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 \\ 0 & 0 & 0 & 0 & B & 0 & 0 & 0 \\ 0 & 0 & 0 & CB^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & B^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & CB & 0 & 0 \\ 0 & 0 & C & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The relation $xi = ix^{-1} = tix$ is also fulfilled.

Set $\tilde{C} := C_{V_{57}}(W)$ with $W =< v_1, v_2 >$ and we compute ν, i, ρ and x on $\tilde{C}$. Recall that $\mathcal{C} = C_{[V_{57},W]}(v_1)$. Since $v_1 = I_9 = v_2$ on $\tilde{C}$, we use the 15-dimensional space $U = \tilde{C} \oplus \mathcal{C}$ in order to have more information. Clearly, ν acts on $\tilde{C}$, thus we can choose a basis of U such that

$$v_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_6 & 0 \\ 0 & 0 & 0 & 0 & J \end{pmatrix} \text{ and } \nu = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & J & 0 \\ 0 & 0 & 0 & 0 & I_6 \end{pmatrix}$$

because, by $\nu \sim_{G_1} v_1$, we have $dimC_{V_{57}}(\nu) = 15$ and $dimC_{[V_{75},W]}(\nu) = 12$ implying that $C_{V_{57}}(S)$ with $S =< v_1, v_2, \nu >$ is three dimensional. Furthermore we have $N_{L_3(7):\mathbb{Z}_2}(S) =< S, i, t, \rho, u >=: N$ acts on $C_{V_{57}}(S)$. Since i inverts both, v_2 and ν, we find that i has the following shape on U:

$$i = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{21} & 0 \\ 0 & 0 & A_{33} \end{pmatrix},$$

where $A_{11} \in GL_3(3)$ and $A_{22}, A_{33} \in GL_6(3)$. Moreover we hold $A_{ii}^2 = 1$ and $J^{A_{22}} = J^{A_{33}} = J^{-1}$. Therefore we choose $A_{22} := C$ and from the previous calculations we have $A_{33} = -C$ because A_{33} determines the action of i on $\mathcal{C}$ computed above. Since $v_2^\rho = v_2^4$, $\nu^\rho = \nu^4$ and $[\rho, i] = 1$ we hold the following for ρ on U:

$$\rho = \begin{pmatrix} R_{11} & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & B \end{pmatrix},$$

where $R_{22} \in GL_6(3)$ was chosen to be B, $R_{11} \in GL_3(3)$ and $[A_{11}, R_{11}] = 1$. Also, by $[i, u] = 1$ and $\mathbb{Z}_7^{1+2} : (\mathbb{Z}_3 \times D_8) \simeq N$, we get that $< i, u, t > \simeq D_8$ and $i \in Z(< i, u, t >)$. Thereby $A_{11} \notin Z(GL_3(3))$[1] . Moreover we hold that t is of the following shape on U:

$$t = \begin{pmatrix} T_{11} & 0 & 0 \\ 0 & T_{22} & 0 \\ 0 & 0 & C \end{pmatrix},$$

where $T_{11} \in GL_3(3)$, $T_{22} \in GL_6(3)$. Since $[u, t] \neq 1$, we get that $T_{11} \notin Z(GL_3(3))$. This yields $R_{11} = I_3$ because $< A_{11}, T_{11} > \simeq \mathbb{Z}_2^2$ and the centralizer of such a group in $GL_3(3)$ is isomorphic to $\mathbb{Z}_2^3$. Since t centralizes ρ, i and ν, we therefore hold $T_{22} \in C_{GL_6(3)}(< C, B, J >) = Z(GL_6(3))$, thus $T_{22} = \pm I_6$ so we choose $T_{22} = -I_6$.

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[1] Note that $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ is not a square in $GL_3(3)$

Now $< A_{11}, T_{11} >$ must fix a vector of the space $GF(3)^3$. Therefore we choose

$$A_{11} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \text{ and } T_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

In order to compute x on $\tilde{C}$, we procced as follows. Let $x = (a_{ij})_{1 \leq i,j \leq 9}$. Exploiting the relations $\rho x = x\rho^{-1}$, $xi = itx$, $x^2 = t$ and $(\nu x)^3 = t$, we get that

$$x = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & \epsilon & \epsilon & 0 & \epsilon & 0 \\ 0 & 0 & 0 & 1 & 2 & 2 & 0 & 2 & 0 \\ 0 & 0 & \epsilon & 0 & 1 & 1 & 2 & 1 & 2 \\ 0 & 0 & \epsilon & 1 & 2 & 2 & 1 & 2 & 0 \\ 0 & -\epsilon & -\epsilon & 1 & 2 & 1 & 0 & 2 & 0 \\ 0 & 0 & \epsilon & 1 & 2 & 2 & 0 & 2 & 1 \\ 0 & -\epsilon & -\epsilon & 1 & 2 & 2 & 0 & 1 & 0 \end{pmatrix}, \ \epsilon \in \{1, 2\}.$$

Thus we choose

$$x = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 & 2 & 1 & 2 & 0 \\ 0 & 2 & 2 & 1 & 2 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 2 & 2 & 0 & 2 & 1 \\ 0 & 2 & 2 & 1 & 2 & 2 & 0 & 1 & 0 \end{pmatrix}.$$

We start to compute u on U. Since u normalizes S, u acts on U and we approach u by

$$u = \begin{pmatrix} A & 0 & 0 \\ 0 & A_{11} & A_{12} \\ 0 & A_{21} & A_{22} \end{pmatrix},$$

where $A \in GL_3(3)$ and $A_{ij} \in GL_6(3)$. Then the relation $v_2 u = u\nu$ yields

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A_{11} & A_{12} \\ 0 & JA_{21} & JA_{22} \end{pmatrix} = \begin{pmatrix} A & 0 & 0 \\ 0 & A_{11}J & A_{12} \\ 0 & A_{21}J & A_{22} \end{pmatrix},$$

hence $A_{11} = A_{22} = 0$. Exploiting $iu = ui$, we get

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{31} & a_{32} + a_{33} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

$CA_{12} + A_{12}C = 0$ and $CA_{21} + A_{21}C = 0$.

Now we use the relation $tu = uti$ and get on $C_{V_{57}}(S)$:

$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ -a_{31} & -a_{32} - a_{33} & 0 \\ 2a_{31} & -a_{32} & -a_{33} \end{pmatrix} = \begin{pmatrix} -a_{11} & a_{12} & 0 \\ -a_{31} & a_{32} + a_{33} & 0 \\ -a_{31} & a_{32} - a_{33} & -a_{33} \end{pmatrix},$$

thus $a_{11} = 0$ and $a_{32} + a_{33} = 0$. Hence we hold

$$A = \begin{pmatrix} 0 & a_{12} & 0 \\ a_{31} & 0 & 0 \\ a_{31} & -a_{33} & a_{33} \end{pmatrix}.$$

Also $u^2 = 1$ implies $A^2 = 1$ which gives us

$$A = \begin{pmatrix} 0 & a_{12} & 0 \\ a_{12} & 0 & 0 \\ a_{12} & -a_{33} & a_{33} \end{pmatrix}.$$

The entries of A will be determined using the Weyl relation. Using $[u, \rho] = 1$, we obtain $[B, A_{12}] = [B, A_{21}] = 1$. Also, $u^2 = 1$ implies $A_{12} = A_{21}^{-1}$. Together with $[J, A_{12}] = 1$ and $CA_{12} = -A_{12}C$ this yields

$$A_{12} = \pm \begin{pmatrix} -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 1 & 1 \\ -1 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & -1 & 1 \end{pmatrix}.$$

Notice that this implies $A_{21} = A_{12}^{-1} = -A_{12}$.

We approach u on $[V_{57}, v_1]$ as follows. Set $u = (a_{ij})_{1 \leq i,j \leq 7}$ where $a_{ij} \in GL_6(3)$. Using the relations $v_2 u = u\nu$ and $\nu u = uv_2$, we hold

$$u = \begin{pmatrix} a & a & a & a & a & a & a \\ a & Ja & J^2a & J^3a & J^4a & J^5a & J^6a \\ a & J^2a & J^4a & J^6a & Ja & J^3a & J^5a \\ a & J^3a & J^6a & J^2a & J^5a & Ja & J^4a \\ a & J^4a & Ja & J^5a & J^2a & J^6a & J^3a \\ a & J^5a & J^3a & Ja & J^6a & J^4a & J^2a \\ a & J^6a & J^5a & J^4a & J^3a & J^2a & Ja \end{pmatrix}, J^a = J^{-1}, a \in GL_6(3).$$

Thereby $u^2 = 1$ implies $a^2 = 1$. Furthermore $tu = uti$ gives $Ca = -aC$ and $\rho u = u\rho$ implies $[B, a] = 1$. Thus we get that

$$a = \pm \begin{pmatrix} 1 & -1 & -1 & 0 & -1 & 0 \\ 1 & 1 & -1 & 1 & -1 & -1 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 & -1 & -1 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

Checking the Weyl relation $(xx^u)^3 = 1$ leads to the following possibilities (for A_{12} and a we just give the sign of the above matrices):

- $a_{12} = -1, a_{33} = -1, A_{12} : +, a : +,$
- $a_{12} = -1, a_{33} = -1, A_{12} : -, a : +,$
- $a_{12} = 1, a_{33} = -1, A_{12} : +, a : +,$
- $a_{12} = 1, a_{33} = -1, A_{12} : -, a : +.$

Of course, multiplication of u with -1 does not change the relation, thus we have eight possibilities. Therefore we choose $a_{12} = a_{33} = -1$ and

$$A_{12} = \begin{pmatrix} -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 1 & 1 \\ -1 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & -1 & 1 \end{pmatrix}, a = \begin{pmatrix} 1 & -1 & -1 & 0 & -1 & 0 \\ 1 & 1 & -1 & 1 & -1 & -1 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 & -1 & -1 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

4.2.1.1 Summary

We summarize the above results. Set

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix},$$

as matrices in $GL_6(3)$. Furthermore we set

$$\nu_9 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & J \end{pmatrix}, \rho_9 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & B \end{pmatrix}, i_9 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & C \end{pmatrix}$$

and

$$x_9 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & -1 & 1 & -1 & 0 \\ 0 & -1 & -1 & 1 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 & 0 & -1 & 1 \\ 0 & -1 & -1 & 1 & -1 & -1 & 0 & 1 & 0 \end{pmatrix}$$

as matrices in $GL_9(3)$. Moreover let

$$\nu_{48} = \begin{pmatrix} I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 \\ 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \rho_{48} = \begin{pmatrix} B & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B^{-1} & 0 & 0 \\ 0 & 0 & B^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & B^{-1} & 0 \\ 0 & 0 & 0 & B^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & B^{-1} \\ 0 & 0 & 0 & 0 & B^{-1} & 0 & 0 & 0 \end{pmatrix},$$

$$i_{48} = \begin{pmatrix} -C & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & -I_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & -I_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & -I_6 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$x_{48} = \begin{pmatrix} 0 & C & 0 & 0 & 0 & 0 & 0 & 0 \\ I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 \\ 0 & 0 & 0 & 0 & B & 0 & 0 & 0 \\ 0 & 0 & 0 & CB^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & B^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & CB & 0 & 0 \\ 0 & 0 & C & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

as matrices in $GL_{48}(3)$. We set

$$v_{1,57} = \begin{pmatrix} I_9 & & & & & & & & \\ & I_6 & & & & & & & \\ & & J & & 0 & & & & \\ & & & J & & & & & \\ & & & & J & & & & \\ & & & & & J & & & \\ & 0 & & & & & J & & \\ & & & & & & & J & \\ & & & & & & & & J \end{pmatrix}, \quad v_{2,57} = \begin{pmatrix} I_9 & & & & & & & & \\ & J & & & & & & & \\ & & I_6 & & & 0 & & & \\ & & & J^6 & & & & & \\ & & & & J^5 & & & & \\ & & & & & J^4 & & & \\ & & 0 & & & & J^3 & & \\ & & & & & & & J^2 & \\ & & & & & & & & J \end{pmatrix},$$

$$\nu_{57} = \begin{pmatrix} \nu_9 & 0 \\ 0 & \nu_{48} \end{pmatrix}, \rho_{57} = \begin{pmatrix} \rho_9 & 0 \\ 0 & \rho_{48} \end{pmatrix}, x_{57} = \begin{pmatrix} x_9 & 0 \\ 0 & x_{48} \end{pmatrix} \text{and } i_{57} = \begin{pmatrix} i_9 & 0 \\ 0 & i_{48} \end{pmatrix}.$$

Then we have proved the following:

Lemma 4.2.1 *We have* $GL_{57}(3) \geq < v_{1,57}, v_{2,57}, \nu_{57}, \rho_{57}, x_{57}, i_{57} > \simeq \mathbb{Z}_7^2 : SL_2(7) : \mathbb{Z}_2$.

□

Now we put

$$A = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ -1 & 1 & -1 \end{pmatrix} \in GL_3(3), A_{12} = \begin{pmatrix} -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 1 & 1 \\ -1 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & -1 & 1 \end{pmatrix} \in GL_6(3).$$

We set

$$u_{15} = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & A_{12} \\ 0 & -A_{12} & 0 \end{pmatrix}.$$

Furthermore

$$a = \begin{pmatrix} 1 & -1 & -1 & 0 & -1 & 0 \\ 1 & 1 & -1 & 1 & -1 & -1 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 & -1 & -1 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{pmatrix} \in GL_6(3)$$

and

$$u_{42} = \begin{pmatrix} a & a & a & a & a & a & a \\ a & Ja & J^2a & J^3a & J^4a & J^5a & J^6a \\ a & J^2a & J^4a & J^6a & Ja & J^3a & J^5a \\ a & J^3a & J^6a & J^2a & J^5a & Ja & J^4a \\ a & J^4a & Ja & J^5a & J^2a & J^6a & J^3a \\ a & J^5a & J^3a & Ja & J^6a & J^4a & J^2a \\ a & J^6a & J^5a & J^4a & J^3a & J^2a & Ja \end{pmatrix}.$$

If we set

$$u_{57} = \begin{pmatrix} u_{15} & 0 \\ 0 & u_{42} \end{pmatrix},$$

then the following holds:

Lemma 4.2.2 *We have* $GL_{57}(3) \geq < v_{1,57}, v_{2,57}, \nu_{57}, \rho_{57}, x_{57}, i_{57}, u_{57} > \simeq L_3(7) : \mathbb{Z}_2$. *Moreover these matrices provide an irreducible 57-dimensional representation of the group* $L_3(7) : \mathbb{Z}_2$.

□

4.2.2 The irreducible 96-dimensional $GF(3)$-module

By [22], we see that $L_3(7) : \mathbb{Z}_2$ has an irreducible 96-dimensional $GF(3)$-module V_{96}. This module splits for $W = O_7(P_1)$ as $V_{96} = C_{V_{96}}(W) \bigoplus_{j=1}^{8} C_{[V_{96},W]}(H_j)$ using the notation introduced above. Since $6 \mid dimC_{[V_{96},W]}(H_j)$, we find that $V_{96} = \bigoplus_{j=1}^{8} C_{V_{96}}(H_j)$ with $dimC_{V_{96}}(H_j) = 12$. We set

$$T := \begin{pmatrix} J^{-1} & 0 \\ 0 & J \end{pmatrix}, Y := \begin{pmatrix} 0 & I_6 \\ I_6 & 0 \end{pmatrix} \text{ and } S := \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}.$$

We set

$$v_{1,96} := \begin{pmatrix} I_{12} & & & & & & & \\ & T & & & & & & \\ & & T & & 0 & & & \\ & & & T & & & & \\ & & & & T & & & \\ & & 0 & & & T & & \\ & & & & & & T & \\ & & & & & & & T \end{pmatrix} \text{ and } v_{2,96} := \begin{pmatrix} T & & & & & & & \\ & I_{12} & & & & & & \\ & & T^6 & & & 0 & & \\ & & & T^5 & & & & \\ & & & & T^4 & & & \\ & & 0 & & & T^3 & & \\ & & & & & & T^2 & \\ & & & & & & & T \end{pmatrix}.$$

Moreover we set $t_{96} = x_{96}^2 := diag(Y)$ and

$$\nu_{96} := \begin{pmatrix} S & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{12} \\ 0 & I_{12} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Furthermore we set

$$\tilde{C} := \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}, D := \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} \text{ and } R := \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}.$$

Now we proceed as for the 57-dimensional module. That is we compute matrices which satisfy the relations for our canonical generators for $L_3(7) : \mathbb{Z}_2$ but we are not considering a particular representation. This means that we construct the matrices successively using all matrices obtained so far. Then we try to determine every matrix as far as possible and choose appropriate matrices when we have more than one choice.

Then similar computations as for the 57-dimensional module lead to

$$i_{96} := \begin{pmatrix} \tilde{C} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & D & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & D \\ 0 & 0 & 0 & 0 & 0 & 0 & D & 0 \\ 0 & 0 & 0 & 0 & 0 & D & 0 & 0 \\ 0 & 0 & 0 & 0 & D & 0 & 0 & 0 \\ 0 & 0 & 0 & D & 0 & 0 & 0 & 0 \\ 0 & 0 & D & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\rho_{96} := \begin{pmatrix} R & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & R^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & R^{-1} & 0 & 0 \\ 0 & 0 & R^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & R^{-1} & 0 \\ 0 & 0 & 0 & R^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & R^{-1} \\ 0 & 0 & 0 & 0 & R^{-1} & 0 & 0 & 0 \end{pmatrix} \text{ and}$$

$$x_{96} := \begin{pmatrix} 0 & I_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & S^{-1} \\ 0 & 0 & 0 & 0 & (RSR)^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & YRSR & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & RS^{-1}R & 0 \\ 0 & 0 & 0 & 0 & 0 & Y(RS^{-1}R)^{-1} & 0 & 0 \\ 0 & 0 & YS & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus we have established the following lemma

Lemma 4.2.3 *We have* $GL_{96}(3) \geq < v_{1,96}, v_{2,96}, \nu_{96}, \rho_{96}, x_{96}, i_{96} > \simeq \mathbb{Z}_7^2 : SL_2(7) : \mathbb{Z}_2$.

□

The matrix u_{96} is also computed similarily. Here we obtain

$$u_{96} := \begin{pmatrix} \tilde{U} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & a & a & a & a & a & a \\ 0 & a & Ta & T^2a & T^3a & T^4a & T^5a & T^6a \\ 0 & a & T^2a & T^4a & T^6a & Ta & T^3a & T^5a \\ 0 & a & T^3a & T^6a & T^2a & T^5a & Ta & T^4a \\ 0 & a & T^4a & Ta & T^5a & T^2a & T^6a & T^3 \\ 0 & a & T^5a & T^3a & Ta & T^6a & T^4a & T^2a \\ 0 & a & T^6a & T^5a & T^4a & T^3a & T^2a & Ta \end{pmatrix},$$

with

$$\tilde{U} := \begin{pmatrix} \pm C & 0 \\ 0 & \pm I_6 \end{pmatrix} \text{ and } a := \begin{pmatrix} bC & b^C \\ b & Cb \end{pmatrix} = \begin{pmatrix} bC & b^3 \\ b & Cb \end{pmatrix}$$

where $b \in C_{GL_6(3)}(< B, J >)$, $o(b) = 8$. Thus we find

$$b = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & -1 & -1 & -1 & 0 \\ 0 & -1 & 0 & -1 & -1 & 0 \end{pmatrix}^i, i \in \{1, 3, 5, 7\}.$$

Checking the Weyl relation leads to the following:

$$\tilde{U} = \begin{pmatrix} C & 0 \\ 0 & I_6 \end{pmatrix} \text{ and}$$

$$b = \begin{pmatrix} -1 & -1 & -1 & 0 & -1 & 0 \\ 0 & -1 & -1 & -1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 & -1 & -1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Of course, multiplication with $-I_{96}$ does not change then relation giving us $-u_{96}$ as a second possibility. Thus we choose

$$u_{96} := \begin{pmatrix} \tilde{U} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & a & a & a & a & a & a \\ 0 & a & Ta & T^2a & T^3a & T^4a & T^5a & T^6a \\ 0 & a & T^2a & T^4a & T^6a & Ta & T^3a & T^5a \\ 0 & a & T^3a & T^6a & T^2a & T^5a & Ta & T^4a \\ 0 & a & T^4a & Ta & T^5a & T^2a & T^6a & T^3 \\ 0 & a & T^5a & T^3a & Ta & T^6a & T^4a & T^2a \\ 0 & a & T^6a & T^5a & T^4a & T^3a & T^2a & Ta \end{pmatrix},$$

with

$$\tilde{U} = \begin{pmatrix} C & 0 \\ 0 & I_6 \end{pmatrix} \text{ and } a = \begin{pmatrix} bC & b^3 \\ b & Cb \end{pmatrix},$$

with

$$b = \begin{pmatrix} -1 & -1 & -1 & 0 & -1 & 0 \\ 0 & -1 & -1 & -1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 & -1 & -1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Then the following lemma holds:

Lemma 4.2.4 *We have* $GL_{96}(3) \geq < v_{1,96}, v_{2,96}, \nu_{96}, \rho_{96}, x_{96}, i_{96}, u_{96} > \simeq L_3(7) : \mathbb{Z}_2$. *Moreover these matrices provide an irreducible 96-dimensional representation of the group* $L_3(7) : \mathbb{Z}_2$.

□

4.2.3 The 154-dimensional representation of $L_3(7) : \mathbb{Z}_2$

We gather the information obtained in the previous subsections and set

$$v_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & v_{1,57} & 0 \\ 0 & 0 & v_{1,96} \end{pmatrix}, v_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & v_{2,57} & 0 \\ 0 & 0 & v_{2,96} \end{pmatrix}, \nu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \nu_{57} & 0 \\ 0 & 0 & \nu_{96} \end{pmatrix}, \rho = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho_{57} & 0 \\ 0 & 0 & \rho_{96} \end{pmatrix},$$

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x_{57} & 0 \\ 0 & 0 & x_{96} \end{pmatrix}, i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i_{57} & 0 \\ 0 & 0 & i_{96} \end{pmatrix} \text{ and } u = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -u_{57} & 0 \\ 0 & 0 & -u_{96} \end{pmatrix}.$$

This form for u is chosen because it ensures that u and x^2 have the same Jordan form. Since we want to construct a representation of $O'N$, this is necessary. Then the following holds:

Lemma 4.2.5 *We have* $GL_{154}(3) \geq < v_1, v_2, \nu, \rho, x, i, u > \simeq L_3(7) : \mathbb{Z}_2$. *Moreover the module splits as* $V_{154} = V_1 \oplus V_{57} \oplus V_{96}$ *with* $dim V_k = k$ *and* V_k *is an irreducible module for* $L_3(7) : \mathbb{Z}_2$ *(*$k = 1, 57, 96$*).*

□

4.3 The construction of the generator a

In this section we construct the remaining generator a. In order to do so, we use the identification of the geometrical generators for $L_3(7) : \mathbb{Z}_2$ coming from the amalgam of the Buekenhout geometry as words in the canonical ones given in Chapter 2. Thus we keep the generator ρ and set $x = (XY)^2$. Then $z = (ut)^{xu}$ where $t = x^2$ and $X = zi$. Furthermore we have $Y = ie$ with $e = (xx^{\nu^3})^{x^\nu}$ and $Z = (x^{-1})^u$. For our further considerations, we need the following two lemmas:

Lemma 4.3.1 *Let* $G_1 =< z, X, Y, Z, \rho > \simeq L_3(7) : \mathbb{Z}_2$ *be as in the amalgam for the Buekenhout geometry. Then* $C_{G_1'}(a) =< zX, x, Z, Z^Y > \simeq L_2(7)$.

Proof. By construction, we have for $\mathbb{Z}_2 \times PGL_2(7) \simeq < z, X, Y, Z >=: H \leq G_1$ that $a^2 = z^{-1}x \in Z(H)$. Moreover H is a subgroup of index two in the parabolic $G_2 \simeq (\mathbb{Z}_4 \times L_2(7)) : \mathbb{Z}_2$ of the Buekenhout geometry, which itself containes a. Thereby we hold that $C_{G_1'}(a)$ is isomorphic to a subgroup of $L_2(7)$ because $Y \in G_1'$ and $a^Y = a^{-1}$. Furthermore the relations of Chapter 2 yield $x, zX, Z \in C_{G_1'}(a)$ and $< x, zX, Z > \simeq S_4$. Also, since $a^Y = a^{-1}$, $Z^Y \in C_{G_1'}(a) - < x, zX, Z >$ because Y is an automorphism of $H' \simeq L_2(7)$ and $o(ZZ^Y) = 3$ proving the assertion. □

We set $G_4 :=< z, X, Y, \rho, a > \simeq \mathbb{Z}_4 L_3(4) : \mathbb{Z}_2$. Then the 154-dimensional module splits for G_4 as $V_{154} = C_{V_{154}}(z) \oplus [V_{154}, z] = C_{V_{154}}(z) \oplus C_{[V_{154},z]}(z^2) \oplus [V_{154}, z^2]$.

Computing the dimensions of the eigenspaces of z shows that we have $dimC_{V_{154}}(z) = 30$, $dimC_{[V_{154},z]}(z^2) = 44$ and $dim[V_{154}, z^2] = 80$. We set $V_{30} := C_{V_{154}}(z)$, $V_{44} := C_{[V_{154},z]}(z^2)$ and $V_{80} := [V_{154}, z^2]$.

Lemma 4.3.2 *Let $G =< z, a, X, Y, Z, \rho >$ be a completion of the amalgam of the Buekenhout geometry. Assume that G has a 154-dimensional $GF(3)G$-module such that the matrices for z, X, Y, Z and ρ are as above. Then z and a have the same Jordan form.*

Proof. Set $G_{14} =< z, \rho, X, Y >$ Using MAGMA [2] to compute the indecomposable summands of V_{154} as a G_{14}-module, we hold that $V_{30}|_{G_{14}} = V_6 \oplus V_9 \oplus V_{15}$, $V_{44}|_{G_{14}} = V_{1,1} \oplus V_{1,2} \oplus V_{12} \oplus V_{15,1} \oplus V_{15,2}$ and $V_{80}|_{G_{14}} = V_8 \oplus V_{24,1} \oplus V_{24,2} \oplus V_{24,3}$ with $dimV_{i,\epsilon} = i$. Moreover one of the 24-dimensional submodules is irreducible. Using the 3-modular characters of $L_3(4)$ as given in [22], we hold therefore that in V_{30} two irreducible 15-dimensional $L_3(4) : \mathbb{Z}_2$-modules are involved, in V_{80} there are an irreducible 8-dimensional (4-dim. over $GF(9)$) and an irreducible 72-dimensional (36-dim. over $GF(9)$) $\mathbb{Z}_4 L_3(4) : \mathbb{Z}_2$-module involved and that V_{44} is an irreducible $\mathbb{Z}_2 L_3(4) : \mathbb{Z}_2$-module (22-dim over $GF(9))^2$.

Using the generators and relations for G_4 as in Chapter 2, we find that a is not a square in G_4 but $z^2 a$ is. Thus the characters we find in [22] are the characters of $z^2 a$. Furthermore the 72-dimensional module admits another involutory automorphism of $L_3(4)$, namely the field automorphism of $GF(4)$. This automorphism fuses two of the classes of elements of order four in $L_3(4)$. This implies that $z^2 a$ is an element of type $4A$ in the notation of [22]. The information gathered so far proves that $dim(V_{154}(a^2, -1) \cap V_{80}) = 40$, $dim(V_{154}(a^2, -1) \cap V_{44}) = 24$, $dim(V_{154}(a^2, -1) \cap V_{30}) = 16$, thus $dimV_{154}(a^2, -1) = 80$ by [22]. Moreover, since $z^2 a$ is a $4A$-element, we hold by [22] that $dim(V_{154}(a, 1) \cap V_{80}) = 16$, $dim(V_{154}(a, 1) \cap V_{44}) = 8$ and $dim(V_{154}(a, -1) \cap V_{80}) = 24$, $dim(V_{154}(a, -1) \cap V_{44}) = 12$. On V_{30} we have that XY is an element of type $4B$ or $4C$. We also get $tr(XY) = -1$ on V_{30}. By [22], this implies that w.l.o.g. $tr(XY) = 3$ on $V_{15,1}$ and $tr(XY) = -1$ on $V_{15,2}$. Since $z = 1$ on $V_{15,1}$, we have $tr(a) = tr(z^2 a)$ on $V_{15,1}$. Therefore we get that $tr(a) = -1$ on $V_{15,1}$ using [22] because $z^2 a$ is of type $4A$. Straightforward calculations yield $dim(V_{154}(a, 1) \cap V_{30}) \in \{3, 5\}$ and $dim(V_{154}(a, -1) \cap V_{30}) \in \{4, 2\}$. Thus either we find $dimV_{154}(a, 1) = 30$ and $dimV_{154}(a, -1) = 44$ or $dimV_{154}(a, 1) = 32$ and $dimV_{154}(a, -1) = 42$. Set $L =< zX, x, Z, Z^Y >= C_{G_1'}(a)$. Then L acts on $C_{V_{154}}(a^2) = V_{154}(a^2, 1)$. We use MAGMA [2] to compute the indecomposable summands of $V_{154}(a^2, 1)|_L$ and obtain $V_{154}(a^2, 1)|_L = W_{7,1} \oplus W_{7,2} \oplus W_{15,1} \oplus W_{15,2} \oplus W_{15,3} \oplus W_{15,4}$ with $dimW_{i,\epsilon} = i$. This proves $dimV_{154}(a, 1) = 30$ and $dimV_{154}(a, -1) = 44$, hence the assertion holds. □

These two lemmas provide a basis to construct the remaining generator a. The construction of a will now consist of constructing a *suitable* a on V_{44} and extend this

---

[2] Recall that G_4 is the group $4_2 L_3(4) : 2_1$ in notation of [8] and [22].

with elements of $C_{G_1'}(a) \simeq L_2(7)$ to the full module.

4.3.1 Computing a on V_{44}

We proceed as follows. Using MAGMA [2], we compute the Jordan form of z by typing
`J, T:=JordanForm(z);`.

The function `JordanForm()` of MAGMA returns two values, the Jordan form J and a transformation matrix T such that $J = TzT^{-1}$. We give the matrix T as MAGMA input in the appendix. Then we have

$$J = \begin{pmatrix} -I_{44} & 0 & 0 \\ 0 & I_{30} & 0 \\ 0 & 0 & J_{80} \end{pmatrix}, \text{ where } J_{80} = \begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0 & \cdots \\ 0 & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix} \in GL_{80}(3).$$

Now, we reduce any matrix $M \in \{X, Y, \rho\}$ and take the upper left 44×44-submatrix of TMT^{-1} by typing
`M44:=Submatrix(T*M*T^-1, 1,1, 44,44);`.

We store these matrices in a sequence Q in MAGMA. Using the presentation of $\mathbb{Z}_2 L_3(4) : \mathbb{Z}_2$ of Chapter 2 (here we set $z^2 = 1$), we can now induce the 44-dimensional Module for $M :=< z, X, Y, \rho >$ to $G :=< z, X, Y, \rho, a >$ as follows:
`W:=GModule(M, Q);`[3]
`V:=Induction(W, G);`

Using the Meataxe-program as installed in MAGMA, we can reduce V and obtain a matrix $a_M \in GL_{44}(3)$ for a, such that $< -I_{44}, X_M, Y_M, \rho_M, a_M > \simeq \mathbb{Z}_2 L_3(4) : \mathbb{Z}_2$.

We conjugate these matrices in $GL_{44}(3)$ on X_{44}, Y_{44}, and ρ_{44}. Set $a_{44,0}$ to be the corresponding conjugate of a_M. We need to construct *suitable* $C_{GL_{44}(3)}(< X_{44}, Y_{44}, \rho_{44} >)$-conjugates. This is achieved as follows. Using the representation for $L_3(7) : \mathbb{Z}_2$, we see that the points (objects stabilized by a $L_3(7) : \mathbb{Z}_2$) of the Buekenhout geometry correspond to certain 1-dimensional subspaces. Now lines of the Buekenhout geometry (objects stabilized by $(\mathbb{Z}_4 \times L_2(7)) : \mathbb{Z}_2$) have exactly two points. Let $G_1 =< z, X, Y, \rho, Z >$ and p_1 the point of the geometry fixed by G_1. Since $a \in G_2 =< a, z, X, Y, Z > \simeq (\mathbb{Z}_4 \times L_2(7)) : \mathbb{Z}_2$, we have that $p_1 a$ is collinear to p_1 and a interchanges p_1 and $p_1 a$. Hence, as a matrix, a has to fuse the two 1-dimensional submodules of G_1 and G_1^a. By construction, the 1-dimensional submodule belonging to G_1 is $< b_1 >$ where b_1 is the first standard basis vector of the 154-dimensional module. We set $M_t := TMT^{-1}$ for $M \in \{X, Y, \rho, Z\}$, $J = TzT^{-1}$, $G_{1,t} :=< J, X_t, Y_t, \rho_t, Z_t >$ and $C :=< JX_t, (X_t Y_t)^2, Z_t, Z_t^{Y_t} > \simeq L_2(7)$. Since C has to centralize a and $C \leq G_{1,t}'$ we get $C \leq (G_{1,t}^{a_t})'$ and the 1-dimensional submodule for $G_{1,t}^{a_t}$ is in $C_{V_{154}}(C)$. Now $C_{V_{154}}(C)$ can easily be computed as the intersection of the eigenspaces $V_{154}(\alpha, 1)$ where

---

[3] The order in Q must be the same as the generators for H, i.e., z, X, Y and ρ, where the $-I_{44}$ is the corresponding matrix for z

$\alpha \in \{JX_t, (X_tY_t)^2, Z_t, Z_t^{Y_t}\}$. This intersection has dimension four and is contained in $V_{44} = V_{154}(J, -1)$. Because $[X, a] = 1$, we must have $X \in G_{1,t}^{a_t} - (G_{1,t}^{a_t})'$ and the relation $Y^a = a^2Y$ implies that $Y_t \in G'_{1,t}$ but $Y_t \in G_{1,t}^{a_t} - (G_{1,t}^{a_t})'$. Thereby we hold that the 1-dimensional submodule for $G_{1,t}^{a_t}$ is inside $V_{154}(X_t, -1) \cap V_{154}(Y_t, -1)$. Thus we compute $W := C_{V_{154}}(C) \cap V_{154}(X_t, -1) \cap V_{154}(Y_t, -1)$ and hold that $W =< b_2, b_3 >$ where $b_2 =$

$$(0, 1, 2, 2, 2, 1, 2, 2, 2, 1, 2, 0, 0, 2, 1, 0, 1, 0, 0, 2, 1, 1, 0, 1, 0, 2, 2, 0, 0, \ldots)$$

and $b_3 =$

$$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 2, 2, 2, 2, 1, 0, 1, 0, 0,\\ 1, 0, 2, 1, 0, 0, \ldots).$$

We can now regard b_1, b_2 and b_3 as vectors in a 44-dimensional space and conjugate $a_{44,0}$ in $C_{GL_{44}(3)}(< X_{44}, Y_{44}, \rho_{44} >)$ such that $b_1 a_{44,0}^k \in < b_2, b_3 >$ using MAGMA. As a result we get eight *suitable* candidates $a_{44,i}$.

The matrices X_{44}, Y_{44}, ρ_{44} and $a_{44,i}$ are given explicitly in the appendix as well as the generators of $C_{GL_{44}(3)}(< X_{44}, Y_{44}, \rho_{44} >)$.

Let us return briefly to W. We find by multiplying with T^{-1} that $b_2T^{-1} =$

$$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 2, 1, 1, 1, 0, 1, 1,\\ 1, 2, 1, 1, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0, 1, 2, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0,\\ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 2, 1, 2, 2,\\ 2, 2, 1, 0, 0, 0, 0, 2, 2, 0, 1, 2, 2, 1, 1, 1, 1, 0, 2, 0, 1, 2, 0, 2, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$

and $b_3T^{-1} =$

$$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0,\\ 2, 2, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 2, 2, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0,\\ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 1,\\ 0, 0, 0, 1, 1, 1, 0, 1, 0, 0, 0, 1, 2, 0, 2, 0, 1, 2, 2, 2, 2, 0, 2, 2, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$

So neither b_2 nor b_3 is contained in the 57-dimensional or 96-dimensional submodule for $G_{1,t}$. Therefore we get an irreducible module for $G =< z, X, Y, Z, \rho, a >$.

4.3.2 Extending a

We give an algorithm to extend the suitable candidates $a_{44,i}$ obtained in the previous subsection.

4.3.2.1 Description of the algorithm

The algorithm constructs a Jordan basis B_a of V_{154} for a and a matrix τ whose j-th row is the j-th vector in B_a. Then $a = T^{-1}\tau^{-1}J\tau T$, where J is the Jordan form of z and T is the transformation matrix from above, i.e., $TzT^{-1} = J$. We work with the transformed matrices $X_t := TXT^{-1}$, $Y_t := TYT^{-1}$, $\rho_t := T\rho T^{-1}$, $Z_t := TZT^{-1}$ and J. Furthermore we set $C :=< JX_t, (X_tY_t)^2, Z_t, Z_t^{Y_t} >\simeq L_2(7)$ which should centralize a by Lemma 4.3.1. Also we construct a 154×154-matrix A_t whose upper left 44×44-submatrix is one of the suitable $a_{44,i}$s obtained above and the rest is simply I_{110}. Moreover V_{44} is now identified with the subspace of V_{154} generated by the first 44 standard basis vectors.

We now compute the eigenspace $E_2 := V_{154}(A_t, -1) \cap V_{44}$ and a basis B_2 of E_2. It turns out that $dimE_2 = 12$. The vectors of B_2 are included as the first 12 vectors in a sequence $B_{a,2}$ which shall become a basis of $V_{154}(a, -1)$. We now extend this sequence in the following way. For a vector $b \in B_2$ and an element $k \in C$ we check whether $bk \in< B_{a,2} >$. If $bk \notin< B_{a,2} >$, we store bk as a new element in $B_{a,2}$. This is repeated until $dim < B_{a,2} >= 44$.

The same process has to be performed for $E_1 := V_{154}(A_t, 1) \cap V_{44}$ with basis B_1 and a sequence $B_{a,1}$. It turns out here that $\mid B_1 \mid= 8$ and we stop if $dim < B_{a,1} >= 30$.

The next step is to construct a partial basis $B_{a,3}$ corresponding to the $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$-boxes. For this we compute $E_3 := V_{154}(X_t, -1) \cap V_{154}(Y_t, 1) \cap V_{44}$ with basis B_3, and $E_4 := V_{154}(X_t, 1) \cap V_{154}(Y_t, -1) \cap V_{44}$ with basis B_4. Since $a^2 = z^{-1}(XY)^2$ has to hold on V_{154} (and holds correspondingly on V_{44}), a cannot have eigenvectors in E_3 and E_4 (since z inverts all vectors in V_{44}). Because $[a_{44,i}, X_{44}] = 1$ and $a_{44,i}^{Y_{44}} = a_{44,i}^{-1}$, we get $E_3A_t = V_{154}(X_t, -1) \cap V_{154}(Y_t, -1) \cap V_{44}$ and $E_4A_t = V_{154}(X_t, 1) \cap V_{154}(Y_t, 1) \cap V_{44}$. Furthermore as a vector space $V_{44} := E_1 \oplus E_2 \oplus E_3 \oplus E_3A_t \oplus E_4 \oplus E_4A_t$, and $dimE_3 = dimE_4 = 6$.

We store in a sequence $B_{a,3}$ firstly the vector pairs $(b, bA_t), b \in B_3$ and secondly the pairs $(b, bA_t), b \in B_4$. Then we run the same loop as above taking the first, third up to the 23-th vector in $B_{a,3}$, thus we check whether for a vector b_i of these and some $k \in C$ we have $b_ik \notin< B_{a,3} >$ and then append b_ik and b_iA_tk to $B_{a,3}$ until $\mid B_{a,3} \mid= 80$.

The last part of the algorithm constructs B_a as a sequence of vectors in V_{154} simply by appending the vectors in $B_{a,2}$, $B_{a,1}$ and $B_{a,3}$ (in this order) to B_a. Then the i-th vector in B_a is put as the i-th row of a matrix τ and we hold $a := T^{-1}\tau^{-1}J\tau T$ such that $< a, z, X, Y, \rho >\simeq \mathbb{Z}_4L_3(4) : \mathbb{Z}_2$ and $[a, Z] = 1$.

Remark. Since this algorithm does not make use of any algorithm in MAGMA more profound than matrix multiplication (one could even store the elements of C as a set of matrices), the extension process of $a_{44,i}$ can be seen as a *computer free* process in the author's opinion.

4.3.2.2 The algorithm

We give the algorithm in MAGMA statements. For $M \in \{X, Y, \rho\}$, `Mt` is the matrix M_t from above, `a44i` is one of the suitable $a_{44,i}$'s.

```
C:=sub<GL(154,3)|J*Xt,(Xt*Yt)^2,Zt, Zt^Yt>;
```

```
V:=VectorSpace(GF(3), 154);
B:=Basis(V);
B44:=[B|];
for k in [1..44] do
   Append(~B44, B[k]);
end for;
V44:=sub<V|B44>;
M154:=MatrixAlgebra(GF(3), 154);
At:=M154!1;
InsertBlock(~At, a44i, 1,1);
Ba:=[V|];
```

This part constructs C, V, the natural basis for V, V_{44}, the matrix A_t and initializes the sequence B_a which shall become the Jordan basis corresponding to a.

```
E2:=Eigenspace(At, 2) meet V44;
B2:=Basis(E2);
Ba2:=[V|];
for k in B2 do
   Append(~Ba2, k);
end for;
for j in [1..12] do
   if # Ba2 ne 44 then
      for k in C do
         if Ba2[j]*k notin sub<V|Ba2> then
            Append(~Ba2, Ba2[j]*k);
         end if;
      end for;
   else break;
   end if;
end for;
```

This part creates the basis $B_{a,2}$ of $V_{154}(a,-1)$.

```
E1:=Eigenspace(At, 1) meet V44;
B1:=Basis(E1);
Ba1:=[V|];
for k in B1 do
   Append(~Ba1, k);
end for;
for j in [1..8] do
   if # Ba1 ne 30 then
      for k in C do
         if Ba1[j]*k notin sub<V|Ba1> then
            Append(~Ba1, Ba1[j]*k);
         end if;
```

```
         end for;
      else break;
      end if;
   end for;
```

This part creates the basis $B_{a,1}$ of $V_{154}(a,1)$.

```
   E3:=Eigenspace(Xt, 2) meet Eigenspace(Yt, 1) meet V44;
   B3:=Basis(E3);
   Ba3:=[V|];
   for k in B3 do
      Append(~Ba3, k);
      Append(~Ba3, k*At);
   end for;
   E4:=Eigenspace(Xt, 1) meet Eigenspace(Yt, 2) meet V44;
   B4:=Basis(E4);
   for k in B4 do
      Append(~Ba3, k);
      Append(~Ba3, k*At);
   end for;
   for j in [0..11] do
      if # Ba3 ne 80 then
         for k in C do
            if Ba3[2*j+1]*k notin sub<V|Ba3> then
               Append(~Ba3, Ba3[2*j+1]*k);
               Append(~Ba3, Ba3[2*j+2]*k);
            end if;
         end for;
      else break;
      end if;
   end for;
```

This part creates the basis $B_{a,3}$ consisting of the vector pairs which belong to the $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$-boxes.

```
   for k in Ba2 do
      Append(~Ba, k);
   end for;
   for k in Ba1 do
      Append(~Ba, k);
   end for;
   for k in Ba3 do
      Append(~Ba, k);
   end for;
```

```
tau:=M154!0;
for k in [1..154] do
   tau[k]:=Ba[k];
end for;
a:=T^-1*tau^-1*J*tau*T;
a:=G!a;
```

This final part constructs the Jordan basis B_a, the matrix τ and the matrix a which is given as a MAGMA input in the appendix. The construction of the matrix a finishes the proof of the following theorem:

Theorem 4.3.3 *Let $\mathcal{A}$ be the amalgam of the Buekenhout geometry for $O'N$. Then every completion G of $\mathcal{A}$ has an irreducible 154-dimensional $GF(3)$-module.*

□