Chapter 5

Construction of the Ivanov-Shpectorov Geometry out of the Buekenhout Geometry

In this chapter we show that a completion of the amalgam related to the Buekenhout geometry is also a completion of the amalgam of the Ivanov-Shpectorov geometry. This is done without using the fact that $O'N$ is a completion of both amalgams.

We fix the following notation. $G :=< a, z, \rho, X, Y, Z >$ and $G_4 :=< a, z, \rho, X, Y >\simeq \mathbb{Z}_4 L_3(4) : 2_1$. Using the generators for G_4 as given in Chapter 2, we have seen that $E_1 :=< f_1, f_2, f_3, f_5, z >\simeq \mathbb{Z}_4 * Q_8 * Q_8$, also $\bar{P_1} :=< E_1, a, \rho_1, f_4 >=< E_1, \tilde{a}, \rho_n, \tau >\simeq (\mathbb{Z}_4 * Q_8 * Q_8) : A_5$ with $E_1 = O_2(\bar{P_1})$. Moreover, if $\bar{P_2} := \bar{P_1}^X$, then $< \bar{P_1}, \bar{P_2} >= G'_4 \simeq \mathbb{Z}_4 L_3(4)$. Thus $O_2(\bar{P_1} \cap \bar{P_2}) \in Syl_2(G'_4)$ and $S :=< O_2(\bar{P_1} \cap \bar{P_2}), X >\in Syl_2(G_4)$. Using the relations of Chapter 2, we get that $O_2(\bar{P_1} \cap \bar{P_2}) =< f_1, f_2, f_3, f_4, f_5, z, a >$ since $X = \beta^{f_3 f_5}$, and therefore $S =< f_1, f_2, f_3, f_4, f_5, z, a, X >$. Furthermore the relation $f_4 = zXY f_2 f_1 z^2$ implies $Y \in S$.

5.1 G has a subgroup $L \simeq \mathbb{Z}_4^3 L_3(2)$

The aim of this section is to establish the following lemma:

Lemma 5.1.1 *Let G be the completion of the amalgam $\mathcal{A}$ related to the Buekenhout geometry for $O'N$. Then G has a subgroup $L \simeq \mathbb{Z}_4^3 L_3(2)$.*

Again, note that this lemma will be proved without using the group $O'N$. We start to prove:

Lemma 5.1.2 *Set $b := f_4 f_5^{-1}$. Then $F :=< a, b, z >\simeq \mathbb{Z}_4^3$ and $F \lhd S$.*

Proof. Since $f_4, f_5 \in G'_4$, we get $[z, b] = 1$. Using the relations of Chapter 2, we obtain $[a, b] = [a, f_5] f_1 f_3 z^{-1}$. By $[f_2, f_5] = 1$, we hold $[a, f_5] = [f_2 f_5 a, f_5] = f_1 f_3 z$ thus F is abelian and $f_3 \in F$. Moreover $o(b) = 4$. The fact $f_3 \notin < z, a >$ implies $F \simeq \mathbb{Z}_4^3$ since $b^2 = f_4^{-1} f_5 f_4 f_5^{-1} = [f_4, f_5^{-1}] = [f_4, f_5] = z f_3$ ($f_5^2 = z^2$). We need to prove that F is normal in S. Clearly, we have $S =< E_1, X >$ and $f_1 \in F$ (since $a^2 = z^{-1} f_1$). Using the relations as given in Chapter 2, we see that $a^X = a^{-1}$, $a^{f_2} = a f_3$ and $a^{f_5} = a f_1 f_3 z$ since $f_3 = [a, f_2]$ and $[a, f_5] = f_1 f_3 z$. Using $[f_2, f_4] = f_1^{-1}$, $f_1^{f_2} = f_1^{f_5} = f_1^{-1}$ and $[f_2, f_5] = 1$, we compute $[b, f_2] = f_1^{-1}$, hence $b^{f_2} = b f_1^{-1}$. Now $[b, f_5] = [f_5, f_4^{-1}] = [f_4, f_5] = f_3 z$, thus $b^{f_5} = b^{-1}$. By $b = f_4 f_5^{-1}$, we hold $b^X = f_5 a f_4 f_3 f_1 z^{-1}$. Using $[f_5, a] = z f_1 f_3^{-1}$, $[f_3, f_4] = 1$ and $[f_1, f_4] = z^2$, we get $b^X = a f_5 f_4 z^2 = a b^{-1} z^2$, proving the lemma. □

Lemma 5.1.3 *Set* $P_1 :=< S, \rho_n >$ *and* $U :=< F, f_2, f_4 >$. *Then* $P_1 \simeq \mathbb{Z}_4^3 S_4$, $F \lhd P_1$ *and* $U = O_2(P_1)$.

Proof. Using our relations, we hold $a^{\rho_n} = a^{-1} b^{-1} z$ and $b^{\rho_n} = a z^2$. Since $[\rho_n, z] = 1$, we have $F \lhd P_1$.

We show $\mid U \mid= 2^8$ and $U \lhd P_1$. Using the relations of Chapter 2, we get $[f_2, f_4] = f_1^{-1} \in F$ and $(f_2 f_4)^2 = [f_2, f_4]$. In particular we hold $o(f_2 f_4) = 8$. Furthermore $(f_2 f_4)^{f_2} = f_4 f_2 = f_4^{-1} f_2^{-1}$, hence $< f_2, f_4 >\simeq Q_{16}$. Since F is abelian and of exponent four, $|< f_2, f_4 > \cap F \mid= 4$ and $\mid U \mid= 2^8$. In particular we have $F \lhd U$ and U/F is elementary abelian of order four.

We show $U \lhd P_1$. Our relations prove the following: $f_2^{\rho_n} = f_5 = b^{-1} f_4$, $f_4^{\rho_n} = b^{-1} a z f_4 f_2$. This shows that $< f_2, f_4, \rho_n > F/F \simeq A_4$. Moreover $Y \in S$ and $\rho_n \rho_n^Y = z^2 a^{-1} b$ and the lemma is proved. □

Lemma 5.1.4 *Set* $g := (ZX)^2$, $x := (b^2)^{g^{-1}}$, $P_2 :=< S, x >$ *and* $W :=< F, X, Y >$. *Then* $(z^2)^x = z^2 a^2$, $(a^2 z^2)^x = z^2$, $P_2 \simeq \mathbb{Z}_4^3 S_4$, $F \lhd P_2$ *and* $W = O_2(P_2)$.

Proof. The action of x on $< z^2, a^2 >$ is verified using the matrices of the previous chapter. Since $o(XY) = 8$, we have $< X, Y >\simeq D_{16}$. Moreover $f_1 = (XY)^2$ and as above we hold $< X, Y > \cap F =< f_1 >$, so $\mid W \mid= 2^8$ and W/F is again elementary abelian of order four.

Using the relations of Chapter 2 and the matrices, we get $X^x = X b^2 a^{-1}$, $Y^x = X Y a^2 b^{-1}$, $z^x = z b^2 a^{-1}$, $b^x = z^2 a b$ and, by construction, $[a, x] = 1$. Thus we have $W \lhd P_2$.

We get $o(x f_2) = 12$. Set $y := (x f_2)^4 \notin S$. Then one verifies $y y^x = z^2 a^2 \in F$ ($x \notin W$), proving the lemma. □

Corollary 5.1.5 *We have* $P_1 \cap P_2 = S$, $S \simeq \mathbb{Z}_4^3 D_8$. *Moreover* U *and* W *are the preimages of the elementary abelian groups of order four in* S/F.

□

Lemma 5.1.6 $L :=< P_1, P_2 >\simeq \mathbb{Z}_4^3 L_3(2)$.

Proof. By the previous lemmas it remains to show that the Weyl relation holds in L. Clearly, $Y^{\rho_n} \notin P_2$, $x \notin P_1$ and we verify, using the matrices, that $o(x Y^{\rho_n}) = 12$, $(x Y^{\rho_n})^3 = z a^{-1} b^{-1}$. □

5.2 Construction of the maximal parabolic groups of the Ivanov-Shpectorov geometry

In this section we construct the maximal parabolic subgroups $\bar{G_1} \simeq J_1$, $\bar{G_2} \simeq M_{11}$ and $\bar{G_5} \simeq (\mathbb{Z}_4 * Q_8 * Q_8) : A_5$. The group L constructed above contains a subgroup $N \simeq \mathbb{Z}_2^3 : \mathbb{Z}_7 : \mathbb{Z}_3$. We construct a group $H \simeq \mathbb{Z}_2 \times A_5$ using $\bar{P_1}$ such that $< N, H > \simeq J_1$.

By the previous section, $\Omega_1(F) \leq O_2(\bar{P_1}) \cap O_2(\bar{P_2})$. Therefore $\Omega_1(F)$ is not contained in any subgroup of $\bar{P_i}$ of shape $\mathbb{Z}_2 \times A_5$. Set $\alpha := \rho_1^X$ then $\alpha \in \bar{P_2} - \bar{P_1}$ and $\bar{G_5} := \bar{P_1}^{\alpha}$. Then $\Omega_1(F) \cap O_2(\bar{G_5}) =< z^2 >$.

Lemma 5.2.1 *Set $i := (f_2 f_5)^X$. Moreover set $A := \tilde{a}$, $B := z^2 i^{\rho_n}$ and $A_n := z^2 i^{\rho_1}$. Then $U :=< A, A_n, B > \simeq \mathbb{Z}_2 \times A_5$, $U' =< z^2 A, z^2 A_n, z^2 B >$ and $U \leq \bar{P_1}$.*

Proof. Obviously, we have $U \leq \bar{P_1}$. Since $[f_2, f_5] = 1$, we have $i^2 = 1$. Using the relations obtained in Chapter 2, we compute $i = a f_2 f_3 f_5 \notin E_1$. Furthermore $A^2 = (a f_2 f_5)^2 = z^2 f_2 f_3 f_5 f_3 f_2 f_5 = (f_2 f_5)^2 = 1$ and $A_n^2 = 1$. Using the matrices, we get $[A, B] = 1$, $o(AA_n) = 5$, $o(BA_n) = 3$ and $o(z^2 A A_n B) = 5$, hence $U \simeq \mathbb{Z}_2 \times A_5$ as in Chapter 3. □

Lemma 5.2.2 $< z^2, A, B >^{\alpha} = \Omega_1(F)$.

Proof. Using the matrices, we verify the following identities: $A^{\alpha} = b^2$ and $B^{\alpha} = a^2 b^2$. Since $[z, \alpha] = 1$, the assertion follows. □

We set $\bar{A} := A^{\alpha}$, $\bar{B} := B^{\alpha}$, $\bar{A_n} := A_n^{\alpha}$, $\bar{G_{15}} := U^{\alpha}$.

Lemma 5.2.3 *Let $y := (x f_2)^4$. Then $N_1 :=< \Omega_1(F), \rho_n, y > \simeq \mathbb{Z}_2^3 : (\mathbb{Z}_7 : \mathbb{Z}_3)$.*

Proof. As in the proof of Lemma 5.1.4, we have $o(y) = 3$. Then the following holds: $s := \rho_n y$ is of order seven, $[\rho_n, y] = s^{-1}$ and $y \rho_n = s^2$, proving the lemma. □

We identify $\bar{A}$ with (12)(34) and $\bar{B}$ with (13)(24) in A_5. Then we can identify $\bar{A_n}$ with (15)(24). Set $d := \bar{B}\bar{A_n}$, then we identify d with (135). With this identification, we have $\bar{d} := d^{(\bar{A}\bar{A_n})^{\bar{B}}} = (234)$. Therefore $K :=< \bar{A}, \bar{B}, \bar{d} > \simeq \mathbb{Z}_2 \times A_4$. Since $\rho_n, \bar{d} \in P_1$, we have $\rho_n \sim_{P_1} \bar{d}$. Set $\delta := X f_2 a b^{-1}$. Then we verify $\rho_n^{\delta} = \bar{d}$, hence we set $N := N_1^{\delta}$. This implies $\bar{G_{15}} \cap N = K$. We prove the following lemma:

Lemma 5.2.4 $\bar{G_1} :=< N, \bar{G_{15}} > \simeq J_1$.

Proof. We show $\bar{G_1} \simeq J_1$ using the generators and relations of the Ivanov geometry as given in Chapter 3. Set $\bar{s} := s^{\alpha}$. Then we have $(z^2)^{\bar{s}} = z^2 \bar{A}\bar{B}$, $(z^2)^{\bar{s}^{-1}} = z^2 \bar{B} \in \bar{G_{15}}'$ and $\bar{B}^{\bar{s}^{-1}} = z^2 \bar{A}$. Moreover we find $o(z^2 \bar{A}\bar{B}\bar{A_n}) = 5$. We set $\tilde{A} := z^2 \bar{A}\bar{B}$, $\tilde{B} := \bar{B}$ and $\tilde{A_n} := \bar{A_n}$. Then $\bar{G_{15}} =< \tilde{A}, \tilde{B}, \tilde{A_n} >$ and $(\tilde{A}, \tilde{B}, \tilde{A_n})$ satisfies all the required relations.

We set $\bar{t} := \tilde{B}\tilde{A_n}$. Since $(z^2)^{\bar{s}^{-1}} = z^2\tilde{B}$, z^2 inverts $\bar{t}^{\bar{s}}$ and $\bar{t}^{\tilde{A}\bar{s}}$. Using the matrices, we find $o([\tilde{A_n}, \bar{t}^{\bar{s}}]) = 5$. Thus we set $\tilde{t} := \bar{t}^{\tilde{A}\bar{s}}$ and compute $[\tilde{A_n}, \tilde{t}] = 1$. Moreover $[\tilde{A}, \tilde{t}] = 1$ by construction, $o(\tilde{B}\tilde{t}) = 5$ and $z^2 = \tilde{B}\tilde{t}[\tilde{B}, \tilde{t}]^2$ finishing the proof. □

Let P be a maximal parabolic subgroup of $L_3(4)$, $j \in P - O_2(P)$ an involution. Then $C_P(j) \simeq \mathbb{Z}_2^4$ and $C_P(j) \cap O_2(P) \simeq \mathbb{Z}_2^2$. In the extension $\mathbb{Z}_4L_3(4)$ the elements j and ij are conjugate, where i denotes the central involution. Thus the centralizer of j in $\mathbb{Z}_4P$ is of order 32. This group has the structure $\mathbb{Z}_2 \times (\mathbb{Z}_4 * D_8)$.

We compute $C := C_{\bar{G_5}}(\tilde{A})$. Clearly, $z \in C$, $\Omega_1(F) \leq C$ and $< z, \Omega_1(F) > \simeq \mathbb{Z}_4 \times \mathbb{Z}_2^2$. Furthermore we find $[\tilde{A}, f_3^\alpha] = 1$, thus $C = < z, \Omega_1(F), f_3^\alpha >$ and $D_8 \simeq < \tilde{B}, f_3^\alpha > \lhd C$.

In order to find the remaining generator $\tilde{v}$ to construct $\bar{G_2} = < \tilde{A_n}, \tilde{B}, \tilde{t}, \tilde{v} > \simeq M_{11}$ (generators as in Chapter 3), we need to compute $C_{<\tilde{B}, f_3^\alpha>}(\tilde{t})$ since the relations $[\tilde{A}, \tilde{v}] = [\tilde{t}, \tilde{v}] = 1$ have to hold. Moreover $< \tilde{B}, f_3^\alpha > = < \tilde{B}, \tilde{v} >$ must be fulfilled.

Lemma 5.2.5 *Set $\tilde{v} := \tilde{B}f_3^\alpha$. Then $\bar{G_2} := < \tilde{A_n}, \tilde{B}, \tilde{t}, \tilde{v} > \simeq M_{11}$.*

Proof. The involutions in $< \tilde{B}, f_3^\alpha >$ are the following: z^2 (the central involution), $\tilde{B}$, $\tilde{B}^{f_3^\alpha}$, $\tilde{B}f_3^\alpha$ and $\tilde{B}^{f_3^\alpha}f_3^\alpha$. Using the matrices, we get $[\tilde{t}, x] = 1$ only for $x = \tilde{B}f_3^\alpha$. Therefore $\tilde{v} \in \{\tilde{B}f_3^\alpha, \tilde{A}\tilde{B}f_3^\alpha\}$. Then we compute $o(\tilde{A_n}\tilde{A}\tilde{B}f_3^\alpha) = 6$ and $o(\tilde{A_n}\tilde{B}f_3^\alpha) = 3$. With respect to the relations as given in Chapter 3, we set $\tilde{v} := \tilde{B}f_3^\alpha$ and the lemma is proved. □

Corollary 5.2.6 *The groups $\bar{G_1}$, $\bar{G_2}$ and $\bar{G_5}$ are the end-parabolic groups of the Ivanov-Shpectorov geometry.*

□

Remark. The amalgam $(G_4, L, \bar{G_1})$ has been used by Lempken to construct $O'N$ [20].