APPROXIMATION AND OPTIMAL CONTROL OF THE STOCHASTIC NAVIER-STOKES EQUATION

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Contents

Introduction **1**

Frequently Used Notations **5**

1 Existence and Approximation of the Solution **7**
- 1.1 Assumptions and formulation of the problem . . . 8
- 1.2 Existence of the solution of the stochastic Navier-Stokes equation by Galerkin approximation . . . 10
- 1.3 A special linear stochastic evolution equation . . . 23
- 1.4 Linear approximation of the solution of the stochastic Navier-Stokes equation . . . 34

2 Optimal Control **41**
- 2.1 Formulation of the control problem . . . 41
- 2.2 Existence of optimal controls . . . 43
- 2.3 The existence of optimal feedback controls . . . 45
- 2.4 Existence of ε-optimal feedback controls . . . 58
- 2.5 A special property . . . 61
- 2.6 The Gateaux derivative of the cost functional . . . 67
- 2.7 A stochastic minimum principle . . . 72
- 2.8 Equation of the adjoint processes . . . 75

3 About the Dynamic Programming Equation **83**
- 3.1 The Markov property . . . 83
- 3.2 Bellman's principle and Bellman's equation for the finite dimensional stochastic Navier-Stokes equation . . . 87

Appendix **93**

A Basic Convergence Results **93**

B Stopping Times **95**

Bibliography **101**

INTRODUCTION

"The Navier-Stokes equation occupy a central position in the study of nonlinear partial differential equations, dynamical systems, and modern scientific computation, as well as classical fluid dynamics. Because of the complexity and variety of fluid dynamical phenomena, and the simplicity and exactitude of the governing equations, a very special depth and beauty is expected in the mathematical theory. Thus, it is a source of pleasure and fascination that many of the most important questions in the theory remain yet to be answered, and seem certain to stimulate contributions of depth, originality and influence far into the future." (J.G. Heywood [15])

The Navier-Stokes equations were formulated by the French physicist C.L.M.H. Navier (1785-1836) in 1822 and the British mathematician and physicist G.G. Stokes (1819-1903) in 1845. Existence and uniqueness theorems for the stationary Navier-Stokes equation were first proved by F. Odquist in 1930 [27] and by J. Leray in 1933-1934 [22], [23]. E. Hopf [17] (1952) was the first who obtained the equation for the characteristic functional of the statistical solution giving a probability description of fluid flows. There is much information about statistical hydromechanics with detailed review of literature in the books written by A.S. Monin and A.M. Jaglom [25] in 1965, 1967. C. Foias investigated in [10] (1972) the questions of existence and uniqueness of spatial statistical solutions. A. Bensoussan and R. Temam [2] (1973) gave for the first time a functional analytical approach for the stochastic Navier-Stokes equations. The research has accelerated during the last twenty five years.

"Researchers are now undertaking the study of flows with free surfaces, flows past obstacles, jets through apertures, heat convection, bifurcation, attractors, turbulence, etc., on the basis of an exact mathematical analysis. At the same time, the advent of high speed computers has made computational fluid dynamics a subject of the greatest practical importance. Hence, the development of computational methods has become another focus of the highest priority for the application of the mathematical theory. It is not surprising, then, that there has been an explosion of activity in recent years, in the diversity of topics being studied, in the number of researchers who are involved, and in the number of countries where they are located." (Preface for "The Navier Stokes Equations II"- Proceedings of the Oberwolfach meeting 1991, [16])

After this short history about the deterministic and stochastic equations of Navier-Stokes type, we give the equation for the stochastic Navier-Stokes equation which describes the behavior of a viscous velocity field of an incompressible liquid. The equation on the domain of flow $G \subset \mathbb{R}^n$ ($n \geq 2$ a natural number) is given by

$$\frac{\partial U}{\partial t} - \nu \Delta U = -(U, \nabla)U + f - \nabla p + \mathcal{C}(U)\frac{\partial w}{\partial t} \tag{0.1}$$

$$\text{div } U = 0, \quad U(0,x) = U_0(x), \quad U(t,x) \mid_{\partial G} = 0, \ t > 0, \ x \in G,$$

where U is the velocity field, ν is the viscosity, Δ is the Laplacian, ∇ is the gradient, f is an external force, p is the pressure, and U_0 is the initial condition. Realistic models for flows should contain a random noise part, because external perturbations and the internal Browninan motion influence the velocity field. For this reason equation (0.1) contains a random noise part $\mathcal{C}(U)\dfrac{\partial w}{\partial t}$.

Here the noise is defined as the distributional derivative of a Wiener process $\Big(w(t)\Big)_{t\in[0,T]}$, whose intensity depends on the state U.

This nonlinear differential equation is only for the simplest examples exactly soluble, usually corresponding to laminar flows. Physical experiments show that turbulence occurs if the outer force f is sufficiently large. In many important applications, including turbulence, the equation must be modified, matched or truncated, or otherwise approximated analytically or numerically in order to obtain any predictions. Sometimes a good approximation can be of equal or greater utility than a complicated exact result.

In the study of equations of Navier-Stokes type one can consider weak solutions of martingal type or strong solutions. Throughout this paper we consider strong solutions ("strong" in the sense of stochastic analysis) of a stochastic equation of Navier-Stokes type (we will call it stochastic Navier-Stokes equation) and define the equation in the generalized sense as an evolution equation, assuming that the stochastic processes are defined on a given complete probability space and the Wiener process is given in advance.

The aim of this dissertation is to prove the existence of the strong solution of the Navier-Stokes equation by approximating it by means of the Galerkin method, i.e., by a sequence of solutions of finite dimensional evolution equations. The Galerkin method involves solving nonlinear equations and often it is difficult to deal with them. For this reason we approximate the solution of the stochastic Navier-Stokes equation by the solutions of a sequence of linear stochastic evolution equations. Another interesting aspect of the stochastic Navier-Stokes equation is to study the behavior of the flow if we act upon the fluid through various external forces. We address the issue of the existence of an optimal action upon the system in order to minimize a given cost functional (for example, the turbulence within the flow). We also derive a stochastic minimum principle and investigate Bellman's equation for the considered control problem.

Chapter 1 is devoted to the proof of the existence of the strong solution of the Navier-Stokes equation using the Galerkin method and then to approximate the solution by a linear method. First we give the assumptions for the considered equation and show how the considered evolution equation can be transformed into (0.1) in the case of $n = 2$. We prove the existence of the solution by the Galerkin method (see Theorem 1.2.2). Important results concerning the theory and numerical analysis of the deterministic Navier-Stokes equation can be found in the book of R. Temam [32]. The author also presents in this book the Galerkin method for this equation, which is one of the well-known methods in the theory of partial differential equations that is used to prove existence properties and to obtain finite dimensional approximations for the solutions of the equations. The Galerkin method for the stochastic Navier-Stokes equation has been investigated for example from A. Bensoussan [4], M. Capinski, N. J. Cutland [6], D. Gatarek [7], A. I. Komech, M. I. Vishik [20], B. Schmalfuß [30], [29], M. Viot [34]. Most of the above-mentioned papers consider weak (statistical) solutions. The techniques used in the proofs are the construction of the Galerkin-type approximations of the solutions and some a priori estimates that allow one to prove compactness properties of the corresponding probability measures and finally to obtain a solution of the equation (using Prokhorov's criterion and Skorokhod's theorem). Since we consider the strong solution (in the sense of stochastic analysis) of the Navier-Stokes equation, we do not need to use the techniques considered in the case of weak solutions. The techniques applied in our paper use in particular the properties of stopping times and some basic convergence principles from functional analysis. An important result is that the Galerkin-type approximations converge in mean square to the solution of the Navier-Stokes equation (see Theorem 1.2.7). There are also other approximation methods for this equation involving, for example, the approximation of the Wiener process by smooth processes (see W. Grecksch, B. Schmalfuß [13]) or time discretizations (see F. Flandoli, V. M. Tortorelli [8]). In this chapter we further approximate the solution of the stochastic Navier-Stokes equation by the solutions of a sequence of linear stochastic evolution equations (see equations $(\hat{P}_n)$), which are easier to study. We also prove the convergence in mean square (see Theorem 1.4.5). Since the approximation method involves linear evolution equations of a special type, we give in Section 1.3 results concerning this type of equations.

Chapter 2 deals with the optimal control of the stochastic Navier-Stokes equation. We investigate the behavior of the flow controlled by different external forces, which are feedback controls and respectively bounded controls. We search for an optimal control that minimize a given cost functional. Whether or not there exist such optimal controls is a common question in optimal control theory and often for the answer one uses the Weierstraß Theorem and assumes that the set of admissible controls is compact. To assure the compactness of this set is sometimes not practicable. Therefore we investigate this problem and prove in Theorem 2.3.4, respectively Theorem 2.4.2, the existence of optimal controls, respectively ε-optimal controls, in the case of feedback controls. In the case of bounded controls this method can not be applied, because it uses the special linear and continuous structure of the feedback controls. Using the ideas from A. Bensoussan [3] and adapting them for the considered Navier-Stokes equation we calculate the Gateaux derivative of the cost functional (see Theorem 2.6.4) and derive a stochastic minimum principle (for the case of bounded controls), which gives us a necessary condition for optimality (see Theorem 2.7.2). We complete the statement of the stochastic minimum principle by giving the equations for the adjoint processes.

Chapter 3 contains some aspects and results of dynamic programming for the stochastic Navier-Stokes equation. First we prove that the solution of the considered equation is a Markov process (see Theorem 3.1.1). This property was proved by B. Schmalfuß [29] for the stochastic Navier-Stokes equation with additive noise. In Section 3.2 we illustrate the dynamic programming approach (called also Bellman's principle) and we give a formal derivation of Bellman's equation. Bellman's principle turns the stochastic control problem into a deterministic control problem about a nonlinear partial differential equation of second order (see equation (3.11)) involving the infinitesimal generator. To round off the results of Chapter 2 we give a sufficient condition for an optimal control (Theorem 3.2.3 and Theorem 3.2.4). This condition requires a suitably behaved solution of the Bellman equation and an admissible control satisfying a certain equation. In this section we consider the finite dimensional stochastic Navier-Stokes equation (i.e., the equations obtained by the Galerkin method). The approach would be very complicate for the infinite dimensional case, because in this case it is difficult to obtain the infinitesimal generator. M.J. Vishik and A.V. Fursikov investigated in [35] also the inverse Kolmogorov equations, which give the inifinitsimal generator of the process being solution of the considered equation, only for the case of $n = 2$ for (0.1).

The final part of the dissertation contains an **Appendix** with useful properties from functional and stochastic analysis. We included them into the paper for the convenience of the reader and because we often make use of them.

The development and implementation of numerical methods for the Navier-Stokes equation remains an open problem for further research: "*...the numerical resolution of the Navier-Stokes equation will require (as in the past) the simultaneous efforts of mathematicians, numerical analysts and specialists in computer science. Several significant problems can already be solved numerically, but much time and effort will be necessary until we master the numerical solution of these equations for realistic values of the physical parameters. Besides the need for the development of appropiate algorithms and codes and the improvement of computers in memory size and computation speed, there is another difficulty of a more mathematical (as well as practical) nature. The solutions of the Navier-Stokes equation under realistic conditions are so highly oscillatory (chaotic behavior) that even if we were able to solve them with a great accuracy we would be faced with too much useless information. One has to find a way, with some kind of averaging, to compute mean values of the solutions and the corresponding desired parameters.*"(R. Temam [33])

Frequently Used Notations

a.e.	almost every
$\rightharpoonup$	weak convergence (in the sense of functional analysis)
I_A	indicator function for the set A
$\mathbb{N}$	set of strictly positive integers
$\mathbb{R}$	set of real numbers
Λ	Lebesgue measure on the interval $[0,T]$
$(\Omega, \mathcal{F}, P)$	complete probability space
EX	mathematical expectation of the random variable X
$(\mathcal{F}_t)_{t\in[0,T]}$	right continuous filtration such that $\mathcal{F}_0$ contains all $\mathcal{F}$-null sets
V^*	dual space of the reflexive Banach space V
$\langle v^*, v\rangle$	the application of $v^* \in V^*$ on $v \in V$
J	duality map $\mathrm{J} : V \to V^*$
$B(V)$	σ-algebra of all Borel measurable sets of V
$C([0,T],V)$	space of all continuous functions $u : [0,T] \to V$
$\mathcal{L}(V)$	space of all linear and continuous operators from the Banach space V to itself
$\mathcal{L}^2_V[0,T]$	space of all $B([0,T])$-measurable functions $u : [0,T] \to V$ with $\int\limits_0^T \|u(t)\|_V^2 dt < \infty$
$\mathcal{L}^2_V(\Omega)$	space of all $\mathcal{F}$-measurable random variables $u : \Omega \to V$ with $E\|u\|_V^2 < \infty$
$\mathcal{L}^2_V(\Omega \times [0,T])$	space of all $\mathcal{F} \times B([0,T])$-measurable processes $u : \Omega \times [0,T] \to V$ that are adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and $E\int\limits_0^T \|u(t)\|_V^2 dt < \infty$
$\mathcal{L}^\infty_V(\Omega \times [0,T])$	space of all $\mathcal{F} \times B([0,T])$-measurable processes $u : \Omega \times [0,T] \to V$ that are adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and for a.e. (ω, t) bounded
$\mathcal{L}^\infty_V(\Omega)$	space of all $\mathcal{F}$-measurable processes $u : \Omega \to V$ that are bounded for a.e. ω
$\mathcal{D}_V(\Omega \times [0,T])$	set of $\xi \in \mathcal{L}^\infty_V(\Omega \times [0,T])$ with $\xi = v\phi, v \in V, \phi \in \mathcal{L}^\infty_{\mathbb{R}}(\Omega \times [0,T])$
$\mathcal{D}_V(\Omega)$	set of $\xi \in \mathcal{L}^\infty_V(\Omega)$ with $\xi = v\phi, v \in V, \phi \in \mathcal{L}^\infty_{\mathbb{R}}(\Omega)$

$\Delta_X(t)$	notation for $\exp\Big\{-\dfrac{b}{\nu}\int\limits_0^t \|X(s)\|_V^2\,ds\Big\}$, where $\Big(X(t)\Big)_{t\in[0,T]}$ is a V-valued stochastic process; b, ν are positive constants
$\mathcal{T}_M^X$	stopping time for the stochastic process $\Big(X(t)\Big)_{t\in[0,T]}$ (for the exact definition see Appendix B)
Π_n	orthogonal projection in a Hilbert space

As usual in the notation of random variables or stochastic processes we generally omit the dependence of $\omega \in \Omega$.

Chapter 1

Existence and Approximation of the Solution

In this chapter we use the Galerkin method to prove the existence of the strong solution of the Navier-Stokes equation. We mean strong solution in the sense of stochastic analysis (see [14], Definition 4.2, p. 104): a complete probability space and a Wiener process are given in advance and the equation is defined in the generalized sense over an evolution triple. The techniques that we used are not the same as in the papers of A. Bensoussan [4], M. Capinski, N. J. Cutland [6], D. Gatarek [7], A. I. Komech, M. I. Vishik [20], B. Schmalfuß [29], [31], M. Viot [34], because in the above-mentioned papers one consider weak (statistical) solutions. The Galerkin-type approximations of the solutions and some a priori estimates allow one to prove compactness properties of the corresponding probability measures and to obtain a solution of the equation. In the paper of B. Schmalfuß [30] are considered strong solutions for the equation with an additive noise (the intensity of the random noise part does not depend on the state). The techniques applied in this dissertation are different from those used in the papers above. We utilize the properties of stopping times and some basic convergence principles from functional analysis. An important result is that the Galerkin-type approximations converge in mean square to the solution of the Navier-Stokes equation (see Theorem 1.2.7). This we can prove by using the property of higher order moments for the solution (see Lemma 1.2.3 and Lemma 1.2.6). The Galerkin method is useful to prove the *existence of the solution*, but it is complicated for numerical developments because it involves nonlinear terms. In Section 1.4 we give another *approximation method* by making use of linear evolution equations (see equations $(\hat{P}_n)$), which are easier to study. We also prove that the approximations converge in mean square to the solution of the stochastic Navier-Stokes equation (see Theorem 1.4.5). Since the approximation method involves linear evolution equations of a special type, we give in Section 1.3 some results concerning this type of equations.

The development and implementation of numerical methods for this type of equations remains an open problem for further research. For numerical solutions of stochastic differential equations we refer the reader to the book of P. Kloeden and E. Platen [19].

1.1 Assumptions and formulation of the problem

First we state the assumptions about the stochastic evolution equation that will be considered.

(i) $(\Omega, \mathcal{F}, P)$ is a complete probability space and $(\mathcal{F}_t)_{t\in[0,T]}$ is a right continuous filtration such that $\mathcal{F}_0$ contains all $\mathcal{F}$-null sets. $(w(t))_{t\in[0,T]}$ is a real valued standard $\mathcal{F}_t$-Wiener process.

(ii) (V, H, V^*) is an evolution triple (see [37], p. 416), where $(V, \|\cdot\|_V)$ and $(H, \|\cdot\|)$ are separable Hilbert spaces, and the embedding operator $V \hookrightarrow H$ is assumed to be compact. We denote by $(\cdot,\cdot)$ the scalar product in H.

(iii) $\mathcal{A} : V \to V^*$ is a linear operator such that $\langle \mathcal{A}v, v\rangle \geq \nu\|v\|_V^2$ for all $v \in V$ and $\langle \mathcal{A}u, v\rangle = \langle \mathcal{A}v, u\rangle$ for all $u, v \in V$, where $\nu > 0$ is a constant and $\langle\cdot,\cdot\rangle$ denotes the dual pairing.

(iv) $\mathcal{B} : V \times V \to V^*$ is a bilinear operator such that $\langle \mathcal{B}(u, v), v\rangle = 0$ for all $u, v \in V$ and for which there exists a positive constant $b > 0$ such that

$$|\langle \mathcal{B}(u, v), z\rangle|^2 \leq b\|z\|_V^2\|u\|\|u\|_V\|v\|\|v\|_V.$$

(v) $\mathcal{C} : [0, T] \times H \to H$ is a mapping such that

- **(a)** $\|\mathcal{C}(t, u) - \mathcal{C}(t, v)\|^2 \leq \lambda\|u - v\|^2$ for all $t \in [0, T]$, $u, v \in H$, where λ is a positive constant;
- **(b)** $\mathcal{C}(t, 0) = 0$ for all $t \in [0, T]$;
- **(c)** $\mathcal{C}(\cdot, v) \in \mathcal{L}^2_H[0, T]$ for all $v \in H$.

(vi) $\Phi : [0, T] \times H \to H$ is a mapping such that

- **(a)** $\|\Phi(t, u) - \Phi(t, v)\|^2 \leq \mu\|u - v\|^2$ for all $t \in [0, T]$, $u, v \in H$, where μ is a positive constant;
- **(b)** $\Phi(t, 0) = 0$ for all $t \in [0, T]$;
- **(c)** $\Phi(\cdot, v) \in \mathcal{L}^2_H[0, T]$ for all $v \in H$.

(vii) x_0 is a H-valued $\mathcal{F}_0$-measurable random variable such that $E\|x_0\|^4 < \infty$.

Definition 1.1.1
We call a process $\Big(U(t)\Big)_{t\in[0,T]}$ from the space $\mathcal{L}^2_V(\Omega \times [0, T])$ with $E\|U(t)\|^2 < \infty$ for all $t \in [0, T]$ a **solution of the stochastic Navier-Stokes equation** if it satisfies the equation:

$$(1.1)\qquad \begin{aligned}(U(t), v) + \int_0^t \langle \mathcal{A}U(s), v\rangle ds &= (x_0, v) + \int_0^t \langle \mathcal{B}(U(s), U(s)), v\rangle ds \\ &+ \int_0^t (\Phi(s, U(s)), v) ds + \int_0^t (\mathcal{C}(s, U(s)), v) dw(s)\end{aligned}$$

for all $v \in V$, $t \in [0, T]$ and a.e. $\omega \in \Omega$, where the stochastic integral is understood in the Ito sense.

Remark 1.1.2

1) Since $\mathcal{A}$ is a linear and monotone operator, it follows that it is continuous (see [37], Proposition 26.4, p. 555), i.e., there exists a constant $c_{\mathcal{A}} > 0$ such that for all $u \in V$ we have

$$\|\mathcal{A}u\|_{V^*}^2 \leq c_{\mathcal{A}} \|u\|_V^2.$$

2) From the properties of the operator $\mathcal{B}$ we can derive the following relation

$$\langle \mathcal{B}(u,v), z\rangle = -\langle \mathcal{B}(u,z), v\rangle \quad \text{ for all } u,v,z \in V,$$

which we will use often in our proofs.

3) The condition $\mathcal{C}(t,0) = 0$ (for all $t \in [0,T]$) is given only to simplify the calculations. It can be omitted, in which case one can use the estimate $\|\mathcal{C}(t,u)\|^2 \leq 2\lambda\|u\|^2 + 2\|\mathcal{C}(t,0)\|^2$ that follows from the Lipschitz condition. The same remark holds for Φ too.

4) If we set $n = 2$, $V = \{u \in \overset{\circ}{W}{}_2^1(G) : \mathrm{div} u = 0\}$, $\quad H = \bar{V}^{L^2(G)}$ and

$$\langle \mathcal{A}u, v\rangle = \int_G \sum_{i=1}^n \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} dx, \quad \langle \mathcal{B}(u,v), z\rangle = -\int_G \sum_{i,j=1}^n u_i \frac{\partial v_j}{\partial x_i} z_j dx, \quad \Phi(t,u) = f(t)$$

for $u,v,z \in V, t \in [0,T]$, then equation (0.1) can be transformed into (1.1); see [32].

For finite dimensional approximations we need some preliminaries. Let $h_1, h_2, \ldots, h_n, \ldots \in H$ be the eigenvectors of the operator $\mathcal{A}$, for which we consider the domain of definition $\mathrm{Dom}(\mathcal{A}) = \{v \in V \mid \mathcal{A}v \in H\}$. These eigenvectors form an orthonormal base in H and they are orthogonal in V (see [24], p. 110). For each $n \in \mathbb{N}$ we consider $H_n := \mathrm{sp}\{h_1, h_2, \ldots, h_n\}$ equipped with the norm induced from H. We write $(H_n, \|\cdot\|_V)$ when we consider H_n equipped with the norm induced from V. We define by $\Pi_n : H \to H_n$ the orthogonal projection of H on H_n

$$\Pi_n h := \sum_{i=1}^n (h, h_i) h_i.$$

Let $\mathcal{A}_n : H_n \to H_n$, $\mathcal{B}_n : H_n \times H_n \to H_n$, $\Phi_n, \mathcal{C}_n : [0,T] \times H_n \to H_n$ be defined respectively by

$$\mathcal{A}_n u = \sum_{i=1}^n \langle \mathcal{A}u, h_i\rangle h_i, \quad \mathcal{B}_n(u,v) = \sum_{i=1}^n \langle \mathcal{B}(u,v), h_i\rangle h_i,$$

$$\mathcal{C}_n(t,u) = \Pi_n \mathcal{C}(t,u), \quad \Phi_n(t,u) = \Pi_n \Phi(t,u), \quad x_{0n} = \Pi_n x_0$$

for all $t \in [0,T]$, $u, v \in H_n$.

Let $\Big(X(t)\Big)_{t\in[0,T]}$ be a process in the space $\mathcal{L}_V^2(\Omega \times [0,T])$ and let $X_n := \Pi_n X$. Using the properties of $\mathcal{A}$ and of its eigenvectors $h_1, h_2, \ldots$ ($\lambda_1, \lambda_2, \ldots$ are the corresponding eigenvalues), we have

$$(1.2) \qquad \|X_n(t)\|_V^2 \leq \|X(t)\|_V^2, \qquad \|X_n(t)\|^2 \leq \|X(t)\|^2, \qquad \|X(t) - X_n(t)\|^2 \leq \|X(t)\|^2,$$

$$\nu\|X(t)-X_n(t)\|_V^2 \leq \langle \mathcal{A}X(t)-\mathcal{A}X_n(t), X(t)-X_n(t)\rangle = \sum_{i=n}^{\infty}\lambda_i (X(t),h_i)^2 \tag{1.3}$$
$$\leq \langle \mathcal{A}X(t), X(t)\rangle \leq c_{\mathcal{A}}\|X(t)\|_V^2.$$

Hence for $P \times [0,T]$ a.e. $(\omega,t)\in\Omega\times[0,T]$ we have

$$\lim_{n\to\infty}\|X(\omega,t)-X_n(\omega,t)\|_V^2 = 0.$$

By the Lebesgue dominated convegence theorem it follows that

$$\lim_{n\to\infty}\int_0^T \|X(t)-X_n(t)\|_V^2 dt = 0 \tag{1.4}$$

and

$$\lim_{n\to\infty} E\int_0^T \|X(t)-X_n(t)\|_V^2 dt = 0. \tag{1.5}$$

If the process $\Big(X(t)\Big)_{t\in[0,T]}$ has almost surely continuous trajectories in H, then

$$\lim_{n\to\infty}\|X(T)-X_n(T)\|^2 = 0 \quad \text{for a.e.} \quad \omega\in\Omega \tag{1.6}$$

and

$$\lim_{n\to\infty} E\|X(T)-X_n(T)\|^2 = 0. \tag{1.7}$$

1.2 Existence of the solution of the stochastic Navier-Stokes equation by Galerkin approximation

We want to prove the existence of the solution of the Navier-Stokes equation (1.1) by approximating it by means of the Galerkin method, i.e., by a sequence of solutions of finite dimensional evolution equations (see equations (P_n)). Since we consider the strong solution of the Navier-Stokes equation, we do not need to use the techniques considered in the case of weak solutions. The techniques applied in our paper use in particular the properties of stopping times and some basic convergence principles from functional analysis. An important result is that the Galerkin-type approximations converge in mean square to the solution of the Navier-Stokes equation (see Theorem 1.2.7).

For each $n = 1,2,3,\dots$ we consider the sequence of finite dimensional evolution equations

$$(P_n) \qquad (U_n(t),v)+\int_0^t(\mathcal{A}_nU_n(s),v)ds = (x_{0n},v)+\int_0^t(\mathcal{B}_n(U_n(s),U_n(s)),v)ds$$
$$+ \int_0^t(\Phi_n(s,U_n(s)),v)ds+\int_0^t(\mathcal{C}_n(s,U_n(s)),v)dw(s),$$

for all $v\in H_n$, $t\in[0,T]$ and a.e. $\omega\in\Omega$.

Theorem 1.2.1 *For each $n \in \mathbb{N}$, equation (P_n) has a solution $U_n \in \mathcal{L}^2_V(\Omega \times [0,T])$, which is unique almost surely and has almost surely continuous trajectories in H.*

PROOF. We use an analogous method as in [31]. Let $\left(\chi_M\right)$ be a family of Lipschitz continuous mappings such that

$$\chi_M(x) = \begin{cases} 1\,, & \text{if} \quad 0 \leq x \leq M, \\ 0\,, & \text{if} \quad x \geq M+1, \\ M+1-x, & \text{if} \quad x \in (M, M+1). \end{cases}$$

For each fixed $n \in \mathbb{N}$ we consider the solution U_n of equation (P_n) approximated by $\left(U_n^M\right)$ $(M = 1, 2, \ldots)$ which is the solution of the equation

$$(P_n^M) \qquad \begin{aligned} (U_n^M(t), v) \;&+\; \int_0^t (\mathcal{A}_n U_n^M(s), v)ds = (x_{0n}, v) \\ &+\; \int_0^t (\chi_M(\|U_n^M(t)\|^2)\mathcal{B}_n(U_n^M(s), U_n^M(s)), v)ds \\ &+\; \int_0^t (\Phi_n(s, U_n^M(s)), v)ds + \int_0^t (\mathcal{C}_n(s, U_n^M(s)), v)dw(s), \end{aligned}$$

for all $v \in H_n$, $t \in [0,T]$, and a.e. $\omega \in \Omega$. For this equation we apply the theory of finite dimensional Ito equations with Lipschitz continuous nonlinearities (see [18], Theorem 3.9, p. 289). Hence there exists $U_n^M \in \mathcal{L}^2_{(H_n, \|\cdot\|_V)}(\Omega \times [0,T])$ almost surely unique solution of (P_n^M) which has continuous trajectories in H.

We consider the stopping times $\mathcal{T}_M := \mathcal{T}_M^{U_n^M}$ (the definition of stopping times is given in Appendix B). By using (P_n^M), the properties of $\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n, \Phi_n$ and Proposition B.2 (for $Q := U_n^M$, $a_0 := x_{0n}, k_1 := 2\nu, k_2 := 2\sqrt{\mu} + \lambda, F_1 = F_2 := 0, F_3 := 2\mathcal{C}_n$) we obtain the following estimate

$$(1.8) \qquad E \sup_{t \in [0,T]} \|U_n^M(t)\|^2 + 2\nu E \int_0^T \|U_n^M(s)\|_V^2 ds \leq cE\|x_0\|^2,$$

where c is a positive constant independent of n and M. From Markov's inequality, the definition of $\mathcal{T}_M$, and (1.8) we have

$$(1.9) \qquad P(\mathcal{T}_M < T) \leq P\Big(\sup_{t \in [0,T]} \|U_n^M(t)\|^2 \geq M\Big) \leq \frac{c}{M} E\|x_0\|^2.$$

Let Ω_n^M be the set of all $\omega \in \Omega$ such that $U_n^M(\omega, \cdot)$ satisfies (P_n^M) for all $t \in [0,T], v \in H_n$ and $U_n^M(\omega, \cdot)$ has continuous trajectories in H. We denote $\Omega' := \bigcap_{M=1}^{\infty} \Omega_n^M$ and have $P(\Omega') = 1$. We also consider

$$S_n := \bigcup_{M=1}^{\infty} \bigcup_{1 \leq K \leq M} \{\omega \in \Omega' | \mathcal{T}_K = T \text{ and } \exists\ t \in [0,T] :\ U_n^K(\omega, t) \neq U_n^M(\omega, t)\}.$$

We get $P(S_n) = 0$, because otherwise there exist two natural numbers M_0, K_0 with $K_0 < M_0$ such that the set

$$S^n_{M_0,K_0} := \{\omega \in \Omega' | \mathcal{T}_{K_0} = T \text{ and } \exists\, t \in [0,T]:\ U_n^{K_0}(\omega,t) \neq U_n^{M_0}(\omega,t)\}$$

has the measure $P(S^n_{M_0,K_0}) > 0$. We define for each $t \in [0,T]$

$$U_n^*(\omega,t) := \begin{cases} U_n^{K_0}(\omega,t) & , \quad \omega \in S^n_{M_0,K_0} \\ U_n^{M_0}(\omega,t) & , \quad \omega \in \Omega' \setminus S^n_{M_0,K_0}. \end{cases}$$

We see that for all $\omega \in S^n_{M_0,K_0}$ there exists $t \in [0,T]$ such that $U_n^*(\omega,t) \neq U_n^{M_0}(\omega,t)$. This contradicts to the almost surely uniqueness of the solution of $(P_n^{M_0})$. Consequently, $P(S_n) = 0$.

Let $\Omega'' := \Omega' \cap \Big(\bigcup\limits_{M=1}^{\infty} \{\mathcal{T}_M = T\} \setminus S_n \Big)$. Using (1.9) and the definition of S we have

$$P(\Omega'') = \lim_{M\to\infty} P\Big(\{\mathcal{T}_M = T\} \setminus S_n\Big) = 1 - \lim_{M\to\infty} P(\mathcal{T}_M < T) = 1.$$

Let $\omega \in \Omega''$. For this ω there exists a natural number M_0 such that $\mathcal{T}_M = T$ for all $M \geq M_0$. Hence $\chi_M(\|U_n^M(s)\|^2) = 1$ for all $s \in [0,T]$ and all $M \geq M_0$. Equation (P_n^M) implies

$$\begin{aligned} (U_n^M(t), v) \quad &+ \int_0^t \langle \mathcal{A}_n U_n^M(s), v\rangle ds = (x_{0n}, v) + \int_0^t \langle \mathcal{B}_n(U_n^M(s), U_n^M(s)), v\rangle ds \\ &+ \int_0^t (\Phi_n(s, U_n^M(s)), v) ds + \int_0^t (\mathcal{C}_n(s, U_n^M(s)), v) dw(s) \end{aligned} \tag{1.10}$$

for all $M \geq M_0$ and all $t \in [0,T], v \in H_n$. For this fixed $\omega \in \Omega''$ and for each $t \in [0,T]$ we define

$$U_n(\omega,t) := U_n^{M_0}(\omega,t) = \lim_{M\to\infty} U_n^M(\omega,t) \tag{1.11}$$

with respect to the H-norm. This definition is correct because $\omega \notin S_n$. Then using (1.10) and (1.11) we obtain

$$\begin{aligned} (U_n(t), v) + \int_0^t (\mathcal{A}_n U_n(s), v) ds \quad &= \quad (x_{0n}, v) + \int_0^t (\mathcal{B}_n(U_n(s), U_n(s)), v) ds \\ &+ \int_0^t (\Phi_n(s, U_n(s)), v) ds + \int_0^t (\mathcal{C}_n(s, U_n(s)), v) dw(s) \end{aligned}$$

for all $\omega \in \Big(\Omega \cap \Omega''\Big) \setminus S_n, t \in [0,T], v \in H_n$. The process $(U_n(t))_{t\in[0,T]}$ is H_n-valued, $\mathcal{F} \times \mathcal{B}_{[0,T]}$-measurable, adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and has almost surely continuous trajectories in H_n, because all U_n^M have this property. Obviously for all $t \in [0,T]$ we have

$$\lim_{M\to\infty} \|U_n^M(t) - U_n(t)\|^2 = 0 \quad \text{for a.e. } \omega \in \Omega \tag{1.12}$$

and

$$\lim_{M\to\infty}\int_0^T \|U_n^M(s)-U_n(s)\|_V^2 ds = 0 \quad \text{for a.e. } \omega\in\Omega.$$

By using (1.8) we obtain the following estimates

$$E\|U_n(t)\|^2 \le \liminf_{M\to\infty} E\|U_n^M(t)\|^2 \le cE\|x_0\|^2 \quad \text{for all } t\in[0,T]$$

and

$$E\int_0^T \|U_n(s)\|_V^2 ds \le \liminf_{M\to\infty} E\int_0^T \|U_n^M(s)\|_V^2 ds \le \frac{c}{2\nu}E\|x_0\|^2.$$

Therefore $U_n \in \mathcal{L}_V^2(\Omega\times[0,T])$.

The uniqueness of the solution can be proved analogously to the case of the stochastic Navier-Stokes equation (see Theorem 1.2.2). ■

One of the **main results** of this chapter is given in the following theorem, in which we state the existence and almost surely uniqueness of the solution U of the Navier-Stokes equation.

Theorem 1.2.2
The Navier-Stokes equation (1.1) *has a solution, which is almost surely unique and has almost surely continuous trajectories in H.*

For the proof of this theorem we need several lemmas.

Lemma 1.2.3
There exists a positive constant c_1 (independent of n) such that for all $n\in\mathbb{N}$

$$E\sup_{t\in[0,T]}\|U_n(t)\|^2 + 2\nu E\int_0^T \|U_n(t)\|_V^2 dt \le c_1 E\|x_0\|^2$$

and each of the following expressions

$$E\sup_{t\in[0,T]}\|U_n(t)\|^4, \quad E\Big(\int_0^T \|U_n(t)\|_V^2 dt\Big)^2$$

is less or equal to $c_1E\|x_0\|^4$.

Proof. Let n be an arbitrary fixed natural number. Equation (P_n) (given at the beginning of this section) can also be written as

$$\begin{aligned}(1.13)\qquad (U_n(t),h_i)+\int_0^t\langle\mathcal{A}U_n(s),h_i\rangle ds &= (x_0,h_i)+\int_0^t\langle\mathcal{B}(U_n(s),U_n(s)),h_i\rangle ds\\ &+ \int_0^t(\Phi(s,U_n(s)),h_i)ds+\int_0^t(\mathcal{C}(s,U_n(s),h_i)dw(s),\end{aligned}$$

for all $i = 1, \dots, n$, $t \in [0,T]$ and a.e. $\omega \in \Omega$. By the Ito formula and by our hypothesis from Section 1.1 we have

$$\|U_n(t)\|^2 + 2\nu \int_0^t \|U_n(s)\|_V^2 ds \le \|x_0\|^2 + (2\sqrt{\mu} + \lambda) \int_0^t \|U_n(s))\|^2 ds + 2 \int_0^t (\mathcal{C}(s, U_n(s)), U_n(s)) dw(s)$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$. Now we apply Proposition B.2 for $Q := U_n, k_1 := 2\nu$, $k_2 := 2\sqrt{\mu} + \lambda, a_0 := x_0, F_1 = F_2 := 0, F_3 := 2\mathcal{C}$. Then we obtain the estimates given in the statement of this lemma. ■

Lemma 1.2.4

(i) *There exist $U \in \mathcal{L}_V^2(\Omega \times [0,T])$, $\mathcal{B}^* \in \mathcal{L}_{V^*}^2(\Omega \times [0,T])$, $\Phi^*, \mathcal{C}^* \in \mathcal{L}_H^2(\Omega \times [0,T])$, and a subsequence (n') of (n) such that for $n' \to \infty$ we have*

$$U_{n'} \rightharpoonup U \quad in \quad \mathcal{L}_V^2(\Omega \times [0,T]),$$

$$\mathcal{B}(U_{n'}, U_{n'}) \rightharpoonup \mathcal{B}^* \quad in \quad \mathcal{L}_{V^*}^2(\Omega \times [0,T]),$$

$$\Phi(\cdot, U_{n'}(\cdot)) \rightharpoonup \Phi^*, \quad \mathcal{C}(\cdot, U_{n'}(\cdot)) \rightharpoonup \mathcal{C}^* \quad in \quad \mathcal{L}_H^2(\Omega \times [0,T]),$$

where $\rightharpoonup$ denotes the weak convergence.

(ii) *For all $v \in V$, $t \in [0,T]$ and a.e. $\omega \in \Omega$ the process $\Big(U(t)\Big)_{t\in[0,T]}$ satisfies the equation:*

$$(U(t), v) + \int_0^t \langle \mathcal{A}U(s), v \rangle ds = (x_0, v) + \int_0^t \langle \mathcal{B}^*(s), v \rangle ds + \int_0^t (\Phi^*(s), v) ds + \int_0^t (\mathcal{C}^*(s), v) dw(s). \tag{1.14}$$

The process $\Big(U(t)\Big)_{t\in[0,T]}$ has almost surely continuous trajectories in H.

(iii) *The function U from* **(ii)** *satisfies $E \sup_{t\in[0,T]} \|U(t)\|^2 < \infty$.*

Proof. **(i)** Taking into account the properties of Φ, $\mathcal{C}$, and the estimates from Lemma 1.2.3 it follows that $\Big(\Phi(\cdot, U_n(\cdot))\Big)$, $\Big(\mathcal{C}(\cdot, U_n(\cdot))\Big)$ are bounded sequences in the space $\mathcal{L}_H^2(\Omega \times [0,T])$. By using the properties of $\mathcal{B}$ we can derive

$$E \int_0^T \|\mathcal{B}(U_n(t), U_n(t))\|_{V^*}^2 dt \le bE \int_0^T \|U_n(t)\|_V^2 \|U_n(t)\|^2 dt \le bc_1 E\|x_0\|^4,$$

so $\Big(\mathcal{B}(U_n,U_n)\Big)$ is a bounded sequence in the space $\mathcal{L}^2_{V^*}(\Omega\times[0,T])$. Applying Proposition A.1 (see Appendix A), it follows that there exist a subsequence (n') of (n) and $\hat{U}\in\mathcal{L}^2_V(\Omega\times[0,T])$, $\mathcal{B}^*\in\mathcal{L}^2_{V^*}(\Omega\times[0,T])$, $\Phi^*,\mathcal{C}^*\in\mathcal{L}^2_H(\Omega\times[0,T])$ such that for $n'\to\infty$

$$U_{n'}\rightharpoonup\hat{U}\quad\text{in } \mathcal{L}^2_V(\Omega\times[0,T]),\qquad \mathcal{B}(U_{n'},U_{n'})\rightharpoonup\mathcal{B}^*\quad\text{in}\quad\mathcal{L}^2_{V^*}(\Omega\times[0,T]),$$

$$\Phi(\cdot,U_{n'}(\cdot))\rightharpoonup\Phi^*,\qquad \mathcal{C}(\cdot,U_{n'}(\cdot))\rightharpoonup\mathcal{C}^*\quad\text{in } \mathcal{L}^2_H(\Omega\times[0,T]).$$

(ii) In (1.13) we take the limit $n'\to\infty$, use the properties of $\mathcal{A}$, the weak convergences from above (also Proposition A.2 and Proposition A.3) and obtain

$$\begin{aligned}(\hat{U}(t),h_i)=(x_0,h_i)-\int_0^t\langle\mathcal{A}\hat{U}(s),h_i\rangle ds\ &+\int_0^t\langle\mathcal{B}^*(s),h_i\rangle ds\\ &+\int_0^t(\Phi^*(s),h_i)ds+\int_0^t(\mathcal{C}^*(s),h_i)dw(s),\end{aligned}\tag{1.15}$$

for a.e. $(\omega,t)\in\Omega\times[0,T]$ and $i\in\mathbb{N}$. Since $\mathrm{sp}\{h_1,h_2,\ldots,h_n,\ldots\}$ is dense in V (because of the properties of the eigenvectors of $\mathcal{A}$) it follows that (1.15) holds also for all $v\in V$.

There exists a $\mathcal{F}_t$-measurable H-valued process which is equal to $\hat{U}(t)$ for $P\times\Lambda$ a.e. $(\omega,t)\in\Omega\times[0,T]$ and is equal to the right side of (1.15) for all $t\in[0,T]$ and a.e. $\omega\in\Omega$. We denote this process by $\Big(U(t)\Big)_{t\in[0,T]}$. Hence

$$(U(t),v)+\int_0^t\langle\mathcal{A}U(s),v\rangle ds=(x_0,v)+\int_0^t\langle\mathcal{B}^*(s),v\rangle ds+\int_0^t(\Phi^*(s),v)ds+\int_0^t(\mathcal{C}^*(s),v)dw(s)$$

for all $v\in V$, $t\in[0,T]$ and a.e. $\omega\in\Omega$; the process $\Big(U(t)\Big)_{t\in[0,T]}$ has in H almost surely continuous trajectories (see [21], Theorem 3.1, p. 88).

(iii) In (1.14) we apply the Ito formula, use the properties of $\mathcal{A}$ and some elementary inequalities. Then we apply Proposition B.2 for $Q:=U, a_0:=x_0, F_1:=\dfrac{1}{\nu}\|\mathcal{B}^*\|^2_{V^*}+\|\Phi^*\|^2+\|\mathcal{C}^*\|^2$, $F_2:=2\mathcal{C}^*$, $F_3:=0$, $k_1:=\nu, k_2:=1$. ■

For each fixed $M\in\mathbb{N}$ we consider $\mathcal{T}_M:=\mathcal{T}^U_M$, where $\Big(U(t)\Big)_{t\in[0,T]}$ is the process obtained in Lemma 1.2.4.

Lemma 1.2.5
The following convergences hold

$$\lim_{n'\to\infty}E\int_0^{\mathcal{T}_M}\|U(s)-U_{n'}(s)\|^2_V ds=0\quad\textit{and}\quad\lim_{n'\to\infty}E\|U(\mathcal{T}_M)-U_{n'}(\mathcal{T}_M)\|^2=0.$$

PROOF. For each $n \in \mathbb{N}$ let $\tilde{U}_n(t) = \Pi_n U$. From (1.14) and (1.13) we have

$$
\begin{aligned}
(U(t) - U_n(t), h_i) &+ \int_0^t \langle \mathcal{A}U(s) - \mathcal{A}U_n(s), h_i\rangle ds = \int_0^t \langle \mathcal{B}^*(s) - \mathcal{B}(U_n(s), U_n(s)), h_i\rangle ds \\
&+ \int_0^t (\Phi^*(s) - \Phi(s, U_n(s)), h_i) ds + \int_0^t (\mathcal{C}^*(s) - \mathcal{C}(s, U_n(s)), h_i) dw(s)
\end{aligned}
$$

for all $t \in [0, T]$, $i = 1, \ldots, n$, a.e. $\omega \in \Omega$. After applying the Ito formula and summing from $i = 1$ to n, we use the properties of $\mathcal{A}$ and obtain

$$
\begin{aligned}
&\|\tilde{U}_n(t) - U_n(t)\|^2 + 2\int_0^t \langle \mathcal{A}\tilde{U}_n(s) - \mathcal{A}U_n(s), \tilde{U}_n(s) - U_n(s)\rangle ds \\
&\quad = 2\int_0^t \langle \mathcal{B}^*(s) - \mathcal{B}(U_n(s), U_n(s)), \tilde{U}_n(s) - U_n(s)\rangle ds \\
&\quad + 2\int_0^t (\Phi^*(s) - \Phi(s, U_n(s)), \tilde{U}_n(s) - U_n(s)) ds \\
&\quad + 2\int_0^t (\mathcal{C}^*(s) - \mathcal{C}(s, U_n(s)), \tilde{U}_n(s) - U_n(s)) dw(s) + \int_0^t \sum_{i=1}^n (\mathcal{C}^*(s) - \mathcal{C}(s, U_n(s)), h_i)^2 ds
\end{aligned}
$$

for all $t \in [0, T]$, $i = 1, \ldots, n$, a.e. $\omega \in \Omega$. Write

$$
e_1(t) = \Delta_U(t) \exp\{-(2\lambda + 2\sqrt{\mu} + 1)t\},
$$

where the notation for Δ_U is given in the paragraph "Frequently Used Notations". By the Ito formula get

$$
\begin{aligned}
&e_1(t)\|\tilde{U}_n(t) - U_n(t)\|^2 + 2\int_0^t e_1(s)\langle \mathcal{A}\tilde{U}_n(s) - \mathcal{A}U_n(s), \tilde{U}_n(s) - U_n(s)\rangle ds \qquad (1.16)\\
= \ & 2\int_0^t e_1(s)\langle \mathcal{B}^*(s) - \mathcal{B}(U_n(s), U_n(s)), \tilde{U}_n(s) - U_n(s)\rangle ds - \frac{b}{\nu}\int_0^t e_1(s)\|U(s)\|_V^2 \|\tilde{U}_n(s) - U_n(s)\|^2 ds \\
- \ & (2\lambda + 2\sqrt{\mu} + 1)\int_0^t e_1(s)\|\tilde{U}_n(s) - U_n(s)\|^2 ds + 2\int_0^t e_1(s)(\Phi^*(s) - \Phi(s, U_n(s)), \tilde{U}_n(s) - U_n(s)) ds \\
+ \ & \int_0^t \sum_{i=1}^n e_1(s)(\mathcal{C}^*(s) - \mathcal{C}(s, U_n(s)), h_i)^2 ds + 2\int_0^t e_1(s)(\mathcal{C}^*(s) - \mathcal{C}(s, U_n(s)), \tilde{U}_n(s) - U_n(s)) dw(s)
\end{aligned}
$$

for all $t \in [0,T]$, $i = 1, \dots, n$, a.e. $\omega \in \Omega$. From the properties of $\mathcal{B}$ and those of $\tilde{U}_n$ (see (1.2) we see that

$$
\begin{aligned}
& \langle \mathcal{B}(U_n(s), U_n(s)), \tilde{U}_n(s) - U_n(s) \rangle = \langle \mathcal{B}(U_n(s), \tilde{U}_n(s)), \tilde{U}_n(s) - U_n(s) \rangle \\
= \; & \langle \mathcal{B}(U_n(s) - \tilde{U}_n(s), \tilde{U}_n(s)), \tilde{U}_n(s) - U_n(s) \rangle + \langle \mathcal{B}(\tilde{U}_n(s), \tilde{U}_n(s)), \tilde{U}_n(s) - U_n(s) \rangle \\
\le \; & \frac{b}{2\nu} \|\tilde{U}_n(s)\|_V^2 \|\tilde{U}_n(s) - U_n(s)\|^2 + \frac{\nu}{2} \|\tilde{U}_n(s) - U_n(s)\|_V^2 + \langle \mathcal{B}(\tilde{U}_n(s), \tilde{U}_n(s)), \tilde{U}_n(s) - U_n(s) \rangle \\
\le \; & \frac{b}{2\nu} \|U(s)\|_V^2 \|\tilde{U}_n(s) - U_n(s)\|^2 + \frac{\nu}{2} \|\tilde{U}_n(s) - U_n(s)\|_V^2 + \langle \mathcal{B}(\tilde{U}_n(s), \tilde{U}_n(s)), \tilde{U}_n(s) - U_n(s) \rangle.
\end{aligned}
$$

The properties of Φ imply

$$
\begin{aligned}
2\Big(\Phi^*(s) - \Phi(s, U_n(s)), \tilde{U}_n(s) - U_n(s)\Big) \; & \le \; 2\Big(\Phi^*(s) - \Phi(s, U(s)), \tilde{U}_n(s) - U_n(s)\Big) \\
& + \; (1 + 2\sqrt{\mu}) \|\tilde{U}_n(s) - U_n(s)\|^2 + \mu \|U(s) - \tilde{U}_n(s)\|^2
\end{aligned}
$$

and from the properties of $\mathcal{C}$ and $\tilde{U}_n$ we get

$$
\begin{aligned}
& \sum_{i=1}^{n} \Big(\mathcal{C}^*(s) - \mathcal{C}(s, U_n(s)), h_i\Big)^2 = \|\mathcal{C}(s, U(s)) - \mathcal{C}(s, U_n(s))\|_{H_n}^2 \\
+ \; & 2\Big(\mathcal{C}^*(s) - \mathcal{C}(s, U_n(s)), \mathcal{C}^*(s) - \mathcal{C}(s, U(s))\Big)_{H_n} - \|\mathcal{C}(s, U(s)) - \mathcal{C}^*(s)\|_{H_n}^2 \\
\le \; & 2\lambda \|U(s) - \tilde{U}_n(s)\|^2 + 2\lambda \|\tilde{U}_n(s) - U_n(s)\|^2 + 2\Big(\mathcal{C}^*(s) - \mathcal{C}(s, U_n(s)), \mathcal{C}^*(s) - \mathcal{C}(s, U(s))\Big)_{H_n} \\
- \; & \|\mathcal{C}(s, U(s)) - \mathcal{C}^*(s)\|_{H_n}^2,
\end{aligned}
$$

where we write $\|x\|_{H_n} := \|\Pi_n x\|$ and $(x,y)_{H_n} := (\Pi_n x, \Pi_n y)$ for $x, y \in H$.

We use these estimates in (1.16) to obtain

$$
\begin{aligned}
& Ee_1(\mathcal{T}_M) \|\tilde{U}_n(\mathcal{T}_M) - U_n(\mathcal{T}_M)\|^2 + \nu E \int_0^{\mathcal{T}_M} e_1(s) \|\tilde{U}_n(s) - U_n(s)\|_V^2 ds \\
+ \; & E \int_0^{\mathcal{T}_M} e_1(s) \|\mathcal{C}(s, U(s)) - \mathcal{C}^*(s)\|_{H_n}^2 ds \\
\le \; & 2E \int_0^{\mathcal{T}_M} e_1(s) \langle \mathcal{B}^*(s) - \mathcal{B}(\tilde{U}_n(s), \tilde{U}_n(s)), \tilde{U}_n(s) - U_n(s) \rangle ds \\
+ \; & (2\lambda + \mu) E \int_0^{\mathcal{T}_M} e_1(s) \|U(s) - \tilde{U}_n(s)\|^2 ds \\
+ \; & 2E \int_0^{\mathcal{T}_M} e_1(s) (\Phi^*(s) - \Phi(s, U(s)), \tilde{U}_n(s) - U_n(s)) ds
\end{aligned} \tag{1.17}
$$

$$+ \quad 2E\int_0^{\mathcal{T}_M} e_1(s)(\mathcal{C}^*(s)-\mathcal{C}(s,U_n(s)),\mathcal{C}^*(s)-\mathcal{C}(s,U(s)))_{H_n}ds$$

where $M \in \mathbb{N}$. Using the properties of $\mathcal{B}$, those of the stopping time $\mathcal{T}_M$ and the fact that $(\tilde{U}_n)$ is the partial sum of the Fourier expansion of $U \in \mathcal{L}_V^2(\Omega \times [0,T])$ (see the properties (1.2) and (1.5) given in the final part of Section 1.1) we have

$$\begin{aligned}
&E\int_0^{\mathcal{T}_M} e_1(s)\|\mathcal{B}(U(s),U(s))-\mathcal{B}(\tilde{U}_n(s),\tilde{U}_n(s))\|_{V^*}^2 ds\\
&\quad\le\ bE\int_0^{\mathcal{T}_M} e_1(s)\Big(\|U(s)\|_V\|U(s)\|+\|\tilde{U}_n(s)\|_V\|\tilde{U}_n(s)\|\Big)\|U(s)-\tilde{U}_n(s)\|_V\|U(s)-\tilde{U}_n(s)\|ds\\
&\quad\le\ 2bE\int_0^{\mathcal{T}_M} e_1(s)\|U(s)\|_V\|U(s)\|^2\|U(s)-\tilde{U}_n(s)\|_V ds\\
&\quad\le\ 2bM\Big(E\int_0^{\mathcal{T}_M}\|U(s)\|_V^2 ds\Big)^{\frac{1}{2}}\Big(E\int_0^{\mathcal{T}_M}\|U(s)-\tilde{U}_n(s)\|_V^2 ds\Big)^{\frac{1}{2}}
\end{aligned}$$

and hence

$$\lim_{n\to\infty} E\int_0^{\mathcal{T}_M} e_1(s)\|\mathcal{B}(U(s),U(s))-\mathcal{B}(\tilde{U}_n(s),\tilde{U}_n(s))\|_{V^*}^2 ds = 0.$$

We have $I_{[0,\mathcal{T}_M]}\mathcal{B}(U,U), \mathcal{B}^* \in \mathcal{L}_{V^*}^2(\Omega\times[0,T])$. For the subsequence (n') of (n) we have proved that $U_{n'} \rightharpoonup U$ in $\mathcal{L}_V^2(\Omega\times[0,T])$ and $\tilde{U}_{n'} \to U$ in $\mathcal{L}_V^2(\Omega\times[0,T])$ (see Lemma 1.2.4 and (1.5) from Section 1.1). Consequently,

$$\begin{aligned}
&\lim_{n'\to\infty} E\int_0^{\mathcal{T}_M} e_1(s)\langle\mathcal{B}^*(s)-\mathcal{B}(\tilde{U}_{n'}(s),\tilde{U}_{n'}(s)),\tilde{U}_{n'}(s)-U_{n'}(s)\rangle ds\\
=\ &\lim_{n'\to\infty} E\int_0^{\mathcal{T}_M} e_1(s)\langle\mathcal{B}^*(s)-\mathcal{B}(U(s),U(s)),\tilde{U}_{n'}(s)-U_{n'}(s)\rangle ds\\
+\ &\lim_{n'\to\infty} E\int_0^{\mathcal{T}_M} e_1(s)\langle\mathcal{B}(U(s),U(s))-\mathcal{B}(\tilde{U}_{n'}(s),\tilde{U}_{n'}(s)),\tilde{U}_{n'}(s)-U_{n'}(s)\rangle ds = 0.
\end{aligned}$$

It also follows that

$$\lim_{n'\to\infty} E\int_0^{\mathcal{T}_M} e_1(s)(\Phi^*(s)-\Phi(s,U(s)),\tilde{U}_{n'}(s)-U_n(s))ds = 0.$$

Since $\mathcal{C}(\cdot, U_{n'}(\cdot)) \rightharpoonup \mathcal{C}^*$ in $\mathcal{L}^2_H(\Omega \times [0,T])$ and $\Pi_n \mathcal{C}^* - \Pi_n \mathcal{C}(\cdot, U(\cdot)) \to \mathcal{C}^* - \mathcal{C}(\cdot, U(\cdot))$, the following convergences hold:

$$\begin{aligned}
&\lim_{n'\to\infty} E\int_0^{\mathcal{T}_M} e_1(s)\Big(\mathcal{C}^*(s) - \mathcal{C}(s, U_{n'}(s)), \mathcal{C}^*(s) - \mathcal{C}(s, U(s))\Big)_{H_{n'}} ds \\
= &\lim_{n'\to\infty} E\int_0^{\mathcal{T}_M} e_1(s)\Big(\mathcal{C}^*(s) - \mathcal{C}(s, U_{n'}(s)), \Pi_n\mathcal{C}^*(s) - \Pi_n\mathcal{C}(s, U(s))\Big) ds = 0
\end{aligned}$$

and

$$\lim_{n'\to\infty} E\int_0^{\mathcal{T}_M} e_1(s)\|\mathcal{C}(s,U(s)) - \mathcal{C}^*(s)\|^2_{H_{n'}} ds = E\int_0^{\mathcal{T}_M} e_1(s)\|\mathcal{C}(s,U(s)) - \mathcal{C}^*(s)\|^2 ds.$$

In view of these results, we see that by taking the limit $n' \to \infty$ in (1.17) the right side of this inequality tends to zero. Therefore

$$\lim_{n'\to\infty} E e_1(\mathcal{T}_M)\|\tilde{U}_{n'}(\mathcal{T}_M) - U_{n'}(\mathcal{T}_M)\|^2 = 0, \qquad \lim_{n'\to\infty} E\int_0^{\mathcal{T}_M} e_1(s)\|\tilde{U}_{n'}(s) - U_{n'}(s)\|^2_V ds = 0$$

and

$$E\int_0^{\mathcal{T}_M} e_1(s)\|\mathcal{C}(s,U(s)) - \mathcal{C}^*(s)\|^2 ds = 0. \tag{1.18}$$

From the properties of e_1 over $[0, \mathcal{T}_M]$ and from (1.5) follows that for each fixed $M \in \mathbb{N}$ we have

$$\lim_{n'\to\infty} E\int_0^{\mathcal{T}_M} \|U(s) - U_{n'}(s)\|^2_V ds = 0 \quad \text{and} \quad \lim_{n'\to\infty} E\|U(\mathcal{T}_M) - U_{n'}(\mathcal{T}_M)\|^2 = 0. \tag{1.19}$$ ■

Proof of Theorem 1.2.2.

From (1.18) we conclude that

$$I_{[0,\mathcal{T}_M]}(s)\mathcal{C}(s, U(s)) = I_{[0,\mathcal{T}_M]}(s)\mathcal{C}^*(s) \quad \text{for a.e.} \quad (\omega, t) \in \Omega \times [0,T]. \tag{1.20}$$

Because $\Phi(\cdot, U_{n'}(\cdot)) \rightharpoonup \Phi^*$ in $\mathcal{L}^2_H(\Omega \times [0,T])$ and Φ is a continuous mapping, it follows from (1.19) that

$$I_{[0,\mathcal{T}_M]}(s)\Phi(s, U(s)) = I_{[0,\mathcal{T}_M]}(s)\Phi^*(s) \quad \text{for a.e.} \quad (\omega, t) \in \Omega \times [0,T]. \tag{1.21}$$

Using (1.19) and the properties of $\mathcal{B}$ it can be proved that

$$\lim_{n'\to\infty} E\int_0^{\mathcal{T}_M} \langle \mathcal{B}(U(s), U(s)) - \mathcal{B}(U_{n'}(s), U_{n'}(s)), x(s)\rangle ds = 0 \quad \text{for all} \quad x \in \mathcal{D}_V(\Omega \times [0,T]).$$

But $\mathcal{B}(U_{n'}, U_{n'}) \rightharpoonup \mathcal{B}^*$ in $\mathcal{L}^2_{V^*}(\Omega \times [0,T])$, so

$$\lim_{n'\to\infty} E \int_0^{\mathcal{T}_M} \langle \mathcal{B}^*(s) - \mathcal{B}(U_{n'}(s), U_{n'}(s)), x(s)\rangle ds = 0 \quad \text{for all} \quad x \in \mathcal{D}_V(\Omega \times [0,T]).$$

Since $\mathcal{D}_V(\Omega \times [0,T])$ is dense in $\mathcal{L}^2_V(\Omega \times [0,T])$, it follows that

$$I_{[0,\mathcal{T}_M]}(s)\mathcal{B}^*(s) = I_{[0,\mathcal{T}_M]}(s)\mathcal{B}(U(s), U(s)) \quad \text{for a.e.} \quad (\omega, t) \in \Omega \times [0,T]. \tag{1.22}$$

Using (1.20), (1.21), and (1.22) in (1.14)

$$\begin{aligned} (U(t\wedge \mathcal{T}_M), v) \quad &+ \int_0^{t\wedge\mathcal{T}_M} \langle \mathcal{A}u(s), v\rangle ds = (x_0, v) + \int_0^{t\wedge\mathcal{T}_M} \langle \mathcal{B}(U(s), U(s)), v\rangle ds \\ &+ \int_0^{t\wedge\mathcal{T}_M} (\Phi(s, U(s)), v) ds + \int_0^{t\wedge\mathcal{T}_M} (\mathcal{C}(s, U(s)), v) dw(s) \end{aligned} \tag{1.23}$$

for all $v \in V, t \in [0,T]$, and a.e. $\omega \in \Omega$.

From the properties of the stopping time $\mathcal{T}_M$ and Proposition B.1 we see that

$$P\Big(\bigcup_{M=1}^{\infty} \{\mathcal{T}_M = T\}\Big) = 1.$$

Let

$$\Omega' := \Big\{\omega \in \Omega : \omega \in \bigcup_{M=1}^{\infty} \{\mathcal{T}_M = T\} \text{ and } U(\omega, t) \text{ satisfies (1.23) for all } v \in V, t \in [0,T]\Big\}.$$

Obviously, we have $P(\Omega') = 1$.

For $\omega \in \Omega'$ there exists a natural number M_0 such that $\mathcal{T}_M(\omega) = T$ for all $M \geq M_0$. From (1.23), we obtain

$$\begin{aligned} (U(t), v) + \int_0^t \langle \mathcal{A}U(s), v\rangle ds = (x_0, v) \quad &+ \int_0^t \langle \mathcal{B}(U(s), U(s)), v\rangle ds \\ &+ \int_0^t (\Phi(s, U(s)), v) ds + \int_0^t (\mathcal{C}(s, U(s)), v) dw(s) \end{aligned} \tag{1.24}$$

for all $v \in V, t \in [0,T]$. Consequently (1.24) holds for all $\omega \in \Omega'$. This means that the process $\Big(U(t)\Big)_{t\in[0,T]}$ satisfies the Navier-Stokes equation (1.1). Taking into account Lemma 1.2.4 it follows that U has almost surely continuous trajectories in H and we have

$$E \sup_{t\in[0,T]} \|U(t)\|^2 < \infty.$$

Hence $\Big(U(t)\Big)_{t\in[0,T]}$ is a solution of the Navier-Stokes equation (1.1).

(ii) In order to prove the uniqueness we assume that $X, Y \in \mathcal{L}^2_V(\Omega\times[0,T])$ are two solutions of equation (1.1), which have in H almost surely continuous trajectories. Let

$$e_2(t) = \Delta_X(t)\exp\{-(\lambda+2\sqrt{\mu})t\}$$

for all $t\in[0,T]$ and a.e. $\omega\in\Omega$. It follows by the Ito formula that

$$\begin{aligned}
&e_2(t)\|X(t)-Y(t)\|^2 + 2\int_0^t e_2(s)\langle \mathcal{A}X(s)-\mathcal{A}Y(s), X(s)-Y(s)\rangle ds\\
= &\; 2\int_0^t e_2(s)\langle \mathcal{B}(X(s),X(s))-\mathcal{B}(Y(s),Y(s)), X(s)-Y(s)\rangle ds\\
- &\; \frac{b}{\nu}\int_0^t e_2(s)\|X(s)\|_V^2\|X(s)-Y(s)\|^2 ds - (\lambda+2\sqrt{\mu})\int_0^t e_2(s)\|X(s)-Y(s)\|^2 ds\\
+ &\; 2\int_0^t e_2(s)(\Phi(s,X(s))-\Phi(s,Y(s)), X(s)-Y(s))ds\\
+ &\; 2\int_0^t e_2(s)(\mathcal{C}(s,X(s))-\mathcal{C}(s,Y(s)), X(s)-Y(s))dw(s) + \int_0^t e_2(s)\|\mathcal{C}(s,X(s))-\mathcal{C}(s,Y(s))\|^2 ds.
\end{aligned}$$

In view of the properties of $\mathcal{B}$ we can write

$$\begin{aligned}
2\langle \mathcal{B}(X(s),X(s))-\mathcal{B}(Y(s),Y(s)), X(s)-Y(s)\rangle &= 2\langle \mathcal{B}(X(s)-Y(s),X(s)), X(s)-Y(s)\rangle\\
&\le \frac{b}{\nu}\|X(s)\|_V^2\|X(s)-Y(s)\|^2 + \nu\|X(s)-Y(s)\|_V^2.
\end{aligned}$$

Now we use the properties of $\mathcal{A}$, Φ, and $\mathcal{C}$ to obtain

$$\begin{aligned}
e_2(t)\|X(t)-Y(t)\|^2 &+ \nu\int_0^t e_2(s)\|X(s)-Y(s)\|_V^2 ds\\
&\le 2\int_0^t e_2(s)(\mathcal{C}(s,X(s))-\mathcal{C}(s,Y(s)), X(s)-Y(s))dw(s)
\end{aligned}$$

for all $t\in[0,T]$ and a.e. $\omega\in\Omega$. This implies (using also the ideas from the proof of Proposition B.2)

$$Ee_2(t)\|X(t)-Y(t)\|^2 = 0 \quad \text{for all} \quad t\in[0,T]$$

and hence $P\Big(X(t)=Y(t)\Big)=1$ for all $t\in[0,T]$. Then for each countable and dense subset $\mathcal{S}\subset[0,T]$ we have

$$P\Big(\sup_{t\in\mathcal{S}}\|X(t)-Y(t)\|=0\Big)=1.$$

But X and Y have almost surely continuous trajectories in H, so

$$P\Big(\sup_{t\in[0,T]} \|X(t)-Y(t)\| = 0\Big) = 1.$$

This means that (1.1) has an almost surely unique solution. ■

Lemma 1.2.6
There exists a positive constant c_2 (depending only on λ, ν, and T) such that

$$E \sup_{t\in[0,T]} \|U(t)\|^4 + E\Big(\int_0^T \|U(s)\|_V^2 ds\Big)^2 \le c_2 E\|x_0\|^4.$$

The proof of Lemma 1.2.6 is analogous to the proof of Lemma 1.2.3 and makes use of Proposition B.2.

Another **important result** of this chapter is the following theorem, in which we state that the Galerkin approximations (U_n) converge in mean square to the solution of the Navier-Stokes equation.

Theorem 1.2.7
The following convergences hold:

$$\lim_{n\to\infty} E\int_0^T \|U(s)-U_n(s)\|_V^2 ds = 0$$

and

$$\lim_{n\to\infty} E\|U(t)-U_n(t)\|^2 = 0 \quad \text{for all } t\in[0,T].$$

Proof. First we apply Proposition B.3 with $\mathcal{T} := T$, $Q_{n'}(\mathcal{T}) := \int_0^{\mathcal{T}} \|U_{n'}(s)-U(s)\|_V^2 ds$, use Lemma 1.2.5, Lemma 1.2.3, and Lemma 1.2.6 to obtain

$$\lim_{n'\to\infty} E\int_0^T \|U(s)-U_{n'}(s)\|_V^2 ds = 0.$$

Let $t\in[0,T]$. Now we apply Proposition B.3 with $\mathcal{T} := t$, $Q_{n'}(\mathcal{T}) := \|U_{n'}(\mathcal{T})-U(\mathcal{T})\|$, use Lemma 1.2.5, Lemma 1.2.3, and Lemma 1.2.6 and get

$$\lim_{n'\to\infty} E\|U_{n'}(t)-U(t)\|^2 = 0.$$

Every subsequence of (U_n) has a further subsequence which converges in the norm of the space $\mathcal{L}_V^2(\Omega\times[0,T])$ to the same limit U, the unique solution of the Navier-Stokes equation (1.1) (because we can repeat all arguments of the results of Section 1.2 for this subsequence). Applying Proposition A.1 it follows that the whole sequence (U_n) converges in mean square to U. By the same argument we can prove that for all $t\in[0,T]$ the whole sequence $(U_n(t))$ converges to $U(t)$ in the norm of the space $\mathcal{L}_H^2(\Omega)$. ■

Remark 1.2.8

1) The results of this section also hold if we consider equation (1.1) starting at s with $s \in [0, T)$ (instead of 0) and we assume that x_0 is a H-valued $\mathcal{F}_s$-measurable random variable such that $E\|x_0\|^4 < \infty$.

2) The results of this section also hold if we consider instead of a mapping Φ, satisfying hypothesis **(vi)** from Section 1.1, a process belonging to the space $\mathcal{L}^2_H(\Omega \times [0, T])$ with $E\int_0^T \|\Phi(t)\|^4 dt < \infty$.

1.3 A special linear stochastic evolution equation

The results presented in this section prepare the investigations for the linear approximation method from Section 1.4.

Let $X, Y \in \mathcal{L}^2_V(\Omega \times [0, T])$ be arbitrary processes with almost surely continuous trajectories in H and

$$E \sup_{t\in[0,T]} \|X(t)\|^2 < \infty, \quad E \sup_{t\in[0,T]} \|Y(t)\|^2 < \infty.$$

For each $M \in \mathbb{N}$ let $\mathcal{T}_M := \min\{\mathcal{T}^X_M, \mathcal{T}^Y_M\}$. From the properties of the stopping times (see Appendix B) it follows that

$$\lim_{M\to\infty} \mathcal{T}_M = T \quad \text{for a.e. } \omega \in \Omega, \tag{1.25}$$

as soon as

$$P\Big(\bigcup_{M=1}^{\infty} \{\mathcal{T}_M = T\}\Big) = 1. \tag{1.26}$$

We define $X^M(t) := X(t \wedge \mathcal{T}_M), Y^M(t) := Y(t \wedge \mathcal{T}_M)$ for all $t \in [0, T]$.

Let $\mathcal{G} : [0, T] \times H \to H$ be a mapping satisfying hypothesis **(v)** from Section 1.1 and we assume that for each $t \in [0, T]$ the mapping $\mathcal{G}(t, \cdot) : H \to H$ is linear. Let a_0 be a H-valued $\mathcal{F}_0$-measurable random variable with $E\|a_0\|^4 < \infty$ and let $\Psi \in \mathcal{L}^2_{V^*}(\Omega \times [0, T]), \Gamma \in \mathcal{L}^2_H(\Omega \times [0, T])$. We consider the linear evolution equation:

$$(P_{\Psi,\Gamma}) \qquad \begin{aligned}(Z_{\Psi,\Gamma}(t), v) &+ \int_0^t \langle \mathcal{A} Z_{\Psi,\Gamma}(s), v\rangle ds = (a_0, v)\\ &+ \int_0^t \langle \mathcal{B}(X(s), Z_{\Psi,\Gamma}(s)) + \mathcal{B}(Z_{\Psi,\Gamma}(s), Y(s)), v\rangle ds\\ &+ \int_0^t \langle \Psi(s), v\rangle ds + \int_0^t (\mathcal{G}(s, Z_{\Psi,\Gamma}(s)), v) dw(s) + \int_0^t (\Gamma(s), v) dw(s)\end{aligned}$$

for all $v \in V$, $t \in [0,T]$, and a.e. $\omega \in \Omega$ and for each $M \in \mathbb{N}$ we consider:

$$
(P^M_{\Psi,\Gamma}) \qquad \begin{aligned}(Z^M_{\Psi,\Gamma}(t), v) &+ \int_0^t \langle \mathcal{A} Z^M_{\Psi,\Gamma}(s), v\rangle ds = (a_0, v)\\
&+ \int_0^t \langle \mathcal{B}(X^M(s), Z^M_{\Psi,\Gamma}(s)) + \mathcal{B}(Z^M_{\Psi,\Gamma}(s), Y^M(s)), v\rangle ds\\
&+ \int_0^t \langle \Psi(s), v\rangle ds + \int_0^t (\mathcal{G}(s, Z^M_{\Psi,\Gamma}(s)), v) dw(s) + \int_0^t (\Gamma(s), v) dw(s)\end{aligned}
$$

for all $v \in V$, $t \in [0,T]$, and a.e. $\omega \in \Omega$.

For each $n \in \mathbb{N}$ we define $\mathcal{G}_n : [0,T] \times H_n \to H_n$ by $\mathcal{G}_n(t,v) := \Pi_n \mathcal{G}(t,v)$ and consider

$$
X_n := \Pi_n X, \quad Y_n := \Pi_n Y, \quad a_{0n} := \Pi_n a_0, \quad X^M_n(t) := X_n(t \wedge \mathcal{T}_M), \quad Y^M_n(t) := Y_n(t \wedge \mathcal{T}_M),
$$

for all $t \in [0,T], v \in H_n$ and a.e. $\omega \in \Omega$.

Let $n \in \mathbb{N}$ and $\psi \in \mathcal{L}^2_{(H_n, \|\cdot\|_V)}(\Omega \times [0,T]), \gamma \in \mathcal{L}^2_{H_n}(\Omega \times [0,T])$. We consider the finite dimensional evolution equations

$$
(P_{n,\psi,\gamma}) \qquad \begin{aligned}(Z_{n,\psi,\gamma}(t), v) &+ \int_0^t (\mathcal{A}_n Z_{n,\psi,\gamma}(s), v) ds = (a_{0n}, v)\\
&+ \int_0^t (\mathcal{B}_n(X_n(s), Z_{n,\psi,\gamma}(s)) + \mathcal{B}_n(Z_{n,\psi,\gamma}(s), Y_n(s)), v) ds\\
&+ \int_0^t (\psi(s), v) ds + \int_0^t (\mathcal{G}_n(s, Z_{n,\psi,\gamma}(s)), v) dw(s) + \int_0^t (\gamma(s), v) dw(s)\end{aligned}
$$

and for each $M \in \mathbb{N}$ let

$$
(P^M_{n,\psi,\gamma}) \qquad \begin{aligned}(Z^M_{n,\psi,\gamma}(t), v) &+ \int_0^t (\mathcal{A}_n Z^M_{n,\psi,\gamma}(s), v) ds = (a_{0n}, v)\\
&+ \int_0^t (\mathcal{B}_n(X^M_n(s), Z_{n,\psi,\gamma}(s)) + \mathcal{B}_n(Z_{n,\psi,\gamma}(s), Y^M_n(s)), v) ds\\
&+ \int_0^t (\psi(s), v) ds + \int_0^t (\mathcal{G}_n(s, Z^M_{n,\psi,\gamma}(s)), v) dw(s) + \int_0^t (\gamma(s), v) dw(s)\end{aligned}
$$

for all $t \in [0,T], v \in H_n$, and a.e. $\omega \in \Omega$.

Theorem 1.3.1

(i) *For each* $\Psi \in \mathcal{L}^2_{V^*}(\Omega \times [0,T]), \Gamma \in \mathcal{L}^2_H(\Omega \times [0,T])$ *there exists a* V*-valued,* $\mathcal{F} \times \mathcal{B}_{[0,T]}$*-measurable process* $\Big(Z_{\Psi,\Gamma}(t)\Big)_{t\in[0,T]}$ *adapted to the filtration* $(\mathcal{F}_t)_{t\in[0,T]}$*, satisfying* $(P_{\Psi,\Gamma})$ *and which has almost surely continuous trajectories in* H*. The solution is almost surely unique, and there exists a positive constant* c_1 *(independent of* a_0, Ψ, Γ*) such that*

$$\begin{aligned} E \sup_{t\in[0,T]} \Delta_Y(t)\|Z_{\Psi,\Gamma}(t)\|^2 \;&+\; E\int_0^T \Delta_Y(t)\|Z_{\Psi,\Gamma}(t)\|^2_V ds \\ &\le\; c_1\Big[E\|a_0\|^2 + E\int_0^T \|\Psi(s)\|^2_{V^*} ds + E\int_0^T \|\Gamma(s)\|^2 ds\Big] \end{aligned}$$

and if $E\int_0^T \|\Psi(t)\|^4_{V^*} dt < \infty$ *and* $E\int_0^T \|\Gamma(t)\|^4 dt < \infty$*, then*

$$\begin{aligned} E \sup_{t\in[0,T]} \Delta^2_Y(t)\|Z_{\Psi,\Gamma}(t)\|^4 \;&+\; E\Big(\int_0^T \Delta_Y(t)\|Z_{\Psi,\Gamma}(t)\|^2_V ds\Big)^2 \\ &\le\; c_1\Big[E\|a_0\|^4 + E\int_0^T \|\Psi(s)\|^4_{V^*} ds + E\int_0^T \|\Gamma(s)\|^4 ds\Big]. \end{aligned}$$

(ii) *For each* $\psi \in \mathcal{L}^2_{(H_n,\|\cdot\|_V)}(\Omega \times [0,T]), \gamma \in \mathcal{L}^2_{H_n}(\Omega \times [0,T])$ *there exists a* V*-valued,* $\mathcal{F} \times \mathcal{B}_{[0,T]}$*-measurable process* $\Big(Z_{n,\psi,\gamma}(t)\Big)_{t\in[0,T]}$ *adapted to the filtration* $(\mathcal{F}_t)_{t\in[0,T]}$*, satisfying* $(P_{n,\psi,\gamma})$ *and which has almost surely continuous trajectories in* H*. The solution is almost surely unique, and there exists a positive constant* c_2 *(independent of* n*,* a_0, ψ, γ*) such that*

$$\begin{aligned} E \sup_{t\in[0,T]} \Delta_Y(t)\|Z_{n,\psi,\gamma}(t)\|^2 \;&+\; E\int_0^T \Delta_Y(t)\|Z_{n,\psi,\gamma}(t)\|^2_V ds \\ &\le\; c_2\Big[E\|a_0\|^2 + E\int_0^T \|\psi(s)\|^2 ds + E\int_0^T \|\gamma(s)\|^2 ds\Big] \end{aligned}$$

and if $E\int_0^T \|\psi(t)\|^4 dt < \infty$ *and* $E\int_0^T \|\gamma(t)\|^4 dt < \infty$*, then*

$$\begin{aligned} E \sup_{t\in[0,T]} \Delta^2_Y(t)\|Z_{n,\psi,\gamma}(t)\|^4 \;&+\; E\Big(\int_0^T \Delta_Y(t)\|Z_{n,\psi,\gamma}(t)\|^2_V ds\Big)^2 \\ &\le\; c_1\Big[E\|a_0\|^4 + E\int_0^T \|\psi(s)\|^4 ds + E\int_0^T \|\gamma(s)\|^4 ds\Big]. \end{aligned}$$

PROOF. **(i)** Let $\Psi \in \mathcal{L}^2_{V^*}(\Omega \times [0,T]), \Gamma \in \mathcal{L}^2_H(\Omega \times [0,T])$. For each $n \in \mathbb{N}$ let $\Psi_n := \sum_{i=1}^{n} \langle \Psi, h_i \rangle h_i$, $\Gamma_n := \Pi_n \Gamma$. For the finite dimensional evolution equation $(P^M_{n,\Psi_n,\Gamma_n})$ we apply the theory of finite dimensional Ito equations with Lipschitz continuous nonlinearities (see [26], Theorem 5.5, p. 45). Hence there exists a solution $Z^M_{n,\Psi_n,\Gamma_n} \in \mathcal{L}^2_{(H_n,\|\cdot\|_V)}(\Omega \times [0,T])$ of $(P^M_{n,\Psi_n,\Gamma_n})$, which has almost surely continuous trajectories in H; this solution is almost surely unique.

For notational simplicity we define $Z^M_n := Z^M_{n,\Psi_n,\Gamma_n}$.

Let $M, n \in \mathbb{N}$. From the equation for Z^M_n and Proposition B.2 we obtain the estimate:

$$
\begin{aligned}
(1.27) \qquad E\Delta_{Y^M_n}(T)\|Z^M_n(T)\|^2 &+ E\int_0^T \Delta_{Y^M_n}(t)\|Z^M_n(t)\|^2_V dt \\
&\leq c\Big[E\|a_0\|^2 + E\int_0^T \|\Psi(s)\|^2_{V^*} ds + E\int_0^T \|\Gamma(s)\|^2 ds\Big],
\end{aligned}
$$

where c is a positive constant independent of M and n, but it depends on ν, λ, T. We can write

$$
\begin{aligned}
(1.28) \qquad E\int_0^T \|Z^M_n(t)\|^2_V dt &\leq E\Delta^{-1}_{Y^M_n}(T)\int_0^T \Delta_{Y^M_n}(t)\|Z^M_n(t)\|^2_V dt \\
&\leq E\Big\{\exp\Big\{\frac{b}{\nu}\int_0^T \|Y_n(t\wedge \mathcal{T}_M)\|^2_V dt\Big\}\int_0^T \Delta_{Y^M_n}(t)\|Z^M_n(t)\|^2_V dt\Big\} \\
&\leq E\Big\{\exp\Big\{\frac{b}{\nu}\int_0^T \|Y(t\wedge \mathcal{T}_M)\|^2_V dt\Big\}\int_0^T \Delta_{Y^M_n}(t)\|Z^M_n(t)\|^2_V dt\Big\} \\
&\leq c\exp\Big\{\frac{bM}{\nu}\Big\}\Big[E\|a_0\|^2 + E\int_0^T \|\Psi(t)\|^2_{V^*} dt + E\int_0^T \|\Gamma(t)\|^2 dt\Big].
\end{aligned}
$$

Hence, for fixed M the sequence $\left(Z^M_n\right)$ is bounded in the space $\mathcal{L}^2_V(\Omega \times [0,T])$. Consequently, there exists a subsequence (n') of (n) and $Z^M \in \mathcal{L}^2_V(\Omega \times [0,T])$ such that for $n' \to \infty$ we have

$$
(1.29) \qquad Z^M_{n'} \rightharpoonup Z^M.
$$

We want to prove that for $n' \to \infty$ the weak convergence $\mathcal{B}_{n'}(X^M_{n'}, Z^M_{n'}) \rightharpoonup \mathcal{B}(X^M, Z^M)$ holds in $\mathcal{L}^2_{V^*}(\Omega \times [0,T])$. Let $v \in V$ and $v_n := \Pi_n v$. We see that

$$
\begin{aligned}
(\mathcal{B}_n(X^M_n, Z^M_n), v) &= (\mathcal{B}_n(X^M_n, Z^M_n), v_n) = \langle \mathcal{B}(X^M_n, Z^M_n), v_n \rangle \\
&= \langle \mathcal{B}(X^M, v) - \mathcal{B}(X^M_n, v_n), Z^M_n \rangle + \langle \mathcal{B}(X^M, v), Z^M - Z^M_n \rangle + \langle \mathcal{B}(X^M, Z^M), v \rangle.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
(1.30) \qquad \langle \mathcal{B}_n(X^M_n, Z^M_n) &- \mathcal{B}(X^M, Z^M), v \rangle \\
&= \langle \mathcal{B}(X^M, v) - \mathcal{B}(X^M_n, v_n), Z^M_n \rangle + \langle \mathcal{B}(X^M, v), Z^M - Z^M_n \rangle.
\end{aligned}
$$

It holds [1]

$$E\int_0^T \|\mathcal{B}(X_n^M(s),v_n)-\mathcal{B}(X^M(s),v)\|_{V^*}^2 ds$$
$$\leq\ bc_{HV}\Big(\|v\|_V^2 E\int_0^T \|X^M(s)-X_n^M(s)\|_V^2 ds+\|v-v_n\|_V^2 E\int_0^T \|X^M(s)\|_V^2 ds\Big).$$

Since v_n and X_n^M are the Fourier expansions of v and X^M, respectively, it follows that

$$\lim_{n\to\infty} E\int_0^T \|\mathcal{B}(X_n^M(s),v_n)-\mathcal{B}(X^M(s),v)\|_{V^*}^2 ds=0. \tag{1.31}$$

Using (1.29), (1.31) in (1.30) we get

$$\lim_{n'\to\infty} E\int_0^T (\mathcal{B}_{n'}(X_{n'}^M(s),Z_{n'}^M(s)),\xi(s))ds = E\int_0^T \langle\mathcal{B}(X^M(s),Z^M(s)),\xi(s)\rangle ds$$

for all $\xi\in\mathcal{D}_V(\Omega\times[0,T])$. Since $\mathcal{B}_{n'}(X_{n'}^M,Z_{n'}^M),\mathcal{B}(X^M,Z^M)\in\mathcal{L}^2_{V^*}(\Omega\times[0,T])$ and $\mathcal{D}_V(\Omega\times[0,T])$ is dense in $\mathcal{L}^2_V(\Omega\times[0,T])$, we have $\mathcal{B}_{n'}(X_{n'}^M,Z_{n'}^M)\rightharpoonup\mathcal{B}(X^M,Z^M)$ for $n'\to\infty$. Analogously we can prove that $\mathcal{B}_{n'}(Z_{n'}^M,Y_{n'}^M)\rightharpoonup\mathcal{B}(Z^M,Y^M)$ for $n'\to\infty$.

We take the limit $n'\to\infty$ in $(P^M_{n',\Psi_{n'},\Gamma_{n'}})$, use the weak convergence (1.29), as soon as the strong convergences of $\left(X_n^M\right)$ to X^M and of $\left(Y_n^M\right)$ to Y^M in the space $\mathcal{L}^2_H(\Omega\times[0,T])$ and Proposition A.3 to obtain

$$\begin{aligned}(Z^M(t),v) &= (a_0,v)-\int_0^t\langle\mathcal{A}Z^M(s),v\rangle ds+\int_0^t\langle\mathcal{B}(X^M(s),Z^M(s))+\mathcal{B}(Z^M(s),Y^M(s)),v\rangle ds\\ &+ \int_0^t\langle\Psi(s),v\rangle ds+\int_0^t(\mathcal{G}(s,Z^M(s)),v)dw(s)+\int_0^t(\Gamma(s),v)dw(s)\end{aligned} \tag{1.32}$$

for all $v\in V$ and $P\times\Lambda$ a.e. $(\omega,t)\in\Omega\times[0,T]$. The right side of (1.32) has a continuous modification (as an H valued process), and this process we identify with $\left(Z^M_{\Psi,\Gamma}(t)\right)_{t\in[0,T]}$ (see [21], Theorem 3.2, p. 91). So, $\left(Z^M_{\Psi,\Gamma}(t)\right)_{t\in[0,T]}$ is a process from the space $\mathcal{L}^2_V(\Omega\times[0,T])$ which has almost surely continuous trajectories in H and satisfies $(P^M_{\Psi,\Gamma})$ (identically with (1.32)) for all $v\in V, t\in[0,T]$ and a.e. $\omega\in\Omega$. By standard methods (see the final part of the proof) we can prove that the solution of $(P^M_{\Psi,\Gamma})$ is almost surely unique.

[1] Since $V\hookrightarrow H$ we have $\|v\|^2\leq c_{HV}\|v\|_V^2$ for all $v\in V$.

Let Ω_K be the set of all $\omega \in \Omega$ such that $Z^K_{\Psi,\Gamma}(\omega,\cdot)$ satisfies $(P^K_{\Psi,\Gamma})$ for all $t \in [0,T], v \in V$ and such that $Z^K_{\Psi,\Gamma}(\omega,\cdot)$ has continuous trajectories in H. We define $\Omega' := \bigcap\limits_{K=1}^{\infty} \Omega_K$. We also consider

$$S := \bigcup_{M=1}^{\infty} \bigcup_{1 \le K \le M} \{\omega \in \Omega' |\ \mathcal{T}_K = T \text{ and } \exists\ t \in [0,T]:\ Z^K_{\Psi,\Gamma}(\omega,t) \neq Z^M_{\Psi,\Gamma}(\omega,t)\}.$$

We have $P(S) = 0$, because otherwise there exist two natural numbers M_0, K_0 with $K_0 < M_0$ such that the set

$$S_{M_0,K_0} := \{\omega \in \Omega' | \mathcal{T}_{K_0} = T \text{ and } \exists\ t \in [0,T]:\ Z^{K_0}_{\Psi,\Gamma}(\omega,t) \neq Z^{M_0}_{\Psi,\Gamma}(\omega,t)\}$$

has the measure $P(S_{M_0,K_0}) > 0$. We define for each $t \in [0,T]$

$$Z^*(\omega,t) := \begin{cases} Z^{K_0}_{\Psi,\Gamma}(\omega,t) & , \quad \omega \in S_{M_0,K_0} \\ Z^{M_0}_{\Psi,\Gamma}(\omega,t) & , \quad \omega \in \Omega' \setminus S_{M_0,K_0}. \end{cases}$$

We see that for all $\omega \in S_{M_0,K_0}$ there exists $t \in [0,T]$ such that $Z^*(\omega,t) \neq Z^{M_0}(\omega,t)$. This contradicts to the almost surely uniqueness of the solution of $(P^{M_0}_{\Psi,\Gamma})$. Consequently, $P(S) = 0$.

We define

$$\Omega'' := \bigcup_{M=1}^{\infty} \{\mathcal{T}_M = T\}.$$

Obviously $P\Big((\Omega' \cap \Omega'') \setminus S\Big) = 1$ (see also (1.26)). Let $\omega \in (\Omega' \cap \Omega'') \setminus S$. For this ω there exists a natural number M_0 such that $\mathcal{T}_M(\omega) = T$ for all $M \ge M_0$. Hence $X^M(s) = X(s)$ and $Y^M(s) = Y(s)$ for all $s \in [0,T]$ and for all $M \ge M_0$. Equation $(P^M_{\Psi,\Gamma})$ implies

$$\begin{aligned} (Z^M_{\Psi,\Gamma}(t),v) \quad &+ \int_0^t \langle \mathcal{A} Z^M_{\Psi,\Gamma}(s), v\rangle ds = (a_0,v) \\ &+ \int_0^t \langle \mathcal{B}(X(s), Z^M_{\Psi,\Gamma}(s)) + \mathcal{B}(Z^M_{\Psi,\Gamma}(s), Y(s)), v\rangle ds \\ &+ \int_0^t \langle \Psi(s), v\rangle ds + \int_0^t (\mathcal{G}(s, Z^M_{\Psi,\Gamma}(s)), v) dw(s) + \int_0^t (\Gamma(s), v) dw(s) \end{aligned} \tag{1.33}$$

for all $M \ge M_0$ and all $t \in [0,T], v \in V$. We have

$$\lim_{M\to\infty} \int_0^T \|Z^M_{\Psi,\Gamma}(t) - Z^{M_0}_{\Psi,\Gamma}(t)\|^2_V dt = 0$$

and

$$\lim_{M\to\infty} \|Z^M_{\Psi,\Gamma}(t) - Z^{M_0}_{\Psi,\Gamma}(t)\|^2 = 0 \qquad \text{for all} \quad t \in [0,T].$$

For each $t \in [0, T]$ we define

$$Z_{\Psi,\Gamma}(\omega, t) := Z_{\Psi,\Gamma}^{M_0}(\omega, t) = \lim_{M\to\infty} Z_{\Psi,\Gamma}^{M}(\omega, t).$$

This definition is correct because $\omega \notin S$. Then (1.33) implies

$$\begin{aligned}(1.34)\qquad (Z_{\Psi,\Gamma}(t), v) &+ \int_0^t (\mathcal{A}Z_{\Psi,\Gamma}(s), v)ds = (a_0, v)\\ &+ \int_0^t \langle \mathcal{B}(X(s), Z_{\Psi,\Gamma}(s)) + \mathcal{B}(Z_{\Psi,\Gamma}(s), Y(s)), v\rangle ds\\ &+ \int_0^t \langle \Psi(s), v\rangle ds + \int_0^t (\mathcal{G}(s, Z_{\Psi,\Gamma}(s)), v)dw(s) + \int_0^t (\Gamma(s), v)dw(s)\end{aligned}$$

for all $\omega \in \Big(\Omega \cap \Omega''\Big) \setminus S, t \in [0, T], v \in V$. The process $(Z_{\Psi,\Gamma}(t))_{t\in[0,T]}$ is V-valued, $\mathcal{F} \times \mathcal{B}_{[0,T]}$-measurable, adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and has almost surely continuous trajectories in H, because all $Z_{\Psi,\Gamma}^M$ have this properties. For $Z_{\Psi,\Gamma}^M$ we can prove an analogous inequality as (1.27). Thus we get

$$\begin{aligned}(1.35)\qquad E\Delta_Y(T)\|Z_{\Psi,\Gamma}(T)\|^2 &+ E\int_0^T \Delta_Y(t)\|Z_{\Psi,\Gamma}(t)\|_V^2 dt\\ &\leq \liminf_{M\to\infty} \Big\{E\Delta_{Y^M}(T)\|Z_{\Psi,\Gamma}^M(T)\|^2 + E\int_0^T \Delta_{Y^M}(t)\|Z_{\Psi,\Gamma}^M(t)\|_V^2 dt\Big\}\\ &\leq c\Big[E\|a_0\|^2 + E\int_0^T \|\Psi(s)\|_{V^*}^2 ds + E\int_0^T \|\Gamma(s)\|^2 ds\Big],\end{aligned}$$

where c is the same constant as in (1.27). We obtain the other estimate by using in $(P_{\Psi,\Gamma})$ the Ito formula and then Proposition B.2.

Now we prove that equation $(P_{\Psi,\Gamma})$ has an almost surely unique solution. Let

$$e_1(t) := \Delta_Y(t) \exp\{-\lambda t\}.$$

We assume that $\tilde{Z}$ and Z are two solutions of $(P_{\Psi,\Gamma})$ which have almost surely continuous trajectories in H. Then for all $t \in [0, T]$ and a.e. $\omega \in \Omega$ we have

$$
\begin{aligned}
& e_1(t)\|\tilde{Z}(t)-Z(t)\|^2+2\int_0^t e_1(s)\langle\mathcal{A}\tilde{Z}(s)-\mathcal{A}Z(s),\tilde{Z}(s)-Z(s)\rangle ds \\
= \;& 2\int_0^t e_1(s)\langle\mathcal{B}(X(s),\tilde{Z}(s)-Z(s))+\mathcal{B}(\tilde{Z}(s)-Z(s),Y(s)),\tilde{Z}(s)-Z(s)\rangle ds \\
- \;& \int_0^t e_1(s)(\lambda+\frac{b}{\nu}\|Y(s)\|_V^2)\|\tilde{Z}(s)-Z(s)\|^2 ds+\int_0^t e_1(s)\|\mathcal{G}(s,\tilde{Z}(s))-\mathcal{G}(s,Z(s))\|^2 ds \\
+ \;& 2\int_0^t e_1(s)(\mathcal{G}(s,\tilde{Z}(s))-\mathcal{G}(s,Z(s)),\tilde{Z}(s)-Z(s))dw(s).
\end{aligned}
$$

Taking into account the properties of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{G}$, it follows that for each $t\in[0,T]$ and a.e. $\omega\in\Omega$

$$
\begin{aligned}
(1.36)\qquad e_1(t)\|\tilde{Z}(t)-Z(t)\|^2 \;&+\; \nu\int_0^t e_1(s)\|\tilde{Z}(s)-Z(s)\|_V^2 ds \\
&\le\; 2\int_0^t e_1(s)(\mathcal{G}(s,\tilde{Z}(s))-\mathcal{G}(s,Z(s)),\tilde{Z}(s)-Z(s))dw(s).
\end{aligned}
$$

This implies (we use the same ideas as in the prove of Proposition B.2)

$$E\int_0^T \Delta_Y(s)\|\tilde{Z}(s)-Z(s)\|_V^2 ds=0.$$

Hence $\tilde{Z}(\omega,t)=Z(\omega,t)$ for $P\times\Lambda$ a.e. $(\omega,t)\in(\Omega\times[0,T])$. Using this result and (1.36) we deduce that

$$E\sup_{t\in[0,T]}\Delta_Y(t)\|\tilde{Z}(t)-Z(t)\|^2=0,$$

which means that $(P_{\Psi,\Gamma})$ has an almost surely unique solution.

(ii) The existence, estimation, and (almost surely) uniqueness of the solution $Z_{n,\psi,\gamma}$ of $(P_{n,\psi,\gamma})$ can be proved analogously to the proof of **(i)**. ■

Lemma 1.3.2
We assume that $E\Delta_Y^{-2}(T)<\infty$. Let (ψ_n), (γ_n) be sequences in $\mathcal{L}_V^2(\Omega\times[0,T])$ and $\mathcal{L}_H^2(\Omega\times[0,T])$, respectively, such that $\psi_n\in\mathcal{L}^2_{(H_n,\|\cdot\|_V)}(\Omega\times[0,T]),\gamma_n\in\mathcal{L}^2_{H_n}(\Omega\times[0,T])$ for each $n\in\mathbb{N}$. If $(\mathrm{J}\psi_n)$ converges weakly to $\Psi\in\mathcal{L}^2_{V^*}(\Omega\times[0,T])$ and (γ_n) converges weakly to $\Gamma\in\mathcal{L}^2_H(\Omega\times[0,T])$, then for $n\to\infty$ we have

$$\Delta_Y Z_{n,\psi_n,\gamma_n}\rightharpoonup\Delta_Y Z_{\Psi,\Gamma}\quad in\quad \mathcal{L}_V^2(\Omega\times[0,T])$$

and

$$\Delta_Y(T)Z_{n,\psi_n,\gamma_n}(T)\rightharpoonup\Delta_Y(T)Z_{\Psi,\Gamma}(T)\quad in\quad \mathcal{L}_H^2(\Omega).$$

PROOF. Because $(\mathrm{J}\psi_n)$ converges weakly to $\Psi \in \mathcal{L}^2_{V^*}(\Omega \times [0,T])$ and (γ_n) converges weakly to $\Gamma \in \mathcal{L}^2_H(\Omega \times [0,T])$, it follows that there exists a constant $c_3 > 0$ such that for all $n \in \mathbb{N}$

$$E\int_0^T \|\psi_n(t)\|_V^2 dt + E\int_0^T \|\gamma_n(t)\|^2 dt \leq c_3.$$

For simplicity we define $Z_n := Z_{n,\psi_n,\gamma_n}$ and $Z_n^M := Z^M_{n,\psi_n,\gamma_n}$. Applying Theorem 1.3.1 we obtain

$$\begin{aligned}(1.37)\qquad &\sup_{1\leq n}\Big\{E\Delta_Y(T)\|Z_n(T) - Z_{\Psi,\Gamma}(T)\|^2 + E\int_0^T \Delta_Y(s)\|Z_n(s) - Z_{\Psi,\Gamma}(s)\|_V^2 ds\Big\} \\ &\leq (c_1+c_2)\Big(c_3 + E\|a_0\|^2 + E\int_0^T \|\Psi(s)\|^2_{V^*} ds + E\int_0^T \|\Gamma(s)\|^2 ds\Big).\end{aligned}$$

Let $\xi \in \mathcal{D}_{V^*}(\Omega \times [0,T])$ be arbitrary, but fixed. We want to prove that

$$\lim_{n\to\infty} E\int_0^T \langle \xi(s), Z_n(s) - Z_{\Psi,\Gamma}(s)\rangle ds = 0.$$

Since $\lim\limits_{M\to\infty} \mathcal{T}_M = T$ (see (1.25)) for a.e. $\omega \in \Omega$, $E\Delta_Y^{-2}(T) < \infty$ and $\xi \in \mathcal{D}_{V^*}(\Omega \times [0,T])$, we get

$$(1.38)\qquad \lim_{M\to\infty} E\int_{\mathcal{T}_M}^T \Delta_Y^{-1}(s)\|\xi(s)\|^2_{V^*} ds = 0.$$

Let $\varepsilon > 0$. There exists a natural number $K = K_\varepsilon$ such that

$$(1.39)\qquad \sup_{1\leq n}\Big\{E\int_0^T \Delta_Y(s)\|Z_n(s) - Z_{\Psi,\Gamma}(s)\|_V^2 ds\Big\} E\int_{\mathcal{T}_K}^T \Delta_Y^{-1}(s)\|\xi(s)\|^2_{V^*} ds < \frac{\varepsilon^2}{4}\ .$$

Relation (1.38) implies

$$\begin{aligned}&E\int_{\mathcal{T}_K}^T \langle \xi(s), Z_n(s) - Z_{\Psi,\Gamma}(s)\rangle ds \\ &\leq \left(\sup_{1\leq n}\Big\{E\int_0^T \Delta_Y(s)\|Z_n(s) - Z_{\Psi,\Gamma}(s)\|_V^2 ds\Big\} E\int_{\mathcal{T}_K}^T \Delta_Y^{-1}(s)\|\xi(s)\|^2_{V^*} ds\right)^{1/2} < \frac{\varepsilon}{2}\end{aligned}$$

for all $n \in \mathbb{N}$.

From the (almost surely) uniqueness of the solutions of $(P^K_{\Psi,\Gamma})$ and $(P^K_{n,\psi_n,\gamma_n})$, respectively, we conclude that

$$E\int_0^{\mathcal{T}_K} \|Z_n(s) - Z_n^K(s)\|_V^2 ds = E\int_0^{\mathcal{T}_K} \|Z^K_{\Psi,\Gamma}(s) - Z_{\Psi,\Gamma}(s)\|_V^2 ds = 0.$$

Then for all $n \in \mathbb{N}$ we have

$$
\begin{aligned}
(1.40) \qquad & \left| E\int_0^T \langle \xi(s), Z_n(s) - Z_{\Psi,\Gamma}(s)\rangle ds \right| \\
\leq \; & \left| E\int_0^{\mathcal{T}_K} \langle \xi(s), Z_n(s) - Z_{\Psi,\Gamma}(s)\rangle ds \right| + \left| E\int_{\mathcal{T}_K}^T \langle \xi(s), Z_n(s) - Z_{\Psi,\Gamma}(s)\rangle ds \right| \\
\leq \; & \left| E\int_0^{\mathcal{T}_K} \langle \xi(s), Z_n(s) - Z_n^K(s)\rangle ds \right| + \left| E\int_0^{\mathcal{T}_K} \langle \xi(s), Z_n^K(s) - Z_{\Psi,\Gamma}^K(s)\rangle ds \right| \\
+ \; & \left| E\int_0^{\mathcal{T}_K} \langle \xi(s), Z_{\Psi,\Gamma}^K(s) - Z_{\Psi,\Gamma}(s)\rangle ds \right| + \frac{\varepsilon}{2} = \left| E\int_0^{\mathcal{T}_K} \langle \xi(s), Z_n^K(s) - Z_{\Psi,\Gamma}^K(s)\rangle ds \right| + \frac{\varepsilon}{2} \,.
\end{aligned}
$$

In the following we prove that there exists an $n_\varepsilon > 0$ such that

$$\left| E\int_0^{\mathcal{T}_K} \langle \xi(s), Z_n^K(s) - Z_{\Psi,\Gamma}^K(s)\rangle ds \right| < \frac{\varepsilon}{2}$$

for all $n \geq n_\varepsilon$. Analogous to (1.28) (see the proof of Theorem 1.3.1) we have

$$E\int_0^T \|Z_n^K(s)\|_V^2 ds \leq c \exp\left\{\frac{bK}{\nu}\right\}\Big[E\|a_0\|^2 + c_3\Big].$$

Hence (Z_n^K) is a bounded sequence from $\mathcal{L}_V^2(\Omega \times [0,T])$. Consequently, by sequential weak compactness there exists a subsequence (n') of (n) and $Z^K \in \mathcal{L}_V^2(\Omega \times [0,T])$ such that for $n' \to \infty$ we have

$$(1.41) \qquad Z_{n'}^K \rightharpoonup Z^K \quad \text{in } \mathcal{L}_V^2(\Omega \times [0,T]).$$

As in the proof of Theorem 1.3.1 we can show that $\mathcal{B}_{n'}(X_{n'}^K, Z_{n'}^K) \rightharpoonup \mathcal{B}(X^K, Z^K)$ and $\mathcal{B}_{n'}(Z_{n'}^K, Y_{n'}^K) \rightharpoonup \mathcal{B}(Z^K, Y^K)$ in $\mathcal{L}_{V^*}^2(\Omega \times [0,T])$ for $n' \to \infty$. We take the limit $n' \to \infty$ in equation $(P_{n',\psi_{n'},\gamma_{n'}}^K)$, use the weak convergences given in the hypothesis and in (1.41), then the strong convergences of $\left(X_{n'}^K\right)$ to X^K and of $\left(Y_{n'}^K\right)$ to Y^K in the space $\mathcal{L}_H^2(\Omega \times [0,T])$ and Proposition A.3. Then we obtain

$$
\begin{aligned}
(Z^K(t), v) + \int_0^t \langle \mathcal{A}Z^K(s), v\rangle ds \;&=\; (a_0, v) + \int_0^t \langle \mathcal{B}(X^K(s), Z^K(s)) + \mathcal{B}(Z^K(s), Y^K(s)), v\rangle ds \\
&+\; \int_0^t \langle \Psi(s), v\rangle ds + \int_0^t (\mathcal{G}(s, Z^K(s)), v) dw(s) + \int_0^t (\Gamma(s), v) dw(s)
\end{aligned}
$$

for all $v \in V$ and $P \times \Lambda$ a.e. $(\omega, t) \in \Omega \times [0,T]$. The (almost surely) uniqueness of the solution of equation $(P_{\Psi,\Gamma}^K)$ implies that

$$Z^K(\omega, t) = Z_{\Psi,\Gamma}^K(\omega, t) \quad \text{for } P \times \Lambda \quad \text{a.e.} \quad (\omega, t) \in \Omega \times [0,T].$$

Hence
$$Z_{n'}^K \rightharpoonup Z_{\Psi,\Gamma}^K \quad \text{in} \quad \mathcal{L}_V^2(\Omega \times [0,T]).$$

We also see that each weakly convergent subsequence of $\left(Z_n^K\right)$ converges weakly to the same limit $Z_{\Psi,\Gamma}^K$. Therefore, the whole sequence $\left(Z_n^K\right)$ converges weakly to $Z_{\Psi,\Gamma}^K$ in $\mathcal{L}_V^2(\Omega \times [0,T])$ (see Proposition A.1).

Hence, there exists $n_\varepsilon > 0$ such that for all $n \geq n_\varepsilon$ we have

$$E\int_0^{\mathcal{T}_K} \langle \xi(s), Z_n^K(s) - Z_{\Psi,\Gamma}^K(s)\rangle ds < \frac{\varepsilon}{2}\ .$$

Using (1.40) we deduce that

$$\left|E\int_0^T \langle \xi(s), Z_n(s) - Z_{\Psi,\Gamma}(s)\rangle ds\right| < \varepsilon \quad \text{for all} \quad n \geq n_\varepsilon$$

and consequently,
$$\lim_{n\to\infty} E\int_0^T \langle \xi(s), Z_n(s) - Z_{\Psi,\Gamma}(s)\rangle ds = 0.$$

Because $\mathcal{D}_{V^*}(\Omega \times [0,T])$ is dense in $\mathcal{L}_{V^*}^2(\Omega \times [0,T])$ and $\Delta_Y Z_n, \Delta_Y Z_{\Psi,\Gamma} \in \mathcal{L}_V^2(\Omega \times [0,T])$ (we do not know whether $Z_n, Z_{\Psi,\Gamma} \in \mathcal{L}_V^2(\Omega \times [0,T])$) we conclude that for $n \to \infty$

$$\Delta_Y Z_n \rightharpoonup \Delta_Y Z_{\Psi,\Gamma} \quad \text{in} \quad \mathcal{L}_V^2(\Omega \times [0,T]). \tag{1.42}$$

We want to prove that for all $\xi \in \mathcal{D}_V(\Omega)$ we have

$$\lim_{n\to\infty} E(Z_{\Psi,\Gamma}(T) - Z_n(T), \xi) = 0. \tag{1.43}$$

Let $\xi = v\phi \in \mathcal{D}_V(\Omega)$, $\varepsilon > 0$. There exists an index $K \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ we get

$$\begin{aligned} & |E(Z_{\Psi,\Gamma}(T) - Z_n(T), v)\phi| \leq |E(Z_{\Psi,\Gamma}(T) - Z_n(T), \Pi_K v)\phi| \\ + \quad & \|v - \Pi_K v\| \left\{E\left(\phi^2 \Delta_Y^{-1}(T)\right) \sup_{1\leq n}\left[E\Delta_Y(T)\|Z_{\Psi,\Gamma}(T) - Z_n(T)\|^2\right]\right\}^{1/2} \\ \leq \quad & |E(Z_{\Psi,\Gamma}(T) - Z_n(T), \Pi_K v)\phi| + \frac{\varepsilon}{2}\ . \end{aligned} \tag{1.44}$$

In the second inequality we have used (1.37). From (P_{n,ψ_n}) and $(P_{\Psi,\Gamma})$ we conclude that

$$
\begin{aligned}
& E\phi(Z_{\Psi,\Gamma}(T) - Z_n(T), \Pi_K v) + E\phi\int_0^T \langle \mathcal{A}Z_{\Psi,\Gamma}(s) - \mathcal{A}Z_n(s), \Pi_K v\rangle ds \\
= \;& E\phi\int_0^T \langle \mathcal{B}(X(s), Z_{\Psi,\Gamma}(s)) - \mathcal{B}(X_n(s), Z_n(s)) + \mathcal{B}(Z_{\Psi,\Gamma}(s), Y(s)) - \mathcal{B}(Z_n(s), Y_n(s)), \Pi_K v\rangle ds \\
+ \;& E\phi\int_0^T \langle \Psi(s) - \mathrm{J}\psi_n(s), \Pi_K v\rangle ds + E\phi\int_0^T (\mathcal{G}(s, Z_{\Psi,\Gamma}(s) - Z_n(s)), \Pi_K v) dw(s) \\
+ \;& E\phi\int_0^T (\Gamma(s) - \gamma_n(s), \Pi_K v) dw(s).
\end{aligned}
$$

In the above equation we take the limit $n \to \infty$, use the weak convergence (1.42) and obtain that there exists an $n_\varepsilon > 0$ such that

$$|E(Z_{\Psi,\Gamma}(T) - Z_n(T), \Pi_K v)\phi| < \frac{\varepsilon}{2} \quad \text{for all } n \geq n_\varepsilon.$$

We use (1.44) to deduce that

$$|E(Z_{\Psi,\Gamma}(T) - Z_n(T), v)\phi| < \varepsilon \quad \text{for all } n \geq n_\varepsilon.$$

Hence (1.43) yields. Since $\mathcal{D}_V(\Omega)$ is dense in $\mathcal{L}^2_V(\Omega) \hookrightarrow \mathcal{L}^2_H(\Omega)$ and since we have $\Delta_Y(T)Z_n(T), \Delta_Y(T)Z_{\Psi,\Gamma}(T) \in \mathcal{L}^2_H(\Omega)$ (note, we do not know whether $Z_n(T), Z_{\Psi,\Gamma}(T) \in \mathcal{L}^2_H(\Omega)$) we conclude that

$$\Delta_Y(T)Z_n(T) \rightharpoonup \Delta_Y(T)Z_{\Psi,\Gamma}(T) \quad \text{in} \quad \mathcal{L}^2_H(\Omega). \qquad \blacksquare$$

Remark 1.3.3

Theorem 1.3.1 and Lemma 1.3.2 hold also if $\mathcal{G}$ is not a mapping satisfying hypothesis **(v)** from Section 1.1, but a stochastic process belonging to the space $\mathcal{L}^2_H(\Omega \times [0,T])$.

1.4 Linear approximation of the solution of the stochastic Navier-Stokes equation

In this section we approximate the solution of the Navier-Stokes equation by the solutions of a sequence of linear equations with additive noise and prove that the approximations $\left(u_n\right)$ converge in mean square to the solution of (1.1).

For each $n = 1, 2, 3, \ldots$ we consider the **linear** evolution equation with additive noise

$$(\hat{P}_n) \qquad (\mathbf{u_n}(t), v) + \int_0^t \langle \mathcal{A}\mathbf{u_n}(s), v\rangle ds = (x_0, v) + \int_0^t \langle \mathcal{B}(\mathbf{u_{n-1}}(s), \mathbf{u_n}(s)), v\rangle ds$$
$$+ \int_0^t (\Phi(s, \mathbf{u_{n-1}}(s)), v) ds + \int_0^t (\mathcal{C}(s, \mathbf{u_{n-1}}(s)), v) dw(s),$$

for all $v \in V$, $t \in [0, T]$, and a.e. $\omega \in \Omega$, where $\mathbf{u_0}(t) = 0$ for all $t \in [0, T]$ and a.e. $\omega \in \Omega$.

Remark 1.4.1
1) In equation $(\hat{P}_n)$, considering that u_{n-1} is known, the operators $\mathcal{A}$ and $\mathcal{B}$ depend linearly on u_n and the noise is additive with respect to u_n.
2) The approximation method given in this section holds also, if the sequence of approximations (u_n) starts with $u_0(t) := x_0$ for all $t \in [0, T]$ and a.e. $\omega \in \Omega$.

Theorem 1.4.2
For each $n \in \mathbb{N}$ equation $(\hat{P}_n)$ has an almost surely unique solution $u_n \in \mathcal{L}^2_V(\Omega \times [0, T])$ with almost surely continuous trajectories in H.

Proof. We prove the statement by induction. We apply successively Theorem 1.3.1 on $Z_{\Psi,\Gamma} := u_n$, $a_0 := x_0$, $X := u_{n-1}, Y := 0, a_0 := x_0$, $\Psi(s) := \Phi(s, u_{n-1}(s))$, $\Gamma := 0$, $\mathcal{G}(s) := \mathcal{C}(s, u_{n-1}(s))$ (for $\mathcal{G}$ we also take into account Remark 1.3.3). ■

Now we establish some properties for the solutions of the equations $(\hat{P}_n)$, $n \in \mathbb{N}$.

Lemma 1.4.3
There exists a positive constant c_1 (depending only on λ, μ, ν, and T) such that each of the expressions

$$\sup_{t\in[0,T]} E\|u_n(t)\|^4, \quad E\Big(\int_0^T \|u_n(s)\|_V^2 ds\Big)^2$$

$(n = 1, 2, \ldots)$ is less than or equal to $c_1 E\|x_0\|^4$.

Proof. Let $n \in \mathbb{N}$. We define $\tilde{z}(t) = e^{-(9\lambda + 5\sqrt{\mu})t}$, $t \in [0, T]$. Using the Ito formula we have

$$\tilde{z}(t)\|u_n(t)\|^2 + 2\int_0^t \tilde{z}(s)\langle \mathcal{A}u_n(s), u_n(s)\rangle ds = \|x_0\|^2 + 2\int_0^t \tilde{z}(s)(\Phi(s, u_{n-1}(s)), u_n(s)) ds$$
$$+ \int_0^t \tilde{z}(s)\|\mathcal{C}(s, u_{n-1}(s))\|^2 ds - (9\lambda + 5\sqrt{\mu})\int_0^t \tilde{z}(s)\|u_n(s)\|^2 ds$$
$$+ 2\int_0^t \tilde{z}(s)(\mathcal{C}(s, u_{n-1}(s)), u_n(s)) dw(s)$$

and

$$\begin{aligned}
\tilde{z}(t)\|u_n(t)\|^4 &+ 4\int_0^t \tilde{z}(s)\langle \mathcal{A}u_n(s), u_n(s)\rangle \|u_n(s)\|^2 ds \\
&= \|x_0\|^4 + 2\int_0^t \tilde{z}(s)\|\mathcal{C}(s, u_{n-1}(s))\|^2\|u_n(s)\|^2 ds - (9\lambda + 5\sqrt{\mu})\int_0^t \tilde{z}(s)\|u_n(s)\|^4 ds \\
&+ 4\int_0^t \tilde{z}(s)(\Phi(s, u_{n-1}(s)), u_n(s))\|u_n(s)\|^2 ds + 4\int_0^t \tilde{z}(s)(\mathcal{C}(s, u_{n-1}(s)), u_n(s))^2 ds \\
&+ 4\int_0^t \tilde{z}(s)(\mathcal{C}(s, u_{n-1}(s)), u_n(s))\|u_n(s)\|^2 dw(s).
\end{aligned}$$

Using the properties of $\mathcal{A}$, Φ, and $\mathcal{C}$ and some elementary calculations, we obtain

$$\begin{aligned}
2\nu\int_0^t \tilde{z}(s)\|u_n(s)\|_V^2 ds &\leq \|x_0\|^2 + (\lambda + \sqrt{\mu})\int_0^t \tilde{z}(s)\|u_{n-1}(s)\|^2 ds \\
&+ 2\int_0^t \tilde{z}(s)(\mathcal{C}(s, u_{n-1}(s)), u_n(s)) dw(s)
\end{aligned} \tag{1.45}$$

and

$$\begin{aligned}
\tilde{z}(t)\|u_n(t)\|^4 &+ 4\nu\int_0^t \tilde{z}(s)\|u_n(s)\|_V^2\|u_n(s)\|^2 ds + 2(3\lambda + \sqrt{\mu})\int_0^t \tilde{z}(s)\|u_n(s)\|^4 ds \\
&\leq \|x_0\|^4 + (3\lambda + \sqrt{\mu})\int_0^t \tilde{z}(s)\|u_{n-1}(s)\|^4 ds \\
&+ 4\int_0^t \tilde{z}(s)(\mathcal{C}(s, u_{n-1}(s)), u_n(s))\|u_n(s)\|^2 dw(s).
\end{aligned} \tag{1.46}$$

Using (1.46) and the ideas from the proof of Proposition B.2 we get

$$\begin{aligned}
&E\tilde{z}(t)\|u_n(t)\|^4 + 2(3\lambda + \sqrt{\mu})E\int_0^t \tilde{z}(s)\|u_n(s)\|^4 ds \\
&\qquad\leq E\|x_0\|^4 + (3\lambda + \sqrt{\mu})E\int_0^t \tilde{z}(s)\|u_{n-1}(s)\|^4 ds.
\end{aligned} \tag{1.47}$$

By successive application of (1.47), we obtain

$$E\tilde{z}(t)\|u_n(t)\|^4 + 2(3\lambda + \sqrt{\mu})E\int_0^t \tilde{z}(s)\|u_n(s)\|^4 ds \leq \Big(1 + \frac{1}{2} + \ldots + \frac{1}{2^{n-1}}\Big)E\|x_0\|^4.$$

Consequently, there exists a constant $c_2 > 0$ (independent of n) such that

$$\sup_{t\in[0,T]} E\|u_n(t)\|^4 + 2(3\lambda + \sqrt{\mu})E\int_0^T \|u_n(s)\|^4 ds \le c_2 E\|x_0\|^4. \tag{1.48}$$

In (1.45) we square both sides of the inequality. Then we use the properties of the stochastic integral and those of the Lebesgue integral to obtain

$$\begin{aligned} 4\nu^2 E\Big(\int_0^T \tilde{z}(s)\|u_n(s)\|_V^2 ds\Big)^2 &\le 3E\|x_0\|^4 + 3(\lambda+\sqrt{\mu})^2 E\Big(\int_0^T \tilde{z}(s)\|u_{n-1}(s)\|^2 ds\Big)^2 \\ + \quad 12E\Big|\int_0^T \tilde{z}(s)(\mathcal{C}(s,u_{n-1}(s)),u_n(s))dw(s)\Big|^2 &\le 3E\|x_0\|^4 + c_3 E\int_0^T \tilde{z}^2(s)\|u_{n-1}(s)\|^4 ds \\ + \quad E\int_0^T \tilde{z}^2(s)\|u_n(s)\|^4 ds, & \end{aligned}$$

where c_3 is a positive constant depending on λ, μ, and T. Taking into account (1.48) and the properties of $\tilde{z}$, it follows that there exists a constant c_4 depending on (λ, μ, ν and T) such that

$$E\Big(\int_0^T \|u_n(s)\|_V^2 ds\Big)^2 \le c_4 E\|x_0\|^4. \qquad \blacksquare$$

We define

$$\tilde{e}(t) = \Delta_U(t)\exp\{-(\lambda+\sqrt{\mu})t\}$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$ and introduce the following notations:

$$s_N(t) = \sum_{n=1}^N \tilde{e}(t)\|u_n(t) - U(t)\|^2, \tag{1.49}$$

$$S_N(t) = \sum_{n=1}^N \tilde{e}(t)\|u_n(t) - U(t)\|_V^2, \tag{1.50}$$

where N is a natural number, $t \in [0,T]$, $\omega \in \Omega$.

Lemma 1.4.4
The following convergences hold:

$$\lim_{n\to\infty} E\int_0^T \tilde{e}(s)\|u_n(s) - U(s)\|_V^2 ds = 0$$

and

$$\lim_{n\to\infty} E\tilde{e}(t)\|u_n(t) - U(t)\|^2 = 0 \quad \textit{for all} \quad t \in [0,T].$$

PROOF. Let $n \in \mathbb{N}$. Using (1.1), $(\hat{P}_n)$ and the Ito formula we obtain

$$\tilde{e}(t)\|u_n(t) - U(t)\|^2 + 2\int_0^t \tilde{e}(s)\langle \mathcal{A}u_n(s) - \mathcal{A}U(s), u_n(s) - U(s)\rangle ds \tag{1.51}$$

$$\begin{aligned}
= \quad & 2\int_0^t \tilde{e}(s)\langle \mathcal{B}(u_{n-1}(s), u_n(s)) - \mathcal{B}(U(s), U(s)), u_n(s) - U(s)\rangle ds \\
- \quad & \frac{b}{\nu}\int_0^t \tilde{e}(s)\|U(s)\|_V^2\|u_n(s) - U(s)\|^2 ds - (\lambda + \sqrt{\mu})\int_0^t \tilde{e}(s)\|u_n(s) - U(s)\|^2 ds \\
+ \quad & 2\int_0^t \tilde{e}(s)(\Phi(s, u_{n-1}(s)) - \Phi(s, U(s)), u_n(s) - U(s)) ds \\
+ \quad & \int_0^t \tilde{e}(s)\|\mathcal{C}(s, u_{n-1}(s)) - \mathcal{C}(s, U(s))\|^2 ds \\
+ \quad & 2\int_0^t \tilde{e}(s)(\mathcal{C}(s, u_{n-1}(s)) - \mathcal{C}(s, U(s)), u_n(s) - U(s)) dw(s),
\end{aligned}$$

for all $t \in [0, T]$ and a.e. $\omega \in \Omega$. From the properties of $\mathcal{B}$ we can derive the following estimate:

$$\begin{aligned}
& 2\langle \mathcal{B}(u_{n-1}(s), u_n(s)) - \mathcal{B}(U(s), U(s)), u_n(s) - U(s)\rangle \\
= \quad & -2\langle \mathcal{B}(u_{n-1}(s) - U(s), u_n(s) - U(s)), U(s)\rangle \\
\leq \quad & 2\sqrt{b}\|U(s)\|_V\|u_{n-1}(s) - U(s)\|_V^{\frac{1}{2}}\|u_{n-1}(s) - U(s)\|^{\frac{1}{2}}\|u_n(s) - U(s)\|_V^{\frac{1}{2}}\|u_n(s) - U(s)\|^{\frac{1}{2}} \\
\leq \quad & \frac{\nu}{2}\|u_{n-1}(s) - U(s)\|_V^2 + \frac{\nu}{2}\|u_n(s) - U(s)\|_V^2 \\
+ \quad & \frac{b}{2\nu}\|U(s)\|_V^2\|u_{n-1}(s) - U(s)\|^2 + \frac{b}{2\nu}\|U(s)\|_V^2\|u_n(s) - U(s)\|^2
\end{aligned}$$

for all $s \in [0, T]$ and a.e. $\omega \in \Omega$. Using this estimation and the continuity of $\mathcal{C}$ in (1.51), we obtain

$$\begin{aligned}
& \tilde{e}(t)\|u_n(t) - U(t)\|^2 + \frac{3\nu}{2}\int_0^t \tilde{e}(s)\|u_n(s) - U(s)\|_V^2 ds + (\lambda + \sqrt{\mu})\int_0^t \tilde{e}(s)\|u_n(s) - U(s)\|^2 ds \\
\leq \quad & \frac{\nu}{2}\int_0^t \tilde{e}(s)\|u_{n-1}(s) - U(s)\|_V^2 ds + 2\int_0^t \tilde{e}(s)(\mathcal{C}(s, u_{n-1}(s)) - \mathcal{C}(s, U(s)), u_n(s) - U(s)) dw(s) \\
+ \quad & \frac{b}{2\nu}\int_0^t \tilde{e}(s)\|U(s)\|_V^2\Big(\|u_{n-1}(s) - U(s)\|^2 - \|u_n(s) - U(s)\|^2\Big) ds
\end{aligned}$$

$$+ \quad (\lambda+\sqrt{\mu})\int_0^t \tilde{e}(s)\|u_{n-1}(s)-U(s)\|^2 ds$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$.

Summing up these relations for $n=1$ to an arbitrary natural number N, we get

$$\begin{aligned} & s_N(t)+\nu\int_0^t S_N(s)ds+\frac{b}{2\nu}\int_0^t \tilde{e}(s)\|U(s)\|_V^2\|u_N(s)-U(s)\|^2 ds \\ + \quad & (\lambda+\sqrt{\mu})\int_0^t \tilde{e}(s)\|u_N(s)-U(s)\|^2 ds \le \frac{\nu}{2}\int_0^t \tilde{e}(s)\|u_0(s)-U(s)\|_V^2 ds \\ + \quad & (\lambda+\sqrt{\mu})\int_0^t \tilde{e}(s)\|u_0(s)-U(s)\|^2 ds+\frac{b}{2\nu}\int_0^t \tilde{e}(s)\|U(s)\|_V^2\|u_0(s)-U(s)\|^2 ds \\ + \quad & 2\sum_{n=1}^N\int_0^t \tilde{e}(s)(\mathcal{C}(s,u_{n-1}(s))-\mathcal{C}(s,U(s)),u_n(s)-U(s))dw(s) \end{aligned}$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$, where s_N and S_N are defined in (1.49) and (1.50). Taking the mathematical expectation we have

$$\begin{aligned} & Es_N(t)+\nu E\int_0^t S_N(s)ds \le \frac{\nu}{2}E\int_0^t \tilde{e}(s)\|u_0(s)-U(s)\|_V^2 ds \\ + \quad & (\lambda+\sqrt{\mu})E\int_0^t \tilde{e}(s)\|u_0(s)-U(s)\|^2 ds+\frac{b}{2\nu}E\int_0^t \tilde{e}(s)\|U(s)\|_V^2\|u_0(s)-U(s)\|^2 ds \end{aligned}$$

for all $t \in [0,T]$. But $0<\tilde{e}(s)\le 1$ and $u_0(s)=0$ for all $s \in [0,T]$, a.e. $\omega \in \Omega$. Applying Lemma 1.2.6 and Lemma 1.4.3 it follows that there exists a positive constant c, which does not depend on N, such that

$$Es_N(t)+\nu E\int_0^t S_N(s)ds \le E\int_0^T \Big(\frac{\nu}{2}\|U(s)\|_V^2+(\lambda+\sqrt{\mu})\|U(s)\|^2+\frac{b}{2\nu}\|U(s)\|_V^2\|U(s)\|^2\Big)ds \le c$$

for all $t \in [0,T]$ and all natural numbers N. Consequently, for all $t \in [0,T]$ we have

$$\sum_{n=1}^{\infty} E\tilde{e}(t)\|u_n(t)-U(t)\|^2+\nu\sum_{n=1}^{\infty} E\int_0^t \tilde{e}(s)\|u_n(s)-U(s)\|_V^2 ds \le c$$

Hence

$$\lim_{n\to\infty} E\int_0^T \tilde{e}(s)\|u_n(s)-U(s)\|_V^2 ds = 0$$

and for all $t \in [0,T]$ we have

$$\lim_{n\to\infty} E\tilde{e}(t)\|u_n(t) - U(t)\|^2 = 0. \qquad \blacksquare$$

The **main result** of this section is the following theorem.

Theorem 1.4.5
The following convergences hold:

$$\lim_{n\to\infty} E\int_0^T \|u_n(s) - U(s)\|_V^2 ds = 0$$

and

$$\lim_{n\to\infty} E\|u_n(t) - U(t)\|^2 = 0 \quad \textit{for all} \quad t \in [0,T].$$

Proof. We take $\mathcal{T}_M := \mathcal{T}_M^U$. From Lemma 1.4.4 it follows that for each fixed natural number M we have

$$\lim_{n\to\infty} E\int_0^{\mathcal{T}_M} \|u_n(s) - U(s)\|_V^2 ds = 0 \quad \text{and} \quad \lim_{n\to\infty} E\|u_n(\mathcal{T}_M) - U(\mathcal{T}_M)\|^2 = 0.$$

First we apply Proposition B.3 for $\mathcal{T} := T$, $Q_n(\mathcal{T}) := \int_0^{\mathcal{T}} \|u_n(s) - U(s)\|^2 ds$, use Lemma 1.4.3 and Lemma 1.2.6 to obtain

$$\lim_{n\to\infty} E\int_0^T \|u_n(s) - U(s)\|_V^2 ds = 0.$$

Let $t \in [0,T]$. Now we apply Proposition B.3 for $\mathcal{T} := t$, $Q_n(\mathcal{T}) := \|u_n(\mathcal{T}) - U(\mathcal{T})\|$, use Lemma 1.4.3 and Lemma 1.2.6 to get

$$\lim_{n\to\infty} E\|u_n(t) - U(t)\|^2 = 0. \qquad \blacksquare$$

Chapter 2

Optimal Control

We consider the stochastic Navier-Stokes equation controlled by linear and continuous feedback controls, respectively, by bounded controls (which are not feedback controls). Since the considered equation is nonlinear, we are dealing with a nonconvex optimization problem. Our purpose is to prove in Section 2.2 , Section 2.3 and Section 2.4 the existence of optimal and ε-optimal controls. In Section 2.5 we investigate a special property for the solution of the stochastic Navier-Stokes equation, which we assume to be fulfilled in the following sections. In Section 2.6 we calculate the Gateaux derivative of the cost functional and in Section 2.7 we formulate a stochastic minimum principle. We complete the statement of the stochastic minimum principle by giving in Section 2.8 the equations for the adjoint processes. In the last three sections we use and adapt the ideas from A. Bensoussan [3] for the case of the stochastic Navier-Stokes equation.

2.1 Formulation of the control problem

First we consider the stochastic Navier-Stokes equation controlled by **linear and continuous feedback controls**. In this case we denote by $\mathcal{U}$ the set of all admissible controls, which we assume to be the set of all functions $\Phi : [0,T] \times H \to H$ satisfying the following conditions: for each $t \in [0,T]$ we have $\Phi(t,\cdot) \in \mathcal{L}(H)$ and

$$\|\Phi(t_1,x_1) - \Phi(t_2,x_2)\|^2 \leq \alpha|t_1 - t_2|^2 + \mu\|x_1 - x_2\|^2 \quad \text{for all } t_1,t_2 \in [0,T], x_1,x_2 \in H$$

where $\alpha, \mu > 0$ are given constants.

Our purpose is to control the solution U_Φ of the Navier-Stokes equation

$$(2.1) \qquad \begin{aligned}(U_\Phi(t),v)+\int_0^t\langle\mathcal{A}U_\Phi(s),v\rangle ds &= (x_0,v)+\int_0^t\langle\mathcal{B}(U_\Phi(s),U_\Phi(s)),v\rangle ds\\ &+ \int_0^t(\Phi(s,U_\Phi(s)),v)ds+\int_0^t(\mathcal{C}(s,U_\Phi(s)),v)dw(s)\end{aligned}$$

for all $v \in V$, $t \in [0,T]$, a.e. $\omega \in \Omega$, by the feedback controls $\Phi \in \mathcal{U}$. We consider the following **cost functional**

$$(2.2) \qquad \mathcal{J}(\Phi) = E\int_0^T \mathcal{L}[s,U_\Phi(s),\Phi(s,U_\Phi(s))]ds + E\mathcal{K}[U_\Phi(T)], \qquad \Phi \in \mathcal{U},$$

where $\mathcal{L} : [0,T]\times H\times H \to \mathbb{R}_+$, $\mathcal{K} : H \to \mathbb{R}_+$ are mappings satisfying the conditions:

$(\mathbf{H_1})$
$$|\mathcal{L}(t,x_1,y_1)-\mathcal{L}(t,x_2,y_2)| \le c_{\mathcal{L}}\Big(\|x_1-x_2\|^2+\|y_1-y_2\|^2\Big),$$
$$|\mathcal{K}(x_1)-\mathcal{K}(x_2)| \le c_{\mathcal{K}}\|x_1-x_2\|^2$$

for all $t \in [0,T]$, $x_1,x_2,y_1,y_2 \in H$, where $c_{\mathcal{L}}, c_{\mathcal{K}}$ are positive constants;

$(\mathbf{H_2})$ for all $x,y \in H$ we assume that $\mathcal{L}(\cdot,x,y) \in \mathcal{L}^2_H[0,T]$.

We denote by $(\mathcal{P})$ the problem of minimizing $\mathcal{J}$ among the admissible controls.

Now we consider the stochastic Navier-Stokes equation controlled by **bounded controls**, which are not feedback controls. In this case we denote by $\mathcal{U}^b$ the set of all admissible controls, which we assume to be the set of all functions $\Phi \in \mathcal{L}^2_H(\Omega\times[0,T])$ satisfying the condition:

$$\|\Phi(\omega,t)\| \le \rho \quad \text{for } P\times\Lambda \text{ a.e. } (\omega,t)\in\Omega\times[0,T],$$

where $\rho > 0$ is a given constant.

In this case the stochastic Navier-Stokes equation has the form

$$(2.3) \qquad \begin{aligned}(U_\Phi(t),v)+\int_0^t\langle\mathcal{A}U_\Phi(s),v\rangle ds &= (x_0,v)+\int_0^t\langle\mathcal{B}(U_\Phi(s),U_\Phi(s)),v\rangle ds\\ &+ \int_0^t(\Phi(s),v)ds+\int_0^t(\mathcal{C}(s,U_\Phi(s)),v)dw(s)\end{aligned}$$

for all $v \in V$, $t \in [0,T]$, a.e. $\omega \in \Omega$, where $\Phi \in \mathcal{U}^b$. The cost functional is in this case given by

$$(2.4) \qquad \mathcal{J}(\Phi) = E\int_0^T \mathcal{L}[s,U_\Phi(s),\Phi(s)]ds + E\mathcal{K}[U_\Phi(T)], \qquad \Phi \in \mathcal{U}^b,$$

where $\mathcal{L}$ and $\mathcal{K}$ satisfy $(\mathbf{H_1})$ and $(\mathbf{H_2})$. We denote by $(\mathcal{P}^b)$ the problem of minimizing $\mathcal{J}$, given in equation (2.4), among the admissible controls of the set $\mathcal{U}^b$.

Remark 2.1.1
In their paper [1] F. Abergel and R. Temam investigate the deterministic Navier-Stokes equation by controlling turbulence inside the flow. They give a cost functional which involves the vorticity in the fluid. For our problem $(\mathcal{P}^b)$ this would be $\mathcal{L}(t, U_\Phi(t), \Phi(t)) := \|\nabla \times U_\Phi(t)\|^2 + \|\Phi(t)\|^2,\ \mathcal{K} := 0$.

2.2 Existence of optimal controls

First we prove some properties of the cost functional $\mathcal{J}$.

Lemma 2.2.1

(i) *Let (Φ_n) be a sequence in $\mathcal{U}$ and let $\Phi \in \mathcal{U}$ be such that*

$$\lim_{n\to\infty} \int_0^T \|\Phi_n(t,\cdot) - \Phi(t,\cdot)\|^2_{\mathcal{L}(H)} dt = 0.$$

Then $\lim_{n\to\infty} \mathcal{J}(\Phi_n) = \mathcal{J}(\Phi)$.

(ii) *Let (Φ_n) be a sequence in $\mathcal{U}^b$ and let $\Phi \in \mathcal{U}^b$ be such that*

$$\lim_{n\to\infty} \int_0^T \|\Phi_n(t) - \Phi(t)\|^2 dt = 0.$$

Then $\lim_{n\to\infty} \mathcal{J}(\Phi_n) = \mathcal{J}(\Phi)$.

Proof. **(i)** Let $U := U_\Phi$ and $e(t) = \Delta_U(t)\exp\{-(\lambda + 2\sqrt{\mu} + 1)t\}$. It follows by the Ito formula that

$$\begin{aligned}
& e(t)\|U(t) - U_{\Phi_n}(t)\|^2 + 2\int_0^t e(s)\langle \mathcal{A}U(s) - \mathcal{A}U_{\Phi_n}(s), U(s) - U_{\Phi_n}(s)\rangle ds \\
= \;& 2\int_0^t e(s)\langle \mathcal{B}(U(s),U(s)) - \mathcal{B}(U_{\Phi_n}(s), U_{\Phi_n}(s)), U(s) - U_{\Phi_n}(s)\rangle ds \\
- \;& \frac{b}{\nu}\int_0^t e(s)\|U(s)\|_V^2 \|U(s) - U_{\Phi_n}(s)\|^2 ds - (\lambda + 2\sqrt{\mu} + 1)\int_0^t e(s)\|U(s) - U_{\Phi_n}(s)\|^2 ds \\
+ \;& 2\int_0^t e(s)(\Phi(s,U(s)) - \Phi_n(s, U_{\Phi_n}(s)), U(s) - U_{\Phi_n}(s)) ds \\
+ \;& \int_0^t e(s)\|\mathcal{C}(s,U(s)) - \mathcal{C}(s, U_{\Phi_n}(s))\|^2 ds \\
+ \;& 2\int_0^t e(s)(\mathcal{C}(s,U(s)) - \mathcal{C}(s, U_{\Phi_n}(s)), U(s) - U_{\Phi_n}(s)) dw(s).
\end{aligned}$$

In view of the properties of $\mathcal{B}$ we can write

$$\begin{aligned} & 2\langle \mathcal{B}(U(s),U(s)) - \mathcal{B}(U_{\Phi_n}(s),U_{\Phi_n}(s)), U(s) - U_{\Phi_n}(s)\rangle \\ & \qquad = 2\langle \mathcal{B}(U(s) - U_{\Phi_n}(s), U(s)), U(s) - U_{\Phi_n}(s)\rangle \\ & \qquad \leq \frac{b}{\nu}\|U(s)\|_V^2\|U(s) - U_{\Phi_n}(s)\|^2 + \nu\|U(s) - U_{\Phi_n}(s)\|_V^2. \end{aligned}$$

Now we use the properties of $\mathcal{A}$, Φ, $\mathcal{C}$, and those of the stochastic integral to obtain

$$\begin{aligned} & E\sup_{s\in[0,t]} e(s)\|U(s) - U_{\Phi_n}(s)\|^2 + \nu E\int_0^t e(s)\|U(s) - U_{\Phi_n}(s)\|_V^2 ds \\ \leq\ & 2E\int_0^t e(s)\|\Phi(s,U(s)) - \Phi_n(s,U(s))\|^2 ds \\ +\ & 4E\sup_{s\in[0,t]}\Big|\int_0^s e(r)(\mathcal{C}(r,U(r)) - \mathcal{C}(r,U_{\Phi_n}(r)), U(r) - U_{\Phi_n}(r))dw(r)\Big| \\ \leq\ & 2E\int_0^t e(s)\|\Phi(s,U(s)) - \Phi_n(s,U(s))\|^2 ds + k_1 E\int_0^t \sup_{r\in[0,s]}\Big\{e(r)\|U(r) - U_{\Phi_n}(r)\|^2\Big\}ds \\ +\ & \frac{1}{2}E\sup_{s\in[0,t]} e(s)\|U(s) - U_{\Phi_n}(s))\|^2, \end{aligned}$$

where k_1 is a positive constant and $t \in [0,T]$. By Gronwall's Lemma we get

$$\begin{aligned} E\sup_{s\in[0,t]} e(s)\|U(s) - U_{\Phi_n}(s)\|^2 &+ 2\nu E\int_0^t e(s)\|U(s) - U_{\Phi_n}(s)\|_V^2 ds \\ &\leq 4e^{2k_1 T}E\int_0^T \|\Phi(s,U(s)) - \Phi_n(s,U(s))\|^2 ds, \end{aligned}$$

for all $t \in [0,T]$.

We take $t := \mathcal{T}_M^U$. Using the hypothesis and the above inequality it follows that for each fixed $M \in \mathbb{N}$ we have

$$\lim_{n\to\infty} E\|U(\mathcal{T}_M^U) - U_{\Phi_n}(\mathcal{T}_M^U)\|^2 = 0, \qquad \lim_{n\to\infty} E\int_0^{\mathcal{T}_M^U} \|U(s) - U_{\Phi_n}(s)\|_V^2 ds = 0.$$

Applying Proposition B.3 for $\mathcal{T} := T$, $\mathcal{T}_M := \mathcal{T}_M^U$, $Q_n(\mathcal{T}) := \|U(\mathcal{T}) - U_{\Phi_n}(\mathcal{T})\|^2$, respectively, $Q_n(\mathcal{T}) := \int\limits_0^{\mathcal{T}} \|U(s) - U_{\Phi_n}(s)\|_V^2 ds$, we obtain

$$\lim_{n\to\infty} E\|U(T) - U_{\Phi_n}(T)\|^2 = 0, \qquad \lim_{n\to\infty} E\int\limits_0^T \|U(s) - U_{\Phi_n}(s)\|_V^2 ds = 0.$$

The continuity properties of $\mathcal{L}$ and $\mathcal{K}$ imply $\lim\limits_{n\to\infty} \mathcal{J}(\Phi_n) = \mathcal{J}(\Phi)$.
(ii) We use the same method as in **(i)**. ■

Using Lemma 2.2.1 and the generalized *Weierstraß Theorem* we obtain the following theorem.

Theorem 2.2.2

(i) *If the admissible controls are in a compact subset of $\mathcal{U}$, then there exist optimal feedback controls for the minimization problem $(\mathcal{P})$.*

(ii) *If the admissible controls are in a compact subset of $\mathcal{U}^b$, then there exist optimal controls for the minimization problem $(\mathcal{P}^b)$.*

2.3 The existence of optimal feedback controls

It is difficult to guarantee the compactness of the set of admissible controls and therefore it is useful to derive the existence of optimal controls using other methods. In this section we prove the existence of optimal feedback controls.

Let (Φ_n) be a sequence in $\mathcal{U}$.

Lemma 2.3.1
There exist a subsequence (n') of (n) and a mapping $\Phi \in \mathcal{U}$ such that for all $t \in [0, T]$, $x, y \in H$ we have

$$\lim_{n'\to\infty} (\Phi_{n'}(t, x), y) = (\Phi(t, x), y). \tag{2.5}$$

Proof. Let $\{t_1, t_2, \ldots\}$ be a dense subset of $[0, T]$ and recall that $\{h_1, h_2, \ldots\}$ is an orthonormal base in H. The sequence $\Big(\Phi_n(t_1, h_1)\Big)_{n\in\mathbb{N}}$ is a bounded sequence in H. Hence there exists a subsequence $(n_k^{1,1})$ of (n) and an element $z_1^1 \in H$ such that for all $y \in H$ we have

$$\lim_{k\to\infty} (\Phi_{n_k^{1,1}}(t_1, h_1), y) = (z_1^1, y).$$

The sequence $\Big(\Phi_n(t_1, h_2)\Big)_{n\in\mathbb{N}}$ is a bounded sequence in H. Hence there exists a subsequence $(n_k^{1,2})$ of $(n_k^{1,1})$ and an element $z_2^1 \in H$ such that for all $y \in H$ we have

$$\lim_{k\to\infty} (\Phi_{n_k^{1,2}}(t_1, h_2), y) = (z_2^1, y).$$

This procedure we repeat for all $h_1, h_2, \ldots$ and then we take the "diagonal sequence" $(n_k^{1,k})_{k\in\mathbb{N}}$, which has the property

$$\lim_{k\to\infty}(\Phi_{n_k^{1,k}}(t_1,h_i),y) = (z_i^1,y)$$

for all $y \in H$ and $i \in \mathbb{N}$. Of course, the subsequence $(n_k^{1,k})$ depends on t_1. Now we repeat the procedure from above for $t_2, t_3, \ldots$ and take again the "diagonal sequence" $(n_k^{k,k})_{k\in\mathbb{N}}$, which we denote by (n'). For all $y \in H$ and $i, j \in \mathbb{N}$, we have

$$\lim_{n'\to\infty}(\Phi_{n'}(t_j,h_i),y) = (z_i^j,y), \tag{2.6}$$

where $z_i^j \in H$.

We want to prove that for each fixed $i, j \in \mathbb{N},\ x \in H$ the sequence $\Big(\Phi_{n'}(t_j,x),h_i)\Big)_{n'\in\mathbb{N}}$ is convergent.

Let $\varepsilon > 0$. There exists $p_\varepsilon \in \mathbb{N}$ such that

$$\|x - x_{p_\varepsilon}\|^2 < \frac{\varepsilon}{3\sqrt{\mu}},$$

where $x_{p_\varepsilon} = \Pi_{p_\varepsilon}x$. Equation (2.6) implies that there exists $n'_0 \in \mathbb{N}$ such that for all $n', m' \geq n'_0$

$$\Big|(\Phi_{n'}(t_j,x_{p_\varepsilon}),h_i) - (\Phi_{m'}(t_j,x_{p_\varepsilon}),h_i)\Big| < \frac{\varepsilon}{3}\ .$$

For all $n', m' \geq n'_0$ we have

$$\begin{aligned}\Big|(\Phi_{n'}(t_j,x),h_i) - (\Phi_{m'}(t_j,x),h_i)\Big| \quad &< \quad \Big|(\Phi_{n'}(t_j,x-x_{p_\varepsilon}),h_i)\Big| + \Big|(\Phi_{m'}(t_j,x-x_{p_\varepsilon}),h_i)\Big| \\ &+ \quad \Big|(\Phi_{n'}(t_j,x_{p_\varepsilon}),h_i) - (\Phi_{m'}(t_j,x_{p_\varepsilon}),h_i)\Big| < \varepsilon.\end{aligned}$$

Hence $\Big(\Phi_{n'}(t_j,x),h_i)\Big)_{n'\in\mathbb{N}}$ is a Cauchy sequence, and we can define the function $f_{i,j} : H \to \mathbb{R}$ by

$$f_{i,j}(x) = \lim_{n'\to\infty}(\Phi_{n'}(t_j,x),h_i).$$

Obviously, $f_{i,j}$ is linear. Let $p \in \mathbb{N}$. We have

$$\sum_{i=1}^{p} f_{i,j}^2(x) = \lim_{n'\to\infty}\sum_{i=1}^{p}(\Phi_{n'}(t_j,x),h_i)^2 \leq \limsup_{n'\to\infty}\|\Phi_{n'}(t_j,x)\|^2 \leq \mu\|x\|^2 < \infty \tag{2.7}$$

and

$$\begin{aligned}\sum_{i=1}^{p}|f_{i,j}(x) - f_{i,k}(x)|^2 \quad &= \quad \lim_{n'\to\infty}\sum_{i=1}^{p}(\Phi_{n'}(t_j,x) - \Phi_{n'}(t_k,x),h_i)^2 \\ &\leq \quad \limsup_{n'\to\infty}\|\Phi_{n'}(t_j,x) - \Phi_{n'}(t_k,x)\|^2 \leq \alpha|t_j - t_k|^2 < \infty\end{aligned} \tag{2.8}$$

for all $j, k \in \mathbb{N},\ x \in H$.

We define $\Phi : \{t_1, t_2, \ldots\} \times H \to H$ as follows:

$$\Phi(t_j, x) = \sum_{i=1}^{\infty} f_{i,j}(x) h_i, \quad j \in \mathbb{N},\ x \in H.$$

By (2.7) and (2.8) we see that for each $j \in \mathbb{N}$ the mapping $\Phi(t_j, \cdot)$ is linear and continuous with

$$\|\Phi(t_j, x)\|^2 = \sum_{i=1}^{\infty} f_{i,j}^2(x) \leq \mu \|x\|^2 \quad \text{for all } x \in H,$$

as soon as for each $j, k \in \mathbb{N}$ we have

$$\|\Phi(t_j, x) - \Phi(t_k, x)\|^2 = \sum_{i=1}^{\infty} |f_{i,j}(x) - f_{i,k}(x)|^2 \leq \alpha |t_j - t_k|^2 \quad \text{for all } x \in H. \tag{2.9}$$

Now we define $\Phi : [0, T] \times H \to H$. Let $t \in [0, T]$. There exists a sequence $(\tilde{t}_n)$ in $\{t_1, t_2, \ldots\}$ such that $\lim\limits_{n\to\infty} \tilde{t}_n = t$, and we put

$$\Phi(t, x) = \lim_{n\to\infty} \Phi(\tilde{t}_n, x).$$

Using (2.9) it can be proved that this definition is independent of the choice of $(\tilde{t}_n)$. Obviously, we have

$$\|\Phi(t_1, x) - \Phi(t_2, x)\|^2 \leq \alpha |t_1 - t_2|^2$$

for all $t_1, t_2 \in [0, T]$ and all $x \in H$. Consequently, $\Phi \in \mathcal{U}$ and by the construction of Φ we deduce that it satisfies (2.5). ■

For convenience, in the following we will denote the subsequence of indices (n') obtained in Lemma 2.3.1 by (n).

For $n = 1, 2, \ldots$ we consider the evolution equation

$$\begin{aligned}
(\tilde{\mathbf{U}}_{\mathbf{\Phi_n}}(t), v) + \int_0^t \langle \mathcal{A} \tilde{\mathbf{U}}_{\mathbf{\Phi_n}}(s), v\rangle ds &= (x_0, v) + \int_0^t \langle \mathcal{B}(U_\Phi(s), \tilde{\mathbf{U}}_{\mathbf{\Phi_n}}(s)), v\rangle ds \\
&+ \int_0^t (\Phi_n(s, U_\Phi(s)), v) ds + \int_0^t (\mathcal{C}(s, U_\Phi(s)), v) dw(s)
\end{aligned} \tag{E_{Φ_n}}$$

for all $t \in [0, T], v \in V$ and a.e. $\omega \in \Omega$. By Theorem 1.3.1, applied for $Z_{\Psi,\Gamma} := \tilde{U}_{\Phi_n}, X := U_\Phi$, $Y := 0$, $a_0 := x_0$, $\Psi(s) := \Phi_n(s, U_\Phi(s))$, $\mathcal{G} := 0$, $\Gamma(s) := \mathcal{C}(s, U_\Phi(s))$ it follows that there exists an almost surely unique solution $\tilde{U}_{\Phi_n} \in \mathcal{L}_V^2(\Omega \times [0, T])$ of (E_{Φ_n}), which has almost surely continuous trajectories in H and

$$E \sup_{t\in[0,T]} \|\tilde{U}_{\Phi_n}(t)\|^4 + E\Big(\int_0^T \|\tilde{U}_{\Phi_n}(s)\|_V^2 ds\Big)^2 \leq \tilde{c}_1 \Big[E\|x_0\|^4 + E\int_0^T \|U_\Phi(s)\|^4 ds\Big],$$

where $\tilde{c}_1 > 0$ is a constant (independent of n).

For $n = 1, 2, \ldots$ we consider the n-dimensional evolution equation

$$(E_{n,\Phi_n}) \quad (\tilde{\mathbf{U}}_{\mathbf{n},\mathbf{\Phi_n}}(t), v) + \int_0^t \langle \mathcal{A}\tilde{\mathbf{U}}_{\mathbf{n},\mathbf{\Phi_n}}(s), v\rangle ds = (x_{0n}, v) + \int_0^t \langle \mathcal{B}_n(\Pi_n U_\Phi(s), \tilde{\mathbf{U}}_{\mathbf{n},\mathbf{\Phi_n}}(s)), v\rangle ds$$
$$+ \int_0^t \Big(\Pi_n \Phi_n(s, U_\Phi(s)), v\Big) ds + \int_0^t \Big(\Pi_n \mathcal{C}(s, U_\Phi(s)), v\Big) dw(s)$$

for all $t \in [0,T], v \in H_n$ and a.e. $\omega \in \Omega$. By Theorem 1.3.1, applied for $Z_{n,\psi,\gamma} := \tilde{U}_{n,\Phi_n}$, $X_n := \Pi_n U_\Phi$, $Y_n := 0$, $a_{0n} := x_{0n}$, $\psi(s) := \Pi_n \Phi_n(s, U_\Phi(s))$, $\mathcal{G} := 0$, $\gamma(s) := \Pi_n \mathcal{C}(s, U_\Phi(s))$ it follows that there exists an almost surely unique solution $\tilde{U}_{n,\Phi_n} \in \mathcal{L}^2_{(H_n, \|\cdot\|_V)}(\Omega \times [0,T])$ of (E_{n,Φ_n}), which has almost surely continuous trajectories in H and

$$E \sup_{t\in[0,T]} \|\tilde{U}_{n,\Phi_n}(t)\|^4 + E\Big(\int_0^T \|\tilde{U}_{n,\Phi_n}(s)\|_V^2 ds\Big)^2 \le \tilde{c}_2 \Big[E\|x_0\|^4 + E\int_0^T \|U_\Phi(s)\|^4 ds\Big],$$

where $\tilde{c}_2 > 0$ is a constant (independent of n).

Theorem 2.3.2
The following convergences hold:

$$\lim_{n\to\infty} E\int_0^T \|U_\Phi(s) - \tilde{U}_{\Phi_n}(s)\|_V^2 ds = 0, \quad \lim_{n\to\infty} E\|U_\Phi(T) - \tilde{U}_{\Phi_n}(T)\|^2 = 0,$$

$$\lim_{n\to\infty} E\int_0^T \|U_\Phi(s) - \tilde{U}_{n,\Phi_n}(s)\|_V^2 ds = 0 \quad \textit{and} \quad \lim_{n\to\infty} E\|U_\Phi(T) - \tilde{U}_{n,\Phi_n}(T)\|^2 = 0.$$

PROOF. We consider the evolution equation

$$(\mathbf{z}(t), v) + \int_0^t \langle \mathcal{A}\mathbf{z}(s), v\rangle ds = (x_0, v) + \int_0^t (\mathcal{C}(s, U_\Phi(s)), v) dw(s) \tag{2.10}$$

for all $t \in [0,T], v \in V$ and a.e. $\omega \in \Omega$. There exists an almost surely unique solution $z \in \mathcal{L}^2_V(\Omega \times [0,T])$ of (2.10), which has almost surely continuous trajectories in H (see [14], Theorem 4.1, p. 105). By using the ideas from Proposition B.2 we can prove that the estimate

$$E \sup_{t\in[0,T]} \|z(t)\|^2 + 2\nu E\int_0^T \|z(s)\|_V^2 ds \le c\Big[E\|x_0\|^2 + E\int_0^T \|U_\Phi(s)\|^2 ds\Big]$$

holds, where c is a positive constant depending on λ. From Theorem 1.2.2 we have

$$E \sup_{t\in[0,T]} \|U_\Phi(t)\|^2 < \infty, \quad E\int_0^T \|U_\Phi(s)\|_V^2 ds < \infty.$$

Hence, there exists $k(\omega) > 0$, independent of n, such that for all $n \in \mathbb{N}$ and a.e. $\omega \in \Omega$

$$\sup_{t\in[0,T]} \|\Pi_n z(t)\|^2 \leq \sup_{t\in[0,T]} \|z(t)\|^2 < k(\omega), \quad \int_0^T \|\Pi_n z(s)\|_V^2 ds \leq \int_0^T \|z(s)\|_V^2 ds < k(\omega), \tag{2.11}$$

$$\sup_{t\in[0,T]} \|\Pi_n U_\Phi(t)\|^2 \leq \sup_{t\in[0,T]} \|U_\Phi(t)\|^2 < k(\omega), \quad \int_0^T \|\Pi_n U_\Phi(s)\|_V^2 ds \leq \int_0^T \|U_\Phi(s)\|_V^2 ds < k(\omega). \tag{2.12}$$

By the properties of the stochastic integral and by the properties of U_Φ (see Lemma 1.2.6) we see that for all $s, t \in [0, T]$

$$\begin{aligned} E\Big\| \int_s^t \mathcal{C}(r, U_\Phi(r)) dw(r) \Big\|_{V^*}^4 &\leq k_1 E\Big(\int_s^t \|\mathcal{C}(r, U_\Phi(r))\|_{V^*}^2 dr \Big)^2 \\ &\leq k_2 (t-s)^2 E \sup_{r\in[0,T]} \|U_\Phi(r)\|^4 \leq c(t-s)^2 E\|x_0\|^4 \end{aligned}$$

and

$$\begin{aligned} E\Big\| \int_s^t \Pi_n \mathcal{C}(r, U_\Phi(r)) dw(r) \Big\|_{V^*}^4 &\leq k_1 E\Big(\int_s^t \Big\|\Pi_n \mathcal{C}(r, U_\Phi(r))\Big\|_{V^*}^2 dr \Big)^2 \\ &\leq k_2 (t-s)^2 E \sup_{r\in[0,T]} \|U_\Phi(r)\|^4 \leq c(t-s)^2 E\|x_0\|^4, \end{aligned}$$

where k_1, k_2 are positive constants. In the above estimates we need the V^*-norm, because we will apply the *Dubinsky Theorem* (see [35], Theorem 4.1, p. 132).

By the *Theorem of Kolmogorov-Centsov* (see [18], Theorem 2.8, p. 53; applied for $\alpha := 4, \beta := 1$ and a process with values in a Hilbert space) it follows that there exist a random variable $\chi(\omega)$ and a positive constant δ such that

$$\Big\| \int_s^t \mathcal{C}(r, U_\Phi(r)) dw(r) \Big\|_{V^*}^2 \leq \delta |t-s|^{2\gamma}, \tag{2.13}$$

$$\Big\| \int_s^t \Pi_n \mathcal{C}(r, U_\Phi(r)) dw(r) \Big\|_{V^*}^2 \leq \delta |t-s|^{2\gamma}, \tag{2.14}$$

for $\gamma \in \Big(0, \dfrac{1}{4}\Big)$ and for every $t, s \in [0, T]$ with $|t - s| < \chi(\omega)$ and a.e. $\omega \in \Omega$.

Let $\tilde\Omega \subseteq \Omega$ with $P(\tilde\Omega) = 1$ be such that for all $\omega \in \tilde\Omega$ we have:

- equations (2.1) and (2.10) hold for all $t \in [0, T]$, $v \in V$;
- for each $n = 1, 2, \ldots$ equations (E_{Φ_n}) and (E_{n,Φ_n}) hold for all $t \in [0, T]$, $v \in V$, respectively $v \in H_n$;
- the inequalities in (2.11), (2.12), (2.13) and (2.14) hold.

From (E_{Φ_n}), (2.10), (2.11), (2.12), and the properties of $\mathcal{A}, \mathcal{B}, \Phi_n$, it follows that for all $\omega \in \tilde{\Omega}$ we have[1]

$$\begin{aligned}\sup_{t\in[0,T]} \|\tilde{U}_{\Phi_n}(t) - z(t)\|^2 \quad &+ \quad \nu\int_0^T \|\tilde{U}_{\Phi_n}(s) - z(s)\|_V^2 ds \leq \frac{4}{\nu}\int_0^T \|\mathcal{B}(U_\Phi(s), z(s))\|_{V^*}^2 ds \\ &+ \quad \frac{4\mu c_{HV}}{\nu}\int_0^T \|U_\Phi(s)\|^2 ds \leq \frac{2b}{\nu}\sup_{t\in[0,T]} \|U_\Phi(t)\|^2 \int_0^T \|U_\Phi(s)\|_V^2 ds \\ &+ \quad \frac{2b}{\nu}\sup_{t\in[0,T]} \|z(t)\|^2 \int_0^t \|z(s)\|_V^2 ds + \frac{4\mu c_{HV}}{\nu}\int_0^T \|U_\Phi(s)\|^2 ds \leq c_1(k^2(\omega) + k(\omega))\end{aligned}$$

where c_1 is a positive constant independent of n and ω. Analogously, using (E_{n,Φ_n}) and (2.10) we have

$$\begin{aligned}&\sup_{t\in[0,T]} \|\tilde{U}_{n,\Phi_n}(t) - \Pi_n z(t)\|^2 + \nu\int_0^T \|\tilde{U}_{n,\Phi_n}(s) - \Pi_n z(s)\|_V^2 ds \\ &\qquad \leq \quad \frac{2b}{\nu}\sup_{t\in[0,T]} \|\Pi_n U_\Phi(t)\|^2 \int_0^T \|\Pi_n U_\Phi(s)\|_V^2 ds + \frac{2b}{\nu}\sup_{t\in[0,T]} \|\Pi_n z(t)\|^2 \int_0^t \|\Pi_n z(s)\|_V^2 ds \\ &\qquad + \quad \frac{4\mu c_{HV}}{\nu}\int_0^T \|U_\Phi(s)\|^2 ds \leq c_1(k^2(\omega) + k(\omega))\end{aligned}$$

where c_1 is the same constant as above.

Hence for all $n \in \mathbb{N}$ we have

$$\sup_{t\in[0,T]} \|\tilde{U}_{\Phi_n}(t)\|^2 + \nu\int_0^T \|\tilde{U}_{\Phi_n}(s)\|_V^2 ds < c_2(\omega) \quad \text{for all } \omega \in \tilde{\Omega} \tag{2.15}$$

and

$$\sup_{t\in[0,T]} \|\tilde{U}_{n,\Phi_n}(t)\|^2 + \nu\int_0^T \|\tilde{U}_{n,\Phi_n}(s)\|_V^2 ds < c_2(\omega) \quad \text{for all } \omega \in \tilde{\Omega}, \tag{2.16}$$

where $c_2(\omega)$ is positive, independent of n.

Let $\omega \in \tilde{\Omega}$. For this ω, we consider the sets

$$S = \left\{\tilde{U}_{\Phi_n}(\omega,\cdot)\middle|\; n = 1,2,\dots\right\}, \quad \tilde{S} = \left\{\tilde{U}_{n,\Phi_n}(\omega,\cdot)\middle|\; n = 1,2,\dots\right\}.$$

For each of these sets we want to apply the *Dubinsky Theorem*. By (2.15) and (2.16) we get that $S \subset \mathcal{L}_V^2[0,T]$ and $\tilde{S} \subset \mathcal{L}_V^2[0,T]$ are bounded. We have to verify that S, respectively $\tilde{S}$, are

[1] Since $V \hookrightarrow H$ we have $\|v\|^2 \leq c_{HV}\|v\|_V^2$ for all $v \in V$.

equi-continuous in $C([0,T],V^*)$. From (E_{Φ_n}) and the Schwarz inequality we have

$$\begin{aligned}\|\tilde{U}_{\Phi_n}(t)-\tilde{U}_{\Phi_n}(s)\|_{V^*}^2 &\leq (t-s)\int_s^t \Big(\|\mathcal{A}\tilde{U}_{\Phi_n}(r)\|_{V^*}^2+\|\mathcal{B}(U_\Phi(r),\tilde{U}_{\Phi_n}(r))\|_{V^*}^2+\|\Phi_n(r,U_\Phi(r))\|_{V^*}^2\Big)dr \\ &+ \Big\|\int_s^t \mathcal{C}(r,U_\Phi(r))dw(r)\Big\|_{V^*}^2\end{aligned}$$

for each $t,s\in[0,T], t>s$. By (2.13), (2.15), and the properties of $\mathcal{A},\mathcal{B},\Phi_n$ we obtain

$$\|\tilde{U}_{\Phi_n}(t)-\tilde{U}_{\Phi_n}(s)\|_{V^*}^2 \leq c_3(\omega)(t-s)+\delta(t-s)^{2\gamma}$$

for $\gamma\in\left(0,\frac{1}{4}\right)$ and for every $t,s\in[0,T]$ with $|t-s|<\chi(\omega)$, where $c_3(\omega)>0$ is independent of n. Consequently, S is equi-continuous in $C([0,T],V^*)$. Analogously we can prove that $\tilde{S}$ is equi-continuous in $C([0,T],V^*)$. Now, using the *Dubinsky Theorem*, it follows that S and $\tilde{S}$ are relatively compact in $\mathcal{L}_H^2[0,T]$ and hence there exists a subsequence (n') of (n) and $\tilde{U},U^*\in\mathcal{L}_H^2[0,T]$ such that

$$\lim_{n'\to\infty}\int_0^T \|\tilde{U}_{\Phi_{n'}}(s)-\tilde{U}(s)\|^2 ds = 0 \tag{2.17}$$

and

$$\lim_{n'\to\infty}\int_0^T \|\tilde{U}_{n',\Phi_{n'}}(s)-U^*(s)\|^2 ds = 0. \tag{2.18}$$

We use $(E_{\Phi_{n'}})$ and (2.1), the generalized chain rule, the properties of $\mathcal{A}$ and $\mathcal{B}$ to obtain

$$\begin{aligned}\|\tilde{U}_{\Phi_{n'}}(T)-U_\Phi(T)\|^2 &+ 2\nu\int_0^T\|\tilde{U}_{\Phi_{n'}}(s)-U_\Phi(s)\|_V^2 ds \\ &\leq 2\int_0^T\Big(\Phi_{n'}(s,U_\Phi(s))-\Phi(s,U_\Phi(s)),\tilde{U}_{\Phi_{n'}}(s)-\tilde{U}(s)\Big)ds \\ &+ 2\int_0^T\Big(\Phi_{n'}(s,U_\Phi(s))-\Phi(s,U_\Phi(s)),\tilde{U}(s)-U_\Phi(s)\Big)ds.\end{aligned}$$

According to Lemma 2.3.1, (2.17), and the properties of $\Phi_n,\Phi\in\mathcal{U}$ we get

$$\lim_{n'\to\infty}\|\tilde{U}_{\Phi_{n'}}(T)-U_\Phi(T)\|^2=0, \quad \lim_{n'\to\infty}\int_0^T\|\tilde{U}_{\Phi_{n'}}(s)-U_\Phi(s)\|_V^2 ds=0. \tag{2.19}$$

Every subsequence of $\left(\tilde{U}_{\Phi_n}(\omega,\cdot)\right)$ has a further subsequence, which converges in the space $\mathcal{L}_V^2[0,T]$ to the same limit $U_\Phi(\omega,\cdot)$ (because we can repeat all arguments of above). Applying Proposition

A.1, it follows that the whole sequence $\left(\tilde{U}_{\Phi_n}(\omega,\cdot)\right)$ converges to $U_\Phi(\omega,\cdot)$ in the space $\mathcal{L}_V^2[0,T]$. Analogously we conclude that the whole sequence $\left(\tilde{U}_{\Phi_n}(\omega,T)\right)$ converges to $U_\Phi(\omega,T)$ in H.

Our arguments from above worked for an arbitrary fixed $\omega \in \tilde{\Omega}$. Hence (2.19) holds for a.e. $\omega \in \Omega$ and for the whole sequence (n). Taking into consideration that the processes $\left(\tilde{U}_{\Phi_n}(t)\right)_{t\in[0,T]}$ and $\left(U_\Phi(t)\right)_{t\in[0,T]}$ are uniformly integrable (see Theorem 1.3.1 and Lemma 1.2.6) it follows that

$$\lim_{n\to\infty} E\int_0^T \|U_\Phi(s) - \tilde{U}_{\Phi_n}(s)\|_V^2 ds = 0$$

and

$$\lim_{n\to\infty} E\|U_\Phi(T) - \tilde{U}_{\Phi_n}(T)\|^2 = 0.$$

Now we prove the convergences for the sequence $(\tilde{U}_{n,\Phi_n})$. We use $(E_{n',\Phi_{n'}})$ and (2.1), the generalized chain rule, the properties of $\mathcal{A}$ and $\mathcal{B}$ to obtain

$$\begin{aligned}(2.20)\quad \|\tilde{U}_{n,\Phi_n}(T) - \Pi_n U_\Phi(T)\|^2 &+ \nu\int_0^T \|\tilde{U}_{n,\Phi_n}(s) - \Pi_n \mathcal{U}_\Phi(s)\|_V^2 ds \\ &\leq \int_0^T \|\mathcal{B}(\Pi_n U_\Phi(s), \Pi_n U_\Phi(s)) - \mathcal{B}(U_\Phi(s), U_\Phi(s))\|_{V^*}^2 ds \\ &+ 2\int_0^T \Big(\Pi_n \Phi_n(s, U_\Phi(s)) - \Phi(s, U_\Phi(s)), \tilde{U}_{n,\Phi_n}(s) - \Pi_n U_\Phi(s)\Big) ds.\end{aligned}$$

By the properties of $\mathcal{B}$ we have

$$\begin{aligned}&\int_0^T \|\mathcal{B}(\Pi_n U_\Phi(s), \Pi_n U_\Phi(s)) - \mathcal{B}(U_\Phi(s), U_\Phi(s))\|_{V^*}^2 ds \\ &\qquad\leq 2b\Big(\sup_{t\in[0,T]} \|U_\Phi(t) - \Pi_n U_\Phi(t)\|^2 \int_0^T \|U_\Phi(s)\|_V^2 ds\Big)^{1/2} \\ &\qquad\times \Big(\sup_{t\in[0,T]} \|U_\Phi(t)\|^2 \int_0^T \|U_\Phi(s) - \Pi_n U_\Phi(s)\|_V^2 ds\Big)^{1/2}\end{aligned}$$

and by (1.4) it follows

$$(2.21)\qquad \lim_{n'\to\infty}\int_0^T \|\mathcal{B}(\Pi_{n'} U_\Phi(s), \Pi_{n'} U_\Phi(s)) - \mathcal{B}(U_\Phi(s), U_\Phi(s))\|_{V^*}^2 ds = 0.$$

We write

$$\begin{aligned}
&\int_0^T \Big(\Pi_n \Phi_n(s, U_\Phi(s)) - \Phi(s, U_\Phi(s)), \tilde{U}_{n,\Phi_n}(s) - \Pi_n U_\Phi(s)\Big) ds \\
= &\int_0^T \Big(\Phi_n(s, U_\Phi(s)) - \Phi(s, U_\Phi(s)), \tilde{U}_{n,\Phi_n}(s) - U^*(s)\Big) ds \\
+ &\int_0^T \Big(\Phi_n(s, U_\Phi(s)) - \Phi(s, U_\Phi(s)), U^*(s) - U_\Phi(s)\Big) ds \\
+ &\int_0^T \Big(\Phi_n(s, U_\Phi(s)) - \Phi(s, U_\Phi(s)), U_\Phi(s) - \Pi_n U_\Phi(s)\Big) ds.
\end{aligned}$$

By using this equality for n', as soon as (1.4), (2.12), (2.16), (2.18), the properties of Φ_n, Φ and Lemma 2.3.1 we get

$$\lim_{n' \to \infty} \int_0^T \Big(\Pi_{n'} \Phi_{n'}(s, U_\Phi(s)) - \Phi(s, U_\Phi(s)), \tilde{U}_{n',\Phi_{n'}}(s) - \Pi_{n'} U_\Phi(s)\Big) ds = 0. \tag{2.22}$$

From (2.21) and (2.22) we obtain that the right side of the inequality in (2.20) tends to zero. Therefore

$$\lim_{n' \to \infty} \|\tilde{U}_{n',\Phi_{n'}}(T) - \Pi_{n'} U_\Phi(T)\|^2 = 0, \quad \lim_{n' \to \infty} \int_0^T \|\tilde{U}_{n',\Phi_{n'}}(s) - \Pi_{n'} U_\Phi(s)\|_V^2 ds = 0.$$

Hence by (1.4) and (1.6) we have

$$\lim_{n' \to \infty} \|\tilde{U}_{n',\Phi_{n'}}(T) - U_\Phi(T)\|^2 = 0, \quad \lim_{n' \to \infty} \int_0^T \|\tilde{U}_{n',\Phi_{n'}}(s) - U_\Phi(s)\|_V^2 ds = 0. \tag{2.23}$$

Every subsequence of $\Big(\tilde{U}_{n,\Phi_n}(\omega, \cdot)\Big)$ has a further subsequence, which converges in the space $\mathcal{L}_V^2[0,T]$ to the same limit $U_\Phi(\omega, \cdot)$ (because we can repeat all arguments of above). Applying Proposition A.1, it follows that the whole sequence $\Big(\tilde{U}_{n,\Phi_n}(\omega, \cdot)\Big)$ converges to $U_\Phi(\omega, \cdot)$ in the space $\mathcal{L}_V^2[0,T]$, respectively. Analogously we conclude that the whole sequence $\Big(\tilde{U}_{n,\Phi_n}(\omega, T)\Big)$ converges in H to $U_\Phi(\omega, T)$.

Our arguments from above worked for an arbitrary fixed $\omega \in \tilde{\Omega}$. Hence (2.23) holds for a.e. $\omega \in \Omega$ and for the whole sequence (n). Taking into consideration that the processes $\Big(\tilde{U}_{n,\Phi_n}(t)\Big)_{t \in [0,T]}$ and $\Big(U_\Phi(t)\Big)_{t \in [0,T]}$ are uniformly integrable (see Theorem 1.3.1 and Lemma 1.2.6) and using (1.5) it follows that the conclusions of this theorem hold. ■

Let $U_{n,\Phi}$ be the solution of (P_n) (see Section 1.2) using the feedback control $\Phi_n := \Pi_n \Phi$. Note that $U_{n,\Phi} = U_{n,\Pi_n \Phi}$ for $\Phi \in \mathcal{U}$ or $\Phi \in \mathcal{U}^b$.

Theorem 2.3.3
The following convergences hold:

$$\lim_{n\to\infty} E\int_0^T \|U_\Phi(s) - U_{\Phi_n}(s)\|_V^2 ds = 0, \quad \lim_{n\to\infty} E\|U_\Phi(T) - U_{\Phi_n}(T)\|^2 = 0,$$

$$\lim_{n\to\infty} E\int_0^T \|U_\Phi(s) - U_{n,\Phi_n}(s)\|_V^2 ds = 0 \quad \textit{and} \quad \lim_{n\to\infty} E\|U_\Phi(T) - U_{n,\Phi_n}(T)\|^2 = 0.$$

Proof. We write $U := U_\Phi$. Let $M \in \mathbb{N}$ and let $\mathcal{T}_M := \mathcal{T}_M^U$ be the stopping time of U. We write

$$e(t) = \Delta_U^2(t)\exp\{-(2\lambda + 2\sqrt{\mu} + 1)t\}.$$

It follows by the Ito formula that for a.e. $\omega \in \Omega$ we have

$$\begin{aligned}
(2.24) \quad & e(\mathcal{T}_M)\|\tilde{U}_{\Phi_n}(\mathcal{T}_M) - U_{\Phi_n}(\mathcal{T}_M)\|^2 + 2\int_0^{\mathcal{T}_M} e(s)\langle \mathcal{A}\tilde{U}_{\Phi_n}(s) - \mathcal{A}U_{\Phi_n}(s), \tilde{U}_{\Phi_n}(s) - U_{\Phi_n}(s)\rangle ds \\
= \; & 2\int_0^{\mathcal{T}_M} e(s)\langle \mathcal{B}(U(s), \tilde{U}_{\Phi_n}(s)) - \mathcal{B}(U_{\Phi_n}(s), U_{\Phi_n}(s)), \tilde{U}_{\Phi_n}(s) - U_{\Phi_n}(s)\rangle ds \\
- \; & \frac{2b}{\nu}\int_0^{\mathcal{T}_M} e(s)\|U(s)\|_V^2\|\tilde{U}_{\Phi_n}(s) - U_{\Phi_n}(s)\|^2 ds - (2\lambda + 2\sqrt{\mu} + 1)\int_0^{\mathcal{T}_M} e(s)\|\tilde{U}_{\Phi_n}(s) - U_{\Phi_n}(s)\|^2 ds \\
+ \; & 2\int_0^{\mathcal{T}_M} e(s)(\Phi_n(s, U(s) - U_{\Phi_n}(s)), \tilde{U}_{\Phi_n}(s) - U_{\Phi_n}(s)) ds \\
+ \; & 2\int_0^{\mathcal{T}_M} e(s)(\mathcal{C}(s, U(s)) - \mathcal{C}(s, U_{\Phi_n}(s)), \tilde{U}_{\Phi_n}(s) - U_{\Phi_n}(s)) dw(s) \\
+ \; & \int_0^{\mathcal{T}_M} e(s)\|\mathcal{C}(s, U(s)) - \mathcal{C}(s, U_{\Phi_n}(s))\|^2 ds.
\end{aligned}$$

In view of the properties of $\mathcal{B}$ we can write

$$\begin{aligned}
& 2\langle \mathcal{B}(U(s), \tilde{U}_{\Phi_n}(s)) - \mathcal{B}(U_{\Phi_n}(s), U_{\Phi_n}(s)), \tilde{U}_{\Phi_n}(s) - U_{\Phi_n}(s)\rangle \\
= \; & 2\langle \mathcal{B}(U(s) - U_{\Phi_n}(s), \tilde{U}_{\Phi_n}(s) - U(s)), \tilde{U}_{\Phi_n}(s) - U_{\Phi_n}(s)\rangle \\
+ \; & 2\langle \mathcal{B}(U(s) - \tilde{U}_{\Phi_n}(s), U(s)), \tilde{U}_{\Phi_n}(s) - U_{\Phi_n}(s)\rangle \\
+ \; & 2\langle \mathcal{B}(\tilde{U}_{\Phi_n}(s) - U_{\Phi_n}(s), U(s)), \tilde{U}_{\Phi_n}(s) - U_{\Phi_n}(s)\rangle
\end{aligned}$$

$$
\begin{aligned}
&\leq 2\sqrt{b}\|\tilde{U}_{\Phi_n}(s)-U(s)\|_V\|U(s)-U_{\Phi_n}(s)\|_V^{\frac{1}{2}}\|U(s)-U_{\Phi_n}(s)\|^{\frac{1}{2}}\\
&\times \|\tilde{U}_{\Phi_n}(s)-U_{\Phi_n}(s)\|_V^{\frac{1}{2}}\|\tilde{U}_{\Phi_n}(s)-U_{\Phi_n}(s)\|^{\frac{1}{2}}\\
&+ \frac{2b}{\nu}\|U(s)\|\|U(s)\|_V\|U(s)-\tilde{U}_{\Phi_n}(s)\|_V\|U(s)-\tilde{U}_{\Phi_n}(s)\|+\nu\|\tilde{U}_{\Phi_n}(s)-U_{\Phi_n}(s)\|_V^2\\
&+ \frac{2b}{\nu}\|U(s)\|_V^2\|\tilde{U}_{\Phi_n}(s)-U_{\Phi_n}(s)\|^2.
\end{aligned}
$$

Using this estimates in (2.24) and after some elementary calculations, we obtain

$$
\begin{aligned}
&Ee(\mathcal{T}_M)\|\tilde{U}_{\Phi_n}(\mathcal{T}_M)-U_{\Phi_n}(\mathcal{T}_M)\|^2+\nu E\int_0^{\mathcal{T}_M} e(s)\|\tilde{U}_{\Phi_n}(s)-U_{\Phi_n}(s)\|_V^2 ds\\
\leq\ & 2\sqrt{b}\Big(E\int_0^T\|\tilde{U}_{\Phi_n}(s)-U(s)\|_V^2 ds\Big)^{\frac{1}{2}}\\
\times\ & \left(E\int_0^T\Big(\|U(s)-U_{\Phi_n}(s)\|_V^2\|U(s)-U_{\Phi_n}(s)\|^2+\|\tilde{U}_{\Phi_n}(s)-U_{\Phi_n}(s)\|_V^2\|\tilde{U}_{\Phi_n}(s)-U_{\Phi_n}(s)\|^2\Big)ds\right)^{\frac{1}{2}}\\
+\ & \frac{2bM}{\nu}\Big(E\int_0^T\|U(s)-\tilde{U}_{\Phi_n}(s)\|_V^2 ds\Big)^{1/2}\Big(E\sup_{t\in[0,T]}\|U(t)-\tilde{U}_{\Phi_n}(t)\|^2\Big)^{1/2}\\
+\ & (\mu+2\lambda)E\int_0^T\|U(s)-\tilde{U}_{\Phi_n}(s)\|^2 ds.
\end{aligned}
$$

Using the above inequalitiy, Theorem 2.3.2 and Lemma 1.2.6 we have

$$
\lim_{n\to\infty} E\|U(\mathcal{T}_M)-U_{\Phi_n}(\mathcal{T}_M)\|^2=0,\qquad \lim_{n\to\infty} E\int_0^{\mathcal{T}_M}\|U(s)-U_{\Phi_n}(s)\|^2 ds=0.
$$

By Proposition B.3, applied on $\mathcal{T} := T$, $\quad Q_n(\mathcal{T}) := \|U(\mathcal{T})-U_{\Phi_n}(\mathcal{T})\|^2$, respectively $Q_n(\mathcal{T}) := \int_0^{\mathcal{T}}\|U(s)-U_{\Phi_n}(s)\|_V^2 ds$, we get

$$
\lim_{n\to\infty} E\int_0^T\|U(s)-U_{\Phi_n}(s)\|_V^2 ds=0 \quad\text{and}\quad \lim_{n\to\infty} E\|U(T)-U_{\Phi_n}(T)\|^2=0.
$$

Now we prove the convergences for the sequence $(\tilde{U}_{n,\Phi_n})$. It follows by the Ito formula and the properties for $\mathcal{A}$ that for a.e. $\omega \in \Omega$ we have

$$e(\mathcal{T}_M)\|\tilde{U}_{n,\Phi_n}(\mathcal{T}_M) - U_{n,\Phi_n}(\mathcal{T}_M)\|^2 + 2\nu \int_0^{\mathcal{T}_M} e(s)\|\tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\|_V^2 ds \tag{2.25}$$

$$\begin{aligned}
= \;& 2\int_0^{\mathcal{T}_M} e(s)\langle \mathcal{B}(\Pi_n U(s), \tilde{U}_{n,\Phi_n}(s)) - \mathcal{B}(U_{n,\Phi_n}(s), U_{n,\Phi_n}(s)), \tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\rangle ds \\
- \;& \frac{2b}{\nu}\int_0^{\mathcal{T}_M} e(s)\|U(s)\|_V^2\|\tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\|^2 ds \\
- \;& (2\lambda + 2\sqrt{\mu} + 1)\int_0^{\mathcal{T}_M} e(s)\|\tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\|^2 ds \\
+ \;& 2\int_0^{\mathcal{T}_M} e(s)\Big(\Pi_n\Phi_n(s, U(s) - U_{n,\Phi_n}(s)), \tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\Big) ds \\
+ \;& 2\int_0^{\mathcal{T}_M} e(s)\Big(\Pi_n\mathcal{C}(s, U(s)) - \Pi_n\mathcal{C}(s, U_{n,\Phi_n}(s)), \tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\Big) dw(s) \\
+ \;& \int_0^{\mathcal{T}_M} e(s)\Big\|\Pi_n\mathcal{C}(s, U(s)) - \Pi_n\mathcal{C}(s, U_{n,\Phi_n}(s))\Big\|^2 ds.
\end{aligned}$$

In view of the properties of $\mathcal{B}$ and (1.2) we can write

$$\begin{aligned}
& 2\langle \mathcal{B}(\Pi_n U(s), \tilde{U}_{n,\Phi_n}(s)) - \mathcal{B}(U_{n,\Phi_n}(s), U_{n,\Phi_n}(s)), \tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\rangle \\
= \;& 2\langle \mathcal{B}(\Pi_n U(s) - U_{n,\Phi_n}(s), \tilde{U}_{n,\Phi_n}(s) - \Pi_n U(s)), \tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\rangle \\
+ \;& 2\langle \mathcal{B}(\Pi_n U(s) - \tilde{U}_{n,\Phi_n}(s), \Pi_n U(s)), \tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\rangle \\
+ \;& 2\langle \mathcal{B}(\tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s), \Pi_n U(s)), \tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\rangle \\
\leq \;& 2\sqrt{b}\|\tilde{U}_{n,\Phi_n}(s) - \Pi_n U(s)\|_V\|\Pi_n U(s) - U_{n,\Phi_n}(s)\|_V^{\frac{1}{2}}\|\Pi_n U(s) - U_{n,\Phi_n}(s)\|^{\frac{1}{2}} \\
\times \;& \|\tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\|_V^{\frac{1}{2}}\|\tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\|^{\frac{1}{2}} \\
+ \;& \frac{2b}{\nu}\|U(s)\|\|U(s)\|_V\|\Pi_n U(s) - \tilde{U}_{n,\Phi_n}(s)\|_V\|\Pi_n U(s) - \tilde{U}_{n,\Phi_n}(s)\| \\
+ \;& \nu\|\tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\|_V^2 + \frac{2b}{\nu}\|U(s)\|_V^2\|\tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\|^2.
\end{aligned}$$

Using this estimates in (2.25) and after some elementary calculations, we obtain

$$Ee(\mathcal{T}_M)\|\tilde{U}_{n,\Phi_n}(\mathcal{T}_M) - U_{n,\Phi_n}(\mathcal{T}_M)\|^2 + \nu E\int_0^{\mathcal{T}_M} e(s)\|\tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\|_V^2 ds$$

$$\begin{aligned}
\leq\ & 2\sqrt{b}\Big(E\int_0^T \|\tilde{U}_{n,\Phi_n}(s) - \Pi_n U(s)\|_V^2 ds\Big)^{\frac{1}{2}} \\
\times\ & \Big(E\int_0^T \Big(\|\Pi_n U(s) - U_{n,\Phi_n}(s)\|_V^2 \|\Pi_n U(s) - U_{n,\Phi_n}(s)\|^2 \\
+\ & \|\tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\|_V^2 \|\tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\|^2\Big) ds\Big)^{\frac{1}{2}} \\
+\ & \frac{2bM}{\nu}\Big(E\int_0^T \|\Pi_n U(s) - \tilde{U}_{n,\Phi_n}(s)\|_V^2 ds\Big)^{1/2}\Big(E\sup_{t\in[0,T]}\|\Pi_n U(t) - \tilde{U}_{n,\Phi_n}(t)\|^2\Big)^{1/2} \\
+\ & (\mu + 2\lambda)E\int_0^T e(s)\|U(s) - \tilde{U}_{n,\Phi_n}(s)\|^2 ds.
\end{aligned}$$

Using Theorem 2.3.2 and Lemma 1.2.6 we have

$$\lim_{n'\to\infty} E\|\tilde{U}_{n,\Phi_n}(\mathcal{T}_M) - U_{n,\Phi_n}(\mathcal{T}_M)\|^2 = 0, \qquad \lim_{n'\to\infty} E\int_0^{\mathcal{T}_M} \|\tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\|^2 ds = 0.$$

By Proposition B.3, applied for $\mathcal{T} := T$, $\quad Q_n(\mathcal{T}) := \|\tilde{U}_{n,\Phi_n}(\mathcal{T}) - U_{n,\Phi_n}(\mathcal{T})\|^2$, respectively $Q_n(\mathcal{T}) := \int_0^{\mathcal{T}} \|\tilde{U}_{n',\Phi_{n'}}(s) - U_{n,\Phi_n}(s)\|_V^2 ds$, we get

$$\lim_{n\to\infty} E\int_0^T \|\tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\|_V^2 ds = 0 \quad \text{and} \quad \lim_{n\to\infty} E\|\tilde{U}_{n,\Phi_n}(T) - U_{n,\Phi_n}(T)\|^2 = 0.$$

Now we use Theorem 2.3.2 to obtain the conclusion of this theorem. ■

The **main result** of this section is the following theorem, in which we prove that there exists at least one optimal feedback control for problem $(\mathcal{P})$.

Theorem 2.3.4
Assume for all $t \in [0,T], x \in H$ that the mappings $\mathcal{L}(t,x,\cdot), \mathcal{K}(\cdot)$ are weakly lower semicontinuous. Then there exists an optimal feedback control for problem $(\mathcal{P})$.

PROOF. Let (Φ_n) be a minimizing sequence for problem $(\mathcal{P})$. We apply Lemma 2.3.1 and Theorem 2.3.3 for this sequence. Therefore there exists a subsequence (n') of (n) and $\Phi \in \mathcal{U}$ such that for all $t \in [0,T]$, $x, y \in H$ and a.e. $\omega \in \Omega$ it holds

$$\lim_{n'\to\infty} (\Phi_{n'}(t, U_{\Phi_{n'}}(t)), y) = (\Phi(t, U_\Phi(t)), y).$$

From $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, and Theorem 2.3.3 we have

$$\begin{aligned}
E\int_0^T \mathcal{L}[t, U_\Phi(t), \Phi(t, U_\Phi(t))]dt &\le \liminf_{n'\to\infty} E\int_0^T \mathcal{L}[t, U_\Phi(t), \Phi_{n'}(t, U_{\Phi_{n'}}(t))]dt \\
&\le \liminf_{n'\to\infty}\left(E\int_0^T \mathcal{L}[t, U_{\Phi_{n'}}(t), \Phi_{n'}(t, U_{\Phi_{n'}}(t))]dt + c_{\mathcal{L}} E\int_0^T \|U_\Phi(t) - U_{\Phi_{n'}}(t)\|_V^2 dt\right) \\
&\le \liminf_{n'\to\infty} E\int_0^T \mathcal{L}[t, U_{\Phi_{n'}}(t), \Phi_{n'}(t, U_{\Phi_{n'}}(t))]dt
\end{aligned}$$

and

$$E\mathcal{K}[U_\Phi(T)] \le \liminf_{n'\to\infty} E\mathcal{K}[U_{\Phi_{n'}}(T)].$$

Consequently,

$$\mathcal{J}(\Phi) \le \liminf_{n'\to\infty} \mathcal{J}(\Phi_{n'}).$$

But (Φ_n) is a minimizing sequence for problem $(\mathcal{P})$. Hence

$$\mathcal{J}(\Phi) = \min_{\Psi\in\mathcal{U}} \mathcal{J}(\Psi)$$

and therefore $\Phi \in \mathcal{U}$ is an optimal feedback control for problem $(\mathcal{P})$. ∎

Remark 2.3.5

We can not use this method in the case of problem $(\mathcal{P}^b)$, because the minimizing sequence (Φ_n) then belongs to the space $\mathcal{L}^2_H(\Omega \times [0,T])$ and we can not find (as in Lemma 2.3.1) a subsequence (n') of (n) *independent of* ω, t such that $(\Phi_{n'}(\omega, t))$ would converge in H to a process $\Phi \in \mathcal{L}^2_H(\Omega \times [0,T])$ for $P \times \Lambda$ a.e. $(\omega, t) \in \Omega \times [0,T]$. The *independence with respect to* ω is essential in the proof of Theorem 2.3.2.

2.4 Existence of ε-optimal feedback controls

For $(\mathcal{P})$ we formulate the corresponding n-dimensional control problem

$$(\mathcal{P}_n) \qquad \begin{cases} \mathcal{J}_n(\Phi_n) \to \min \\ \Phi_n \in \mathcal{U}_n \end{cases}$$

where

$$\mathcal{J}_n(\Phi_n) = E\int_0^T \mathcal{L}[s, U_{n,\Phi_n}(s), \Phi_n(s, U_{n,\Phi_n}(s))]ds + E\mathcal{K}[U_{n,\Phi_n}(T)]$$

and

$$\mathcal{U}_n := \Big\{\Phi_n : [0,T] \times H_n \to H_n | \Phi_n = \Pi_n\Phi, \Phi \in \mathcal{U}\Big\}.$$

Here U_{n,Φ_n} is the solution of (P_n) using the feedback control $\Phi_n \in \mathcal{U}_n$.

Analogously we define the n-dimensional control problem corresponding to $(\mathcal{P}^b)$. We denote this problem by $(\mathcal{P}^b_n)$.

Theorem 2.4.1

Let (Φ_n) be a sequence in $\mathcal{U}$ such that $\Phi_n \in \mathcal{U}_n$ for each $n \in \mathbb{N}$. There exists a subsequence (n') of (n) such that

$$\lim_{n'\to\infty} E\int_0^T \|U_{\Phi_{n'}}(s) - U_{n',\Phi_{n'}}(s)\|^2_V ds = 0 \quad \textit{and} \quad \lim_{n'\to\infty} E\|U_{\Phi_{n'}}(T) - U_{n',\Phi_{n'}}(T)\|^2 = 0,$$

where $U_{\Phi_{n'}}$ and $U_{n',\Phi_{n'}}$ are the solutions of (2.1), respectively $(P_{n'})$, using the feedback control $\Phi_{n'}$.

Proof. First we apply Lemma 2.3.1 on the sequence (Φ_n). Consequently, there exist a subsequence (n') of (n) and $\Phi \in \mathcal{U}$ such that for all $t \in [0,T]$, $x, y \in H$

$$\lim_{n'\to\infty} (\Phi_{n'}(t,x), y) = (\Phi(t,x), y).$$

By Theorem 2.3.3 it follows that

$$\lim_{n'\to\infty} E\int_0^T \|U_{\Phi_{n'}}(s) - U_\Phi(s)\|^2_V ds = 0, \quad \lim_{n'\to\infty} E\|U_{\Phi_{n'}}(T) - U_\Phi(T)\|^2 = 0 \tag{2.26}$$

and

$$\lim_{n'\to\infty} E\int_0^T \|U_{n',\Phi_{n'}}(s) - U_\Phi(s)\|^2_V ds = 0, \quad \lim_{n'\to\infty} E\|U_{n',\Phi_{n'}}(T) - U_\Phi(T)\|^2 = 0. \tag{2.27}$$

We see that

$$E\int_0^T \|U_{\Phi_{n'}}(s) - U_{n',\Phi_{n'}}(s)\|^2_V ds \leq 2E\int_0^T \|U_{\Phi_{n'}}(s) - U_\Phi(s)\|^2_V ds + 2E\int_0^T \|U_\Phi(s) - U_{n',\Phi_{n'}}(s)\|^2_V ds.$$

By using (2.26) and (2.27) we have

$$\lim_{n'\to\infty} E\int_0^T \|U_{\Phi_{n'}}(s) - U_{n',\Phi_{n'}}(s)\|^2_V ds = 0.$$

Analogously we deduce

$$\lim_{n'\to\infty} E\|U_{\Phi_{n'}}(T) - U_{n',\Phi_{n'}}(T)\|^2 = 0. \qquad \blacksquare$$

Theorem 2.4.2
Assume that for sufficiently large n there exists optimal controls for problem $(\mathcal{P}_n)$. *If* $\Phi^* \in \mathcal{U}$ *is an optimal control for problem* $(\mathcal{P})$ *and* $\varepsilon > 0$ *is arbitrary fixed, then there exists* $n_\varepsilon \in \mathbb{N}$ *such that for all* $n \geq n_\varepsilon$ $\Phi_n^* \in \mathcal{U}_n$ *is an optimal control for the* n*-dimensional control problem* $(\mathcal{P}_n)$ *and*

$$|\mathcal{J}_n(\Phi_n^*) - \mathcal{J}(\Phi^*)| < \varepsilon, \quad \mathcal{J}(\Phi_n^*) - \mathcal{J}(\Phi^*) < \varepsilon,$$

hence ${\Phi^*}_n$ *is an* ε*-optimal control for problem* $(\mathcal{P})$.

Proof. Let $\varepsilon > 0$ and take

$$\varepsilon^* := \frac{\varepsilon}{2(c_\mathcal{L} + c_\mathcal{K})},$$

where $c_\mathcal{L}, c_\mathcal{K}$ are that constants that occure in $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ from Section 2.1.

For each $m \in \mathbb{N}$ let $\tilde{\Phi}_m := \Pi_m \Phi^*$. From Theorem 1.2.7 and the properties (1.5) (from Section 1.1) it follows that there exists an $m_\varepsilon > 0$ such that for all $m \geq m_\varepsilon$ it holds

$$E\int_0^T \|U_{m,\Phi^*}(s) - U_{\Phi^*}(s)\|^2 ds + E\|U_{m,\Phi^*}(T) - U_{\Phi^*}(T)\|^2 < \varepsilon^*$$

and

$$E\int_0^T \|\tilde{\Phi}_m(s, U_{m,\Phi^*}(s)) - \Phi^*(s, U_{\Phi^*}(s))\|^2 ds < \varepsilon^*.$$

Let $n \geq m_\varepsilon$ and let Φ_n^* be an optimal control for the n-dimensional control problem $(\mathcal{P}_n)$. By Theorem 2.4.1, applied on (Φ_n^*), there exists a subsequence (n') of (n) and $n'_\varepsilon \geq m_\varepsilon$ such that for all $n' \geq n'_\varepsilon$ we have

$$E\int_0^T \|U_{\Phi^*_{n'}}(s) - U_{n',\Phi^*_{n'}}(s)\|^2 ds + E\|U_{\Phi^*_{n'}}(T) - U_{n',\Phi^*_{n'}}(T)\|^2 < \varepsilon^*. \tag{2.28}$$

First case: $\mathcal{J}_{n'}(\Phi^*_{n'}) - \mathcal{J}(\Phi^*) \geq 0$. Then by using the properties of $\mathcal{L}$ and $\mathcal{K}$ (given in Section 2.1), we have for all $n \geq n'_\varepsilon$

$$\begin{aligned}
0 &\leq \mathcal{J}_{n'}(\Phi^*_{n'}) - \mathcal{J}(\Phi^*) \leq \mathcal{J}_{n'}(\tilde{\Phi}_{n'}) - \mathcal{J}(\Phi^*) \\
&\leq c_\mathcal{L}\Big(E\int_0^T \|U_{n',\Phi^*}(s) - U_{\Phi^*}(s)\|^2 ds + E\int_0^T \|\tilde{\Phi}_{n'}(s, U_{n',\Phi^*}(s)) - \Phi^*(s, U_{\Phi^*}(s))\|^2 ds\Big) \\
&+ c_\mathcal{K} E\|U_{n',\Phi^*}(T) - U_{\Phi^*}(T)\|^2 < (c_\mathcal{L} + c_\mathcal{K})\varepsilon^* < \frac{\varepsilon}{2}.
\end{aligned}$$

Second case: $\mathcal{J}_{n'}(\Phi^*_{n'}) - \mathcal{J}(\Phi^*) < 0$. We write

$$\begin{aligned}
0 &< \mathcal{J}(\Phi^*) - \mathcal{J}_{n'}(\Phi^*_{n'}) \leq \mathcal{J}(\Phi^*_{n'}) - \mathcal{J}_{n'}(\Phi^*_{n'}) \\
&\leq c_{\mathcal{L}} E \int_0^T \|U_{n',\Phi^*_{n'}}(s) - U_{\Phi^*_{n'}}(s)\|^2 ds + c_{\mathcal{K}} E \|U_{n',\Phi^*_{n'}}(T) - U_{\Phi^*_{n'}}(T)\|^2 \\
&< (c_{\mathcal{L}} + c_{\mathcal{K}})\varepsilon^* < \frac{\varepsilon}{2}.
\end{aligned}$$

Hence for all $n' \geq n'_\varepsilon$ we have for all $n' \geq n'_\varepsilon$

$$|\mathcal{J}_{n'}(\Phi^*_{n'}) - \mathcal{J}(\Phi^*)| < \frac{\varepsilon}{2} < \varepsilon.$$

Using this inequality and (2.28) we get

$$\begin{aligned}
0 &\leq \mathcal{J}(\Phi^*_{n'}) - \mathcal{J}(\Phi^*) \leq |\mathcal{J}(\Phi^*_{n'}) - \mathcal{J}_n(\Phi^*_{n'})| + |\mathcal{J}_{n'}(\Phi^*_{n'}) - \mathcal{J}(\Phi^*)| \\
&\leq c_{\mathcal{L}} E \int_0^T \|U_{\Phi^*_{n'}}(s) - U_{n',\Phi^*_{n'}}(s)\|^2 ds + c_{\mathcal{K}} E \|U_{\Phi^*_{n'}}(T) - U_{n,\Phi^*_{n'}}(T)\|^2 + \frac{\varepsilon}{2} \\
&< (c_{\mathcal{L}} + c_{\mathcal{K}})\varepsilon^* + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

The sequence (n'), $n'_\varepsilon \in \mathbb{N}$ and ${\Phi^*}_{n'}$ obtained above satisfy the conclusion of the theorem. ■

2.5 A special property

We will prove a special property for the solution U_Φ of the Navier-Stokes equation (2.1) (respectively (2.3) in the case of bounded controls):

$$E \exp\Big\{\beta \int_0^T \|U_\Phi(s)\|_V^2 ds\Big\} < K < \infty \tag{2.29}$$

where $\beta > 0$ satisfies certain conditions and K is a positive constant independent of Φ. We need this property because of the special structure of the Navier-Stokes equation. If (2.29) is not satisfied, we have to consider in the cost functional $\mathcal{J}$ together with the expression of the state U_Φ the expression Δ_{U_Φ} too (as a discount factor). The computations are in this case more complicated. If $\beta := \dfrac{b}{\nu}$ then (2.29) becomes

$$E\Delta^{-1}_{U_\Phi}(T) < K < \infty.$$

We want to show that there exist situations for which (2.29) is satisfied. We formulate some conditions which assure that (2.29) holds. Of course, these conditions are not the only possible ones.

For the stochastic Navier-Stokes equation we assume that the supplementary assumptions hold
(v') $\mathcal{C}$ satisfies assumption **(v)** from Section 1.1 and

$$\gamma := \sup_{\substack{x\in H\\ t\in[0,T]}} \|\mathcal{C}(t,x)\|^2 < \infty;$$

(vii') $x_0 \in H$ (it does not depend on ω).

Let $\Phi \in \mathcal{U}$ or $\Phi \in \mathcal{U}^b$ (we recall the definition of $\mathcal{U}$ and $\mathcal{U}^b$ from Section 2.1). We make the convention: if $\Phi \in \mathcal{U}$ then we take $\rho := 0$ and if $\Phi \in \mathcal{U}^b$ then we take $\mu := 0$.

We consider the conditions:

($\mathbf{C}_1$) $\dfrac{(\nu - \sqrt{\mu}c_{HV})^2}{2\gamma c_{HV}} > \beta;$

($\mathbf{C}_2$) $1 - 2\sqrt{\mu}T > 0, \quad \dfrac{2\nu(\nu - \sqrt{\mu}c_{HV})}{\gamma c_{HV}} > \beta, \quad \dfrac{\nu(1-2\sqrt{\mu}\gamma T)^2}{4\gamma T} > \beta;$

($\mathbf{C}_3$) $\dfrac{\nu^2 e^{-2\sqrt{\mu}T}}{\gamma c_{HV}} > \beta.$

A possible interpretation for this conditions in the case $\beta := \frac{4b}{\nu}$ (which will be used in the following sections) is given at the end of this section in Remark 2.5.2.

Theorem 2.5.1
Assume that hypotheses **(i)-(iv)**, **(v')**, **(vii')** *are fulfilled and* $\Phi \in \mathcal{U}$ *or* $\Phi \in \mathcal{U}^b$. *If one of the conditions* ($\mathbf{C}_1$), ($\mathbf{C}_2$) *or* ($\mathbf{C}_3$) *holds, then there exists a positive constant* K *independent of* Φ *such that inequality* (2.29) *is satisfied.*

PROOF. Applying the Ito formula for $U := U_\Phi$ and using the properties of $\mathcal{A}, \mathcal{B}$ and Φ we obtain

$$\begin{aligned} \|U(t)\|^2 \;&+\; 2\nu\int_0^t \|U(s)\|_V^2 ds \le \|x_0\|^2 + \frac{\rho^2}{2\varepsilon} + 2(\sqrt{\mu}+\varepsilon)\int_0^t \|U(s)\|^2 ds \\ &+\; 2\int_0^t (\mathcal{C}(s,U(s)),U(s))dw(s) + \int_0^t \|\mathcal{C}(s,U(s))\|^2 ds \end{aligned} \tag{2.30}$$

with $\varepsilon > 0$.

We assume that ($\mathbf{C}_1$) is fulfilled: There exists a sufficiently small $\varepsilon > 0$ such that

$$\frac{(\nu - \sqrt{\mu}c_{HV} - \varepsilon c_{HV})^2}{2\gamma c_{HV}} \ge \beta. \tag{2.31}$$

We find an $\eta > 0$ such that

$$2\eta(\nu - \sqrt{\mu}c_{HV} - \varepsilon c_{HV} - \gamma c_{HV}\eta) = \beta. \tag{2.32}$$

By (2.30) and assumption **(v')** we get

$$\eta\|U(t)\|^2 + 2\eta\nu\int_0^t \|U(s)\|_V^2 ds \le \eta\|x_0\|^2 + \eta\gamma T + \frac{\eta\rho^2}{2\varepsilon} + 2(\sqrt{\mu}+\varepsilon)\eta\int_0^t \|U(s)\|^2 ds \tag{2.33}$$

$$+ \quad 2\eta^2\int_0^t (\mathcal{C}(s,U(s)),U(s))^2 ds + 2\eta\int_0^t (\mathcal{C}(s,U(s)),U(s))dw(s) - 2\eta^2\int_0^t (\mathcal{C}(s,U(s)),U(s))^2 ds,$$

which implies that

$$\begin{aligned}\eta\|U(t)\|^2 \quad &+ \quad 2\eta(\nu - \sqrt{\mu}c_{HV} - \varepsilon c_{HV} - \gamma\eta c_{HV})\int_0^t \|U(s)\|_V^2 ds \le \eta\|x_0\|^2 + \eta\gamma T + \frac{\eta\rho^2}{2\varepsilon} \\ &+ \quad 2\eta\int_0^t (\mathcal{C}(s,U(s)),U(s))dw(s) - 2\eta^2\int_0^t (\mathcal{C}(s,U(s)),U(s))^2 ds.\end{aligned}$$

Hence

$$\begin{aligned}& E\exp\Big\{2\eta(\nu - \sqrt{\mu}c_{HV} - \varepsilon c_{HV} - \gamma\eta c_{HV})\int_0^t \|U(s)\|_V^2 ds\Big\} \\ \le \quad & \exp\{\eta\|x_0\|^2 + \eta\gamma T + \frac{\eta\rho^2}{2\varepsilon}\} \\ \times \quad & E\exp\Big\{2\eta\int_0^t (\mathcal{C}(s,U(s)),U(s))dw(s) - 2\eta^2\int_0^t (\mathcal{C}(s,U(s)),U(s))^2 ds\Big\}.\end{aligned}$$

By Levi's inequality (see [12], p. 331) it then follows that

$$E\exp\Big\{2\eta(\nu - \sqrt{\mu}c_{HV} - \varepsilon c_{HV} - \gamma\eta c_{HV})\int_0^t \|U(s)\|_V^2 ds\Big\} \le \exp\Big\{\eta\|x_0\|^2 + \eta\gamma T + \frac{\eta\rho^2}{2\varepsilon}\Big\}.$$

By using ($\mathbf{C}_1$), (2.31) and (2.32), we can find a positive constant K independent of Φ such that

$$E\exp\Big\{\beta\int_0^t \|U(s)\|_V^2 ds\Big\} \le \exp\Big\{\eta\|x_0\|^2 + \eta\gamma T + \frac{\eta\rho^2}{2\varepsilon}\Big\} < K < \infty.$$

Now we assume that ($\mathbf{C}_2$) is fulfilled: There exists a sufficiently small $\varepsilon > 0$ such that

$$1 - 2\sqrt{\mu}T - 2\varepsilon T > 0, \qquad \frac{\nu(1-2\sqrt{\mu}T - 2\varepsilon T)^2}{4\gamma T} \ge \beta. \tag{2.34}$$

By the Ito formula and the property of $\mathcal{B}$ we have

$$\exp\{\eta\|U(t)\|^2\} + 2\eta\int_0^t \langle \mathcal{A}U(s),U(s)\rangle \exp\{\eta\|U(s)\|^2\}ds \le \exp\{\eta\|x_0\|^2\}$$

$$+ \quad \eta\int_0^t \Big(2(\Phi(s,U(s)),U(s)) + \|\mathcal{C}(s,U(s))\|^2\Big)\exp\{\eta\|U(s)\|^2\}ds$$
$$+ \quad 2\eta\int_0^t(\mathcal{C}(s,U(s)),U(s))\exp\{\eta\|U(s)\|^2\}dw(s) + 2\eta^2\int_0^t(\mathcal{C}(s,U(s)),U(s))^2\exp\{\eta\|U(s)\|^2\}ds,$$

with $\eta > 0$. Now we use the properties of $\mathcal{A},\Phi,\mathcal{C}$ to obtain

$$\exp\{\eta\|U(t)\|^2\} + 2\eta\Big(\frac{\nu}{c_{HV}} - \sqrt{\mu} - \varepsilon - \eta\gamma\Big)\int_0^t\|U(s)\|^2\exp\{\eta\|U(s)\|^2\}ds \le \exp\{\eta\|x_0\|^2\}$$
$$+ \quad \eta\Big(\gamma + \frac{\rho^2}{2\varepsilon}\Big)\int_0^t\exp\{\eta\|U(s)\|^2\}ds + 2\eta\int_0^t(\mathcal{C}(s,U(s)),U(s))\exp\{\eta\|U(s)\|^2\}dw(s)$$

for all $t \in [0,T]$. We chose η such that $\nu - \sqrt{\mu}c_{HV} - \varepsilon c_{HV} - \eta\gamma c_{HV} > 0$ (see (2.38)). Then by Proposition B.2 (applied for real valued processes) we get

$$E\exp\{\eta\|U(t)\|^2\} \le c\exp\{\eta\|x_0\|^2\} \quad \text{for all } t \in [0,T], \tag{2.35}$$

where $c > 0$ is a constant depending on $\eta,\gamma,T,\rho,\varepsilon$. Taking into account the Hölder and the Levi inequality in (2.33), we have

$$E\exp\Big\{2\eta\nu\int_0^T\|U(s)\|_V^2ds\Big\} \le \exp\Big\{\eta\|x_0\|^2 + \eta\gamma T + \frac{\eta\rho^2}{2\varepsilon}\Big\} \tag{2.36}$$
$$\times \quad \Big(E\exp\Big\{2\eta p\int_0^T(\mathcal{C}(s,U(s)),U(s))dw(s) - 2\eta^2p^2\int_0^T(\mathcal{C}(s,U(s)),U(s))^2ds\Big\}\Big)^{\frac{1}{p}}$$
$$\times \quad \Big(E\exp\Big\{2\eta q(\sqrt{\mu}+\varepsilon+\eta\gamma p)\int_0^T\|U(s)\|^2ds\Big\}\Big)^{\frac{1}{q}}$$
$$\le \quad \exp\Big\{\eta\|x_0\|^2 + \eta\gamma T + \frac{\eta\rho^2}{2\varepsilon}\Big\}\Big(E\exp\Big\{2\eta q(\sqrt{\mu}+\varepsilon+\eta\gamma p)\int_0^T\|U(s)\|^2ds\Big\}\Big)^{\frac{1}{q}}$$

where $p,q > 1, \dfrac{1}{p} + \dfrac{1}{q} = 1$ (the exact value of p is given in (2.38)). By Jensen's inequality we obtain

$$E\exp\Big\{2\eta q(\sqrt{\mu}+\varepsilon+\eta\gamma p)\int_0^T\|U(s)\|^2ds\Big\} \le \frac{1}{T}E\int_0^T\exp\Big\{2\eta qT(\sqrt{\mu}+\varepsilon+\eta\gamma p)\|U(s)\|^2\Big\}ds. \tag{2.37}$$

We set

$$\eta := \frac{\beta}{2\nu} \quad \text{and} \quad p = \frac{2}{1 - 2\sqrt{\mu}T - 2\varepsilon T}\ . \tag{2.38}$$

Using (2.34) and (2.38) we write the condition for β as follows:

$$\beta \leq \nu \frac{(1-2\sqrt{\mu}T - 2\varepsilon T)p - 1}{\gamma T p^2}.$$

This implies

$$2qT(\sqrt{\mu} + \varepsilon + \eta\gamma p) \leq 1.$$

By applying these estimates in (2.35), (2.36), and (2.37) we get

$$E\exp\Big\{2\eta\nu\int_0^T \|U(s)\|_V^2 ds\Big\} \leq T^{-\frac{1}{q}} \exp\Big\{\eta\|x_0\|^2 + \eta\gamma T + \frac{\eta\rho^2}{2\varepsilon}\Big\}\Bigg(\int_0^T E\exp\Big\{\eta\|U(s)\|^2\Big\}ds\Bigg)^{\frac{1}{q}} < \infty.$$

Hence there exists a positive constant K independent of Φ such that (2.29) holds.

We assume that $(\mathbf{C}_3)$ is fulfilled: There exists a sufficiently small $\varepsilon > 0$ such that

$$\frac{\nu^2 e^{-2(\sqrt{\mu}+\varepsilon)T}}{2\gamma c_{HV}} \geq \beta \tag{2.39}$$

and $\eta > 0$ such that

$$2\eta e^{-2(\sqrt{\mu}+\varepsilon)T}(\nu - \gamma c_{HV}\eta) = \beta. \tag{2.40}$$

By the Ito formula and the properties of $\mathcal{A}, \mathcal{B}, \Phi$ we obtain

$$\begin{aligned} e^{-2(\sqrt{\mu}+\varepsilon)t}\|U(t)\|^2 &+ 2\nu\int_0^t e^{-2(\sqrt{\mu}+\varepsilon)s}\|U(s)\|_V^2 ds \leq \|x_0\|^2 + \frac{\rho^2}{2\varepsilon} \\ &+ 2\int_0^t e^{-2(\sqrt{\mu}+\varepsilon)s}(\mathcal{C}(s,U(s)),U(s))dw(s) + \int_0^t e^{-2(\sqrt{\mu}+\varepsilon)s}\|\mathcal{C}(s,U(s))\|^2 ds. \end{aligned}$$

For an arbitrary fixed $\eta > 0$ we write

$$\begin{aligned} 2\eta(\nu - \gamma\eta c_{HV})\int_0^t e^{-2(\sqrt{\mu}+\varepsilon)s}\|U(s)\|_V^2 ds &\leq \eta\|x_0\|^2 + \eta\gamma T + \frac{\eta\rho^2}{2\varepsilon} \\ + \ 2\eta\int_0^t e^{-2(\sqrt{\mu}+\varepsilon)s}(\mathcal{C}(s,U(s)),U(s))dw(s) &- 2\eta^2\int_0^t e^{-4(\sqrt{\mu}+\varepsilon)s}(\mathcal{C}(s,U(s)),U(s))^2 ds. \end{aligned}$$

Hence

$$\begin{aligned} &E\exp\Big\{2\eta e^{-2(\sqrt{\mu}+\varepsilon)T}(\nu - \gamma\eta c_{HV})\int_0^T \|U(s)\|_V^2 ds\Big\} \leq \exp\Big\{\eta\|x_0\|^2 + \eta\gamma T + \frac{\eta\rho^2}{2\varepsilon}\Big\} \\ \times \ &E\exp\Big\{2\eta\int_0^T e^{-2(\sqrt{\mu}+\varepsilon)s}(\mathcal{C}(s,U(s)),U(s))dw(s) - 2\eta^2\int_0^T e^{-4(\sqrt{\mu}+\varepsilon)s}(\mathcal{C}(s,U(s)),U(s))^2 ds\Big\}. \end{aligned}$$

Using Levi's inequality we have

$$E\exp\Big\{2\eta e^{-2(\sqrt{\mu}+\varepsilon)T}(\nu-\gamma\eta c_{\scriptscriptstyle HV})\int\limits_0^t\|U(s)\|_V^2ds\Big\}\le\exp\Big\{\eta\|x_0\|^2+\eta\gamma T+\frac{\eta\rho^2}{2\varepsilon}\Big\}.$$

By (2.40) it follows that there exists a positive constant K independent of Φ such that

$$E\exp\Big\{\beta\int\limits_0^t\|U(s)\|_V^2ds\Big\}\le\exp\Big\{\eta\|x_0\|^2+\eta\gamma T+\eta\frac{\rho^2}{2\varepsilon}\Big\}<K<\infty. \qquad \blacksquare$$

Remark 2.5.2
1) In the following sections of this chapter we need the condition

$$E\Delta_{U_\Phi}^{-4}(T)=E\exp\Big\{4\frac{b}{\nu}\int\limits_0^T\|U_\Phi(s)\|_V^2ds\Big\}<K<\infty. \tag{2.41}$$

By taking $\beta:=\frac{4b}{\nu}$ the conditions mentioned at the beginning of this section become

($\mathbf{C}_1$) $\dfrac{\nu(\nu-\sqrt{\mu}c_{\scriptscriptstyle HV})^2}{8\gamma c_{\scriptscriptstyle HV}}>b;$

($\mathbf{C}_2$) $1-2\sqrt{\mu}T>0, \quad \dfrac{2\nu^2(\nu-\sqrt{\mu}c_{\scriptscriptstyle HV})}{4\gamma c_{\scriptscriptstyle HV}}>b, \quad \dfrac{\nu^2(1-2\sqrt{\mu}T)^2}{16\gamma T}>b;$

($\mathbf{C}_3$) $\dfrac{\nu^3e^{-2\sqrt{\mu}T}}{4\gamma c_{\scriptscriptstyle HV}}>b.$

If one of these conditions is fulfilled then (2.41) holds.

2) If $\Phi\in\mathcal{U}^b$, then by the convention from the beginning of this section we have $\mu:=0$ and the three conditions from above can be written as follows:

($\mathbf{C}_1^b$) $\dfrac{\nu^3}{8\gamma c_{\scriptscriptstyle HV}}>b;$

($\mathbf{C}_2^b$) $\dfrac{\nu^3}{2\gamma c_{\scriptscriptstyle HV}}>b, \quad \dfrac{\nu^2}{16\gamma T}>b;$

($\mathbf{C}_3^b$) $\dfrac{\nu^3}{4\gamma c_{\scriptscriptstyle HV}}>b.$

If one of these conditions is fulfilled then (2.41) holds.

3) These conditions seem to be very complicate, but they can be interpreted as follows: if ν, involving the viscosity, is large (we have a "very viscous fluid") then we can act with large external forces (μ can be chosen large) and we can have a "strong" influence of the Brownian motion (γ can be chosen large). The inequality $1-2\sqrt{\mu}T>0$ is satisfied if we have large external forces and a small interval $[0,T]$ or conversely, a large interval and small external forces.

2.6 The Gateaux derivative of the cost functional

For the mappings that occur in the expression of the cost functional $\mathcal{J}$ (see (2.2)) we will assume further some supplementary conditions:

($\mathbf{H_3}$) the mappings $\mathcal{L}(t,\cdot,\cdot), \mathcal{K}(\cdot)$ are Fréchet differentiable for each fixed $t \in [0,T]$;

($\mathbf{H_4}$) the mappings $\mathcal{L}_x(t,\cdot,\cdot),\ \mathcal{L}_y(t,\cdot,\cdot), \mathcal{K}'(\cdot)$ are Lipschitz continuous and

$$\|\mathcal{L}_x(t,x,y)\| + \|\mathcal{L}_y(t,x,y)\| \le k_{\mathcal{L}}(1+\|x\|+\|y\|) \qquad \text{and} \qquad \|\mathcal{K}'(x)\| \le k_{\mathcal{K}}(1+\|x\|)$$

for all $t \in [0,T], x, y \in H$, where $k_{\mathcal{L}}, k_{\mathcal{K}}$ are positive constants;

($\mathbf{H_5}$) $\mathcal{L}_x(\cdot,x,y), \mathcal{L}_y(\cdot,x,y) \in \mathcal{L}^2_H[0,T]$ for all $x, y \in H$.

For the stochastic Navier-Stokes equation we assume that the supplementary condition holds:

(v") $\mathcal{C}$ satisfies assumption **(v)** from Section 1.1 and for each $t \in [0,T]$ the mapping $\mathcal{C}(t,\cdot)$ is Fréchet differentiable and $\mathcal{C}'(t,x) \in \mathcal{L}^2(H,H)$, the Fréchet derivative of $\mathcal{C}(t,\cdot)$ at the point x, satisfies

$$\|\mathcal{C}'(t,x)(y)\| \le k_{\mathcal{C}'}\|y\| \quad \text{for all } t \in [0,T], x, y \in H$$

where $k_{\mathcal{C}'}$ is a positive constant independent of t and x.

Using the properties of $\mathcal{B}$ it can be proved that the mapping $x \in V \mapsto \mathcal{B}(x,x) \in V^*$ is Fréchet differentiable and

$$\mathcal{B}'(x)(y) = \mathcal{B}(x,y) + \mathcal{B}(y,x) \quad \text{for all } x, y \in V.$$

We consider the case of **bounded controls**. Let $\Phi, \Upsilon \in \mathcal{U}^b$ such that for sufficiently small $\theta > 0$ we have $\Phi + \theta\Upsilon \in \mathcal{U}^b$. We denote by

$$X_\theta := \frac{U_{\Phi+\theta\Upsilon} - U_\Phi - \theta Z_\Upsilon}{\theta}. \tag{2.42}$$

Throughout this section we assume that $\beta, \nu, \gamma, \rho, T$ are chosen in such a way that

$$E\Delta^{-4}_{U_\Phi}(T) < K < \infty.$$

We recall here the results mentioned in Remark 2.5.2.

Let $\Upsilon \in \mathcal{L}^2_H(\Omega \times [0,T])$. We consider the stochastic evolution equation

$$\begin{aligned}(Z_\Upsilon(t), v) \quad &+ \int_0^t \langle \mathcal{A}Z_\Upsilon(s), v\rangle ds = \int_0^t \langle \mathcal{B}'(U_\Phi(s))(Z_\Upsilon(s)), v\rangle ds \\ &+ \int_0^t (\Upsilon(s), v) ds + \int_0^t (\mathcal{C}'(s, U_\Phi(s))(Z_\Upsilon(s)), v) dw(s)\end{aligned} \tag{2.43}$$

for all $v \in V$, $t \in [0,T]$ and a.e. $\omega \in \Omega$.

Lemma 2.6.1
There exists a V-valued, $\mathcal{F}\times\mathcal{B}_{[0,T]}$-measurable process $\Big(Z_\Upsilon(t)\Big)_{t\in[0,T]}$ adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$, satisfying (2.43) and which has almost surely continuous trajectories in H. The solution is almost surely unique and there exists a constant $c>0$ (independent of Υ) such that

$$E\Delta_{U_\Phi}(T)\|Z_\Upsilon(T)\|^2+E\int_0^T\Delta_{U_\Phi}(t)\|Z_\Upsilon(t)\|_V^2dt\le cE\int_0^T\|\Upsilon(t)\|^2dt$$

and

$$E\Delta_{U_\Phi}^2(T)\|Z_\Upsilon(T)\|^4+E\Big(\int_0^T\Delta_{U_\Phi}(t)\|Z_\Upsilon(t)\|_V^2dt\Big)^2\le cE\int_0^T\|\Upsilon(t)\|^4dt.$$

Proof. We apply Theorem 1.3.1 on $X=Y:=U_\Phi, a_0:=0, \Psi:=\Upsilon, \Gamma:=0, \mathcal{G}(s,h):=\mathcal{C}'(s,U_\Phi(s))(h)$, $Z_{\Psi,\Gamma}:=Z_\Upsilon$. ■

Lemma 2.6.2

(i) *There exists a positive constant c independent of θ such that*

$$E\Delta_{U_\Phi}^2(T)\Big\|\frac{U_{\Phi+\theta\Upsilon}(T)-U_\Phi(T)}{\theta}\Big\|^4+E\Big(\int_0^T\Delta_{U_\Phi}(s)\Big\|\frac{U_{\Phi+\theta\Upsilon}(s)-U_\Phi(s)}{\theta}\Big\|_V^2ds\Big)^2\le cE\int_0^T\|\Upsilon(s)\|^4ds$$

and

$$\lim_{\theta\searrow 0}\frac{1}{\theta^2}E\int_0^T\Delta_{U_\Phi}^2(s)\|U_{\Phi+\theta\Upsilon}(s)-U_\Phi(s)\|_V^2\|U_{\Phi+\theta\Upsilon}(s)-U_\Phi(s)\|^2ds=0.$$

(ii) *The following convergences hold*

$$\lim_{\theta\searrow 0}E\|X_\theta(T)\|^2=0\quad\textit{and}\quad\lim_{\theta\searrow 0}E\int_0^T\|X_\theta(s)\|_V^2ds=0,$$

where X_θ is defined in (2.42).

Proof. For all $t\in[0,T]$ and a.e. $\omega\in\Omega$ let

$$e_1(t)=\Delta_{U_\Phi}(t)\exp\{-(2\lambda+1)t\}.$$

(i) We use the Ito formula and the properties of $\mathcal{A},\mathcal{B},\mathcal{C}$ to get

$$\begin{aligned}(2.44)\qquad & e_1(t)\|U_{\Phi+\theta\Upsilon}(t)-U_\Phi(t)\|^2+\nu\int_0^t e_1(s)\|U_{\Phi+\theta\Upsilon}(s)-U_\Phi(s)\|_V^2ds\\ \le\ & \theta^2\int_0^t e_1(s)\|\Upsilon(s)\|^2ds+2\int_0^t e_1(s)(\mathcal{C}(s,U_{\Phi+\theta\Upsilon}(s))-\mathcal{C}(s,U_\Phi(s)),U_{\Phi+\theta\Upsilon}(s)-U_\Phi(s))dw(s)\end{aligned}$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$. Using Proposition B.2 we obatin

$$E\Big\{\sup_{t\in[0,T]} \Delta^2_{U_\Phi}(t)\Big\|\frac{U_{\Phi+\theta\Upsilon}(t)-U_\Phi(t)}{\theta}\Big\|^4\Big\} \ + \ \nu^2 E\Big(\int_0^T \Delta_{U_\Phi}(s)\Big\|\frac{U_{\Phi+\theta\Upsilon}(s)-U_\Phi(s)}{\theta}\Big\|_V^2 ds\Big)^2$$
$$\leq \ cE\int_0^T \|\Upsilon(s)\|^4 ds$$

where c is a positive constant independent of θ. We write

$$E\int_0^T \Delta^2_{U_\Phi}(s)\|U_{\Phi+\theta\Upsilon}(s)-U_\Phi(s)\|_V^2\|U_{\Phi+\theta\Upsilon}(s)-U_\Phi(s)\|^2 ds$$
$$\leq \ \theta^4\Big[E\Big\{\sup_{t\in[0,T]} \Delta^2_{U_\Phi}(t)\Big\|\frac{U_{\Phi+\theta\Upsilon}(t)-U_\Phi(t)}{\theta}\Big\|^4\Big\}E\Big(\int_0^T \Delta_{U_\Phi}(s)\Big\|\frac{U_{\Phi+\theta\Upsilon}(s)-U_\Phi(s)}{\theta}\Big\|_V^2 ds\Big)^2\Big]^{\frac{1}{2}}.$$

Consequently,

$$\lim_{\theta\searrow 0} \frac{1}{\theta^2}E\int_0^T \Delta^2_{U_\Phi}(s)\|U_{\Phi+\theta\Upsilon}(s)-U_\Phi(s)\|_V^2\|U_{\Phi+\theta\Upsilon}(s)-U_\Phi(s)\|^2 ds = 0.$$

(ii) By the Ito formula and the properties of $\mathcal{A}, \Phi, \Upsilon$ we get

$$e_1(T)\|X_\theta(T)\|^2 + 2\nu\int_0^T e_1(s)\|X_\theta(s)\|_V^2 ds$$
$$\leq \ 2\int_0^T e_1(s)\langle \mathcal{B}(X_\theta(s), U_\Phi(s)) + \frac{1}{\theta}\mathcal{B}(U_{\Phi+\theta\Upsilon}(s)-U_\Phi(s), U_{\Phi+\theta\Upsilon}(s)-U_\Phi(s)), X_\theta(s)\rangle ds$$
$$- \ (2\lambda+1)\int_0^T e_1(s)\|X_\theta(s)\|^2 ds - \frac{b}{\nu}\int_0^T e_1(s)\|U_\Phi(s)\|^2\|X_\theta(s)\|^2 ds$$
$$+ \ 2\int_0^T \frac{e_1(s)}{\theta}\Big(\mathcal{C}(s, U_{\Phi+\theta\Upsilon}(s)) - \mathcal{C}(s, U_\Phi(s)) - \theta\mathcal{C}'(s, U_\Phi(s))(Z_\Upsilon(s)), X_\theta(s)\Big)dw(s)$$
$$+ \ \int_0^T \frac{e_1(s)}{\theta^2}\Big\|\mathcal{C}(s, U_{\Phi+\theta\Upsilon}(s)) - \mathcal{C}(s, U_\Phi(s)) - \theta\mathcal{C}'(s, U_\Phi(s))(Z_\Upsilon(s))\Big\|^2 ds.$$

Using the properties of $\mathcal{B}, \Phi, \Upsilon, \mathcal{C}$, it follows that

$$Ee_1(T)\|X_\theta(T)\|^2 + \frac{3\nu}{4}E\int_0^T e_1(s)\|X_\theta(s)\|_V^2 ds$$

$$\leq \frac{4b}{\nu\theta^2} E\int_0^T e_1(s)\|U_{\Phi+\theta\Upsilon}(s) - U_\Phi(s)\|^2 \|U_{\Phi+\theta\Upsilon}(s) - U_\Phi(s))\|_V^2 ds$$

$$+ \frac{2}{\theta^2} E\int_0^T e_1(s)\|\mathcal{C}(s, U_\Phi(s) + \theta Z_\Upsilon(s)) - \mathcal{C}(s, U_\Phi(s)) - \theta\mathcal{C}'(s, U_\Phi(s))(Z_\Upsilon(s))\|^2 ds.$$

Applying **(i)**, the properties of $\mathcal{C}$ and the Lebesgue Theorem we conclude

$$\lim_{\theta\searrow 0} E\Delta_{U_\Phi}(T)\|X_\theta(T)\|^2 = 0, \quad \lim_{\theta\searrow 0} E\int_0^T \Delta_{U_\Phi}(s)\|X_\theta(s)\|_V^2 ds = 0. \tag{2.45}$$

We see that

$$E\Delta_{U_\Phi}^2(T)\|X_\theta(T)\|^4 + E\Big(\int_0^T \Delta_{U_\Phi}(s)\|X_\theta(s)\|_V^2 ds\Big)^2 \tag{2.46}$$

$$\leq 8\left[E\Delta_{U_\Phi}^2(T)\left\|\frac{U_{\Phi+\theta\Upsilon}(T) - U_\Phi(T)}{\theta}\right\|^4 + E\Delta_{U_\Phi}^2(T)\|Z_\Upsilon(T)\|^4\right.$$

$$\left. + E\left(\int_0^T \Delta_{U_\Phi}(s)\left\|\frac{U_{\Phi+\theta\Upsilon}(s) - U_\Phi(s)}{\theta}\right\|_V^2 ds\right)^2 + E\Big(\int_0^T \Delta_{U_\Phi}(s)\|Z_\Upsilon(s)\|_V^2 ds\Big)^2\right].$$

By using the Schwarz inequality we obtain

$$E\|X_\theta(T)\|^2 \leq \Big(E\Delta_{U_\Phi}(T)\|X_\theta(T)\|^2\Big)^{\frac{1}{2}} \Big(E\Delta_{U_\Phi}^{-4}(T)\Big)^{\frac{1}{4}} \Big(E\Delta_{U_\Phi}^2(T)\|X_\theta(T)\|^4\Big)^{\frac{1}{4}}.$$

Taking into account (2.45), (2.46), **(i)**, Theorem 1.3.1 and the condition $E\Delta_{U_\Phi}^{-4}(T) < K < \infty$ it follows that

$$\lim_{\theta\searrow 0} E\|X_\theta(T)\|^2 = 0.$$

Analogously we can prove that

$$\lim_{\theta\searrow 0} E\int_0^T \|X_\theta(s)\|_V^2 ds = 0. \qquad \blacksquare$$

Remark 2.6.3

For the proof of **(i)** in Lemma 2.6.2 we do not need the condition $E\Delta_{U_\Phi}^{-4}(T) < \infty$.

Theorem 2.6.4
The cost functional $\mathcal{J}$ is Gateaux differentiable with

$$\text{(2.47)} \qquad \frac{\mathrm{d}\mathcal{J}(\Phi+\theta\Upsilon)}{\mathrm{d}\theta}\bigg|_{\theta=0} = E\int_0^T \Big(\mathcal{L}_x[t,U_\Phi(t),\Phi(t)], Z_\Upsilon(t)\Big)dt + E\int_0^T \Big(\mathcal{L}_y[t,U_\Phi(t),\Phi(t)], \Upsilon(t)\Big)dt + E\Big(\mathcal{K}'[U_\Phi(T)], Z_\Upsilon(T)\Big).$$

Proof. We have

$$\text{(2.48)} \qquad \mathcal{K}(x)-\mathcal{K}(\tilde{x}) = \int_0^1 \Big(\mathcal{K}'[\tilde{x}+r(x-\tilde{x}))], x-\tilde{x}\Big)dr$$

and

$$\text{(2.49)} \qquad \mathcal{L}(t,x,y)-\mathcal{L}(t,\tilde{x},\tilde{y}) = \int_0^1 \Big(\mathcal{L}_x[t,\tilde{x}+r(x-\tilde{x}),\tilde{y}+r(y-\tilde{y})], x-\tilde{x}\Big)dr + \int_0^1 \Big(\mathcal{L}_y[t,\tilde{x}+r(x-\tilde{x}),\tilde{y}+r(y-\tilde{y})], y-\tilde{y}\Big)dr$$

for all $x,\tilde{x},y,\tilde{y}\in H,\ t\in[0,T]$. Equation (2.48) implies

$$\mathcal{K}[U_{\Phi+\theta\Upsilon}(T)]-\mathcal{K}[U_\Phi(T)] = \int_0^1 \theta\Big(\mathcal{K}'[U_\Phi(T)+r\theta(X_\theta(T)+Z_\Upsilon(T))], X_\theta(T)+Z_\Upsilon(T)\Big)dr.$$

Using (2.49) we obtain

$$\begin{aligned}
&\mathcal{L}[t,U_{\Phi+\theta\Upsilon}(t),(\Phi+\theta\Upsilon)(t)]-\mathcal{L}[t,U_\Phi(t),\Phi(t)]\\
&= \int_0^1 \Big\{\theta\Big(\mathcal{L}_x[t,U_\Phi(t)+r\theta(X_\theta(t)+Z_\Upsilon(t)),\Phi(t)+r\theta\Upsilon(t)], X_\theta(t)+Z_\Upsilon(t)\Big)\\
&+ \Big(\mathcal{L}_y[t,U_\Phi(t)+r\theta(X_\theta(t)+Z_\Upsilon(t)),\Phi(t)+r\theta\Upsilon(t)], \theta\Upsilon(t)\Big)dr\\
&= \int_0^1 \theta\Big\{\Big(\mathcal{L}_x[t,U_\Phi(t)+r\theta(X_\theta(t)+Z_\Upsilon(t)),\Phi(t)+r\theta\Upsilon(t)], X_\theta(t)+Z_\Upsilon(t)\Big)\\
&+ \Big(\mathcal{L}_y[t,U_\Phi(t)+r\theta(X_\theta(t)+Z_\Upsilon(t)),\Phi(t)+r\theta\Upsilon(t)], \Upsilon(t)\Big)\Big\}dr,
\end{aligned}$$

for all $t\in[0,T]$. Using the properties of $\mathcal{L},\mathcal{K}$, and Lemma 2.6.2 it follows that relation (2.47) holds. ■

Remark 2.6.5
Now we consider the case of feedback controls. Let $\Phi, \Upsilon \in \mathcal{U}$ such that for sufficiently small $\theta > 0$ we have $\Phi + \theta\Upsilon \in \mathcal{U}$.

We assume that $\beta, \nu, \gamma, \rho, T$ are chosen in such a way that

$$E\Delta_{U_\Phi}^{-4}(T) < K < \infty.$$

We recall here the results mentioned in Remark 2.5.2.

Analogously to Theorem 2.6.4 it can be proved that the cost functional $\mathcal{J}$ is Gateaux differentiable with

$$\frac{\mathrm{d}\mathcal{J}(\Phi+\theta\Upsilon)}{\mathrm{d}\theta}\bigg|_{\theta=0} = E\int_0^T \Big(\mathcal{L}_x[t, U_\Phi(t), \Phi(t, U_\Phi(t)], Z_\Upsilon(t)\Big)dt + \tag{2.50}$$

$$+E\int_0^T \Big(\mathcal{L}_y[t, U_\Phi(t), \Phi(t, U_\Phi(t)], \Upsilon(t, U_\Phi(t)) + \Phi(t, Z_\Upsilon(t))\Big)dt + E\Big(\mathcal{K}'[U_\Phi(T)], Z_\Upsilon(T)\Big),$$

where Z_Υ is the solution of the evolution equation

$$\begin{aligned}(Z_\Upsilon(t), v) + \int_0^t \langle \mathcal{A}Z_\Upsilon(s), v\rangle ds \;&=\; \int_0^t \langle \mathcal{B}'(U_\Phi)(Z_\Upsilon(s)), v\rangle ds + \int_0^t (\Upsilon(s, U_\Phi(s)) \\ &+\; \Phi(s, Z_\Upsilon(s)), v)ds + \int_0^t (\mathcal{C}'(s, U_\Phi(s))(Z_\Upsilon(s)), v)dw(s)\end{aligned}$$

for all $v \in V$, $t \in [0,T]$ and a.e. $\omega \in \Omega$. To establish the existence and almost surely uniqueness of the solution of this equation we use the same methods as in Theorem 1.3.1.

2.7 A stochastic minimum principle

We will state a stochastic minimum principle in the case of problem $(\mathcal{P}^b)$. Let $\Phi^* \in \mathcal{U}^b$ be an optimal control with $E\Delta_{U_{\Phi^*}}^{-4}(T) < \infty$, $\Psi \in \mathcal{L}^2_{V^*}(\Omega \times [0,T]), \Gamma \in \mathcal{L}^2_H(\Omega \times [0,T])$ and let $Z_{\Psi,\Gamma}$ be the solution of

$$\begin{aligned}(Z_{\Psi,\Gamma}(t), v) \;&+\; \int_0^t \langle \mathcal{A}Z_{\Psi,\Gamma}(s), v\rangle ds = \int_0^t \langle \mathcal{B}'(U_{\Phi^*}(s))(Z_{\Psi,\Gamma}(s)), v\rangle ds \\ &+\; \int_0^t \langle \Psi(s), v\rangle ds + \int_0^t (\mathcal{C}'(s, U_{\Phi^*}(s))(Z_{\Psi,\Gamma}(s)), v)dw(s) + \int_0^t (\Gamma(s), v)dw(s)\end{aligned} \tag{2.51}$$

for all $v \in V$, $t \in [0,T]$ and a.e. $\omega \in \Omega$. This equation is $(P_{\Psi,\Gamma})$ from Section 1.3 applied for $X = Y := U_{\Phi^*}, a_0 := 0, \mathcal{G}(s,h) := \mathcal{C}'(s, U_{\Phi^*}(s))(h)$.

The mapping

$$
\begin{aligned}
(\Psi,\Gamma) \quad &\in \quad \mathcal{L}^2_{V^*}(\Omega\times[0,T])\times\mathcal{L}^2_H(\Omega\times[0,T]) \mapsto \\
&\mapsto \quad E\int_0^T\Big(\mathcal{L}_x[t,U_{\Phi^*}(t),\Phi^*(t)],Z_{\Psi,\Gamma}(t)\Big)dt + E\Big(\mathcal{K}'[U_{\Phi^*}(T)],Z_{\Psi,\Gamma}(T)\Big)\in\mathbb{R}
\end{aligned}
$$

is linear and continuous, because

$$(\Psi,\Gamma)\in\mathcal{L}^2_{V^*}(\Omega\times[0,T])\times\mathcal{L}^2_H(\Omega\times[0,T])\mapsto Z_{\Psi,\Gamma}\in\mathcal{L}^2_V(\Omega\times[0,T])$$

and

$$(\Psi,\Gamma)\in\mathcal{L}^2_{V^*}(\Omega\times[0,T])\times\mathcal{L}^2_H(\Omega\times[0,T])\mapsto Z_{\Psi,\Gamma}(T)\in\mathcal{L}^2_H(\Omega)$$

are linear. By using the properties for $\mathcal{L},\mathcal{K},U_{\Phi^*},\Phi^*$ and Theorem 1.3.1 we get

$$
\begin{aligned}
&\Big|E\int_0^T\Big(\mathcal{L}_x[t,U_{\Phi^*}(t),\Phi^*(t)],Z_{\Psi,\Gamma}(t)\Big)dt + E\Big(\mathcal{K}'[U_{\Phi^*}(T)],Z_{\Psi,\Gamma}(T)\Big)\Big| \le \Big(E\Delta^{-2}_{U_{\Phi^*}}(T)\Big)^{1/4} \\
\times &\quad \Big\{\Big[E\Big(\int_0^T\|\mathcal{L}_x[t,U_{\Phi^*}(t),\Phi^*(t)]\|^2dt\Big)^2\Big]^{1/4} + \Big(E\|\mathcal{K}'[U_{\Phi^*}(T)]\|^4\Big)^{1/4}\Big\} \\
\times &\quad \Big\{\Big(E\int_0^T\Delta_{U_{\Phi^*}}(t)\|Z_{\Psi,\Gamma}(t)\|^2dt\Big)^{1/2} + \Big(E\Delta_{U_{\Phi^*}}(T)\|Z_{\Psi,\Gamma}(T)\|^2\Big)^{1/2}\Big\} \\
\le &\quad \tilde{c}\Big(E\Delta^{-2}_{U_{\Phi^*}}(T)\Big)^{1/4}\Big(E\int_0^T\|\Psi(t)\|^2_{V^*}dt + E\int_0^T\|\Gamma(t)\|^2dt\Big)^{1/2}
\end{aligned}
$$

where $\tilde{c}$ is a positive constant independent of Ψ and Γ. By the Riesz Theorem it follows that there exist in a unique way processes

$$p\in\mathcal{L}^2_V(\Omega\times[0,T]),\quad q\in\mathcal{L}^2_H(\Omega\times[0,T])$$

such that

$$
\begin{aligned}
(2.52)\quad E\int_0^T\langle\Psi(t),p(t)\rangle dt \quad &+ \quad E\int_0^T\Big(\Gamma(t),q(t)\Big)dt \\
&= \quad E\int_0^T\Big(\mathcal{L}_x[t,U_{\Phi^*}(t),\Phi^*(t)],Z_{\Psi,\Gamma}(t)\Big)dt + E\Big(\mathcal{K}'[U_{\Phi^*}(T)],Z_{\Psi,\Gamma}(T)\Big)
\end{aligned}
$$

for all $\Psi\in\mathcal{L}^2_{V^*}(\Omega\times[0,T]),\Gamma\in\mathcal{L}^2_H(\Omega\times[0,T])$.

Let

$$\mathcal{H}(t,v,\tilde{v},x,y) := \mathcal{L}(t,x,y) + \langle -\mathcal{A}x+\mathcal{B}(x,x),v\rangle + (\mathcal{C}(t,x),\tilde{v}) + (y,v)$$

for $v,x\in V, v,\tilde{v},y\in H$.

Lemma 2.7.1
For all $\Upsilon \in \mathcal{U}^b$ we have

$$\left.\frac{\mathrm{d}\mathcal{J}(\Phi^* + \theta(\Upsilon - \Phi^*))}{\mathrm{d}\theta}\right|_{\theta=0} = E\int\limits_0^T \Big(\mathcal{H}_y[t, p(t), q(t), U_{\Phi^*}(t), \Phi^*(t)], \Upsilon(t) - \Phi^*(t)\Big)dt \geq 0. \tag{2.53}$$

Proof. Since $\mathcal{U}^b$ is convex it follows that $\Phi^* + \theta(\Upsilon - \Phi^*) \in \mathcal{U}^b$ for all $\theta \in [0, 1]$. Equation (2.52) and Theorem 2.6.4 implies

$$\left.\frac{\mathrm{d}\mathcal{J}(\Phi^* + \theta(\Upsilon - \Phi^*))}{\mathrm{d}\theta}\right|_{\theta=0} = E\int\limits_0^T \Big(\Upsilon(t) - \Phi^*(t), p(t)\Big)dt + E\int\limits_0^T \Big(\mathcal{L}_y[t, U_{\Phi^*}(t), \Phi^*(t)], \Upsilon(t) - \Phi^*(t)\Big)dt.$$

Since Φ^* is an optimal control, we have

$$\left.\frac{\mathrm{d}\mathcal{J}(\Phi^* + \theta(\Upsilon - \Phi^*))}{\mathrm{d}\theta}\right|_{\theta=0} \geq 0.$$

Taking into account the definition of $\mathcal{H}$, it follows that (2.53) holds. ■

The statement of **the stochastic minimum principle** is contained in the following theorem.

Theorem 2.7.2
If $\Phi^* \in \mathcal{U}^b$ is an optimal control, then for all $h \in H$ with $\|h\| \leq \rho$ the inequality

$$\Big(\mathcal{L}_y[t, U_{\Phi^*}(t), \Phi^*(t)] + p(t), h - \Phi^*(t)\Big) \geq 0 \tag{2.54}$$

holds for $P \times \Lambda$ a.e. $(\omega, t) \in \Omega \times [0, T]$.

Proof. Let $h \in H$ with $\|h\| \leq \rho$. We denote by

$$\xi(\omega, t) := \Big(\mathcal{H}_y[t, p(t), q(t), U_{\Phi^*}(t), \Phi^*(t)], h - \Phi^*(t)\Big), \quad \mathcal{S} := \{(\omega, t) \in \Omega \times [0, T] | \xi(\omega, t) < 0\},$$

and $\mathcal{S}_t := \{\omega \in \Omega | \xi(\omega, t) < 0\}$ for each $t \in [0, T]$. Obviously, for each $t \in [0, T]$ the set $\mathcal{S}_t$ is $\mathcal{F}_t$-measurable. We take

$$\Upsilon(\omega, t) = \begin{cases} h & , \quad \omega \in \mathcal{S}_t \\ \Phi^*(\omega, t) & , \quad \omega \notin \mathcal{S}_t. \end{cases}$$

We see that $\Upsilon \in \mathcal{U}^b$ and

$$\Big(\mathcal{H}_y[t, p(t), q(t), U_{\Phi^*}(t), \Phi^*(t)], \Upsilon(t) - \Phi^*(t)\Big) = I_{\mathcal{S}_t}(\omega)\xi(\omega, t) \leq 0. \tag{2.55}$$

From Lemma 2.7.1 and (2.55) it follows

$$\begin{aligned} & 0 \leq E\int\limits_0^T \Big(\mathcal{H}_y[t, p(t), q(t), U_{\Phi^*}(t), \Phi^*(t)], \Upsilon(t) - \Phi^*(t)\Big)dt \\ = & \int\limits_{\mathcal{S}} \Big(\mathcal{H}_y[t, p(t), q(t), U_{\Phi^*}(t), \Phi^*(t)], h - \Phi^*(t)\Big)d(P \times \Lambda) = \int\limits_{\mathcal{S}} \xi(\omega, t)d(P \times \Lambda) \leq 0. \end{aligned}$$

Consequently, $(P \times \Lambda)(\mathcal{S}) = 0$ and therefore

$$\Big(\mathcal{H}_y[t, p(t), q(t), U_{\Phi^*}(t), \Phi^*(t)], h - \Phi^*(t)\Big) = \Big(\mathcal{L}_y[t, U_{\Phi^*}(t), \Phi^*(t)] + p(t), h - \Phi^*(t)\Big) \geq 0$$

for $P \times \Lambda$ a.e. $(\omega, t) \in \Omega \times [0, T]$ and all $h \in H$ with $\|h\| \leq \rho$. ■

2.8 Equation of the adjoint processes

To complete the statement of the stochastic minimum principle, we need to derive the equation for the processes $\Big(p(t)\Big)_{t\in[0,T]}$ and $\Big(q(t)\Big)_{t\in[0,T]}$, called **adjoint equation**. We will use an approximation procedure and we will derive the equation for the approximation processes $\Big(p_n(t)\Big)_{t\in[0,T]}$, $\Big(q_n(t)\Big)_{t\in[0,T]}$ $(n \in \mathbb{N})$.

In this section we specialize the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ namely by $\Big(\mathcal{F}_{[w(r):r\leq t]}\Big)_{t\in[0,T]}$, which is the filtration generated by the Wiener process $\Big(w(t)\Big)_{t\in[0,T]}$.

Let $n \in \mathbb{N}$, $\psi \in \mathcal{L}^2_{(H_n,\|\cdot\|_V)}(\Omega \times [0, T]), \Gamma \in \mathcal{L}^2_{H_n}(\Omega \times [0, T])$ and let $\Phi^* \in \mathcal{U}^b$ be an optimal control with $E\Delta^{-4}_{U_{\Phi^*}}(T) < \infty$ (Remark 2.5.2 contains sufficient conditions for this inequality). We consider $Z_{n,\psi,\gamma}$ to be the solution of

$$\begin{aligned}(2.56) \qquad (Z_{n,\psi,\gamma}(t), v) \; &+ \; \int_0^t (\mathcal{A}_n Z_{n,\psi,\gamma}(s), v) ds = \int_0^t (\mathcal{B}'_n(U^n_{\Phi^*}(s))(Z_{n,\psi,\gamma}(s)), v) ds \\ &+ \; \int_0^t (\psi(s), v) ds + \int_0^t (\mathcal{C}'_n(s, U_{\Phi^*}(s))(Z_{n,\psi,\gamma}(s)), v) dw(s) + \int_0^t (\gamma(s), v) dw(s)\end{aligned}$$

for all $v \in H_n$, $t \in [0, T]$ and a.e. $\omega \in \Omega$, where $\mathcal{B}'_n(x)(y) := \sum_{i=1}^n \langle \mathcal{B}'(x)(y), h_i \rangle h_i$ for all $x, y \in V$, $U^n_{\Phi^*} := \Pi_n U_{\Phi^*}$, $\mathcal{C}'_n := \Pi_n \mathcal{C}'$. This equation is $(P_{n,\psi,\gamma})$ from Section 1.3 applied on $a_0 := 0$, $X = Y := U_{\Phi^*}, \mathcal{G}_n(s, h) := \mathcal{C}'_n(s, U_{\Phi^*}(s))(h)$.

The mapping

$$\begin{aligned}(\psi, \gamma) \quad &\in \quad \mathcal{L}^2_{(H_n,\|\cdot\|_V)}(\Omega \times [0, T]) \times \mathcal{L}^2_{H_n}(\Omega \times [0, T]) \mapsto \\ &\mapsto \quad E\int_0^T \Big(\mathcal{L}_x[t, U_{\Phi^*}(t), \Phi^*(t)], Z_{n,\psi,\gamma}(t)\Big) dt + E\Big(\mathcal{K}'[U_{\Phi^*}(T)], Z_{n,\psi,\gamma}(T)\Big) \in \mathbb{R}\end{aligned}$$

is linear and continuous (by the same arguments as in the infinite dimensional case from Section 2.7).

By the Riesz Theorem it follows that there exist in a unique way processes

$$p_n \in \mathcal{L}^2_{(H_n,\|\cdot\|_V)}(\Omega \times [0,T]), q_n \in \mathcal{L}^2_{H_n}(\Omega \times [0,T])$$

such that

$$\begin{aligned} (2.57)\quad E\int_0^T \Big(\psi(t), p_n(t)\Big)dt \quad &+ \quad E\int_0^T \Big(\gamma(t), q_n(t)\Big)dt \\ &= \quad E\int_0^T \Big(\mathcal{L}_x[t, U_{\Phi^*}(t), \Phi^*(t)], Z_{n,\psi,\gamma}(t)\Big)dt + E\Big(\mathcal{K}'[U_{\Phi^*}(T)], Z_{n,\psi,\gamma}(T)\Big) \end{aligned}$$

for all $\psi \in \mathcal{L}^2_{(H_n,\|\cdot\|_V)}(\Omega \times [0,T]), \gamma \in \mathcal{L}^2_{H_n}(\Omega \times [0,T])$.

Let $\Psi \in \mathcal{L}^2_{V^*}(\Omega \times [0,T]), \Gamma \in \mathcal{L}^2_H(\Omega \times [0,T])$ and set

$$\Psi_n := \sum_{i=1}^n \langle \Psi, h_i \rangle h_i, \qquad \Gamma_n := \Pi_n \Gamma.$$

We have

$$\begin{aligned} (2.58)\qquad E\int_0^T \langle \Psi(t), p_n(t)\rangle dt \quad &+ \quad E\int_0^T \Big(\Gamma(t), q_n(t)\Big)dt \\ &= \quad E\int_0^T \Big(\Psi_n(t), p_n(t)\Big)dt + E\int_0^T \Big(\Gamma_n(t), q_n(t)\Big)dt \\ &= \quad E\int_0^T \Big(\Delta^{-1}_{U_{\Phi^*}}(t)\mathcal{L}_x[t, U_{\Phi^*}(t), \Phi^*(t)], \Delta_{U_{\Phi^*}}(t) Z_{n,\Psi_n,\Gamma_n}(t)\Big)dt \\ &+ \quad E\Big(\Delta^{-1}_{U_{\Phi^*}}(T)\mathcal{K}'[U_{\Phi^*}(T)], \Delta_{U_{\Phi^*}}(T) Z_{n,\Psi_n,\Gamma_n}(T)\Big). \end{aligned}$$

From the properties of the solution of the Navier-Stokes equation (see Lemma 1.2.6) and from the hypothesis on $\mathcal{L}$ and $\mathcal{K}$ we can deduce that

$$\Delta^{-1}_{U_{\Phi^*}}(t)\mathcal{L}_x[t, U_{\Phi^*}(t), \Phi^*(t)] \in \mathcal{L}^2_H(\Omega \times [0,T]), \quad \Delta^{-1}_{U_{\Phi^*}}(T)\mathcal{K}'[U_{\Phi^*}(T)] \in \mathcal{L}^2_H(\Omega).$$

We have $\Psi = \lim_{n\to\infty} \Psi_n$ in the space $\mathcal{L}^2_{V^*}(\Omega \times [0,T])$ and $\Gamma = \lim_{n\to\infty} \Gamma_n$ in the space $\mathcal{L}^2_H(\Omega \times [0,T])$. Now we use Lemma 1.3.2 and (2.52) in (2.58) to obtain

$$\begin{aligned}
\lim_{n\to\infty} \Big\{ E\int_0^T \langle \Psi(t), p_n(t)\rangle dt + E\int_0^T \Big(\Gamma(t), q_n(t)\Big) dt \Big\} \\
&= E\int_0^T \Big(\mathcal{L}_x[t, U_{\Phi^*}(t), \Phi^*(t)], Z_{\Psi,\Gamma}(t)\Big) dt + +E\Big(\mathcal{K}'[U_{\Phi^*}(T)], Z_{\Psi,\Gamma}(T)\Big) \\
&= E\int_0^T \langle \Psi(t), p(t)\rangle dt + E\int_0^T \Big(\Gamma(t), q(t)\Big) dt,
\end{aligned}$$

for all $\Psi \in \mathcal{L}^2_{V^*}(\Omega\times[0,T]), \Gamma\in\mathcal{L}^2_H(\Omega\times[0,T])$. Hence, for $n\to\infty$ we have

$$p_n \rightharpoonup p \quad \text{in} \quad \mathcal{L}^2_V(\Omega\times[0,T]) \quad \text{and} \quad q_n \rightharpoonup q \quad \text{in} \quad \mathcal{L}^2_H(\Omega\times[0,T]). \tag{2.59}$$

In (2.57) we take $\psi := p_n, \gamma := q_n$, use the weak convergence from above and Lemma 1.3.2. Then

$$\begin{aligned}
\lim_{n\to\infty} \Big\{ E\int_0^T \|p_n(t)\|^2 dt \quad + \quad E\int_0^T \|q_n(t)\|^2 dt \Big\} = E\int_0^T \Big(\mathcal{L}_x[t, U_{\Phi^*}(t), \Phi^*(t)], Z_{\mathrm{J}p,q}(t)\Big) dt \\
+ \quad E\Big(\mathcal{K}'[U_{\Phi^*}(T)], Z_{\mathrm{J}p,q}(T)\Big) = E\int_0^T \|p(t)\|^2_V dt + E\int_0^T \|q(t)\|^2 dt.
\end{aligned} \tag{2.60}$$

From (2.59) and (2.60) it follows that the following strong convergences hold:

$$\lim_{n\to\infty} p_n = p \quad \text{in } \mathcal{L}^2_V(\Omega\times[0,T]) \quad \text{and} \quad \lim_{n\to\infty} q_n = q \quad \text{in } \mathcal{L}^2_H(\Omega\times[0,T]). \tag{2.61}$$

Now we derive the equations for $\Big(p_n(t)\Big)_{t\in[0,T]}$ and $\Big(q_n(t)\Big)_{t\in[0,T]}$ $(n\in\mathbb{N})$ and then by passing to the limit obtain the equation for $\Big(p(t)\Big)_{t\in[0,T]}$ and $\Big(q(t)\Big)_{t\in[0,T]}$.

We consider the following matrices:

$$\tilde{\mathcal{A}}_n := \Big(\langle \mathcal{A}h_j, h_i\rangle\Big)_{i,j=1,n}, \; I_n := \Big(\delta_{i,j}\Big)_{i,j=1,n},$$

$$\tilde{\mathcal{B}}_n(s) := \Big(\langle \mathcal{B}'(U^n_{\Phi^*}(s))(h_j), h_i\rangle\Big)_{i,j=1,n}, \; \tilde{\mathcal{C}}_n(s) := \Big((\mathcal{C}'(s, U_{\Phi^*}(s))(h_j), h_i)\Big)_{i,j=1,n}.$$

The last two matrices depend on s and ω and are $\mathcal{F}_s$-measurable.

For each natural number n we introduce the $n\times n$ matrix processes

$$\Big(X_n(t)\Big)_{t\in[0,T]} = \Big(\Big(X^{i,j}_n(t)\Big)_{i,j=1,n}\Big)_{t\in[0,T]}, \qquad \Big(Y_n(t)\Big)_{t\in[0,T]} = \Big(\Big(Y^{i,j}_n(t)\Big)_{i,j=1,n}\Big)_{t\in[0,T]}$$

as the solutions of the stochastic matrix equations

$$X_n(t) + \int_0^t \tilde{\mathcal{A}}_n X_n(s) ds = I_n + \int_0^t \tilde{\mathcal{B}}_n(s) X_n(s) ds + \int_0^t \tilde{\mathcal{C}}_n(s) X_n(s) dw(s) \tag{2.62}$$

and

$$(2.63)\quad Y_n(t) - \int_0^t Y_n(s)\tilde{\mathcal{A}}_n(s)ds = I_n - \int_0^t Y_n(s)\tilde{\mathcal{B}}_n(s)ds + \int_0^t Y_n(s)\tilde{\mathcal{C}}_n(s)\tilde{\mathcal{C}}_n(s)ds - \int_0^t Y_n(s)\tilde{\mathcal{C}}_n(s)dw(s)$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$. To prove the existence and (almost surely) uniqueness of the solution of (2.62) and (2.63), respectively, we consider the above equations as linear evolution equations with the unknown variable X_n, respectively Y_n. Their coeficients may depend on ω and s (see $\tilde{\mathcal{B}}_n, \tilde{\mathcal{C}}_n$). We use the same techniques as in the investigation of equations $(P_{\Psi,\Gamma})$, $(P_{n,\psi,\gamma})$ in Section 1.3. For each $i,j \in \{1,\ldots,n\}$ the process $\left(Y_n^{i,j}(t)\right)_{t\in[0,T]}$ has continuous trajectories in $\mathbb{R}$.

Using the Ito formula we obtain

$$(2.64)\qquad Y_n(t)X_n(t) = I_n \quad \text{for all } t \in [0,T],\ \text{a.e. } \omega \in \Omega$$

and hence

$$(2.65)\qquad X_n(t)Y_n(t) = I_n \quad \text{for all } t \in [0,T],\ \text{a.e. } \omega \in \Omega.$$

If $M := \left(M_{i,j}\right)_{i,j=1,n}$ is a matrix of real numbers and $h \in H_n$, then we write

$$Mh := \sum_{i,j=1}^{n} M_{i,j}(h,h_j)h_i.$$

We write $\widehat{M}$ for the transposed matrix of M.

Theorem 2.8.1
The processes $\left(p(t)\right)_{t\in[0,T]}$ and $\left(q(t)\right)_{t\in[0,T]}$ satisfy the adjoint equation

$$\begin{aligned}(\mathcal{K}'[U_{\Phi^*}(T)] - p(t), v) \quad &- \quad \int_t^T \langle \mathcal{A}v, p(s)\rangle ds \\ &= \quad -\int_t^T \langle \mathcal{B}(U_{\Phi^*}(s), v) + \mathcal{B}(v, U_{\Phi^*}(s)), p(s)\rangle ds - \int_t^T (\mathcal{L}_x[s, U_{\Phi^*}(s), \Phi^*(s)], v)ds \\ &- \quad \int_t^T (\mathcal{C}'(s, U_{\Phi^*}(s))(v), q(s))ds + \int_t^T (q(s), v)dw(s),\end{aligned}$$

for all $t \in [0,T], v \in V$ and a.e. $\omega \in \Omega$. The processes $\left(p(t)\right)_{t\in[0,T]}$ and $\left(q(t)\right)_{t\in[0,T]}$ are uniquely characterized by this equation.

Proof. Let $\psi \in \mathcal{D}_V(\Omega \times [0,T])$, $\gamma \in \mathcal{D}_H(\Omega \times [0,T])$ and we define for each $K \in \mathbb{N}$ the stopping time

$$\mathcal{T}_K^n := \min\{\mathcal{T}_K^{Y_n^{i,j}} \ :\ 1 \le i,j \le n\}$$

and we take

$$\psi_K := I_{[0,\mathcal{T}_K^n]}\psi,\ \ \gamma_K := I_{[0,\mathcal{T}_K^n]}\gamma.$$

We consider the H_n-valued process

$$W_n^K(t) := Y_n(t)Z_{n,\psi_K,\gamma_K}(t)$$

where the process $\Big(Z_{n,\psi_K,\gamma_K}(t)\Big)_{t\in[0,T]}$ is the solution of (2.56) (with ψ_K and γ_K instead of ψ and γ). Using (2.65) we obtain for all $t \in [0,T]$ and a.e. $\omega \in \Omega$ that

$$Z_{n,\psi_K,\gamma_K}(t) = X_n(t)W_n^K(t). \tag{2.66}$$

Using (2.56), (2.63), and the Ito formula it follows that the process $\Big(W_n^K(t)\Big)_{t\in[0,T]}$ satisfies

$$(W_n^K(t),h) = \int_0^t (Y_n(s)\psi_K(s),h)ds - \int_0^t (Y_n(s)\tilde{\mathcal{C}}_n(s)\gamma_K(s),h)ds + \int_0^t (Y_n(s)\gamma_K(s),h)dw(s) \tag{2.67}$$

for all $t \in [0,T]$, $h \in H_n$ and a.e. $\omega \in \Omega$.

We use (2.57) and (2.66) to obtain

$$\begin{aligned} & E\int_0^T \Big(\psi_K(t),p_n(t)\Big)dt + E\int_0^T \Big(\gamma_K(t),q_n(t)\Big)dt \\ = \;& E\int_0^T \Big(\mathcal{L}_x[t,U_{\Phi^*}(t),\Phi^*(t)],Z_{n,\psi_K,\gamma_K}(t)\Big)dt + E\Big(\mathcal{K}'[U_{\Phi^*}(T)],Z_{n,\psi_K,\gamma_K}(T)\Big) \\ = \;& E\int_0^T \Big(\widehat{X}_n(t)\mathcal{L}_x^n[t,U_{\Phi^*}(t),\Phi^*(t)],W_n^K(t)\Big)dt + E\Big(\widehat{X}_n(T)\mathcal{K}_n'[U_{\Phi^*}(T)],W_n^K(T)\Big) \end{aligned} \tag{2.68}$$

where $\mathcal{L}_x^n(t,x,y) := \Pi_n\mathcal{L}_x(t,x,y), \mathcal{K}_n'(x) := \Pi_n\mathcal{K}'(x),\ t \in [0,T], x,y \in H$. Let us define the H_n-valued random variable

$$\xi_n = \widehat{X}_n(T)\mathcal{K}_n'[U_{\Phi^*}(T)] + \int_0^T \widehat{X}_n(t)\mathcal{L}_x^n[t,U_{\Phi^*}(t),\Phi^*(t)]dt, \tag{2.69}$$

and the H_n-valued process

$$\zeta_n(t) = -\int_0^t \widehat{X}_n(t)\mathcal{L}_x^n[t,U_{\Phi^*}(t),\Phi^*(t)]dt + E(\xi_n|\mathcal{F}_t) \tag{2.70}$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$. By the representation theorem of Levy (see [18], Theorem 4.15, p. 182 and Problem 4.17, p. 184) we have

$$E(\xi_n|\mathcal{F}_t) = E\xi_n + \int_0^t G_n(s)dw(s) \tag{2.71}$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$ where $G_n \in \mathcal{L}^2_{H_n}(\Omega \times [0,T])$. Without loss of generality we can assume that the process $\Big(\zeta_n(t)\Big)_{t\in[0,T]}$ has continuous trajectories in H. We see that $\zeta_n(T) = \widehat{X}_n(T)\mathcal{K}_n'[U_{\Phi^*}(T)]$ for a.e. $\omega \in \Omega$.

By using (2.69), (2.70), and (2.71) we deduce by Ito's calculus that

$$E\Big(\zeta_n(T), W_n^K(T)\Big) = -E\int\limits_0^T \Big(\widehat{X}_n(t)\mathcal{L}_x^n[t, U_{\Phi^*}(t), \Phi^*(t)], W_n^K(t)\Big)dt$$
$$+E\int\limits_0^T \Big\{(\zeta_n(t), Y_n(t)\psi_K(t) - Y_n(t)\tilde{\mathcal{C}}_n(t)\gamma_K(t)) + (G_n(t), Y_n(t)\gamma_K(t))\Big\}dt.$$

Here we have omitted to write explicitly an intermediate step: To consider stopping times for G_n. After taking the mathematical expectation in the above relation (with $\mathcal{T}_M^{G_n}$ instead of T) we let these stopping times to tend to T and use the almost surely continuity of the trajectories of ζ_n and W_n^K. Then we obtain the above equality.

Hence,

$$E\Big(\widehat{X}_n(T)\mathcal{K}_n'[U_{\Phi^*}(T)], W_n^K(T)\Big) + E\int\limits_0^T \Big(\widehat{X}_n(t)\mathcal{L}_x^n[t, U_{\Phi^*}(t), \Phi^*(t)], W_n^K(t)\Big)dt$$
$$= E\int\limits_0^T \Big(\widehat{Y}_n(t)\zeta_n(t), \psi_K(t)\Big)dt + E\int\limits_0^T \Big(\widehat{Y}_n(t)G_n(t) - \widehat{\tilde{\mathcal{C}}}_n(t)\widehat{Y}_n(t)\zeta_n(t), \gamma_K(t)\Big)dt.$$

The processes ψ, γ were arbitrary fixed, and by (2.57) and (2.66) it follows that

$$I_{[0,\mathcal{T}_K^n]}(t)p_n(t) = I_{[0,\mathcal{T}_K^n]}(t)\widehat{Y}_n(t)\zeta_n(t) \quad \text{for} \quad P\times\Lambda \text{ a.e. } (\omega,t)\in\Omega\times[0,T] \tag{2.72}$$

and

$$I_{[0,\mathcal{T}_K^n]}(t)q_n(t) = I_{[0,\mathcal{T}_K^n]}(t)\Big(\widehat{Y}_n(t)G_n(t) - \widehat{\tilde{\mathcal{C}}}_n(t)\widehat{Y}_n(t)\zeta_n(t)\Big) = I_{[0,\mathcal{T}_K^n]}(t)\Big(\widehat{Y}_n(t)G_n(t) - \widehat{\tilde{\mathcal{C}}}_n(t)p_n(t)\Big)$$

for $P\times\Lambda$ a.e. $(\omega,t)\in\Omega\times[0,T]$. Since $\lim\limits_{K\to\infty}\mathcal{T}_K^n = T$ for a.e. $\omega\in\Omega$ (see Proposition B.1) and by using (2.72) we have

$$0 = \lim_{K\to\infty} E\int\limits_0^{\mathcal{T}_K^n} \|p_n(t) - \widehat{Y}_n(t)\zeta_n(t)\|dt = E\int\limits_0^T \|p_n(t) - \widehat{Y}_n(t)\zeta_n(t)\|dt.$$

This implies

$$p_n(t) = \widehat{Y}_n(t)\zeta_n(t) \quad \text{for} \quad P\times\Lambda \text{ a.e. } (\omega,t)\in\Omega\times[0,T].$$

Analogously we obtain

$$q_n(t) = \widehat{Y}_n(t)G_n(t) - \widehat{\tilde{\mathcal{C}}}_n(t)p_n(t)\Big) \quad \text{for} \quad P\times\Lambda \text{ a.e. } (\omega,t)\in\Omega\times[0,T].$$

We can identify $\Big(p_n(t)\Big)_{t\in[0,T]}$ with a process which has continuous trajectories in H. Then for all $t \in [0,T]$ we have

$$p_n(t) = \widehat{Y}_n(t)\zeta_n(t) \quad \text{for all } t \in [0,T] \quad \text{and} \quad p_n(T) = \mathcal{K}_n'[U_{\Phi^*}(T)] \quad \text{for a.e} \quad \omega \in \Omega.$$

By using the equations for $\Big(\widehat{Y}_n(t)\Big)_{t\in[0,T]}$ and $\Big(\zeta_n(t)\Big)_{t\in[0,T]}$ it follows by the Ito calculus that $\Big(p_n(t)\Big)_{t\in[0,T]}$ satisfies for all $t \in [0,T]$ and a.e. $\omega \in \Omega$ the n-dimensional evolution equation

$$\begin{aligned}
(2.73) \qquad p_n(T) - p_n(t) - \int_t^T \widehat{\widetilde{\mathcal{A}}_n} p_n(s)ds &= -\int_t^T \Big\{\widehat{\widetilde{\mathcal{B}}_n}(s)p_n(s) + \mathcal{L}_x^n[s, U_{\Phi^*}(s), \Phi^*(s)]\Big\}ds \\
&- \int_t^T \widehat{\widetilde{\mathcal{C}}_n}(s)q_n(s)ds + \int_t^T q_n(s)dw(s)
\end{aligned}$$

with $p_n(T) = \mathcal{K}_n'[U_{\Phi^*}(T)]$. Equation (2.73) can be written equivalently as

$$\begin{aligned}
(p_n(T) - p_n(t), v) &- \int_t^T \langle \mathcal{A}v, p_n(s)\rangle ds \\
&= -\int_t^T \langle \mathcal{B}(U_{\Phi^*}^n(s), v) + \mathcal{B}(v, U_{\Phi^*}^n(s)), p_n(s)\rangle ds - \int_t^T (\mathcal{L}_x^n[s, U_{\Phi^*}(s), \Phi^*(s)], v)ds \\
&- \int_t^T \Big(\mathcal{C}'(s, U_{\Phi^*}(s))(v), q_n(s)\Big)ds + \int_t^T (q_n(s), v)dw(s),
\end{aligned}$$

for all $t \in [0,T], v \in H_n$ and a.e. $\omega \in \Omega$. In this equation we take the limit for $n \to \infty$, use (2.61), and obtain

$$\begin{aligned}
(2.74) \qquad (\mathcal{K}'[U_{\Phi^*}(T)] - p(t), v) &- \int_t^T \langle \mathcal{A}v, p(s)\rangle ds \\
&= -\int_t^T \langle \mathcal{B}(U_{\Phi^*}(s), v) + \mathcal{B}(v, U_{\Phi^*}(s)), p(s)\rangle ds - \int_t^T (\mathcal{L}_x[s, U_{\Phi^*}(s), \Phi^*(s)], h)ds \\
&- \int_t^T \Big(\mathcal{C}'(s, U_{\Phi^*}(s))(v), q(s)\Big)ds + \int_t^T (q(s), v)dw(s),
\end{aligned}$$

for $P \times \Lambda$ a.e. $(\omega, t) \in \Omega \times [0,T]$ and all $v \in V$. We can identify $\Big(p(t)\Big)_{t\in[0,T]}$ with a process which has continuous trajectories in H and satisfies (2.74) for all $t \in [0,T]$ and a.e. $\omega \in \Omega$.

In order to show that (2.74) characterize in a unique way the adjoint processes $\Big(p(t)\Big)_{t\in[0,T]}$ and $\Big(q(t)\Big)_{t\in[0,T]}$, let us take any processes $\Big(p(t)\Big)_{t\in[0,T]}$ and $\Big(q(t)\Big)_{t\in[0,T]}$ which satisfy (2.74). Let

$\Psi \in \mathcal{L}^2_{V^*}(\Omega \times [0,T])$, $\Gamma \in \mathcal{L}^2_H(\Omega \times [0,T])$ and let $Z_{\Psi,\Gamma}$ be the solution of (2.51). Then we have

$$E\Big(p(T), Z_{\Psi,\Gamma}(T)\Big) = E\int\limits_0^T \Big\{\langle \mathcal{A}Z_{\Psi,\Gamma}(t), p(t)\rangle - \langle \mathcal{B}'(U_{\Phi^*}(t))(Z_{\Psi,\Gamma}(t)), p(t)\rangle$$
$$- \quad \Big(\mathcal{L}_x[t, U_{\Phi^*}(t), \Phi^*(t)], Z_{\Psi,\Gamma}(t)\Big) - \Big(\mathcal{C}'(t, U_{\Phi^*}(t))(Z_{\Psi,\Gamma}(t)), q(t)\Big) - \langle \mathcal{A}Z_{\Psi,\Gamma}(t), p(t)\rangle$$
$$+ \quad \langle \mathcal{B}'(U_{\Phi^*}(t))(Z_{\Psi,\Gamma}(t)), p(t)\rangle + \langle \Psi(t), p(t)\rangle + \Big(\mathcal{C}'(t, U_{\Phi^*}(t))(Z_{\Psi,\Gamma}(t)), q(t)\Big) + (\Gamma(t), q(t))\Big\}dt.$$

Hence for all $\Psi \in \mathcal{L}^2_{V^*}(\Omega \times [0,T]), \Gamma \in \mathcal{L}^2_H(\Omega \times [0,T])$ we get

$$\begin{aligned} E\Big(\mathcal{K}'[U_{\Phi^*}(T)], Z_{\Psi,\Gamma}(T)\Big) \quad &+ \quad E\int\limits_0^T \Big(\mathcal{L}_x[t, U_{\Phi^*}(t), \Phi^*(t)], Z_{\Psi,\Gamma}(t)\Big)dt \\ &= \quad E\int\limits_0^T \langle \Psi(t), p(t)\rangle dt + E\int\limits_0^T (\Gamma(t), q(t))dt. \end{aligned}$$

Therefore, $\Big(p(t)\Big)_{t\in[0,T]}$ and $\Big(q(t)\Big)_{t\in[0,T]}$ must be the processes that are uniquely defined in (2.52). ■

Chapter 3

About the Dynamic Programming Equation

In Section 3.1 of this chapter we prove that the solution of the stochastic Navier-Stokes equation is a Markov process (see Theorem 3.1.1). In Section 3.2 we illustrate the dynamic programming approach (called also Bellman's principle) and we give a formal derivation of Bellman's equation. Bellman's principle turns the stochastic control problem into a deterministic control problem of a nonlinear partial differential equation of second order (see equation (3.11)) involving the infinitesimal generator. To round off the results of Chapter 2 we give a sufficient condition for an optimal control (Theorem 3.2.3 and Theorem 3.2.4). This condition requires a suitably behaved solution of the Bellman equation and an admissible control satisfying a certain equation. In this section we consider the finite dimensional stochastic Navier-Stokes equation, i.e., the equations (P_n) used in the Galerkin method in Section 1.2. The approach would be very complicate for the infinite dimensional case, because in this case it is difficult to obtain the infinitesimal generator. M.J. Vishik and A.V. Fursikov investigated in Chapter 11 of [35] the inverse Kolmogorov equations, which give the inifinitesimal generator of the process being solution of the considered equation, only for the case of $n = 2$ for (0.1). We take into account ideas and results on optimal control of Markov diffusion processes from the book of W.H. Fleming and R.W. Rishel [9] and adapt them for our problem.

3.1 The Markov property

An important property used in the dynamic programming approach is the Markov property of the solution of the Navier-Stokes equation. We will prove this property in this section.

Let us introduce the following σ-algebras

$$\sigma_{[U(s)]} := \sigma\{U(s)\}, \quad \sigma_{[U(r):r\leq s]} := \sigma\{U(r) : r \leq s\}$$

and the event

$$\sigma_{[U(s)=y]} := \{\omega : U(s) = y\}.$$

We define for the solution $U := U_\Phi$ of the Navier-Stokes equation (2.1), where $\Phi \in \mathcal{U}$, the **transition function**

$$\bar{P}(s, x, t, A) := P(U(t) \in A | \sigma_{[U(s)=x]})$$

with $s, t \in [0, T], s < t, x \in H, A \in B(H)$. In the following theorem we prove that **the solution of the Navier-Stokes equation is a Markov process**. This means that the state $U(s)$ at time s must contain all probabilistic information relevant to the evolution of the process for times $t > s$.

Theorem 3.1.1

(i) *For fixed $s, t \in [0, T], s < t, A \in B(H)$ the mapping*

$$y \in H \mapsto \bar{P}(s, y, t, A) \in \mathbb{R}$$

is measurable.

(ii) *The following equalities hold*

$$P(U(t) \in A | \mathcal{F}_s) = P(U(t) \in A | \sigma_{[U(s)]})$$

and

$$P\Big(U(t) \in A \Big| \sigma_{[U(r): r \le s]}\Big) = P\Big(U(t) \in A \Big| \sigma_{[U(s)]}\Big)$$

for all $s, t \in [0, T], s < t, y \in H, A \in B(H)$.

Proof. **(i)** Let $s, t \in [0, T], s < t, y \in H$. We denote by $\Big(\tilde{U}(t, s, y)\Big)_{t \in [s,T]}$ the solution of the Navier-Stokes equation starting in s with the initial value y, i.e. $\tilde{U}(s, s, y) = y$ for a.e. $\omega \in \Omega$.

Let $A \in B(H)$. Without loss of generality we can consider the set A to be closed. Let (a_n) be a sequence of continuous and uniformly bounded functions $a_n : H \to \mathbb{R},\ n \in \mathbb{N}$ such that

$$\lim_{n \to \infty} \|a_n(y) - I_A(y)\| = 0 \quad \text{for all } y \in H. \tag{3.1}$$

By the uniqueness of the solution of the Navier-Stokes equation and from the definition of the transition function we have

$$\bar{P}(s, y, t, A) = E\Big(I_A(U(t)) \Big| \sigma_{[U(s)=y]}\Big) = E\Big(I_A(\tilde{U}(t, s, y))\Big).$$

We consider an arbitrary sequence (y_n) in H such that $\lim\limits_{n \to \infty} \|y_n - y\| = 0$. Using the same method as in the proof of Lemma 2.2.1 we can prove that

$$\lim_{n \to \infty} E\|\tilde{U}(t, s, y_n) - \tilde{U}(t, s, y)\|^2 = 0. \tag{3.2}$$

Therefore $\Big(\tilde{U}(t, s, y_n)\Big)$ converges in probability to $\tilde{U}(t, s, y)$. Using (3.2) and the Lebesgue Theorem it follows that for all $k \in \mathbb{N}$

$$\lim_{n \to \infty} Ea_k\Big(\tilde{U}(t, s, y_n)\Big) = Ea_k\Big(\tilde{U}(t, s, y)\Big).$$

We conclude that for each $k \in \mathbb{N}$ the mapping

$$y \in H \mapsto Ea_k\Big(\tilde{U}(t,s,y)\Big) \in \mathbb{R}$$

is continuous. Hence it is measurable. By the Lebesgue Theorem and (3.1) we deduce that for all $y \in H$

$$\lim_{k\to\infty} Ea_k\Big(\tilde{U}(t,s,y)\Big) = EI_A\Big(\tilde{U}(t,s,y)\Big).$$

Consequently, $\bar{P}(s,\cdot,t,A) = EI_A\Big(\tilde{U}(t,s,\cdot)\Big)$ is measurable, because it is the pointwise limit of measurable functions.

(ii) First we prove that for each fixed $s,t \in [0,T], s < t, y \in H$ the random variable $\tilde{U}(t,s,y)$ (considered as a H-valued random variable) is independent of $\mathcal{F}_s$. By relation (1.12) from Section 1.2 we have

$$\lim_{M\to\infty} \|\tilde{U}_n^M(t,s,y) - \tilde{U}_n(t,s,y)\| = 0 \quad \text{for each } n \in \mathbb{N} \text{ and a.e. } \omega \in \Omega, \tag{3.3}$$

and by Theorem 1.2.7 it follows that there exists a subsequence (n') of (n) such that

$$\lim_{n'\to\infty} \|\tilde{U}_{n'}(t,s,y) - \tilde{U}(t,s,y)\| = 0 \quad \text{for a.e. } \omega \in \Omega \tag{3.4}$$

where $\Big(\tilde{U}_n^M(t,s,y)\Big)_{t\in[s,T]}$ and $\Big(\tilde{U}_n(t,s,y)\Big)_{t\in[s,T]}$ are the solutions of (P_n^M) and (P_n), respectively, if we start in s with the initial value y (see Section 1.2). Since for fixed n, M the random variable $\tilde{U}_n^M(t,s,y)$ is approximated by Picard-iteration and each Picard-approximation is independent of $\mathcal{F}_s$ (as a H-valued random variable), it follows by Proposition B.4 that $\tilde{U}_n(t,s,y)$ is independent of $\mathcal{F}_s$. Using (3.3), (3.4), and Proposition B.4 we conclude that $\tilde{U}(t,s,y)$ is independent of $\mathcal{F}_s$.

Let $A \in B(H)$. Now we apply Proposition B.5 for $\widehat{\mathcal{F}} := \mathcal{F}_s, f(y,\omega) := I_A\Big(\tilde{U}(t,s,y)\Big)$, $\xi(\omega) := U(s)$. Hence

$$E\Big(I_A\Big(\tilde{U}(t,s,U(s))\Big)\Big|\mathcal{F}_s\Big) = E\Big(I_A\Big(\tilde{U}(t,s,U(s))\Big)\Big|\sigma_{[U(s)]}\Big). \tag{3.5}$$

Since the solution of the Navier-Stokes equation is (almost surely) unique it follows that

$$\tilde{U}(t,s,U(s)) = U(t) \quad \text{for all } t \in [s,T] \text{ and a.e. } \omega \in \Omega.$$

Then relation (3.5) becomes

$$E\Big(I_A\Big(U(t)\Big)\Big|\mathcal{F}_s\Big) = E\Big(I_A\Big(U(t)\Big)\Big|\sigma_{[U(s)]}\Big).$$

Consequently,

$$P\Big(U(t) \in A\Big|\mathcal{F}_s\Big) = P\Big(U(t) \in A\Big|\sigma_{[U(s)]}\Big). \tag{3.6}$$

We know

$$\sigma_{[U(s)]} \subseteq \sigma_{[U(r):r\le s]} \subseteq \mathcal{F}_s.$$

Taking into account the properties of the conditional expectation and (3.6) we deduce that

$$\begin{aligned} P\Big(U(t) \in A \Big| \sigma_{[U(r):r\leq s]}\Big) &= E\Big(E\Big(U(t) \in A\Big|\mathcal{F}_s\Big)\Big|\sigma_{[U(r):r\leq s]}\Big) \\ &= E\Big(E\Big(U(t) \in A\Big|\sigma_{[U(s)]}\Big)\Big|\sigma_{[U(r):r\leq s]}\Big) = P\Big(U(t) \in A\Big|\sigma_{[U(s)]}\Big). \qquad \blacksquare \end{aligned}$$

Corollary 3.1.2 ([11], Chapter 3, Section 9, pp. 59)

(i) *For fixed $s, t \in [0, T], s < t, y \in H$ the mapping*

$$A \in B(H) \mapsto \bar{P}(s, y, t, \cdot) \in \mathbb{R}$$

is a probability measure.

(ii) *The Chapman-Kolmogorov equation*

$$\bar{P}(s, y, t, A) = \int_H \bar{P}(r, x, t, A)\bar{P}(s, y, r, dx)$$

holds for any $r, s, t \in [0, T], s < r < t, y \in H, A \in B(H)$.

Remark 3.1.3

1) We have the **autonomous version** of the stochastic Navier-Stokes equation if for $t \in [0, T]$, $h \in H$ we have $\mathcal{C}(t, h) = \mathcal{C}(h)$ and $\Phi(t, h) = \Phi(h)$ for $\Phi \in \mathcal{U}$. In this case $\Big(U_\Phi(t)\Big)_{t\in[0,T]}$ is a **homogeneous Markov process**, i.e., we have

$$\bar{P}(0, y, t - s, A) = \bar{P}(s, y, t, A) \tag{3.7}$$

for all $s, t \in [0, T], s < t, y \in H, A \in B(H)$.

We prove the above property for $\Phi \in \mathcal{U}^a$, where $\mathcal{U}^a$ is the set of all autonomous feedback controls. Let $s, t \in [0, T], s < t, y \in H$. The solution U_Φ of the Navier-Stokes equation, which starts in s with the initial value y satisfies

$$\begin{aligned} (U_\Phi(t), v) + \int_s^t \langle \mathcal{A}U_\Phi(r), v\rangle dr &= (y, v) + \int_s^t \langle \mathcal{B}(U_\Phi(r), U_\Phi(r)), v\rangle dr \\ &+ \int_s^t (\Phi(U_\Phi(r)), v)dr + \int_s^t (\mathcal{C}(U_\Phi(r)), v)dw(r) \end{aligned}$$

for all $v \in V$ and a.e. $\omega \in \Omega$. We take $\tilde{U}(r) = U_\Phi(s + r), \tilde{w}(r) := w(s + r) - w(s)$ for $r \in [0, t - s]$. Then for $\tilde{U}(t - s)$ we have

$$\begin{aligned} (\tilde{U}(t - s), v) + \int_0^{t-s} \langle \mathcal{A}\tilde{U}(r), v\rangle dr &= (y, v) + \int_0^{t-s} \langle \mathcal{B}(\tilde{U}(r), \tilde{U}(r)), v\rangle dr \\ &+ \int_0^{t-s} (\Phi(\tilde{U}(r)), v)dr + \int_0^{t-s} (\mathcal{C}(\tilde{U}(r)), v)d\tilde{w}(r) \end{aligned}$$

for all $v \in V$ and a.e. $\omega \in \Omega$. Since $\Big(\tilde{w}(r)\Big)_{r\in[0,t-s]}$ and $\Big(w(r)\Big)_{r\in[s,t]}$ have the same distribution and because of the uniqueness of the solution of the Navier-Stokes equation, it follows that $\tilde{U}(t-s)$ and $U_\Phi(t)$ have the same distribution. Hence (3.7) holds.

2) The Galerkin approximations (the solutions of the equations (P_n) from Section 1.2) of the Navier-Stokes equation are also Markov processes.

3.2 Bellman's principle and Bellman's equation for the finite dimensional stochastic Navier-Stokes equation

Before we illustrate the dynamic programming approach (also called Bellman's principle) for our control problem, we need the definition of the infinitesimal generator associated to a process. This infinitesimal generator is a partial differential operator of second order (see Lemma 3.2.2) and it occurs in Bellman's equation.

Definition 3.2.1

Let $\Big(X(t)\Big)_{t\in[0,T]}$ be a process in the space $\mathcal{L}^2_H(\Omega\times[0,T])$ and let $t \in [0,T]$. The function $F : H \to \mathbb{R}$ is said to belong to the domain $\mathcal{D}_{\mathbf{A}_X(t)}$ of the **infinitesimal generator** $\mathbf{A}_X$ of $\Big(X(t)\Big)_{t\in[0,T]}$ if the limit

$$\mathbf{A}_X(t)F(y) := \lim_{\theta\searrow 0} \frac{1}{\theta}\Big[E\Big(F(X(t+\theta))\Big|\sigma_{[X(t)=y]}\Big) - F(y)\Big], \tag{3.8}$$

exists and is finite for all $y \in H$.

We define $C^2(H)$ to be the set of all mappings $F : H \to \mathbb{R}$ which are twice continuously Fréchet differentiable in each point of H and which satisfy the conditions:

(i) F, F', F'' are locally bounded;

(ii) for each $h \in H$

$$\|F'(h)\| \le c_F(1+\|h\|), \quad \Big|\Big(F''(h)h_1, h_2\Big)\Big| \le c_F\|h_1\|\|h_2\|(1+\|h\|),$$

where c_F is a positive constant.

We define $C^{1,2}([0,T]\times H)$ to be the set of all mappings $G : [0,T]\times H \to \mathbb{R}$ such that

(i) for each fixed $t \in [0,T]$ we have $G(t,\cdot) \in C^2(H)$;

(ii) there exists the partial derivative G_t which is assumed to be continuous on $[0,T]$ and

$$|G_t(t,x)| \le c_G\|x\|$$

for all $t \in [0,T]$ and $x \in H$.

In this section we consider the n-dimensional stochastic Navier-Stokes equation

$$(P_n) \qquad \begin{aligned}(U_{n,\Phi}(t),v) &+ \int_0^t (\mathcal{A}_n U_{n,\Phi}(s),v)ds = (x_0,v) + \int_0^t (\mathcal{B}_n(U_{n,\Phi}(s),U_{n,\Phi}(s)),v)ds \\ &+ \int_0^t (\Phi(s,U_{n,\Phi}(s)),v)ds + \int_0^t (\mathcal{C}_n(s,U_{n,\Phi}(s)),v)dw(s),\end{aligned}$$

for all $v \in H_n$, $t \in [0,T]$ and a.e. $\omega \in \Omega$, controlled by *feedback controls* $\Phi \in \mathcal{U}_n$ (we proceed analogously in the case $\Phi \in \mathcal{U}_n^b$), where the set $\mathcal{U}_n$ (respectively $\mathcal{U}_n^b$) is defined in Section 2.4. We denote by

$$E_{t,y}(\cdot) := E\Big(\cdot \,\Big|\sigma_{[U_{n,\Phi}(t)=y]}\Big)$$

where $t \in [0,T], y \in H$.

We assume that the mappings $\mathcal{C}(\cdot,x), \mathcal{L}(\cdot,x,y)$ are continuous on $[0,T]$ for each $x,y \in H$. The formula of the infinitesimal generator for the process $\Big(U_{n,\Phi}(t)\Big)_{t\in[0,T]}$ is given in the following lemma.

Lemma 3.2.2
The infinitesimal generator of $\Big(U_{n,\Phi}(t)\Big)_{t\in[0,T]}$ *satisfies*

$$\mathbf{A}_{U_{n,\Phi}}(s)G(s,y) = G_t(s,y) + \Big(G_x(s,y), -\mathcal{A}_n y + \mathcal{B}(y,y) + \Phi(s,y)\Big) + \frac{1}{2}\Big(G_{xx}(s,y)\mathcal{C}_n(s,y), \mathcal{C}_n(s,y)\Big)$$

for all $s \in [0,T], y \in H_n, G \in C^{1,2}([0,T]\times H_n), \Phi \in \mathcal{U}_n$.
In the autonomous version of problem (P_n) *the infinitesimal generator of* $\Big(U_{n,\Phi}(t)\Big)_{t\in[0,T]}$ *satisfies*

$$\mathbf{A}_{U_{n,\Phi}}F(y) = \Big(F_x(y), -\mathcal{A}_n y + \mathcal{B}(y,y) + \Phi(y)\Big) + \frac{1}{2}\Big(F_{xx}(y)\mathcal{C}_n(y), \mathcal{C}_n(y)\Big)$$

for all $y \in H_n, F \in C^2(H_n), \Phi \in \mathcal{U}_n^a$.

PROOF. Let $G \in C^{1,2}([0,T]\times H_n)$. We write $\Phi(r)$ instead of $\Phi(r,U_{n,\Phi}(r))$. By the Ito formula it follows that

$$\begin{aligned} &G(s+h,U_{n,\Phi}(s+h)) - G(s,U_{n,\Phi}(s)) \\ &\quad= \int_s^{s+h} G_t(r,U_{n,\Phi}(r)) + \Big(G_x(r,U_{n,\Phi}(r)), -\mathcal{A}_n U_{n,\Phi}(r) + \mathcal{B}_n(U_{n,\Phi}(r),U_{n,\Phi}(r)) + \Phi(r)\Big)dr \\ &\quad+ \frac{1}{2}\int_s^{s+h} \Big(G_{xx}(r,U_{n,\Phi}(r))\mathcal{C}_n(r,U_{n,\Phi}(r)), \mathcal{C}_n(r,U_{n,\Phi}(r))\Big)dr \\ &\quad+ \int_s^{s+h} \Big(G_x(r,U_{n,\Phi}(r)), \mathcal{C}_n(r,U_{n,\Phi}(r))\Big)dw(r),\end{aligned}$$

for each $h, s \in [0,T]$ with $s+h \le T$. In the above relation we take the conditional expectation $E_{s,y}$. We obtain

$$\begin{aligned}
&\frac{1}{h}\Big[E_{s,y}\Big(G(s+h, U_{n,\Phi}(s+h))\Big) - G(s,y)\Big] \\
= \quad & E_{s,y}\Big\{\frac{1}{h}\int_s^{s+h} G_t(r, U_{n,\Phi}(r)) + \Big(G_x(r, U_{n,\Phi}(r)), -\mathcal{A}_n U_{n,\Phi}(r) + \mathcal{B}_n(U_{n,\Phi}(r), U_{n,\Phi}(r)) + \Phi(r)\Big)dr\Big\} \\
+ \quad & \frac{1}{2}E_{s,y}\Big\{\frac{1}{h}\int_s^{s+h} \Big(G_{xx}(r, U_{n,\Phi}(r))\mathcal{C}_n(r, U_{n,\Phi}(r)), \mathcal{C}_n(r, U_{n,\Phi}(r))\Big)dr\Big\}.
\end{aligned}$$

We take $h \searrow 0$, use the properties of the process $\Big(U_{n,\Phi}(t)\Big)_{t\in[0,T]}$ (see Theorem 1.2.1 and Lemma 1.2.3) and those of $G, \Phi, \mathcal{C}_n$. Then, for each $t \in [0,T], y \in H_n$ we have

$$\mathbf{A}_{U_{n,\Phi}}(s)G(s,y) = G_t(s,y) + \Big(G_x(s,y), -\mathcal{A}_n y + \mathcal{B}(y,y) + \Phi(s,y)\Big) + \frac{1}{2}\Big(G_{xx}(s,y)\mathcal{C}_n(s,y), \mathcal{C}_n(s,y)\Big).$$

We proceed similarly in the autonomous case. ■

We consider the cost functional

$$\mathcal{J}(s,y,\Phi) := E_{s,y}\Big\{\int_s^T \mathcal{L}[r, U_{n,\Phi}(r), \Phi(r, U_{n,\Phi}(r))]dr + \mathcal{K}[U_{n,\Phi}(T)]\Big\}$$

where $s \in [0,T], y \in H_n$ and the feedback control $\Phi \in \mathcal{U}_n$.

To illustrate the dynamic programming approach we give a *formal derivation of Bellman's equation*, our arguments are of heuristic nature. *Bellman's principle* turns the stochastic control problem $(\mathcal{P}_n)$ into a problem about a nonlinear differential equation of second order (see equation (3.11)).

In dynamic programming the optimal expected system performance is considered as a function of the initial data

$$W(s,y) = \inf_{\Phi\in\mathcal{U}_n} \mathcal{J}(s,y,\Phi).$$

If $W \in C^{1,2}([0,T]\times H_n)$, then by using (P_n), the Ito formula, and Lemma 3.2.2, it follows that

$$E_{s,y}W(t, U_{n,\Phi}(t)) - W(s,y) = E_{s,y}\int_s^t \Big(W_t(r, U_{n,\Phi}(r)) + \mathbf{A}_{U_{n,\Phi}}(r)W(r, U_{n,\Phi}(r))\Big)dr. \tag{3.9}$$

Suppose that the controller uses Φ for times $s \le r \le t$ and uses an optimal control Φ^* after time t. His expectet performance cannot be less than $W(s,y)$. Thus for all $y \in H_n$ let

$$\tilde{\Phi}(r,y) = \begin{cases} \Phi(r,y) & \text{for} \quad s \le r \le t \\ \Phi^*(r,y) & \text{for} \quad t < r \le T. \end{cases}$$

By the Chapman-Kolmogorov equation (see Corollary 3.1.2) and the properties of the conditional expectation we have

$$\mathcal{J}(s,y,\tilde{\Phi}) = E_{s,y}\int_s^t \mathcal{L}[r,U_{n,\Phi}(r),\Phi(r,U_{n,\Phi}(r))]dr + E_{s,y}\mathcal{J}(t,U_{n,\Phi^*}(t),\Phi^*).$$

Because $\Phi^* \in \mathcal{U}_n$ is an optimal control, then for all $t \in [0,T], y \in H_n$ we have

$$W(t,U_{n,\Phi^*}(t)) = \mathcal{J}(t,U_{n,\Phi^*}(t),\Phi^*), \quad W(s,y) \leq \mathcal{J}(s,y,\Phi)$$

and

$$W(s,y) \leq E_{s,y}\int_s^t \mathcal{L}[r,U_{n,\Phi}(r),\Phi(r,U_{n,\Phi}(r))]dr + E_{s,y}W(t,U_{n,\Phi^*}(t)) \tag{3.10}$$

In (3.10) we have equality if an optimal control $\Phi := \Phi^*$ is used during $[s,t]$. By (3.9) and (3.10) we obtain

$$0 \leq E_{s,y}\int_s^t \Big\{\mathcal{L}[r,U_{n,\Phi}(r),\Phi(r,U_{n,\Phi}(r))] + W_t(r,U_{n,\Phi}(r)) + \mathbf{A}_{U_{n,\Phi}}(r)W(r,U_{n,\Phi}(r))\Big\}dr.$$

In the above inequality we divide by $t-s$, take $t \searrow s$, use the continuity properties of $\Big(U_{n,\Phi}(t)\Big)_{t\in[0,T]}$ (see Theorem 1.2.1 and Lemma 1.2.3) and those of $W, \mathbf{A}_{U_{n,\Phi}}, \mathcal{L}$. Thus

$$0 \leq \mathcal{L}[s,y,\Phi(s,y)] + W_t(s,y) + \mathbf{A}_{U_{n,\Phi}}(s)W(s,y).$$

Equality holds above, if $\Phi = \Phi^*$. For W we have derived formally the **continuous-time dynamic programming equation of optimal stochastic control theory**, also called **Bellman's equation**

$$0 = W_t(s,y) + \min_{\Phi\in\mathcal{U}_n}\Big\{\mathcal{L}[s,y,\Phi(s,y)] + \mathbf{A}_{U_{n,\Phi}}(s)W(s,y)\Big\} \quad s \in [0,T],\ y \in H_n \tag{3.11}$$

with the boundary condition

$$W(T,y) = \mathcal{K}(y), \quad y \in H_n.$$

The **main result** of this section is a sufficient condition for a minimum (Theorem 3.2.3 and for the autonomous case Theorem 3.2.4). The sufficient condition requires a suitably behaved solution W of the Bellman equation (3.11) and an admissible control Φ^* satisfying (3.14). Such a result is called *verification theorem.*

Theorem 3.2.3
Let W be the solution of Bellman's equation

$$0 = W_t(s,y) + \inf_{\Phi\in\mathcal{U}_n}\Big\{\mathcal{L}[s,y,\Phi(s,y)] + \mathbf{A}_{U_{n,\Phi}}(s)W(s,y)\Big\} \tag{3.12}$$

for all $(s,y) \in [0,T] \times H_n$, satisfying the boundary condition

$$W(T,U_{n,\Phi}(T)) = \mathcal{K}(U_{n,\Phi}(T)) \quad for\ all\ \Phi \in \mathcal{U}. \tag{3.13}$$

If $W \in C^{1,2}([0,T] \times H_n)$, then:

(i) *$W(s,y) \le \mathcal{J}(s,y,\Phi)$ for any $\Phi \in \mathcal{U}_n$, $s \in [0,T], y \in H_n$.*

(ii) *If $\Phi^* \in \mathcal{U}_n$ is a feedback control such that*

$$\mathcal{L}[s,y,\Phi^*(s,y)] + \mathbf{A}_{U_{n,\Phi^*}}(s)W(s,y) = \min_{\Phi \in \mathcal{U}_n} \Big\{ \mathcal{L}[s,y,\Phi(s,y)] + \mathbf{A}_{U_{n,\Phi}}(s)W(s,y) \Big\} \tag{3.14}$$

for all $s \in [0,T]$ and $y \in H_n$, then $W(s,y) = \mathcal{J}(s,y,\Phi^*)$ for all $s \in [0,T], y \in H_n$. Thus Φ^* is an optimal feedback control.

PROOF. **(i)** Let $\Phi \in \mathcal{U}_n, s \in [0,T], y \in H_n$. From (3.12) it follows that

$$0 \le W_t(r,U_{n,\Phi}(r)) + \mathcal{L}[r,U_{n,\Phi}(r),\Phi(r,U_{n,\Phi}(r))] + \mathbf{A}_{U_{n,\Phi}}(r)W(r,U_{n,\Phi}(r)), \quad r \in [0,T].$$

We integrate from s to T, use (3.9), take the conditional expectation $E_{s,y}$ and have

$$W(s,y) \le E_{s,y}W(T,U_{n,\Phi}(T)) + E_{s,y}\int_s^T \mathcal{L}[r,U_{n,\Phi}(r),\Phi(r,U_{n,\Phi}(r))]dr.$$

Now we use the boundary condition (3.13) and hence

$$W(s,y) \le \mathcal{J}(s,y,\Phi).$$

(ii) We use the same arguments as above. Instead of Φ we take Φ^*, and instead of $\le$ we take $=$. ■

Let us state a corresponding verification theorem for the **autonomous version** of the problem, formulated at the end of Section 3.1. The cost functional is given by

$$\mathcal{J}(y,\Phi) = E_y\Big\{\int_0^T \mathcal{L}[U_{n,\Phi}(r),\Phi(U_{n,\Phi}(r))]dr + \mathcal{K}[U_{n,\Phi}(T)]\Big\},$$

with $y \in H_n, \Phi \in \mathcal{U}_n^a$ (see Remark 3.1.3) and $E_y(\cdot) = E\Big(\cdot|\sigma_{[U_{n,\Phi}(0)=y]}\Big)$. The mapping $\mathcal{L}$ that occurs in the expression of the cost functional does not depend on $r \in [0,T]$ and satisfies the conditions $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ from Section 2.1.

Analogously to Theorem 3.2.3 we can prove the following verification theorem.

Theorem 3.2.4
Let W be the solution of Bellman's equation

$$0 = \inf_{\Phi \in \mathcal{U}_n^a} \Big\{ \mathcal{L}[y,\Phi(y)] + \mathbf{A}_{U_{n,\Phi}}W(y) \Big\} \quad for\ all \quad y \in H_n$$

with the boundary condition

$$W(U_{n,\Phi}(T)) = \mathcal{K}(U_{n,\Phi}(T)) \quad for\ all\ \Phi \in \mathcal{U}.$$

If $W \in C^2(H_n)$, then:

(i) $W(y) \leq \mathcal{J}(y, \Phi)$ *for any* $\Phi \in \mathcal{U}_n^a$ *and any initial data* $y \in H_n$.

(ii) *If* $\Phi^* \in \mathcal{U}_n^a$ *is a feedback control such that*

$$\mathcal{L}[y, \Phi^*(y)] + \mathbf{A}_{U_{n,\Phi^*}} W(y) = \min_{\Phi \in \mathcal{U}_n^a} \Big\{ \mathcal{L}[y, \Phi(y)] + \mathbf{A}_{U_{n,\Phi}} W(y) \Big\} \quad for\ all\ y \in H_n,$$

then $W(y) = \mathcal{J}(y, \Phi^*)$ *for all* $y \in H_n$. *Thus* Φ^* *is an optimal feedback control.*

Appendix A

Basic Convergence Results

For the convenience of the reader we recall some basic convergence results.

Proposition A.1 ([36], Proposition 10.13, p. 480).
Let (x_n) be a sequence in a Banach space S. Then the following assertions hold:

(i) *If S is reflexive and (x_n) is bounded, then (x_n) has a weakly convergent subsequence. If, in addition, every weakly convergent subsequence of (x_n) has the same limit $x \in S$, then (x_n) converges weakly to x.*

(ii) *If every subsequence of (x_n) has a subsequence which converges strongly to the same limit $x \in S$, then $x_n \to x$.*

Proposition A.2 ([37], Proposition 21.27, p.261).
Let S_1 and S_2 be Banach spaces and let $L : S_1 \to S_2$ be a continuous linear operator. If (x_n) is a sequence in S_1 such that $x_n \rightharpoonup x$ (where $x \in S_1$), then $L(x_n) \rightharpoonup L(x)$.

Proposition A.3
If S is a Banach space and if (x_n) is a sequence from $\mathcal{L}^2_S(\Omega \times [0,T])$ which converges weakly to $x \in \mathcal{L}^2_S(\Omega \times [0,T])$, then for $n \to \infty$ the following assertions are true:

(i) $$\int_0^t x_n(s)dw(s) \rightharpoonup \int_0^t x(s)dw(s) \quad \textit{and} \quad \int_0^t x_n(s)ds \rightharpoonup \int_0^t x(s)ds \quad \textit{in} \quad \mathcal{L}^2_S(\Omega \times [0,T]);$$

(ii) $$\int_0^T x_n(s)dw(s) \rightharpoonup \int_0^T x(s)dw(s) \quad \textit{and} \quad \int_0^T x_n(s)ds \rightharpoonup \int_0^T x(s)ds \quad \textit{in} \quad \mathcal{L}^2_S(\Omega).$$

Proof. We apply Proposition A.2 on $S_1 = S_2 := \mathcal{L}^2_S(\Omega\times[0,T])$, $L : \mathcal{L}^2_S(\Omega\times[0,T]) \to \mathcal{L}^2_S(\Omega\times[0,T])$, where

$$L(x) := \int_0^t x(s)dw(s).$$

Obviously, is L a linear mapping. By the properties of the stochastic integral we have

$$\begin{aligned}
\|L(x)\|^2_{\mathcal{L}^2_S(\Omega\times[0,T])} &= E\int_0^T \Big\|\int_0^t x(s)dw(s)\Big\|^2_S dt \leq TE \sup_{t\in[0,T]} \Big\|\int_0^t x(s)dw(s)\Big\|^2_S \\
&\leq 4TE\int_0^T \|x(t)\|^2_S dt = 4T\|x\|^2_{\mathcal{L}^2_S(\Omega\times[0,T])}.
\end{aligned}$$

Hence L is continuous and we can apply Proposition A.2. The other convergences are proved analogously. ■

Appendix B

Stopping Times

Let $\Big(Q(t)\Big)_{t\in[0,T]}$ be a V-valued process with

$$\int_0^T \|Q(s)\|_V^2 ds < \infty \quad \text{and} \quad \sup_{t\in[0,T]} \|Q(t)\|^2 < \infty$$

for a.e. $\omega \in \Omega$. For each $M \in \mathbb{N}$ we introduce the following **stopping times**

$$\widetilde{\mathcal{T}}_M^Q = \begin{cases} T\ , & \text{if} \quad \sup\limits_{t\in[0,T]} \|Q(t)\|^2 < M \\ \inf\Big\{t \in [0,T] : \|Q(t)\|^2 \geq M\Big\}, & \text{otherwise,} \end{cases}$$

$$\widehat{\mathcal{T}}_M^Q = \begin{cases} T\ , & \text{if} \quad \int_0^T \|Q(s)\|_V^2 ds < M \\ \inf\Big\{t \in [0,T] : \int_0^t \|Q(s)\|_V^2 ds \geq M\Big\}, & \text{otherwise.} \end{cases}$$

We define

$$\mathcal{T}_M^Q := \min\{\widetilde{\mathcal{T}}_M^Q, \widehat{\mathcal{T}}_M^Q\}.$$

We see that for all $t \in [0,T]$ and a.e. $\omega \in \Omega$ we have

$$\|Q(t \wedge \mathcal{T}_M^Q)\|^2 \leq M, \qquad \int_0^{t\wedge\mathcal{T}_M^Q} \|Q(s)\|_V^2 ds \leq M.$$

Proposition B.1
The following convergences hold:

$$\lim_{M\to\infty} P(\mathcal{T}_M^Q < T) = 0$$

and

$$\lim_{M\to\infty} \mathcal{T}_M^Q = T \quad \textit{for a.e. } \omega \in \Omega.$$

Proof. Using some elementary inequalities we obtain

$$\begin{aligned}
&\lim_{M\to\infty} P(\mathcal{T}_M^Q < T) \le \lim_{M\to\infty} P(\widetilde{\mathcal{T}}_M^Q < T) + \lim_{M\to\infty} P(\widehat{\mathcal{T}}_M^Q < T) \\
\le\ & \lim_{M\to\infty} P\Big(\sup_{t\in[0,T]} \|Q(t)\|^2 \ge M\Big) + \lim_{M\to\infty} P\Big(\int_0^T \|Q(s)\|_V^2 ds\Big) \ge M\Big) \\
\le\ & P\Big(\bigcap_{M=1}^{\infty} \Big\{\sup_{t\in[0,T]} \|Q(t)\|^2 \ge M\Big\}\Big) + P\Big(\bigcap_{M=1}^{\infty} \Big\{\int_0^T \|Q(s)\|_V^2 \ge M\Big\}\Big) = 0.
\end{aligned}$$

The sequence $\Big(T - \mathcal{T}_M^Q\Big)$ is monotone decreasing (for a.e. $\omega \in \Omega$). We have proved above that it converges in probability to zero. Therefore it converges to zero for almost every $\omega \in \Omega$. ■

Proposition B.2
We assume that the following assumptions are fulfilled:
(1) $k_1, k_2 > 0$ *are real numbers;*
(2) a_0 *is a* H*-valued* $\mathcal{F}_0$*-measurable random variable with* $E\|a_0\|^4 < \infty$*;*
(3) $F_1 \in \mathcal{L}^1_{\mathbb{R}}(\Omega \times [0,T]), F_2 \in \mathcal{L}^2_H(\Omega \times [0,T])$.
(4) $F_3 : [0,T] \times H \to H$ *is a mapping such that for all* $t \in [0,T], x \in H$ *we have* $\|F_3(t,x)\| \le k_{F_3}\|x\|$ *with* k_{F_3} *a positive constant and* $F_3(\cdot, x) \in \mathcal{L}^2_H[0,T]$ *for all* $x \in H$*;*
(5) $\Big(Q(t)\Big)_{t\in[0,T]}$ *is a* V*-valued process with*

$$\int_0^T \|Q(s)\|_V^2 ds < \infty \quad \textit{and} \quad \sup_{t\in[0,T]} \|Q(t)\|^2 < \infty \quad \textit{for a.e. } \omega \in \Omega,$$

satisfying the inequality

$$\begin{aligned}
\|Q(t)\|^2 + k_1 \int_0^t \|Q(s)\|_V^2 ds \ \le\ & \|a_0\|^2 + k_2 \int_0^t \|Q(s)\|^2 ds \\
+\ & \int_0^t |F_1(s)| ds + \int_0^t (F_2(s) + F_3(s, Q(s)), Q(s)) dw(s)
\end{aligned}$$

for all $t \in [0,T]$ *and a.e.* $\omega \in \Omega$. *Then there exists a positive constant* c *(depending on* k_1, k_2, k_{F_3}, T*) such that*

$$E \sup_{t\in[0,T]} \|Q(t)\|^2 + E\int_0^T \|Q(s)\|_V^2 ds \le c\Big[E\|a_0\|^2 + E\int_0^T |F_1(s)|ds + E\int_0^T \|F_2(s)\|^2 ds\Big] \tag{B.1}$$

and if $E\int_0^T |F_1(s)|^2 ds < \infty,\ E\int_0^T \|F_2(s)\|^4 ds < \infty$ *then*

$$E \sup_{t\in[0,T]} \|Q(t)\|^4 + E\Big(\int_0^T \|Q(s)\|_V^2 ds\Big)^2 \le c\Big[E\|a_0\|^4 + E\int_0^T |F_1(s)|^2 ds + E\int_0^T \|F_2(s)\|^4 ds\Big]. \tag{B.2}$$

Proof. We consider the stopping times $\mathcal{T}_M := \mathcal{T}_M^Q$, $M \in \mathbb{N}$. Using **(5)** it follows that for all $t \in [0,T]$

$$\begin{aligned}\sup_{s\in[0,t\wedge\mathcal{T}_M]} \|Q(s)\|^2 \quad &+ \quad k_1 \int_0^{t\wedge\mathcal{T}_M} \|Q(s)\|_V^2 ds \le 2\|a_0\|^2 + 2k_2 \int_0^{t\wedge\mathcal{T}_M} \|Q(s)\|^2 ds \\ &+ \quad 2\int_0^{t\wedge\mathcal{T}_M} |F_1(s)|ds + 2 \sup_{s\in[0,t\wedge\mathcal{T}_M]} \Big|\int_0^s (F_2(r) + F_3(r,Q(r)), Q(r))dw(r)\Big|.\end{aligned}$$

and

$$\begin{aligned}\sup_{s\in[0,t\wedge\mathcal{T}_M]} \|Q(s)\|^4 \quad &+ \quad k_1^2\Big(\int_0^{t\wedge\mathcal{T}_M} \|Q(s)\|_V^2 ds\Big)^2 \le 16\|a_0\|^4 + 16k_2^2\Big(\int_0^{t\wedge\mathcal{T}_M} \|Q(s)\|^2 ds\Big)^2 \\ &+ \quad 16\Big(\int_0^{t\wedge\mathcal{T}_M} |F_1(s)|ds\Big)^2 + 16 \sup_{s\in[0,t\wedge\mathcal{T}_M]} \Big|\int_0^s (F_2(r) + F_3(r,Q(r)), Q(r))dw(r)\Big|^2.\end{aligned}$$

Now we use the Burkholder inequality (see [18], p. 166) and the Schwarz inequality to obtain

$$\begin{aligned}E \sup_{s\in[0,t\wedge\mathcal{T}_M]} \|Q(s)\|^2 + k_1 E \int_0^{t\wedge\mathcal{T}_M} \|Q(s)\|_V^2 ds &\le 2E\|a_0\|^2 + 2k_2 E \int_0^{t\wedge\mathcal{T}_M} \|Q(s)\|^2 ds \\ + \quad 2E \int_0^{t\wedge\mathcal{T}_M} |F_1(s)|ds + \frac{1}{2}E \sup_{s\in[0,t\wedge\mathcal{T}_M]} &\|Q(s)\|^2 + c_1 E \int_0^{t\wedge\mathcal{T}_M} \|F_2(s) + F_3(s,Q(s))\|^2 ds\end{aligned}$$

and

$$\begin{aligned}E \sup_{s\in[0,t\wedge\mathcal{T}_M]} \|Q(s)\|^4 + k_1^2 E\Big(\int_0^{t\wedge\mathcal{T}_M} \|Q(s)\|_V^2 ds\Big)^2 &\le 16E\|a_0\|^4 + 16k_2^2 TE \int_0^{t\wedge\mathcal{T}_M} \|Q(s)\|^4 ds \\ + \quad 16TE \int_0^{t\wedge\mathcal{T}_M} |F_1(s)|^2 ds + \frac{1}{2}E \sup_{s\in[0,t\wedge\mathcal{T}_M]} &\|Q(s)\|^4 + c_2 E \int_0^{t\wedge\mathcal{T}_M} \|F_2(s) + F_3(s,Q(s))\|^4 ds\end{aligned}$$

for all $t \in [0,T]$, where c_1, c_2 are positive constants. Consequently, for all $t \in [0,T]$, we have

$$E \sup_{s\in[0,t]} I_{[0,\mathcal{T}_M]}(s)\|Q(s)\|^2 + 2k_1 E\int_0^t I_{[0,\mathcal{T}_M]}(s)\|Q(s)\|_V^2 ds \le 4E\|a_0\|^2$$
$$+ \quad 4(k_2 + c_1 k_{F_3})E\int_0^t \sup_{r\in[0,s]} I_{[0,\mathcal{T}_M]}(r)\|Q(r)\|^2 dr + 4E\int_0^T |F_1(s)|ds + 4c_1 E\int_0^T \|F_2(s)\|^2 ds$$

and

$$E \sup_{s\in[0,t]} I_{[0,\mathcal{T}_M]}(s)\|Q(s)\|^4 + 2k_1^2 E\Big(\int_0^t I_{[0,\mathcal{T}_M]}(s)\|Q(s)\|_V^2 ds\Big)^2 \le 32E\|a_0\|^4$$
$$+ \quad (32k_2^2 T + 16c_2 k_{F_3})E\int_0^t \sup_{r\in[0,s]} I_{[0,\mathcal{T}_M]}(r)\|Q(r)\|^4 dr + 32TE\int_0^T |F_1(s)|^2 ds + 16c_2 E\int_0^T \|F_2(s)\|^4 ds.$$

By Gronwall's Lemma it follows that there exists a positive constant c^* (independent of M) such that

$$E \sup_{s\in[0,T\wedge\mathcal{T}_M]} \|Q(s)\|^2 + 2k_1 E \int_0^{T\wedge\mathcal{T}_M} \|Q(s)\|_V^2 ds \le c^*\Big[E\|a_0\|^2 + E\int_0^T |F_1(s)|ds + E\int_0^T \|F_2(s)\|^2 ds\Big]$$

and

$$E \sup_{s\in[0,T\wedge\mathcal{T}_M]} \|Q(s)\|^4 + 2k_1^2 E\Big(E \int_0^{T\wedge\mathcal{T}_M} \|Q(s)\|_V^2 ds\Big)^2 \le c^*\Big[E\|a_0\|^4 + E\int_0^T |F_1(s)|^2 ds + E\int_0^T \|F_2(s)\|^4 ds\Big].$$

Now we use Proposition B.1, take the limit $M \to \infty$ in the above inequalities to obtain (B.1) and (B.2). ■

Proposition B.3
Let $(\mathcal{T}_M)$ and $\mathcal{T}$ be stopping times, such that

$$\lim_{M\to\infty} P(\mathcal{T}_M < \mathcal{T}) = 0.$$

Let (Q_n) be a sequence of processes from the space $\mathcal{L}^2_{\mathbb{R}}([0,T]\times\Omega)$ such that for each fixed M we have

$$\lim_{n\to\infty} E|Q_n(\mathcal{T}_M)| = 0$$

and there exists a positive constant c independent of n such that

$$E|Q_n(\mathcal{T})|^2 < c \quad \text{for all} \quad n \in \mathbb{N}.$$

Then

$$\lim_{n\to\infty} E|Q_n(\mathcal{T})| = 0.$$

PROOF. Let $\varepsilon, \delta > 0$. There exists $M_0 \in \mathbb{N}$ such that

$$P(\mathcal{T}_{M_0} < \mathcal{T}) \leq \frac{\varepsilon}{2}.$$

By the hypothesis it follows that for this M_0 we have $\lim_{n\to\infty} E|Q_n(\mathcal{T}_{M_0})| = 0$. Consequently, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{\delta} E|Q_n(\mathcal{T}_{M_0})| \leq \frac{\varepsilon}{2}$$

for all $n \geq n_0$. We write

$$\begin{aligned} P\Big(|Q_n(\mathcal{T})| \geq \delta\Big) &\leq P\Big(\mathcal{T}_{M_0} < \mathcal{T}\Big) + P\Big(\{\mathcal{T}_{M_0} = \mathcal{T}\} \wedge \Big\{|Q_n(\mathcal{T})| \geq \delta\Big\}\Big) \\ &\leq \frac{\varepsilon}{2} + P\Big(|Q_n(\mathcal{T}_{M_0})| \geq \delta\Big) \leq \frac{\varepsilon}{2} + \frac{1}{\delta} E|Q_n(\mathcal{T}_{M_0})| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for all $n \geq n_0$. Hence for all $\delta > 0$ we get $\lim_{n\to\infty} P\Big(|Q_n(\mathcal{T})| \geq \delta\Big) = 0$. Therefore, the sequence $\Big(|Q_n(\mathcal{T})|\Big)$ converges in probability to zero. From the hypothesis it follows that this sequence is uniformly integrable (with respect to $\omega \in \Omega$). Hence it converges also in mean to zero

$$\lim_{n\to\infty} E|Q_n(\mathcal{T})| = 0. \qquad \blacksquare$$

Proposition B.4
Let $\widehat{\mathcal{F}} \subseteq \mathcal{F}$ be a σ-algebra, (Q_n) be a sequence of H-valued random variables which converges for a.e. $\omega \in \Omega$ to Q. If each random variable Q_n is independent of $\widehat{\mathcal{F}}$, then Q is independent of $\widehat{\mathcal{F}}$.

PROOF. The random variable Q is independent of $\widehat{\mathcal{F}}$ if

$$P(\{\|Q\| < x\} \cap A) = P(\|Q\| < x)P(A) \tag{B.3}$$

for all $x \in \mathbb{R}, A \in \widehat{\mathcal{F}}$. The hypothesis implies that the sequence $\Big(\|Q_n\|\Big)$ converge in probability to $\|Q\|$. Therefore, the sequence of their distribution functions is convergent

$$\lim_{n\to\infty} F_{\|Q_n\|}(x) = F_{\|Q\|}(x) \tag{B.4}$$

for each $x \in \mathbb{R}$ which is continuity point of $F_{\|Q\|}$.

Let $x \in \mathbb{R}, A \in \widehat{\mathcal{F}}, \delta > 0$. First we consider that $F_{\|Q\|}$ is continuous in x. Then using the hypothesis and (B.4) we get

$$\lim_{n\to\infty} P(\{\|Q_n\| < x\} \cap A) = \lim_{n\to\infty} P(\|Q_n\| < x)P(A) = P(\|Q\| < x)P(A). \tag{B.5}$$

We write

$$\begin{aligned} P(\{\|Q\| < x - \delta\} \cap A) &\leq P(\{\|Q\| < x - \delta\} \cap \{\|Q_n\| < x\} \cap A) \\ &+ P(\{\|Q\| < x - \delta\} \cap \{\|Q_n\| \geq x\} \cap A) \\ &\leq P(\{\|Q_n\| < x\} \cap A) + P\Big(\Big|\|Q\| - \|Q_n\|\Big| > \delta\Big). \end{aligned}$$

Analogously we have

$$P(\{\|Q_n\| < x\} \cap A) \leq P(\{\|Q\| < x+\delta\} \cap A) + P\Big(\Big|\|Q\| - \|Q_n\|\Big| > \delta\Big).$$

Consequently,

$$\begin{aligned} P(\{\|Q\| < x-\delta\} \cap A) - P\Big(\Big|\|Q\| - \|Q_n\|\Big| > \delta\Big) &\leq P(\|Q_n\| < x)P(A) \\ &\leq \quad P(\{\|Q\| < x+\delta\} \cap A) + P\Big(\Big|\|Q\| - \|Q_n\|\Big| > \delta\Big). \end{aligned}$$

In the inequalities above we take the limit $n \to \infty$ and use (B.5) to obtain

$$P(\{\|Q\| < x-\delta\} \cap A) \leq P(\|Q\| < x)P(A) \leq P(\{\|Q\| < x+\delta\} \cap A).$$

Let $\delta \searrow 0$ in the inequalities above. Then

$$P(\{\|Q\| \leq x\} \cap A) \leq P(\|Q\| < x)P(A) \leq P(\{\|Q\| \leq x\} \cap A).$$

Since x is a point of continuity for $F_{\|Q\|}$ we have

$$P(\{\|Q\| \leq x\} \cap A) = P(\{\|Q\| < x\} \cap A).$$

Consequently, (B.3) holds and Q is independent of $\widehat{\mathcal{F}}$.

Now we consider that x is not a point of continuity of $F_{\|Q\|}$. Let (x_n) be a monotone increasing sequence of continuity points of $F_{\|Q\|}$ which converges to x. Then

$$\lim_{n\to\infty} F_{\|Q\|}(x_n) = F_{\|Q\|}(x),$$

and because x_n is a point of continuity for $F_{\|Q\|}$, we have

$$P(\{\|Q\| < x_n\} \cap A) = P(\|Q\| < x_n)P(A).$$

Now we take the limit $n \to \infty$ and conclude that (B.3) holds. Hence Q is independent of $\widehat{\mathcal{F}}$. ■

Proposition B.5
Let $\widehat{\mathcal{F}} \subseteq \mathcal{F}$ be a σ-algebra, $f : H \times \Omega \to H$ be a mapping such that for each $x \in H$ the random variable $f(x,\cdot)$ is bounded, measurable and independent of $\widehat{\mathcal{F}}$. Let ξ be a H-valued $\widehat{\mathcal{F}}$-measurable random variable. Then

$$E(f(\xi,\omega)|\widehat{\mathcal{F}}) = E(f(\xi,\omega)|\sigma_{[\xi]}),$$

where $\sigma_{[\xi]}$ is the σ-algebra generated by the random variable ξ.

This Proposition can be proved analogously to Lemma 1, p. 63 in [11]. ■

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APPROXIMATION UND OPTIMALE STEUERUNG DER STOCHASTISCHEN NAVIER-STOKES-GLEICHUNG

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Zusammenfassung der Dissertationsschrift

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1. In der Hydromechanik hat die Navier-Stokes-Gleichung wichtige Anwendungen. Sie beschreibt das Verhalten eines inkompressiblen Strömungsfeldes in einem gegebenen Strömungsgebiet. Äußere zufällige Einflüsse sowie die innere Brownsche Bewegung beeinflussen das Verhalten der Flüssigkeit. Daher enthalten realistischere Modelle auch stochastische Terme und die Lösung der Gleichung ist ein stochastischer Prozeß. Die Behandlung solcher Gleichungen bettet sich in die Theorie der stochastischen Evolutionsgleichungen ein.

Die vorliegende Arbeit ist der Untersuchung der Eigenschaften der stochastischen Navier-Stokes-Gleichung gewidmet: Es werden Existenz- und Eindeutigkeitssätze für die Lösung bewiesen, Approximationsmethoden angegeben sowie Aussagen zur optimalen Steuerung der Gleichung bezüglich des Einflusses der äußeren Kräfte hergeleitet.

Die Arbeit besteht aus den Kapiteln: "Existenz und Approximation der Lösung“, "Optimale Steuerung“, "Zur Gleichung der dynamischen Optimierung“.

2. In der Arbeit wird der starke Lösungsbegriff (im Sinne der stochastischen Analysis) der stochastischen Navier-Stokes-Gleichung zugrundegelegt, und die Gleichung wird im verallgemeinerten Sinne als eine Evolutionsgleichung über einem Evolutionstripel $\Big((V,\|\cdot\|_V),(H,\|\cdot\|_H),(V^*,\|\cdot\|_{V^*})\Big)$ betrachtet. Die zufälligen Variablen sind auf einem gegebenen vollständigen Wahrscheinlichkeitsraum $(\Omega,\mathcal{F},P)$ definiert, $(\mathcal{F}_t)_{t\in[0,T]}$ ist eine rechtsstetige Filtration, und es wird ein reellwertiger $\mathcal{F}_t$-Wiener-Prozeß $\Big(w(t)\Big)_{t\in[0,T]}$ als gegeben vorausgesetzt. Wir nennen den zur Filtration $(\mathcal{F}_t)_{t\in[0,T]}$-adaptierten, V-wertigen stochastischen Prozeß $\Big(U(t)\Big)_{t\in[0,T]}$ Lösung der stochastischen Navier-Stokes-Gleichung, wenn $E\int\limits_0^T\|U(t)\|_V^2dt<\infty$, $E\|U(t)\|_H^2<\infty$ für alle $t \in [0,T]$, und

$$(2.6) \qquad (U(t),v)_H + \int_0^t \langle \mathcal{A}U(s),v\rangle ds = (x_0,v)_H + \int_0^t \langle \mathcal{B}(U(s),U(s)),v\rangle ds + \int_0^t (\Phi(s,U(s)),v)_H ds + \int_0^t (\mathcal{C}(s,U(s)),v)_H dw(s)$$

für alle $t \in [0,T], v \in V$ und fast alle $\omega \in \Omega$.

Der Operator $\mathcal{A} : V \to V^*$ ist linear, symmetrisch und koerzitiv, x_0 ist eine H-wertige $\mathcal{F}_0$-meßbare Zufallsgröße mit $E\|x_0\|^4 < \infty$. Der bilineare Operator $\mathcal{B} : V \times V \to V^*$ erfüllt die Bedingungen $\langle \mathcal{B}(u,v),v\rangle = 0$ und $|\langle \mathcal{B}(u,v),z\rangle|^2 \le b\|z\|_V^2 \|u\|_H \|u\|_V \|v\|_H \|v\|_V$ für alle $u,v,z \in V$ (b ist eine positive Konstante). Die Abbildungen $\Phi, \mathcal{C} : [0,T] \times H \to H$ sind lipschitzstetig bezüglich der zweiten Variablen, und es werden solche Voraussetzungen gewählt, daß die deterministischen Integrale und das stochastische Integral (im Sinne von Ito) in (2.6) existieren.

3. Als eigenständiges Resultat wurde die Existenz der Lösung von (2.6) mit Hilfe der Galerkin-Methode bewiesen. Die Lösung des unendlichdimensionalen Problems ergibt sich als Grenzwert im quadratischen Mittel der Galerkin-Approximationen, indem man a priori Abschätzungen für die Galerkin-Approximationen herleitet, Stoppzeiten für stochastische Prozesse einführt und Konvergenzprinzipien der Funktionalanalysis anwendet. Es wird auch die Eindeutigkeit (mit Wahrscheinlichkeit 1) der Lösung von (2.6) bewiesen.

4. Die Galerkin-Approximationen sind ebenfalls Lösungen von nichtlinearen Gleichungen, und diese sind für numerische Simulationen aufwendig. Dabei wurde eine neue Linearisierungsmethode entwickelt. Für jede natürliche Zahl n sei der Prozeß $\Big(u_n(t)\Big)_{t\in[0,T]}$ Lösung der folgenden linearen Evolutionsgleichung

$$(\hat{P}_n) \qquad (u_n(t),v)_H + \int_0^t \langle \mathcal{A}u_n(s),v\rangle ds = (x_0,v)_H + \int_0^t \langle \mathcal{B}(u_{n-1}(s),u_n(s)),v\rangle ds + \int_0^t (\Phi(s,u_{n-1}(s)),v)_H ds + \int_0^t (\mathcal{C}(s,u_{n-1}(s)),v)_H dw(s)$$

für alle $t \in [0,T], v \in V$ und fast alle $\omega \in \Omega$, wobei $u_0 := 0$ ist. Es werden die Existenz und Eindeutigkeit der Lösung dieser Gleichungen untersucht und folgende Konvergenzeigenschaften bewiesen:

$$\lim_{n\to\infty} E\int_0^T \|u_n(t) - U(t)\|_V^2 dt = 0$$

und für alle $t \in [0,T]$

$$\lim_{n\to\infty} E\|u_n(t) - U(t)\|_H^2 = 0.$$

5. Im zweiten Teil der Arbeit wird das Verhalten des Strömungsfeldes untersucht, wenn verschiedene äußere Kräfte Φ als Steuerungen wirken, wobei sowohl lineare und stetige Rückkopplungssteuerungen als auch beschränkte Steuerungen als zulässige Steuerungen betrachtet werden. Das Problem der optimalen Steuerung besteht in der Minimierung des Kostenfunktionals

$$\mathcal{J}(\Phi) = E\int_0^T \mathcal{L}[t, U_\Phi(t), \Phi(t, U_\Phi(t))]dt + E\mathcal{K}[U_\Phi(T)],$$

bezüglich der eingeführten zulässigen Steuerungen, wobei $\mathcal{L} : [0,T] \times H \times H \to \mathbb{R}_+$, $\mathcal{K} : H \to \mathbb{R}_+$ bestimmte Stetigkeits- bzw. Differenzierbarkeitsbedingungen erfüllen. Es gelang, die Existenz von optimalen und ε-optimalen Rückkopplungssteuerungen zu beweisen, wobei die Kompaktheitseigenschaft der Menge der zulässigen Steuerungen in diesen Fällen nicht vorausgesetzt werden muß.

6. Eine notwendige Optimalitätsbedingung für das Problem der optimalen Steuerung wird in Form eines stochastischen Minimumprinzips hergeleitet. Dazu wird die Ableitung im Sinne von Gateaux des Kostenfunktionals berechnet. Weiterhin werden Gleichungen für die adjungierten Prozesse hergeleitet und Näherungen durch endlichdimensionale Approximationen ermittelt.

7. Um die Aussagen für das Steuerproblem abzurunden, wurde die Bellmansche Funktionalgleichung für die endlichdimensionalen Galerkin-Approximationen hergeleitet. Der unendlichdimensionale Fall kann nur in Spezialfällen behandelt werden, da die Existenz des infinitesimalen Generators vorausgesetzt werden muß. Durch das Bellmansche Prinzip wird das stochastische Steuerproblem in ein deterministisches Steuerproblem bezüglich einer nichtlinearen partiellen Differentialgleichung zweiter Ordnung überführt. Die Bellmansche Funktionalgleichung liefert hier eine hinreichende Bedingung für die Existenz optimaler Steuerungen.

8. Es wird auch bewiesen, daß die Lösung der Gleichung (2.6) und die zugehörigen Galerkin-Approximationen die Markov-Eigenschaft besitzen.

9. Im Anhang der Arbeit werden Aussagen der Funktionalanalysis sowie der stochastischen Analysis angegeben und ein Teil davon auch bewiesen.

Selbständigkeitserklärung

Hiermit erkläre ich, daß ich die vorliegende Arbeit selbständig und ohne fremde Hilfe verfaßt habe. Andere als die angegebenen Quellen und Hilfsmittel habe ich nicht benutzt. Wörtlich oder inhaltlich anderen Werken entnommene Stellen habe ich als solche kenntlich gemacht.

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