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Frequently Used Notations

a.e.	almost every
\rightharpoonup	weak convergence (in the sense of functional analysis)
I_A	indicator function for the set A
\mathbb{N}	set of strictly positive integers
\mathbb{R}	set of real numbers
Λ	Lebesgue measure on the interval $[0, T]$
(Ω, \mathcal{F}, P)	complete probability space
EX	mathematical expectation of the random variable X
$(\mathcal{F}_t)_{t \in [0, T]}$	right continuous filtration such that \mathcal{F}_0 contains all \mathcal{F} -null sets
V^*	dual space of the reflexive Banach space V
$\langle v^*, v \rangle$	the application of $v^* \in V^*$ on $v \in V$
J	duality map $J : V \rightarrow V^*$
$B(V)$	σ -algebra of all Borel measurable sets of V
$C([0, T], V)$	space of all continuous functions $u : [0, T] \rightarrow V$
$\mathcal{L}(V)$	space of all linear and continuous operators from the Banach space V to itself
$\mathcal{L}_V^2[0, T]$	space of all $B([0, T])$ -measurable functions $u : [0, T] \rightarrow V$ with $\int_0^T \ u(t)\ _V^2 dt < \infty$
$\mathcal{L}_V^2(\Omega)$	space of all \mathcal{F} -measurable random variables $u : \Omega \rightarrow V$ with $E\ u\ _V^2 < \infty$
$\mathcal{L}_V^2(\Omega \times [0, T])$	space of all $\mathcal{F} \times B([0, T])$ -measurable processes $u : \Omega \times [0, T] \rightarrow V$ that are adapted to the filtration $(\mathcal{F}_t)_{t \in [0, T]}$ and $E \int_0^T \ u(t)\ _V^2 dt < \infty$
$\mathcal{L}_V^\infty(\Omega \times [0, T])$	space of all $\mathcal{F} \times B([0, T])$ -measurable processes $u : \Omega \times [0, T] \rightarrow V$ that are adapted to the filtration $(\mathcal{F}_t)_{t \in [0, T]}$ and for a.e. (ω, t) bounded
$\mathcal{L}_V^\infty(\Omega)$	space of all \mathcal{F} -measurable processes $u : \Omega \rightarrow V$ that are bounded for a.e. ω
$\mathcal{D}_V(\Omega \times [0, T])$	set of $\xi \in \mathcal{L}_V^\infty(\Omega \times [0, T])$ with $\xi = v\phi$, $v \in V$, $\phi \in \mathcal{L}_\mathbb{R}^\infty(\Omega \times [0, T])$
$\mathcal{D}_V(\Omega)$	set of $\xi \in \mathcal{L}_V^\infty(\Omega)$ with $\xi = v\phi$, $v \in V$, $\phi \in \mathcal{L}_\mathbb{R}^\infty(\Omega)$

- $\Delta_X(t)$ notation for $\exp\left\{-\frac{b}{\nu} \int_0^t \|X(s)\|_V^2 ds\right\}$, where $(X(t))_{t \in [0, T]}$ is a V -valued stochastic process; b, ν are positive constants
- \mathcal{T}_M^X stopping time for the stochastic process $(X(t))_{t \in [0, T]}$ (for the exact definition see Appendix B)
- Π_n orthogonal projection in a Hilbert space

As usual in the notation of random variables or stochastic processes we generally omit the dependence of $\omega \in \Omega$.