# INTRODUCTION

*"The Navier-Stokes equation occupy a central position in the study of nonlinear partial differential equations, dynamical systems, and modern scientific computation, as well as classical fluid dynamics. Because of the complexity and variety of fluid dynamical phenomena, and the simplicity and exactitude of the governing equations, a very special depth and beauty is expected in the mathematical theory. Thus, it is a source of pleasure and fascination that many of the most important questions in the theory remain yet to be answered, and seem certain to stimulate contributions of depth, originality and influence far into the future."* (J.G. Heywood [15])

The Navier-Stokes equations were formulated by the French physicist C.L.M.H. Navier (1785-1836) in 1822 and the British mathematician and physicist G.G. Stokes (1819-1903) in 1845. Existence and uniqueness theorems for the stationary Navier-Stokes equation were first proved by F. Odquist in 1930 [27] and by J. Leray in 1933-1934 [22], [23]. E. Hopf [17] (1952) was the first who obtained the equation for the characteristic functional of the statistical solution giving a probability description of fluid flows. There is much information about statistical hydromechanics with detailed review of literature in the books written by A.S. Monin and A.M. Jaglom [25] in 1965, 1967. C. Foias investigated in [10] (1972) the questions of existence and uniqueness of spatial statistical solutions. A. Bensoussan and R. Temam [2] (1973) gave for the first time a functional analytical approach for the stochastic Navier-Stokes equations. The research has accelerated during the last twenty five years.

*"Researchers are now undertaking the study of flows with free surfaces, flows past obstacles, jets through apertures, heat convection, bifurcation, attractors, turbulence, etc., on the basis of an exact mathematical analysis. At the same time, the advent of high speed computers has made computational fluid dynamics a subject of the greatest practical importance. Hence, the development of computational methods has become another focus of the highest priority for the application of the mathematical theory. It is not surprising, then, that there has been an explosion of activity in recent years, in the diversity of topics being studied, in the number of researchers who are involved, and in the number of countries where they are located."* (Preface for "The Navier Stokes Equations II"- Proceedings of the Oberwolfach meeting 1991, [16])

After this short history about the deterministic and stochastic equations of Navier-Stokes type, we give the equation for the stochastic Navier-Stokes equation which describes the behavior of a viscous velocity field of an incompressible liquid. The equation on the domain of flow $G \subset \mathbb{R}^n$ ($n \geq 2$ a natural number) is given by

$$\frac{\partial U}{\partial t} - \nu \Delta U = -(U, \nabla) U + f - \nabla p + \mathcal{C}(U) \frac{\partial w}{\partial t} \tag{0.1}$$

$$\text{div } U = 0, \quad U(0,x) = U_0(x), \quad U(t,x) \mid_{\partial G} = 0, \ t > 0, \ x \in G,$$

where $U$ is the velocity field, $\nu$ is the viscosity, $\Delta$ is the Laplacian, $\nabla$ is the gradient, $f$ is an external force, $p$ is the pressure, and $U_0$ is the initial condition. Realistic models for flows should contain a random noise part, because external perturbations and the internal Browninan motion influence the velocity field. For this reason equation (0.1) contains a random noise part $\mathcal{C}(U) \frac{\partial w}{\partial t}$.

Here the noise is defined as the distributional derivative of a Wiener process $\Big(w(t)\Big)_{t\in[0,T]}$, whose intensity depends on the state $U$.

This nonlinear differential equation is only for the simplest examples exactly soluble, usually corresponding to laminar flows. Physical experiments show that turbulence occurs if the outer force $f$ is sufficiently large. In many important applications, including turbulence, the equation must be modified, matched or truncated, or otherwise approximated analytically or numerically in order to obtain any predictions. Sometimes a good approximation can be of equal or greater utility than a complicated exact result.

In the study of equations of Navier-Stokes type one can consider weak solutions of martingal type or strong solutions. Throughout this paper we consider strong solutions ("strong" in the sense of stochastic analysis) of a stochastic equation of Navier-Stokes type (we will call it stochastic Navier-Stokes equation) and define the equation in the generalized sense as an evolution equation, assuming that the stochastic processes are defined on a given complete probability space and the Wiener process is given in advance.

The aim of this dissertation is to prove the existence of the strong solution of the Navier-Stokes equation by approximating it by means of the Galerkin method, i.e., by a sequence of solutions of finite dimensional evolution equations. The Galerkin method involves solving nonlinear equations and often it is difficult to deal with them. For this reason we approximate the solution of the stochastic Navier-Stokes equation by the solutions of a sequence of linear stochastic evolution equations. Another interesting aspect of the stochastic Navier-Stokes equation is to study the behavior of the flow if we act upon the fluid through various external forces. We address the issue of the existence of an optimal action upon the system in order to minimize a given cost functional (for example, the turbulence within the flow). We also derive a stochastic minimum principle and investigate Bellman's equation for the considered control problem.

**Chapter 1** is devoted to the proof of the existence of the strong solution of the Navier-Stokes equation using the Galerkin method and then to approximate the solution by a linear method. First we give the assumptions for the considered equation and show how the considered evolution equation can be transformed into (0.1) in the case of $n = 2$. We prove the existence of the solution by the Galerkin method (see Theorem 1.2.2). Important results concerning the theory and numerical analysis of the deterministic Navier-Stokes equation can be found in the book of R. Temam [32]. The author also presents in this book the Galerkin method for this equation, which is one of the well-known methods in the theory of partial differential equations that is used to prove existence properties and to obtain finite dimensional approximations for the solutions of the equations. The Galerkin method for the stochastic Navier-Stokes equation has been investigated for example from A. Bensoussan [4], M. Capinski, N. J. Cutland [6], D. Gatarek [7], A. I. Komech, M. I. Vishik [20], B. Schmalfuß [30], [29], M. Viot [34]. Most of the above-mentioned papers consider weak (statistical) solutions. The techniques used in the proofs are the construction of the Galerkin-type approximations of the solutions and some a priori estimates that allow one to prove compactness properties of the corresponding probability measures and finally to obtain a solution of the equation (using Prokhorov's criterion and Skorokhod's theorem). Since we consider the strong solution (in the sense of stochastic analysis) of the Navier-Stokes equation, we do not need to use the techniques considered in the case of weak solutions. The techniques applied in our paper use in particular the properties of stopping times and some basic convergence principles from functional analysis. An important result is that the Galerkin-type approximations converge in mean square to the solution of the Navier-Stokes equation (see Theorem 1.2.7). There are also other approximation methods for this equation involving, for example, the approximation of the Wiener process by smooth processes (see W. Grecksch, B. Schmalfuß [13]) or time discretizations (see F. Flandoli, V. M. Tortorelli [8]). In this chapter we further approximate the solution of the stochastic Navier-Stokes equation by the solutions of a sequence of linear stochastic evolution equations (see equations $(\hat{P}_n)$), which are easier to study. We also prove the convergence in mean square (see Theorem 1.4.5). Since the approximation method involves linear evolution equations of a special type, we give in Section 1.3 results concerning this type of equations.

**Chapter 2** deals with the optimal control of the stochastic Navier-Stokes equation. We investigate the behavior of the flow controlled by different external forces, which are feedback controls and respectively bounded controls. We search for an optimal control that minimize a given cost functional. Whether or not there exist such optimal controls is a common question in optimal control theory and often for the answer one uses the Weierstraß Theorem and assumes that the set of admissible controls is compact. To assure the compactness of this set is sometimes not practicable. Therefore we investigate this problem and prove in Theorem 2.3.4, respectively Theorem 2.4.2, the existence of optimal controls, respectively $\varepsilon$-optimal controls, in the case of feedback controls. In the case of bounded controls this method can not be applied, because it uses the special linear and continuous structure of the feedback controls. Using the ideas from A. Bensoussan [3] and adapting them for the considered Navier-Stokes equation we calculate the Gateaux derivative of the cost functional (see Theorem 2.6.4) and derive a stochastic minimum principle (for the case of bounded controls), which gives us a necessary condition for optimality (see Theorem 2.7.2). We complete the statement of the stochastic minimum principle by giving the equations for the adjoint processes.

**Chapter 3** contains some aspects and results of dynamic programming for the stochastic Navier-Stokes equation. First we prove that the solution of the considered equation is a Markov process (see Theorem 3.1.1). This property was proved by B. Schmalfuß [29] for the stochastic Navier-Stokes equation with additive noise. In Section 3.2 we illustrate the dynamic programming approach (called also Bellman's principle) and we give a formal derivation of Bellman's equation. Bellman's principle turns the stochastic control problem into a deterministic control problem about a nonlinear partial differential equation of second order (see equation (3.11)) involving the infinitesimal generator. To round off the results of Chapter 2 we give a sufficient condition for an optimal control (Theorem 3.2.3 and Theorem 3.2.4). This condition requires a suitably behaved solution of the Bellman equation and an admissible control satisfying a certain equation. In this section we consider the finite dimensional stochastic Navier-Stokes equation (i.e., the equations obtained by the Galerkin method). The approach would be very complicate for the infinite dimensional case, because in this case it is difficult to obtain the infinitesimal generator. M.J. Vishik and A.V. Fursikov investigated in [35] also the inverse Kolmogorov equations, which give the inifinitsimal generator of the process being solution of the considered equation, only for the case of $n = 2$ for (0.1).

The final part of the dissertation contains an **Appendix** with useful properties from functional and stochastic analysis. We included them into the paper for the convenience of the reader and because we often make use of them.

The development and implementation of numerical methods for the Navier-Stokes equation remains an open problem for further research: "*...the numerical resolution of the Navier-Stokes equation will require (as in the past) the simultaneous efforts of mathematicians, numerical analysts and specialists in computer science. Several significant problems can already be solved numerically, but much time and effort will be necessary until we master the numerical solution of these equations for realistic values of the physical parameters. Besides the need for the development of appropiate algorithms and codes and the improvement of computers in memory size and computation speed, there is another difficulty of a more mathematical (as well as practical) nature. The solutions of the Navier-Stokes equation under realistic conditions are so highly oscillatory (chaotic behavior) that even if we were able to solve them with a great accuracy we would be faced with too much useless information. One has to find a way, with some kind of averaging, to compute mean values of the solutions and the corresponding desired parameters.*" (R. Temam [33])