Chapter 1

Existence and Approximation of the Solution

In this chapter we use the Galerkin method to prove the existence of the strong solution of the Navier-Stokes equation. We mean strong solution in the sense of stochastic analysis (see [14], Definition 4.2, p. 104): a complete probability space and a Wiener process are given in advance and the equation is defined in the generalized sense over an evolution triple. The techniques that we used are not the same as in the papers of A. Bensoussan [4], M. Capinski, N. J. Cutland [6], D. Gatarek [7], A. I. Komech, M. I. Vishik [20], B. Schmalfuß [29], [31], M. Viot [34], because in the above-mentioned papers one consider weak (statistical) solutions. The Galerkin-type approximations of the solutions and some a priori estimates allow one to prove compactness properties of the corresponding probability measures and to obtain a solution of the equation. In the paper of B. Schmalfuß [30] are considered strong solutions for the equation with an additive noise (the intensity of the random noise part does not depend on the state). The techniques applied in this dissertation are different from those used in the papers above. We utilize the properties of stopping times and some basic convergence principles from functional analysis. An important result is that the Galerkin-type approximations converge in mean square to the solution of the Navier-Stokes equation (see Theorem 1.2.7). This we can prove by using the property of higher order moments for the solution (see Lemma 1.2.3 and Lemma 1.2.6). The Galerkin method is useful to prove the *existence of the solution*, but it is complicated for numerical developments because it involves nonlinear terms. In Section 1.4 we give another *approximation method* by making use of linear evolution equations (see equations $(\hat{P}_n)$), which are easier to study. We also prove that the approximations converge in mean square to the solution of the stochastic Navier-Stokes equation (see Theorem 1.4.5). Since the approximation method involves linear evolution equations of a special type, we give in Section 1.3 some results concerning this type of equations.

The development and implementation of numerical methods for this type of equations remains an open problem for further research. For numerical solutions of stochastic differential equations we refer the reader to the book of P. Kloeden and E. Platen [19].

1.1 Assumptions and formulation of the problem

First we state the assumptions about the stochastic evolution equation that will be considered.

(i) $(\Omega, \mathcal{F}, P)$ is a complete probability space and $(\mathcal{F}_t)_{t\in[0,T]}$ is a right continuous filtration such that $\mathcal{F}_0$ contains all $\mathcal{F}$-null sets. $(w(t))_{t\in[0,T]}$ is a real valued standard $\mathcal{F}_t$-Wiener process.

(ii) (V, H, V^*) is an evolution triple (see [37], p. 416), where $(V, \|\cdot\|_V)$ and $(H, \|\cdot\|)$ are separable Hilbert spaces, and the embedding operator $V \hookrightarrow H$ is assumed to be compact. We denote by $(\cdot,\cdot)$ the scalar product in H.

(iii) $\mathcal{A} : V \to V^*$ is a linear operator such that $\langle \mathcal{A}v, v\rangle \geq \nu\|v\|_V^2$ for all $v \in V$ and $\langle \mathcal{A}u, v\rangle = \langle \mathcal{A}v, u\rangle$ for all $u, v \in V$, where $\nu > 0$ is a constant and $\langle\cdot,\cdot\rangle$ denotes the dual pairing.

(iv) $\mathcal{B} : V \times V \to V^*$ is a bilinear operator such that $\langle \mathcal{B}(u, v), v\rangle = 0$ for all $u, v \in V$ and for which there exists a positive constant $b > 0$ such that

$$|\langle \mathcal{B}(u, v), z\rangle|^2 \leq b\|z\|_V^2 \|u\|\|u\|_V \|v\|\|v\|_V .$$

(v) $\mathcal{C} : [0, T] \times H \to H$ is a mapping such that

- **(a)** $\|\mathcal{C}(t, u) - \mathcal{C}(t, v)\|^2 \leq \lambda\|u - v\|^2$ for all $t \in [0, T]$, $u, v \in H$, where λ is a positive constant;
- **(b)** $\mathcal{C}(t, 0) = 0$ for all $t \in [0, T]$;
- **(c)** $\mathcal{C}(\cdot, v) \in \mathcal{L}_H^2[0, T]$ for all $v \in H$.

(vi) $\Phi : [0, T] \times H \to H$ is a mapping such that

- **(a)** $\|\Phi(t, u) - \Phi(t, v)\|^2 \leq \mu\|u - v\|^2$ for all $t \in [0, T]$, $u, v \in H$, where μ is a positive constant;
- **(b)** $\Phi(t, 0) = 0$ for all $t \in [0, T]$;
- **(c)** $\Phi(\cdot, v) \in \mathcal{L}_H^2[0, T]$ for all $v \in H$.

(vii) x_0 is a H-valued $\mathcal{F}_0$-measurable random variable such that $E\|x_0\|^4 < \infty$.

Definition 1.1.1
We call a process $\big(U(t)\big)_{t\in[0,T]}$ from the space $\mathcal{L}_V^2(\Omega \times [0, T])$ with $E\|U(t)\|^2 < \infty$ for all $t \in [0, T]$ a **solution of the stochastic Navier-Stokes equation** if it satisfies the equation:

$$(1.1)\qquad \begin{aligned}(U(t), v) + \int_0^t \langle \mathcal{A}U(s), v\rangle ds &= (x_0, v) + \int_0^t \langle \mathcal{B}(U(s), U(s)), v\rangle ds \\ &+ \int_0^t (\Phi(s, U(s)), v) ds + \int_0^t (\mathcal{C}(s, U(s)), v) dw(s)\end{aligned}$$

for all $v \in V$, $t \in [0, T]$ and a.e. $\omega \in \Omega$, where the stochastic integral is understood in the Ito sense.

Remark 1.1.2

1) Since $\mathcal{A}$ is a linear and monotone operator, it follows that it is continuous (see [37], Proposition 26.4, p. 555), i.e., there exists a constant $c_{\mathcal{A}} > 0$ such that for all $u \in V$ we have

$$\|\mathcal{A}u\|_{V^*}^2 \leq c_{\mathcal{A}} \|u\|_V^2.$$

2) From the properties of the operator $\mathcal{B}$ we can derive the following relation

$$\langle \mathcal{B}(u,v), z \rangle = -\langle \mathcal{B}(u,z), v \rangle \quad \text{for all } u, v, z \in V,$$

which we will use often in our proofs.

3) The condition $\mathcal{C}(t,0) = 0$ (for all $t \in [0,T]$) is given only to simplify the calculations. It can be omitted, in which case one can use the estimate $\|\mathcal{C}(t,u)\|^2 \leq 2\lambda \|u\|^2 + 2\|\mathcal{C}(t,0)\|^2$ that follows from the Lipschitz condition. The same remark holds for Φ too.

4) If we set $n = 2$, $V = \{u \in \overset{\circ}{W}{}_2^1(G) : \text{div} u = 0\}$, $\quad H = \bar{V}^{L^2(G)}$ and

$$\langle \mathcal{A}u, v \rangle = \int_G \sum_{i=1}^n \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx, \quad \langle \mathcal{B}(u,v), z \rangle = -\int_G \sum_{i,j=1}^n u_i \frac{\partial v_j}{\partial x_i} z_j dx, \quad \Phi(t,u) = f(t)$$

for $u, v, z \in V, t \in [0,T]$, then equation (0.1) can be transformed into (1.1); see [32].

For finite dimensional approximations we need some preliminaries. Let $h_1, h_2, \ldots, h_n, \ldots \in H$ be the eigenvectors of the operator $\mathcal{A}$, for which we consider the domain of definition $\text{Dom}(\mathcal{A}) = \{v \in V \mid \mathcal{A}v \in H\}$. These eigenvectors form an orthonormal base in H and they are orthogonal in V (see [24], p. 110). For each $n \in \mathbb{N}$ we consider $H_n := \text{sp}\{h_1, h_2, \ldots, h_n\}$ equipped with the norm induced from H. We write $(H_n, \|\cdot\|_V)$ when we consider H_n equipped with the norm induced from V. We define by $\Pi_n : H \to H_n$ the orthogonal projection of H on H_n

$$\Pi_n h := \sum_{i=1}^n (h, h_i) h_i.$$

Let $\mathcal{A}_n : H_n \to H_n$, $\mathcal{B}_n : H_n \times H_n \to H_n$, $\Phi_n, \mathcal{C}_n : [0,T] \times H_n \to H_n$ be defined respectively by

$$\mathcal{A}_n u = \sum_{i=1}^n \langle \mathcal{A}u, h_i \rangle h_i, \quad \mathcal{B}_n(u,v) = \sum_{i=1}^n \langle \mathcal{B}(u,v), h_i \rangle h_i,$$

$$\mathcal{C}_n(t,u) = \Pi_n \mathcal{C}(t,u), \quad \Phi_n(t,u) = \Pi_n \Phi(t,u), \quad x_{0n} = \Pi_n x_0$$

for all $t \in [0,T]$, $u, v \in H_n$.

Let $\Big(X(t)\Big)_{t \in [0,T]}$ be a process in the space $\mathcal{L}_V^2(\Omega \times [0,T])$ and let $X_n := \Pi_n X$. Using the properties of $\mathcal{A}$ and of its eigenvectors $h_1, h_2, \ldots$ ($\lambda_1, \lambda_2, \ldots$ are the corresponding eigenvalues), we have

$$\|X_n(t)\|_V^2 \leq \|X(t)\|_V^2, \qquad \|X_n(t)\|^2 \leq \|X(t)\|^2, \qquad \|X(t) - X_n(t)\|^2 \leq \|X(t)\|^2, \tag{1.2}$$

$$\begin{aligned}(1.3)\qquad \nu\|X(t)-X_n(t)\|_V^2 &\leq \langle \mathcal{A}X(t)-\mathcal{A}X_n(t), X(t)-X_n(t)\rangle = \sum_{i=n}^{\infty}\lambda_i(X(t),h_i)^2\\ &\leq \langle \mathcal{A}X(t), X(t)\rangle \leq c_{\mathcal{A}}\|X(t)\|_V^2.\end{aligned}$$

Hence for $P\times[0,T]$ a.e. $(\omega,t)\in\Omega\times[0,T]$ we have

$$\lim_{n\to\infty}\|X(\omega,t)-X_n(\omega,t)\|_V^2=0.$$

By the Lebesgue dominated convegence theorem it follows that

$$\lim_{n\to\infty}\int_0^T\|X(t)-X_n(t)\|_V^2dt=0 \tag{1.4}$$

and

$$\lim_{n\to\infty}E\int_0^T\|X(t)-X_n(t)\|_V^2dt=0. \tag{1.5}$$

If the process $\Big(X(t)\Big)_{t\in[0,T]}$ has almost surely continuous trajectories in H, then

$$\lim_{n\to\infty}\|X(T)-X_n(T)\|^2=0 \quad \text{for a.e.} \quad \omega\in\Omega \tag{1.6}$$

and

$$\lim_{n\to\infty}E\|X(T)-X_n(T)\|^2=0. \tag{1.7}$$

1.2 Existence of the solution of the stochastic Navier-Stokes equation by Galerkin approximation

We want to prove the existence of the solution of the Navier-Stokes equation (1.1) by approximating it by means of the Galerkin method, i.e., by a sequence of solutions of finite dimensional evolution equations (see equations (P_n)). Since we consider the strong solution of the Navier-Stokes equation, we do not need to use the techniques considered in the case of weak solutions. The techniques applied in our paper use in particular the properties of stopping times and some basic convergence principles from functional analysis. An important result is that the Galerkin-type approximations converge in mean square to the solution of the Navier-Stokes equation (see Theorem 1.2.7).

For each $n=1,2,3,\dots$ we consider the sequence of finite dimensional evolution equations

$$\begin{aligned}(P_n)\qquad (U_n(t),v)+\int_0^t(\mathcal{A}_nU_n(s),v)ds &= (x_{0n},v)+\int_0^t(\mathcal{B}_n(U_n(s),U_n(s)),v)ds\\ &+ \int_0^t(\Phi_n(s,U_n(s)),v)ds+\int_0^t(\mathcal{C}_n(s,U_n(s)),v)dw(s),\end{aligned}$$

for all $v\in H_n$, $t\in[0,T]$ and a.e. $\omega\in\Omega$.

Theorem 1.2.1 *For each $n \in \mathbb{N}$, equation (P_n) has a solution $U_n \in \mathcal{L}^2_V(\Omega \times [0,T])$, which is unique almost surely and has almost surely continuous trajectories in H.*

Proof. We use an analogous method as in [31]. Let $\Big(\chi_M\Big)$ be a family of Lipschitz continuous mappings such that

$$\chi_M(x) = \begin{cases} 1\ , & \text{if} \quad 0 \le x \le M, \\ 0\ , & \text{if} \quad x \ge M+1, \\ M+1-x, & \text{if} \quad x \in (M, M+1). \end{cases}$$

For each fixed $n \in \mathbb{N}$ we consider the solution U_n of equation (P_n) approximated by $\Big(U_n^M\Big)$ $(M = 1, 2, \ldots)$ which is the solution of the equation

$$(P_n^M) \qquad \begin{aligned} (U_n^M(t), v) \ &+ \int_0^t (\mathcal{A}_n U_n^M(s), v) ds = (x_{0n}, v) \\ &+ \int_0^t (\chi_M(\|U_n^M(t)\|^2) \mathcal{B}_n(U_n^M(s), U_n^M(s)), v) ds \\ &+ \int_0^t (\Phi_n(s, U_n^M(s)), v) ds + \int_0^t (\mathcal{C}_n(s, U_n^M(s)), v) dw(s), \end{aligned}$$

for all $v \in H_n$, $t \in [0,T]$, and a.e. $\omega \in \Omega$. For this equation we apply the theory of finite dimensional Ito equations with Lipschitz continuous nonlinearities (see [18], Theorem 3.9, p. 289). Hence there exists $U_n^M \in \mathcal{L}^2_{(H_n, \|\cdot\|_V)}(\Omega \times [0,T])$ almost surely unique solution of (P_n^M) which has continuous trajectories in H.

We consider the stopping times $\mathcal{T}_M := \mathcal{T}_M^{U_n^M}$ (the definition of stopping times is given in Appendix B). By using (P_n^M), the properties of $\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n, \Phi_n$ and Proposition B.2 (for $Q := U_n^M$, $a_0 := x_{0n}, k_1 := 2\nu, k_2 := 2\sqrt{\mu} + \lambda, F_1 = F_2 := 0, F_3 := 2\mathcal{C}_n$) we obtain the following estimate

$$(1.8) \qquad E \sup_{t \in [0,T]} \|U_n^M(t)\|^2 + 2\nu E \int_0^T \|U_n^M(s)\|_V^2 ds \le cE\|x_0\|^2,$$

where c is a positive constant independent of n and M. From Markov's inequality, the definition of $\mathcal{T}_M$, and (1.8) we have

$$(1.9) \qquad P(\mathcal{T}_M < T) \le P\Big(\sup_{t \in [0,T]} \|U_n^M(t)\|^2 \ge M\Big) \le \frac{c}{M} E\|x_0\|^2.$$

Let Ω_n^M be the set of all $\omega \in \Omega$ such that $U_n^M(\omega, \cdot)$ satisfies (P_n^M) for all $t \in [0,T], v \in H_n$ and $U_n^M(\omega, \cdot)$ has continuous trajectories in H. We denote $\Omega' := \bigcap_{M=1}^{\infty} \Omega_n^M$ and have $P(\Omega') = 1$. We also consider

$$S_n := \bigcup_{M=1}^{\infty} \bigcup_{1 \le K \le M} \{\omega \in \Omega' | \mathcal{T}_K = T \text{ and } \exists\ t \in [0,T] :\ U_n^K(\omega, t) \ne U_n^M(\omega, t)\}.$$

We get $P(S_n) = 0$, because otherwise there exist two natural numbers M_0, K_0 with $K_0 < M_0$ such that the set

$$S^n_{M_0,K_0} := \{\omega \in \Omega' | \mathcal{T}_{K_0} = T \text{ and } \exists\, t \in [0,T] : \; U_n^{K_0}(\omega,t) \neq U_n^{M_0}(\omega,t)\}$$

has the measure $P(S^n_{M_0,K_0}) > 0$. We define for each $t \in [0,T]$

$$U_n^*(\omega,t) := \begin{cases} U_n^{K_0}(\omega,t) & , \quad \omega \in S^n_{M_0,K_0} \\ U_n^{M_0}(\omega,t) & , \quad \omega \in \Omega' \setminus S^n_{M_0,K_0}. \end{cases}$$

We see that for all $\omega \in S^n_{M_0,K_0}$ there exists $t \in [0,T]$ such that $U_n^*(\omega,t) \neq U_n^{M_0}(\omega,t)$. This contradicts to the almost surely uniqueness of the solution of $(P_n^{M_0})$. Consequently, $P(S_n) = 0$.

Let $\Omega'' := \Omega' \cap \Big(\bigcup\limits_{M=1}^{\infty} \{\mathcal{T}_M = T\} \setminus S_n \Big)$. Using (1.9) and the definition of S we have

$$P(\Omega'') = \lim_{M\to\infty} P\Big(\{\mathcal{T}_M = T\} \setminus S_n\Big) = 1 - \lim_{M\to\infty} P(\mathcal{T}_M < T) = 1.$$

Let $\omega \in \Omega''$. For this ω there exists a natural number M_0 such that $\mathcal{T}_M = T$ for all $M \geq M_0$. Hence $\chi_M(\|U_n^M(s)\|^2) = 1$ for all $s \in [0,T]$ and all $M \geq M_0$. Equation (P_n^M) implies

$$\begin{aligned} (U_n^M(t),v) \quad &+ \int_0^t \langle \mathcal{A}_n U_n^M(s), v\rangle ds = (x_{0n},v) + \int_0^t \langle \mathcal{B}_n(U_n^M(s),U_n^M(s)),v\rangle ds \\ &+ \int_0^t (\Phi_n(s,U_n^M(s)),v)ds + \int_0^t (\mathcal{C}_n(s,U_n^M(s)),v)dw(s) \end{aligned} \tag{1.10}$$

for all $M \geq M_0$ and all $t \in [0,T], v \in H_n$. For this fixed $\omega \in \Omega''$ and for each $t \in [0,T]$ we define

$$U_n(\omega,t) := U_n^{M_0}(\omega,t) = \lim_{M\to\infty} U_n^M(\omega,t) \tag{1.11}$$

with respect to the H-norm. This definition is correct because $\omega \notin S_n$. Then using (1.10) and (1.11) we obtain

$$\begin{aligned} (U_n(t),v) + \int_0^t (\mathcal{A}_n U_n(s),v)ds \quad &= \quad (x_{0n},v) + \int_0^t (\mathcal{B}_n(U_n(s),U_n(s)),v)ds \\ &\quad + \int_0^t (\Phi_n(s,U_n(s)),v)ds + \int_0^t (\mathcal{C}_n(s,U_n(s)),v)dw(s) \end{aligned}$$

for all $\omega \in \Big(\Omega \cap \Omega''\Big) \setminus S_n, t \in [0,T], v \in H_n$. The process $(U_n(t))_{t\in[0,T]}$ is H_n-valued, $\mathcal{F} \times \mathcal{B}_{[0,T]}$-measurable, adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and has almost surely continuous trajectories in H_n, because all U_n^M have this property. Obviously for all $t \in [0,T]$ we have

$$\lim_{M\to\infty} \|U_n^M(t) - U_n(t)\|^2 = 0 \quad \text{for a.e. } \omega \in \Omega \tag{1.12}$$

and

$$\lim_{M\to\infty}\int_0^T \|U_n^M(s)-U_n(s)\|_V^2 ds = 0 \quad \text{for a.e. } \omega\in\Omega.$$

By using (1.8) we obtain the following estimates

$$E\|U_n(t)\|^2 \le \liminf_{M\to\infty} E\|U_n^M(t)\|^2 \le cE\|x_0\|^2 \quad \text{for all } t\in[0,T]$$

and

$$E\int_0^T \|U_n(s)\|_V^2 ds \le \liminf_{M\to\infty} E\int_0^T \|U_n^M(s)\|_V^2 ds \le \frac{c}{2\nu}E\|x_0\|^2.$$

Therefore $U_n \in \mathcal{L}_V^2(\Omega\times[0,T])$.

The uniqueness of the solution can be proved analogously to the case of the stochastic Navier-Stokes equation (see Theorem 1.2.2). ■

One of the **main results** of this chapter is given in the following theorem, in which we state the existence and almost surely uniqueness of the solution U of the Navier-Stokes equation.

Theorem 1.2.2
The Navier-Stokes equation (1.1) *has a solution, which is almost surely unique and has almost surely continuous trajectories in* H.

For the proof of this theorem we need several lemmas.

Lemma 1.2.3
There exists a positive constant c_1 *(independent of* n*) such that for all* $n\in\mathbb{N}$

$$E\sup_{t\in[0,T]}\|U_n(t)\|^2 + 2\nu E\int_0^T \|U_n(t)\|_V^2 dt \le c_1 E\|x_0\|^2$$

and each of the following expressions

$$E\sup_{t\in[0,T]}\|U_n(t)\|^4, \quad E\Big(\int_0^T \|U_n(t)\|_V^2 dt\Big)^2$$

is less or equal to $c_1E\|x_0\|^4$.

Proof. Let n be an arbitrary fixed natural number. Equation (P_n) (given at the beginning of this section) can also be written as

$$(1.13)\qquad (U_n(t),h_i) + \int_0^t \langle \mathcal{A}U_n(s),h_i\rangle ds = (x_0,h_i) + \int_0^t \langle \mathcal{B}(U_n(s),U_n(s)),h_i\rangle ds + \int_0^t (\Phi(s,U_n(s)),h_i)ds + \int_0^t (\mathcal{C}(s,U_n(s),h_i)dw(s),$$

for all $i = 1, \dots, n$, $t \in [0,T]$ and a.e. $\omega \in \Omega$. By the Ito formula and by our hypothesis from Section 1.1 we have

$$\|U_n(t)\|^2 + 2\nu \int_0^t \|U_n(s)\|_V^2 ds \le \|x_0\|^2 + (2\sqrt{\mu} + \lambda) \int_0^t \|U_n(s))\|^2 ds + 2 \int_0^t (\mathcal{C}(s, U_n(s)), U_n(s)) dw(s)$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$. Now we apply Proposition B.2 for $Q := U_n, k_1 := 2\nu$, $k_2 := 2\sqrt{\mu} + \lambda, a_0 := x_0, F_1 = F_2 := 0, F_3 := 2\mathcal{C}$. Then we obtain the estimates given in the statement of this lemma. ∎

Lemma 1.2.4

(i) *There exist* $U \in \mathcal{L}_V^2(\Omega \times [0,T])$, $\mathcal{B}^* \in \mathcal{L}_{V^*}^2(\Omega \times [0,T])$, $\Phi^*, \mathcal{C}^* \in \mathcal{L}_H^2(\Omega \times [0,T])$, *and a subsequence* (n') *of* (n) *such that for* $n' \to \infty$ *we have*

$$U_{n'} \rightharpoonup U \quad in \quad \mathcal{L}_V^2(\Omega \times [0,T]),$$

$$\mathcal{B}(U_{n'}, U_{n'}) \rightharpoonup \mathcal{B}^* \quad in \quad \mathcal{L}_{V^*}^2(\Omega \times [0,T]),$$

$$\Phi(\cdot, U_{n'}(\cdot)) \rightharpoonup \Phi^*, \quad \mathcal{C}(\cdot, U_{n'}(\cdot)) \rightharpoonup \mathcal{C}^* \quad in \quad \mathcal{L}_H^2(\Omega \times [0,T]),$$

where $\rightharpoonup$ *denotes the weak convergence.*

(ii) *For all* $v \in V$, $t \in [0,T]$ *and a.e.* $\omega \in \Omega$ *the process* $\Big(U(t)\Big)_{t\in[0,T]}$ *satisfies the equation:*

$$(U(t), v) + \int_0^t \langle \mathcal{A}U(s), v\rangle ds = (x_0, v) + \int_0^t \langle \mathcal{B}^*(s), v\rangle ds + \int_0^t (\Phi^*(s), v) ds + \int_0^t (\mathcal{C}^*(s), v) dw(s). \tag{1.14}$$

The process $\Big(U(t)\Big)_{t\in[0,T]}$ *has almost surely continuous trajectories in* H.

(iii) *The function* U *from* **(ii)** *satisfies* $E \sup_{t\in[0,T]} \|U(t)\|^2 < \infty$.

Proof. **(i)** Taking into account the properties of Φ, $\mathcal{C}$, and the estimates from Lemma 1.2.3 it follows that $\Big(\Phi(\cdot, U_n(\cdot))\Big)$, $\Big(\mathcal{C}(\cdot, U_n(\cdot))\Big)$ are bounded sequences in the space $\mathcal{L}_H^2(\Omega \times [0,T])$. By using the properties of $\mathcal{B}$ we can derive

$$E\int_0^T \|\mathcal{B}(U_n(t), U_n(t))\|_{V^*}^2 dt \le bE\int_0^T \|U_n(t)\|_V^2 \|U_n(t)\|^2 dt \le bc_1 E\|x_0\|^4,$$

so $\Big(\mathcal{B}(U_n, U_n)\Big)$ is a bounded sequence in the space $\mathcal{L}^2_{V^*}(\Omega \times [0,T])$. Applying Proposition A.1 (see Appendix A), it follows that there exist a subsequence (n') of (n) and $\hat{U} \in \mathcal{L}^2_V(\Omega \times [0,T])$, $\mathcal{B}^* \in \mathcal{L}^2_{V^*}(\Omega \times [0,T])$, $\Phi^*, \mathcal{C}^* \in \mathcal{L}^2_H(\Omega \times [0,T])$ such that for $n' \to \infty$

$$U_{n'} \rightharpoonup \hat{U} \quad \text{in } \mathcal{L}^2_V(\Omega \times [0,T]), \qquad \mathcal{B}(U_{n'}, U_{n'}) \rightharpoonup \mathcal{B}^* \quad \text{in} \quad \mathcal{L}^2_{V^*}(\Omega \times [0,T]),$$

$$\Phi(\cdot, U_{n'}(\cdot)) \rightharpoonup \Phi^*, \qquad \mathcal{C}(\cdot, U_{n'}(\cdot)) \rightharpoonup \mathcal{C}^* \quad \text{in } \mathcal{L}^2_H(\Omega \times [0,T]).$$

(ii) In (1.13) we take the limit $n' \to \infty$, use the properties of $\mathcal{A}$, the weak convergences from above (also Proposition A.2 and Proposition A.3) and obtain

$$(1.15) \qquad (\hat{U}(t), h_i) = (x_0, h_i) - \int_0^t \langle \mathcal{A}\hat{U}(s), h_i \rangle ds \; + \; \int_0^t \langle \mathcal{B}^*(s), h_i \rangle ds + \int_0^t (\Phi^*(s), h_i) ds + \int_0^t (\mathcal{C}^*(s), h_i) dw(s),$$

for a.e. $(\omega, t) \in \Omega \times [0,T]$ and $i \in \mathbb{N}$. Since $\mathrm{sp}\{h_1, h_2, \ldots, h_n, \ldots\}$ is dense in V (because of the properties of the eigenvectors of $\mathcal{A}$) it follows that (1.15) holds also for all $v \in V$.

There exists a $\mathcal{F}_t$-measurable H-valued process which is equal to $\hat{U}(t)$ for $P \times \Lambda$ a.e. $(\omega, t) \in \Omega \times [0,T]$ and is equal to the right side of (1.15) for all $t \in [0,T]$ and a.e. $\omega \in \Omega$. We denote this process by $\Big(U(t)\Big)_{t \in [0,T]}$. Hence

$$(U(t), v) + \int_0^t \langle \mathcal{A}U(s), v \rangle ds = (x_0, v) + \int_0^t \langle \mathcal{B}^*(s), v \rangle ds + \int_0^t (\Phi^*(s), v) ds + \int_0^t (\mathcal{C}^*(s), v) dw(s)$$

for all $v \in V$, $t \in [0,T]$ and a.e. $\omega \in \Omega$; the process $\Big(U(t)\Big)_{t \in [0,T]}$ has in H almost surely continuous trajectories (see [21], Theorem 3.1, p. 88).

(iii) In (1.14) we apply the Ito formula, use the properties of $\mathcal{A}$ and some elementary inequalities. Then we apply Proposition B.2 for $Q := U, a_0 := x_0, F_1 := \dfrac{1}{\nu}\|\mathcal{B}^*\|^2_{V^*} + \|\Phi^*\|^2 + \|\mathcal{C}^*\|^2$, $F_2 := 2\mathcal{C}^*$, $F_3 := 0$, $k_1 := \nu, k_2 := 1$. ■

For each fixed $M \in \mathbb{N}$ we consider $\mathcal{T}_M := \mathcal{T}^U_M$, where $\Big(U(t)\Big)_{t \in [0,T]}$ is the process obtained in Lemma 1.2.4.

Lemma 1.2.5

The following convergences hold

$$\lim_{n' \to \infty} E \int_0^{\mathcal{T}_M} \|U(s) - U_{n'}(s)\|^2_V ds = 0 \quad \textit{and} \quad \lim_{n' \to \infty} E\|U(\mathcal{T}_M) - U_{n'}(\mathcal{T}_M)\|^2 = 0.$$

Proof. For each $n \in \mathbb{N}$ let $\tilde{U}_n(t) = \Pi_n U$. From (1.14) and (1.13) we have

$$(U(t) - U_n(t), h_i) + \int_0^t \langle \mathcal{A}U(s) - \mathcal{A}U_n(s), h_i\rangle ds = \int_0^t \langle \mathcal{B}^*(s) - \mathcal{B}(U_n(s), U_n(s)), h_i\rangle ds$$
$$+ \int_0^t (\Phi^*(s) - \Phi(s, U_n(s)), h_i) ds + \int_0^t (\mathcal{C}^*(s) - \mathcal{C}(s, U_n(s)), h_i) dw(s)$$

for all $t \in [0, T]$, $i = 1, \dots, n$, a.e. $\omega \in \Omega$. After applying the Ito formula and summing from $i = 1$ to n, we use the properties of $\mathcal{A}$ and obtain

$$\|\tilde{U}_n(t) - U_n(t)\|^2 + 2\int_0^t \langle \mathcal{A}\tilde{U}_n(s) - \mathcal{A}U_n(s), \tilde{U}_n(s) - U_n(s)\rangle ds$$
$$= 2\int_0^t \langle \mathcal{B}^*(s) - \mathcal{B}(U_n(s), U_n(s)), \tilde{U}_n(s) - U_n(s)\rangle ds$$
$$+ 2\int_0^t (\Phi^*(s) - \Phi(s, U_n(s)), \tilde{U}_n(s) - U_n(s)) ds$$
$$+ 2\int_0^t (\mathcal{C}^*(s) - \mathcal{C}(s, U_n(s)), \tilde{U}_n(s) - U_n(s)) dw(s) + \int_0^t \sum_{i=1}^n (\mathcal{C}^*(s) - \mathcal{C}(s, U_n(s)), h_i)^2 ds$$

for all $t \in [0, T]$, $i = 1, \dots, n$, a.e. $\omega \in \Omega$. Write

$$e_1(t) = \Delta_U(t) \exp\{-(2\lambda + 2\sqrt{\mu} + 1)t\},$$

where the notation for Δ_U is given in the paragraph "Frequently Used Notations". By the Ito formula get

$$e_1(t)\|\tilde{U}_n(t) - U_n(t)\|^2 + 2\int_0^t e_1(s)\langle \mathcal{A}\tilde{U}_n(s) - \mathcal{A}U_n(s), \tilde{U}_n(s) - U_n(s)\rangle ds \tag{1.16}$$
$$= 2\int_0^t e_1(s)\langle \mathcal{B}^*(s) - \mathcal{B}(U_n(s), U_n(s)), \tilde{U}_n(s) - U_n(s)\rangle ds - \frac{b}{\nu}\int_0^t e_1(s)\|U(s)\|_V^2 \|\tilde{U}_n(s) - U_n(s)\|^2 ds$$
$$- (2\lambda + 2\sqrt{\mu} + 1)\int_0^t e_1(s)\|\tilde{U}_n(s) - U_n(s)\|^2 ds + 2\int_0^t e_1(s)(\Phi^*(s) - \Phi(s, U_n(s)), \tilde{U}_n(s) - U_n(s)) ds$$
$$+ \int_0^t \sum_{i=1}^n e_1(s)(\mathcal{C}^*(s) - \mathcal{C}(s, U_n(s)), h_i)^2 ds + 2\int_0^t e_1(s)(\mathcal{C}^*(s) - \mathcal{C}(s, U_n(s)), \tilde{U}_n(s) - U_n(s)) dw(s)$$

for all $t \in [0,T]$, $i = 1,\dots,n$, a.e. $\omega \in \Omega$. From the properties of $\mathcal{B}$ and those of $\tilde{U}_n$ (see (1.2) we see that

$$\begin{aligned}
&\langle \mathcal{B}(U_n(s),U_n(s)),\tilde{U}_n(s)-U_n(s)\rangle = \langle \mathcal{B}(U_n(s),\tilde{U}_n(s)),\tilde{U}_n(s)-U_n(s)\rangle \\
= \;& \langle \mathcal{B}(U_n(s)-\tilde{U}_n(s),\tilde{U}_n(s)),\tilde{U}_n(s)-U_n(s)\rangle + \langle \mathcal{B}(\tilde{U}_n(s),\tilde{U}_n(s)),\tilde{U}_n(s)-U_n(s)\rangle \\
\le \;& \frac{b}{2\nu}\|\tilde{U}_n(s)\|_V^2\|\tilde{U}_n(s)-U_n(s)\|^2 + \frac{\nu}{2}\|\tilde{U}_n(s)-U_n(s)\|_V^2 + \langle \mathcal{B}(\tilde{U}_n(s),\tilde{U}_n(s)),\tilde{U}_n(s)-U_n(s)\rangle \\
\le \;& \frac{b}{2\nu}\|U(s)\|_V^2\|\tilde{U}_n(s)-U_n(s)\|^2 + \frac{\nu}{2}\|\tilde{U}_n(s)-U_n(s)\|_V^2 + \langle \mathcal{B}(\tilde{U}_n(s),\tilde{U}_n(s)),\tilde{U}_n(s)-U_n(s)\rangle.
\end{aligned}$$

The properties of Φ imply

$$\begin{aligned}
2\Big(\Phi^*(s)-\Phi(s,U_n(s)),\tilde{U}_n(s)-U_n(s)\Big) \;&\le\; 2\Big(\Phi^*(s)-\Phi(s,U(s)),\tilde{U}_n(s)-U_n(s)\Big) \\
&+\; (1+2\sqrt{\mu})\|\tilde{U}_n(s)-U_n(s)\|^2 + \mu\|U(s)-\tilde{U}_n(s)\|^2
\end{aligned}$$

and from the properties of $\mathcal{C}$ and $\tilde{U}_n$ we get

$$\begin{aligned}
&\sum_{i=1}^n \Big(\mathcal{C}^*(s)-\mathcal{C}(s,U_n(s)),h_i\Big)^2 = \|\mathcal{C}(s,U(s))-\mathcal{C}(s,U_n(s))\|_{H_n}^2 \\
+\;& 2\Big(\mathcal{C}^*(s)-\mathcal{C}(s,U_n(s)),\mathcal{C}^*(s)-\mathcal{C}(s,U(s))\Big)_{H_n} - \|\mathcal{C}(s,U(s))-\mathcal{C}^*(s)\|_{H_n}^2 \\
\le\;& 2\lambda\|U(s)-\tilde{U}_n(s)\|^2 + 2\lambda\|\tilde{U}_n(s)-U_n(s)\|^2 + 2\Big(\mathcal{C}^*(s)-\mathcal{C}(s,U_n(s)),\mathcal{C}^*(s)-\mathcal{C}(s,U(s))\Big)_{H_n} \\
-\;& \|\mathcal{C}(s,U(s))-\mathcal{C}^*(s)\|_{H_n}^2,
\end{aligned}$$

where we write $\|x\|_{H_n} := \|\Pi_n x\|$ and $(x,y)_{H_n} := (\Pi_n x,\Pi_n y)$ for $x,y \in H$.

We use these estimates in (1.16) to obtain

$$\begin{aligned}
&Ee_1(\mathcal{T}_M)\|\tilde{U}_n(\mathcal{T}_M)-U_n(\mathcal{T}_M)\|^2 + \nu E\int_0^{\mathcal{T}_M} e_1(s)\|\tilde{U}_n(s)-U_n(s)\|_V^2 ds \\
+\;& E\int_0^{\mathcal{T}_M} e_1(s)\|\mathcal{C}(s,U(s))-\mathcal{C}^*(s)\|_{H_n}^2 ds \\
\le\;& 2E\int_0^{\mathcal{T}_M} e_1(s)\langle \mathcal{B}^*(s)-\mathcal{B}(\tilde{U}_n(s),\tilde{U}_n(s)),\tilde{U}_n(s)-U_n(s)\rangle ds \\
+\;& (2\lambda+\mu)E\int_0^{\mathcal{T}_M} e_1(s)\|U(s)-\tilde{U}_n(s)\|^2 ds \\
+\;& 2E\int_0^{\mathcal{T}_M} e_1(s)(\Phi^*(s)-\Phi(s,U(s)),\tilde{U}_n(s)-U_n(s))ds
\end{aligned} \tag{1.17}$$

$$+ \quad 2E\int_0^{\mathcal{T}_M} e_1(s)(\mathcal{C}^*(s)-\mathcal{C}(s,U_n(s)),\mathcal{C}^*(s)-\mathcal{C}(s,U(s)))_{H_n}ds$$

where $M \in \mathbb{N}$. Using the properties of $\mathcal{B}$, those of the stopping time $\mathcal{T}_M$ and the fact that $(\tilde{U}_n)$ is the partial sum of the Fourier expansion of $U \in \mathcal{L}^2_V(\Omega\times[0,T])$ (see the properties (1.2) and (1.5) given in the final part of Section 1.1) we have

$$\begin{aligned}
&E\int_0^{\mathcal{T}_M} e_1(s)\|\mathcal{B}(U(s),U(s))-\mathcal{B}(\tilde{U}_n(s),\tilde{U}_n(s))\|^2_{V^*}ds\\
&\leq \; bE\int_0^{\mathcal{T}_M} e_1(s)\Big(\|U(s)\|_V\|U(s)\|+\|\tilde{U}_n(s)\|_V\|\tilde{U}_n(s)\|\Big)\|U(s)-\tilde{U}_n(s)\|_V\|U(s)-\tilde{U}_n(s)\|ds\\
&\leq \; 2bE\int_0^{\mathcal{T}_M} e_1(s)\|U(s)\|_V\|U(s)\|^2\|U(s)-\tilde{U}_n(s)\|_V ds\\
&\leq \; 2bM\Big(E\int_0^{\mathcal{T}_M}\|U(s)\|^2_V ds\Big)^{\frac{1}{2}}\Big(E\int_0^{\mathcal{T}_M}\|U(s)-\tilde{U}_n(s)\|^2_V ds\Big)^{\frac{1}{2}}
\end{aligned}$$

and hence

$$\lim_{n\to\infty} E\int_0^{\mathcal{T}_M} e_1(s)\|\mathcal{B}(U(s),U(s))-\mathcal{B}(\tilde{U}_n(s),\tilde{U}_n(s))\|^2_{V^*}ds=0.$$

We have $I_{[0,\mathcal{T}_M]}\mathcal{B}(U,U),\mathcal{B}^*\in\mathcal{L}^2_{V^*}(\Omega\times[0,T])$. For the subsequence (n') of (n) we have proved that $U_{n'}\rightharpoonup U$ in $\mathcal{L}^2_V(\Omega\times[0,T])$ and $\tilde{U}_{n'}\to U$ in $\mathcal{L}^2_V(\Omega\times[0,T])$ (see Lemma 1.2.4 and (1.5) from Section 1.1). Consequently,

$$\begin{aligned}
&\lim_{n'\to\infty} E\int_0^{\mathcal{T}_M} e_1(s)\langle\mathcal{B}^*(s)-\mathcal{B}(\tilde{U}_{n'}(s),\tilde{U}_{n'}(s)),\tilde{U}_{n'}(s)-U_{n'}(s)\rangle ds\\
= \;&\lim_{n'\to\infty} E\int_0^{\mathcal{T}_M} e_1(s)\langle\mathcal{B}^*(s)-\mathcal{B}(U(s),U(s)),\tilde{U}_{n'}(s)-U_{n'}(s)\rangle ds\\
+ \;&\lim_{n'\to\infty} E\int_0^{\mathcal{T}_M} e_1(s)\langle\mathcal{B}(U(s),U(s))-\mathcal{B}(\tilde{U}_{n'}(s),\tilde{U}_{n'}(s)),\tilde{U}_{n'}(s)-U_{n'}(s)\rangle ds=0.
\end{aligned}$$

It also follows that

$$\lim_{n'\to\infty} E\int_0^{\mathcal{T}_M} e_1(s)(\Phi^*(s)-\Phi(s,U(s)),\tilde{U}_{n'}(s)-U_n(s))ds=0.$$

Since $\mathcal{C}(\cdot, U_{n'}(\cdot)) \rightharpoonup \mathcal{C}^*$ in $\mathcal{L}^2_H(\Omega \times [0,T])$ and $\Pi_n \mathcal{C}^* - \Pi_n \mathcal{C}(\cdot, U(\cdot)) \to \mathcal{C}^* - \mathcal{C}(\cdot, U(\cdot))$, the following convergences hold:

$$\lim_{n'\to\infty} E \int_0^{\mathcal{T}_M} e_1(s)\Big(\mathcal{C}^*(s) - \mathcal{C}(s, U_{n'}(s)), \mathcal{C}^*(s) - \mathcal{C}(s, U(s))\Big)_{H_{n'}} ds$$

$$= \lim_{n'\to\infty} E \int_0^{\mathcal{T}_M} e_1(s)\Big(\mathcal{C}^*(s) - \mathcal{C}(s, U_{n'}(s)), \Pi_n \mathcal{C}^*(s) - \Pi_n \mathcal{C}(s, U(s))\Big) ds = 0$$

and

$$\lim_{n'\to\infty} E \int_0^{\mathcal{T}_M} e_1(s) \|\mathcal{C}(s, U(s)) - \mathcal{C}^*(s)\|^2_{H_{n'}} ds = E \int_0^{\mathcal{T}_M} e_1(s) \|\mathcal{C}(s, U(s)) - \mathcal{C}^*(s)\|^2 ds.$$

In view of these results, we see that by taking the limit $n' \to \infty$ in (1.17) the right side of this inequality tends to zero. Therefore

$$\lim_{n'\to\infty} E e_1(\mathcal{T}_M) \|\tilde{U}_{n'}(\mathcal{T}_M) - U_{n'}(\mathcal{T}_M)\|^2 = 0, \quad \lim_{n'\to\infty} E \int_0^{\mathcal{T}_M} e_1(s) \|\tilde{U}_{n'}(s) - U_{n'}(s)\|^2_V ds = 0$$

and

$$E \int_0^{\mathcal{T}_M} e_1(s) \|\mathcal{C}(s, U(s)) - \mathcal{C}^*(s)\|^2 ds = 0. \tag{1.18}$$

From the properties of e_1 over $[0, \mathcal{T}_M]$ and from (1.5) follows that for each fixed $M \in \mathbb{N}$ we have

$$\lim_{n'\to\infty} E \int_0^{\mathcal{T}_M} \|U(s) - U_{n'}(s)\|^2_V ds = 0 \quad \text{and} \quad \lim_{n'\to\infty} E \|U(\mathcal{T}_M) - U_{n'}(\mathcal{T}_M)\|^2 = 0. \tag{1.19}$$

■

Proof of Theorem 1.2.2.

From (1.18) we conclude that

$$I_{[0,\mathcal{T}_M]}(s)\mathcal{C}(s, U(s)) = I_{[0,\mathcal{T}_M]}(s)\mathcal{C}^*(s) \quad \text{for a.e.} \quad (\omega, t) \in \Omega \times [0,T]. \tag{1.20}$$

Because $\Phi(\cdot, U_{n'}(\cdot)) \rightharpoonup \Phi^*$ in $\mathcal{L}^2_H(\Omega \times [0,T])$ and Φ is a continuous mapping, it follows from (1.19) that

$$I_{[0,\mathcal{T}_M]}(s)\Phi(s, U(s)) = I_{[0,\mathcal{T}_M]}(s)\Phi^*(s) \quad \text{for a.e.} \quad (\omega, t) \in \Omega \times [0,T]. \tag{1.21}$$

Using (1.19) and the properties of $\mathcal{B}$ it can be proved that

$$\lim_{n'\to\infty} E \int_0^{\mathcal{T}_M} \langle \mathcal{B}(U(s), U(s)) - \mathcal{B}(U_{n'}(s), U_{n'}(s)), x(s)\rangle ds = 0 \quad \text{for all} \quad x \in \mathcal{D}_V(\Omega \times [0,T]).$$

But $\mathcal{B}(U_{n'}, U_{n'}) \rightharpoonup \mathcal{B}^*$ in $\mathcal{L}^2_{V^*}(\Omega \times [0,T])$, so

$$\lim_{n'\to\infty} E \int_0^{\mathcal{T}_M} \langle \mathcal{B}^*(s) - \mathcal{B}(U_{n'}(s), U_{n'}(s)), x(s)\rangle ds = 0 \quad \text{for all} \quad x \in \mathcal{D}_V(\Omega \times [0,T]).$$

Since $\mathcal{D}_V(\Omega \times [0,T])$ is dense in $\mathcal{L}^2_V(\Omega \times [0,T])$, it follows that

$$I_{[0,\mathcal{T}_M]}(s)\mathcal{B}^*(s) = I_{[0,\mathcal{T}_M]}(s)\mathcal{B}(U(s), U(s)) \quad \text{for a.e.} \quad (\omega, t) \in \Omega \times [0,T]. \tag{1.22}$$

Using (1.20), (1.21), and (1.22) in (1.14)

$$\begin{aligned}(U(t\wedge\mathcal{T}_M), v) \; &+ \int_0^{t\wedge\mathcal{T}_M} \langle \mathcal{A}u(s), v\rangle ds = (x_0, v) + \int_0^{t\wedge\mathcal{T}_M} \langle \mathcal{B}(U(s), U(s)), v\rangle ds \\ &+ \int_0^{t\wedge\mathcal{T}_M} (\Phi(s, U(s)), v) ds + \int_0^{t\wedge\mathcal{T}_M} (\mathcal{C}(s, U(s)), v) dw(s)\end{aligned} \tag{1.23}$$

for all $v \in V, t \in [0,T]$, and a.e. $\omega \in \Omega$.

From the properties of the stopping time $\mathcal{T}_M$ and Proposition B.1 we see that

$$P\Big(\bigcup_{M=1}^{\infty} \{\mathcal{T}_M = T\}\Big) = 1.$$

Let

$$\Omega' := \Big\{\omega \in \Omega : \omega \in \bigcup_{M=1}^{\infty} \{\mathcal{T}_M = T\} \text{ and } U(\omega, t) \text{ satisfies (1.23) for all } v \in V, t \in [0,T]\Big\}.$$

Obviously, we have $P(\Omega') = 1$.

For $\omega \in \Omega'$ there exists a natural number M_0 such that $\mathcal{T}_M(\omega) = T$ for all $M \geq M_0$. From (1.23), we obtain

$$\begin{aligned}(U(t), v) + \int_0^t \langle \mathcal{A}U(s), v\rangle ds = (x_0, v) \; &+ \int_0^t \langle \mathcal{B}(U(s), U(s)), v\rangle ds \\ &+ \int_0^t (\Phi(s, U(s)), v) ds + \int_0^t (\mathcal{C}(s, U(s)), v) dw(s)\end{aligned} \tag{1.24}$$

for all $v \in V, t \in [0,T]$. Consequently (1.24) holds for all $\omega \in \Omega'$. This means that the process $\Big(U(t)\Big)_{t\in[0,T]}$ satisfies the Navier-Stokes equation (1.1). Taking into account Lemma 1.2.4 it follows that U has almost surely continuous trajectories in H and we have

$$E \sup_{t\in[0,T]} \|U(t)\|^2 < \infty.$$

Hence $\Big(U(t)\Big)_{t\in[0,T]}$ is a solution of the Navier-Stokes equation (1.1).

(ii) In order to prove the uniqueness we assume that $X, Y \in \mathcal{L}^2_V(\Omega\times[0,T])$ are two solutions of equation (1.1), which have in H almost surely continuous trajectories. Let

$$e_2(t) = \Delta_X(t)\exp\{-(\lambda+2\sqrt{\mu})t\}$$

for all $t\in[0,T]$ and a.e. $\omega\in\Omega$. It follows by the Ito formula that

$$\begin{aligned}
&e_2(t)\|X(t)-Y(t)\|^2 + 2\int_0^t e_2(s)\langle \mathcal{A}X(s)-\mathcal{A}Y(s), X(s)-Y(s)\rangle ds \\
= \;& 2\int_0^t e_2(s)\langle \mathcal{B}(X(s),X(s))-\mathcal{B}(Y(s),Y(s)), X(s)-Y(s)\rangle ds \\
- \;& \frac{b}{\nu}\int_0^t e_2(s)\|X(s)\|_V^2\|X(s)-Y(s)\|^2 ds - (\lambda+2\sqrt{\mu})\int_0^t e_2(s)\|X(s)-Y(s)\|^2 ds \\
+ \;& 2\int_0^t e_2(s)(\Phi(s,X(s))-\Phi(s,Y(s)), X(s)-Y(s))ds \\
+ \;& 2\int_0^t e_2(s)(\mathcal{C}(s,X(s))-\mathcal{C}(s,Y(s)), X(s)-Y(s))dw(s) + \int_0^t e_2(s)\|\mathcal{C}(s,X(s))-\mathcal{C}(s,Y(s))\|^2 ds.
\end{aligned}$$

In view of the properties of $\mathcal{B}$ we can write

$$\begin{aligned}
2\langle \mathcal{B}(X(s),X(s))-\mathcal{B}(Y(s),Y(s)), X(s)-Y(s)\rangle &= 2\langle \mathcal{B}(X(s)-Y(s),X(s)), X(s)-Y(s)\rangle \\
&\le \frac{b}{\nu}\|X(s)\|_V^2\|X(s)-Y(s)\|^2 + \nu\|X(s)-Y(s)\|_V^2.
\end{aligned}$$

Now we use the properties of $\mathcal{A}$, Φ, and $\mathcal{C}$ to obtain

$$\begin{aligned}
e_2(t)\|X(t)-Y(t)\|^2 &+ \nu\int_0^t e_2(s)\|X(s)-Y(s)\|_V^2 ds \\
&\le 2\int_0^t e_2(s)(\mathcal{C}(s,X(s))-\mathcal{C}(s,Y(s)), X(s)-Y(s))dw(s)
\end{aligned}$$

for all $t\in[0,T]$ and a.e. $\omega\in\Omega$. This implies (using also the ideas from the proof of Proposition B.2)

$$Ee_2(t)\|X(t)-Y(t)\|^2 = 0 \quad \text{for all} \quad t\in[0,T]$$

and hence $P\Big(X(t)=Y(t)\Big)=1$ for all $t\in[0,T]$. Then for each countable and dense subset $\mathcal{S}\subset[0,T]$ we have

$$P\Big(\sup_{t\in\mathcal{S}}\|X(t)-Y(t)\|=0\Big)=1.$$

But X and Y have almost surely continuous trajectories in H, so

$$P\Big(\sup_{t\in[0,T]} \|X(t) - Y(t)\| = 0\Big) = 1.$$

This means that (1.1) has an almost surely unique solution. ■

Lemma 1.2.6
There exists a positive constant c_2 (depending only on λ, ν, and T) such that

$$E \sup_{t\in[0,T]} \|U(t)\|^4 + E\Big(\int_0^T \|U(s)\|_V^2 ds\Big)^2 \le c_2 E\|x_0\|^4.$$

The proof of Lemma 1.2.6 is analogous to the proof of Lemma 1.2.3 and makes use of Proposition B.2.

Another **important result** of this chapter is the following theorem, in which we state that the Galerkin approximations (U_n) converge in mean square to the solution of the Navier-Stokes equation.

Theorem 1.2.7
The following convergences hold:

$$\lim_{n\to\infty} E\int_0^T \|U(s) - U_n(s)\|_V^2 ds = 0$$

and

$$\lim_{n\to\infty} E\|U(t) - U_n(t)\|^2 = 0 \quad for\ all\ t \in [0,T].$$

Proof. First we apply Proposition B.3 with $\mathcal{T} := T$, $Q_{n'}(\mathcal{T}) := \int_0^{\mathcal{T}} \|U_{n'}(s) - U(s)\|_V^2 ds$, use Lemma 1.2.5, Lemma 1.2.3, and Lemma 1.2.6 to obtain

$$\lim_{n'\to\infty} E\int_0^T \|U(s) - U_{n'}(s)\|_V^2 ds = 0.$$

Let $t \in [0,T]$. Now we apply Proposition B.3 with $\mathcal{T} := t$, $Q_{n'}(\mathcal{T}) := \|U_{n'}(\mathcal{T}) - U(\mathcal{T})\|$, use Lemma 1.2.5, Lemma 1.2.3, and Lemma 1.2.6 and get

$$\lim_{n'\to\infty} E\|U_{n'}(t) - U(t)\|^2 = 0.$$

Every subsequence of (U_n) has a further subsequence which converges in the norm of the space $\mathcal{L}_V^2(\Omega\times[0,T])$ to the same limit U, the unique solution of the Navier-Stokes equation (1.1) (because we can repeat all arguments of the results of Section 1.2 for this subsequence). Applying Proposition A.1 it follows that the whole sequence (U_n) converges in mean square to U. By the same argument we can prove that for all $t \in [0,T]$ the whole sequence $(U_n(t))$ converges to $U(t)$ in the norm of the space $\mathcal{L}_H^2(\Omega)$. ■

Remark 1.2.8

1) The results of this section also hold if we consider equation (1.1) starting at s with $s \in [0, T)$ (instead of 0) and we assume that x_0 is a H-valued $\mathcal{F}_s$-measurable random variable such that $E\|x_0\|^4 < \infty$.

2) The results of this section also hold if we consider instead of a mapping Φ, satisfying hypothesis **(vi)** from Section 1.1, a process belonging to the space $\mathcal{L}^2_H(\Omega \times [0, T])$ with $E\int_0^T \|\Phi(t)\|^4 dt < \infty$.

1.3 A special linear stochastic evolution equation

The results presented in this section prepare the investigations for the linear approximation method from Section 1.4.

Let $X, Y \in \mathcal{L}^2_V(\Omega \times [0, T])$ be arbitrary processes with almost surely continuous trajectories in H and

$$E \sup_{t\in[0,T]} \|X(t)\|^2 < \infty, \quad E \sup_{t\in[0,T]} \|Y(t)\|^2 < \infty.$$

For each $M \in \mathbb{N}$ let $\mathcal{T}_M := \min\{\mathcal{T}^X_M, \mathcal{T}^Y_M\}$. From the properties of the stopping times (see Appendix B) it follows that

$$\lim_{M\to\infty} \mathcal{T}_M = T \quad \text{for a.e. } \omega \in \Omega, \tag{1.25}$$

as soon as

$$P(\bigcup_{M=1}^{\infty} \{\mathcal{T}_M = T\}) = 1. \tag{1.26}$$

We define $X^M(t) := X(t \wedge \mathcal{T}_M), Y^M(t) := Y(t \wedge \mathcal{T}_M)$ for all $t \in [0, T]$.

Let $\mathcal{G} : [0, T] \times H \to H$ be a mapping satisfying hypothesis **(v)** from Section 1.1 and we assume that for each $t \in [0, T]$ the mapping $\mathcal{G}(t, \cdot) : H \to H$ is linear. Let a_0 be a H-valued $\mathcal{F}_0$-measurable random variable with $E\|a_0\|^4 < \infty$ and let $\Psi \in \mathcal{L}^2_{V^*}(\Omega \times [0, T]), \Gamma \in \mathcal{L}^2_H(\Omega \times [0, T])$. We consider the linear evolution equation:

$$\begin{aligned}
(P_{\Psi,\Gamma}) \qquad (Z_{\Psi,\Gamma}(t), v) &+ \int_0^t \langle \mathcal{A}Z_{\Psi,\Gamma}(s), v\rangle ds = (a_0, v) \\
&+ \int_0^t \langle \mathcal{B}(X(s), Z_{\Psi,\Gamma}(s)) + \mathcal{B}(Z_{\Psi,\Gamma}(s), Y(s)), v\rangle ds \\
&+ \int_0^t \langle \Psi(s), v\rangle ds + \int_0^t (\mathcal{G}(s, Z_{\Psi,\Gamma}(s)), v) dw(s) + \int_0^t (\Gamma(s), v) dw(s)
\end{aligned}$$

for all $v \in V$, $t \in [0,T]$, and a.e. $\omega \in \Omega$ and for each $M \in \mathbb{N}$ we consider:

$$(P^M_{\Psi,\Gamma}) \qquad \begin{aligned}(Z^M_{\Psi,\Gamma}(t),v) &+ \int_0^t \langle \mathcal{A} Z^M_{\Psi,\Gamma}(s), v\rangle ds = (a_0, v)\\ &+ \int_0^t \langle \mathcal{B}(X^M(s), Z^M_{\Psi,\Gamma}(s)) + \mathcal{B}(Z^M_{\Psi,\Gamma}(s), Y^M(s)), v\rangle ds\\ &+ \int_0^t \langle \Psi(s), v\rangle ds + \int_0^t (\mathcal{G}(s, Z^M_{\Psi,\Gamma}(s)), v) dw(s) + \int_0^t (\Gamma(s), v) dw(s)\end{aligned}$$

for all $v \in V$, $t \in [0,T]$, and a.e. $\omega \in \Omega$.

For each $n \in \mathbb{N}$ we define $\mathcal{G}_n : [0,T] \times H_n \to H_n$ by $\mathcal{G}_n(t,v) := \Pi_n \mathcal{G}(t,v)$ and consider

$$X_n := \Pi_n X, \quad Y_n := \Pi_n Y, \quad a_{0n} := \Pi_n a_0, \quad X^M_n(t) := X_n(t \wedge \mathcal{T}_M), \quad Y^M_n(t) := Y_n(t \wedge \mathcal{T}_M),$$

for all $t \in [0,T], v \in H_n$ and a.e. $\omega \in \Omega$.

Let $n \in \mathbb{N}$ and $\psi \in \mathcal{L}^2_{(H_n, \|\cdot\|_V)}(\Omega \times [0,T]), \gamma \in \mathcal{L}^2_{H_n}(\Omega \times [0,T])$. We consider the finite dimensional evolution equations

$$(P_{n,\psi,\gamma}) \qquad \begin{aligned}(Z_{n,\psi,\gamma}(t),v) &+ \int_0^t (\mathcal{A}_n Z_{n,\psi,\gamma}(s), v) ds = (a_{0n}, v)\\ &+ \int_0^t (\mathcal{B}_n(X_n(s), Z_{n,\psi,\gamma}(s)) + \mathcal{B}_n(Z_{n,\psi,\gamma}(s), Y_n(s)), v) ds\\ &+ \int_0^t (\psi(s), v) ds + \int_0^t (\mathcal{G}_n(s, Z_{n,\psi,\gamma}(s)), v) dw(s) + \int_0^t (\gamma(s), v) dw(s)\end{aligned}$$

and for each $M \in \mathbb{N}$ let

$$(P^M_{n,\psi,\gamma}) \qquad \begin{aligned}(Z^M_{n,\psi,\gamma}(t),v) &+ \int_0^t (\mathcal{A}_n Z^M_{n,\psi,\gamma}(s), v) ds = (a_{0n}, v)\\ &+ \int_0^t (\mathcal{B}_n(X^M_n(s), Z_{n,\psi,\gamma}(s)) + \mathcal{B}_n(Z_{n,\psi,\gamma}(s), Y^M_n(s)), v) ds\\ &+ \int_0^t (\psi(s), v) ds + \int_0^t (\mathcal{G}_n(s, Z^M_{n,\psi,\gamma}(s)), v) dw(s) + \int_0^t (\gamma(s), v) dw(s)\end{aligned}$$

for all $t \in [0,T], v \in H_n$, and a.e. $\omega \in \Omega$.

Theorem 1.3.1

(i) *For each* $\Psi \in \mathcal{L}^2_{V^*}(\Omega \times [0,T]), \Gamma \in \mathcal{L}^2_H(\Omega \times [0,T])$ *there exists a* V*-valued,* $\mathcal{F} \times \mathcal{B}_{[0,T]}$*-measurable process* $\Big(Z_{\Psi,\Gamma}(t)\Big)_{t\in[0,T]}$ *adapted to the filtration* $(\mathcal{F}_t)_{t\in[0,T]}$*, satisfying* $(P_{\Psi,\Gamma})$ *and which has almost surely continuous trajectories in* H*. The solution is almost surely unique, and there exists a positive constant* c_1 *(independent of* a_0, Ψ, Γ*) such that*

$$\begin{aligned} E\sup_{t\in[0,T]} \Delta_Y(t)\|Z_{\Psi,\Gamma}(t)\|^2 &+ E\int_0^T \Delta_Y(t)\|Z_{\Psi,\Gamma}(t)\|_V^2 ds \\ &\le c_1\Big[E\|a_0\|^2 + E\int_0^T \|\Psi(s)\|_{V^*}^2 ds + E\int_0^T \|\Gamma(s)\|^2 ds\Big] \end{aligned}$$

and if $E\int_0^T \|\Psi(t)\|_{V^*}^4 dt < \infty$ *and* $E\int_0^T \|\Gamma(t)\|^4 dt < \infty$*, then*

$$\begin{aligned} E\sup_{t\in[0,T]} \Delta_Y^2(t)\|Z_{\Psi,\Gamma}(t)\|^4 &+ E\Big(\int_0^T \Delta_Y(t)\|Z_{\Psi,\Gamma}(t)\|_V^2 ds\Big)^2 \\ &\le c_1\Big[E\|a_0\|^4 + E\int_0^T \|\Psi(s)\|_{V^*}^4 ds + E\int_0^T \|\Gamma(s)\|^4 ds\Big]. \end{aligned}$$

(ii) *For each* $\psi \in \mathcal{L}^2_{(H_n,\|\cdot\|_V)}(\Omega \times [0,T]), \gamma \in \mathcal{L}^2_{H_n}(\Omega \times [0,T])$ *there exists a* V*-valued,* $\mathcal{F} \times \mathcal{B}_{[0,T]}$*-measurable process* $\Big(Z_{n,\psi,\gamma}(t)\Big)_{t\in[0,T]}$ *adapted to the filtration* $(\mathcal{F}_t)_{t\in[0,T]}$*, satisfying* $(P_{n,\psi,\gamma})$ *and which has almost surely continuous trajectories in* H*. The solution is almost surely unique, and there exists a positive constant* c_2 *(independent of* n*,* a_0, ψ, γ*) such that*

$$\begin{aligned} E\sup_{t\in[0,T]} \Delta_Y(t)\|Z_{n,\psi,\gamma}(t)\|^2 &+ E\int_0^T \Delta_Y(t)\|Z_{n,\psi,\gamma}(t)\|_V^2 ds \\ &\le c_2\Big[E\|a_0\|^2 + E\int_0^T \|\psi(s)\|^2 ds + E\int_0^T \|\gamma(s)\|^2 ds\Big] \end{aligned}$$

and if $E\int_0^T \|\psi(t)\|^4 dt < \infty$ *and* $E\int_0^T \|\gamma(t)\|^4 dt < \infty$*, then*

$$\begin{aligned} E\sup_{t\in[0,T]} \Delta_Y^2(t)\|Z_{n,\psi,\gamma}(t)\|^4 &+ E\Big(\int_0^T \Delta_Y(t)\|Z_{n,\psi,\gamma}(t)\|_V^2 ds\Big)^2 \\ &\le c_1\Big[E\|a_0\|^4 + E\int_0^T \|\psi(s)\|^4 ds + E\int_0^T \|\gamma(s)\|^4 ds\Big]. \end{aligned}$$

PROOF. **(i)** Let $\Psi \in \mathcal{L}^2_{V^*}(\Omega \times [0,T]), \Gamma \in \mathcal{L}^2_H(\Omega \times [0,T])$. For each $n \in \mathbb{N}$ let $\Psi_n := \sum_{i=1}^{n} \langle \Psi, h_i \rangle h_i$, $\Gamma_n := \Pi_n \Gamma$. For the finite dimensional evolution equation $(P^M_{n,\Psi_n,\Gamma_n})$ we apply the theory of finite dimensional Ito equations with Lipschitz continuous nonlinearities (see [26], Theorem 5.5, p. 45). Hence there exists a solution $Z^M_{n,\Psi_n,\Gamma_n} \in \mathcal{L}^2_{(H_n, \|\cdot\|_V)}(\Omega \times [0,T])$ of $(P^M_{n,\Psi_n,\Gamma_n})$, which has almost surely continuous trajectories in H; this solution is almost surely unique.

For notational simplicity we define $Z^M_n := Z^M_{n,\Psi_n,\Gamma_n}$.

Let $M, n \in \mathbb{N}$. From the equation for Z^M_n and Proposition B.2 we obtain the estimate:

$$
\begin{aligned}
(1.27) \qquad E\Delta_{Y^M_n}(T)\|Z^M_n(T)\|^2 &+ E\int_0^T \Delta_{Y^M_n}(t)\|Z^M_n(t)\|^2_V dt \\
&\leq c\Big[E\|a_0\|^2 + E\int_0^T \|\Psi(s)\|^2_{V^*} ds + E\int_0^T \|\Gamma(s)\|^2 ds\Big],
\end{aligned}
$$

where c is a positive constant independent of M and n, but it depends on ν, λ, T. We can write

$$
\begin{aligned}
(1.28) \qquad & E\int_0^T \|Z^M_n(t)\|^2_V dt \leq E\Delta^{-1}_{Y^M_n}(T)\int_0^T \Delta_{Y^M_n}(t)\|Z^M_n(t)\|^2_V dt \\
\leq\ & E\Bigg\{\exp\Big\{\frac{b}{\nu}\int_0^T \|Y_n(t\wedge \mathcal{T}_M)\|^2_V dt\Big\}\int_0^T \Delta_{Y^M_n}(t)\|Z^M_n(t)\|^2_V dt\Bigg\} \\
\leq\ & E\Bigg\{\exp\Big\{\frac{b}{\nu}\int_0^T \|Y(t\wedge \mathcal{T}_M)\|^2_V dt\Big\}\int_0^T \Delta_{Y^M_n}(t)\|Z^M_n(t)\|^2_V dt\Bigg\} \\
\leq\ & c\exp\Big\{\frac{bM}{\nu}\Big\}\Bigg[E\|a_0\|^2 + E\int_0^T \|\Psi(t)\|^2_{V^*} dt + E\int_0^T \|\Gamma(t)\|^2 dt\Bigg].
\end{aligned}
$$

Hence, for fixed M the sequence $\left(Z^M_n\right)$ is bounded in the space $\mathcal{L}^2_V(\Omega \times [0,T])$. Consequently, there exists a subsequence (n') of (n) and $Z^M \in \mathcal{L}^2_V(\Omega \times [0,T])$ such that for $n' \to \infty$ we have

$$
(1.29) \qquad Z^M_{n'} \rightharpoonup Z^M.
$$

We want to prove that for $n' \to \infty$ the weak convergence $\mathcal{B}_{n'}(X^M_{n'}, Z^M_{n'}) \rightharpoonup \mathcal{B}(X^M, Z^M)$ holds in $\mathcal{L}^2_{V^*}(\Omega \times [0,T])$. Let $v \in V$ and $v_n := \Pi_n v$. We see that

$$
\begin{aligned}
(\mathcal{B}_n(X^M_n, Z^M_n), v) &= (\mathcal{B}_n(X^M_n, Z^M_n), v_n) = \langle \mathcal{B}(X^M_n, Z^M_n), v_n\rangle \\
&= \langle \mathcal{B}(X^M, v) - \mathcal{B}(X^M_n, v_n), Z^M_n\rangle + \langle \mathcal{B}(X^M, v), Z^M - Z^M_n\rangle + \langle \mathcal{B}(X^M, Z^M), v\rangle.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
(1.30) \qquad \langle \mathcal{B}_n(X^M_n, Z^M_n) &- \mathcal{B}(X^M, Z^M), v\rangle \\
&= \langle \mathcal{B}(X^M, v) - \mathcal{B}(X^M_n, v_n), Z^M_n\rangle + \langle \mathcal{B}(X^M, v), Z^M - Z^M_n\rangle.
\end{aligned}
$$

It holds [1]

$$E\int_0^T \|\mathcal{B}(X_n^M(s),v_n)-\mathcal{B}(X^M(s),v)\|_{V^*}^2 ds$$

$$\leq\ bc_{HV}\Big(\|v\|_V^2 E\int_0^T \|X^M(s)-X_n^M(s)\|_V^2 ds+\|v-v_n\|_V^2 E\int_0^T \|X^M(s)\|_V^2 ds\Big).$$

Since v_n and X_n^M are the Fourier expansions of v and X^M, respectively, it follows that

$$\lim_{n\to\infty} E\int_0^T \|\mathcal{B}(X_n^M(s),v_n)-\mathcal{B}(X^M(s),v)\|_{V^*}^2 ds=0. \tag{1.31}$$

Using (1.29), (1.31) in (1.30) we get

$$\lim_{n'\to\infty} E\int_0^T (\mathcal{B}_{n'}(X_{n'}^M(s),Z_{n'}^M(s)),\xi(s))ds=E\int_0^T \langle\mathcal{B}(X^M(s),Z^M(s)),\xi(s)\rangle ds$$

for all $\xi\in\mathcal{D}_V(\Omega\times[0,T])$. Since $\mathcal{B}_{n'}(X_{n'}^M,Z_{n'}^M),\mathcal{B}(X^M,Z^M)\in\mathcal{L}_{V^*}^2(\Omega\times[0,T])$ and $\mathcal{D}_V(\Omega\times[0,T])$ is dense in $\mathcal{L}_V^2(\Omega\times[0,T])$, we have $\mathcal{B}_{n'}(X_{n'}^M,Z_{n'}^M)\rightharpoonup\mathcal{B}(X^M,Z^M)$ for $n'\to\infty$. Analogously we can prove that $\mathcal{B}_{n'}(Z_{n'}^M,Y_{n'}^M)\rightharpoonup\mathcal{B}(Z^M,Y^M)$ for $n'\to\infty$.

We take the limit $n'\to\infty$ in $(P_{n',\Psi_{n'},\Gamma_{n'}}^M)$, use the weak convergence (1.29), as soon as the strong convergences of $\left(X_n^M\right)$ to X^M and of $\left(Y_n^M\right)$ to Y^M in the space $\mathcal{L}_H^2(\Omega\times[0,T])$ and Proposition A.3 to obtain

$$\begin{aligned}(Z^M(t),v)\ &=\ (a_0,v)-\int_0^t\langle\mathcal{A}Z^M(s),v\rangle ds+\int_0^t\langle\mathcal{B}(X^M(s),Z^M(s))+\mathcal{B}(Z^M(s),Y^M(s)),v\rangle ds\\ &+\ \int_0^t\langle\Psi(s),v\rangle ds+\int_0^t(\mathcal{G}(s,Z^M(s)),v)dw(s)+\int_0^t(\Gamma(s),v)dw(s)\end{aligned} \tag{1.32}$$

for all $v\in V$ and $P\times\Lambda$ a.e. $(\omega,t)\in\Omega\times[0,T]$. The right side of (1.32) has a continuous modification (as an H valued process), and this process we identify with $\left(Z_{\Psi,\Gamma}^M(t)\right)_{t\in[0,T]}$ (see [21], Theorem 3.2, p. 91). So, $\left(Z_{\Psi,\Gamma}^M(t)\right)_{t\in[0,T]}$ is a process from the space $\mathcal{L}_V^2(\Omega\times[0,T])$ which has almost surely continuous trajectories in H and satisfies $(P_{\Psi,\Gamma}^M)$ (identically with (1.32)) for all $v\in V,t\in[0,T]$ and a.e. $\omega\in\Omega$. By standard methods (see the final part of the proof) we can prove that the solution of $(P_{\Psi,\Gamma}^M)$ is almost surely unique.

[1] Since $V\hookrightarrow H$ we have $\|v\|^2\leq c_{HV}\|v\|_V^2$ for all $v\in V$.

Let Ω_K be the set of all $\omega \in \Omega$ such that $Z^K_{\Psi,\Gamma}(\omega,\cdot)$ satisfies $(P^K_{\Psi,\Gamma})$ for all $t \in [0,T], v \in V$ and such that $Z^K_{\Psi,\Gamma}(\omega,\cdot)$ has continuous trajectories in H. We define $\Omega' := \bigcap_{K=1}^{\infty} \Omega_K$. We also consider

$$S := \bigcup_{M=1}^{\infty} \bigcup_{1\le K\le M} \{\omega \in \Omega' |\ \mathcal{T}_K = T \text{ and } \exists\ t \in [0,T]:\ Z^K_{\Psi,\Gamma}(\omega,t) \neq Z^M_{\Psi,\Gamma}(\omega,t)\}.$$

We have $P(S) = 0$, because otherwise there exist two natural numbers M_0, K_0 with $K_0 < M_0$ such that the set

$$S_{M_0,K_0} := \{\omega \in \Omega' | \mathcal{T}_{K_0} = T \text{ and } \exists\ t \in [0,T]:\ Z^{K_0}_{\Psi,\Gamma}(\omega,t) \neq Z^{M_0}_{\Psi,\Gamma}(\omega,t)\}$$

has the measure $P(S_{M_0,K_0}) > 0$. We define for each $t \in [0,T]$

$$Z^*(\omega,t) := \begin{cases} Z^{K_0}_{\Psi,\Gamma}(\omega,t) & , \quad \omega \in S_{M_0,K_0} \\ Z^{M_0}_{\Psi,\Gamma}(\omega,t) & , \quad \omega \in \Omega' \setminus S_{M_0,K_0}. \end{cases}$$

We see that for all $\omega \in S_{M_0,K_0}$ there exists $t \in [0,T]$ such that $Z^*(\omega,t) \neq Z^{M_0}(\omega,t)$. This contradicts to the almost surely uniqueness of the solution of $(P^{M_0}_{\Psi,\Gamma})$. Consequently, $P(S) = 0$.

We define

$$\Omega'' := \bigcup_{M=1}^{\infty} \{\mathcal{T}_M = T\}.$$

Obviously $P\Big((\Omega' \cap \Omega'') \setminus S\Big) = 1$ (see also (1.26)). Let $\omega \in (\Omega' \cap \Omega'') \setminus S$. For this ω there exists a natural number M_0 such that $\mathcal{T}_M(\omega) = T$ for all $M \ge M_0$. Hence $X^M(s) = X(s)$ and $Y^M(s) = Y(s)$ for all $s \in [0,T]$ and for all $M \ge M_0$. Equation $(P^M_{\Psi,\Gamma})$ implies

$$\begin{aligned} (Z^M_{\Psi,\Gamma}(t),v) &+ \int_0^t \langle \mathcal{A} Z^M_{\Psi,\Gamma}(s), v\rangle ds = (a_0,v) \\ &+ \int_0^t \langle \mathcal{B}(X(s), Z^M_{\Psi,\Gamma}(s)) + \mathcal{B}(Z^M_{\Psi,\Gamma}(s), Y(s)), v\rangle ds \\ &+ \int_0^t \langle \Psi(s), v\rangle ds + \int_0^t (\mathcal{G}(s, Z^M_{\Psi,\Gamma}(s)), v) dw(s) + \int_0^t (\Gamma(s), v) dw(s) \end{aligned} \tag{1.33}$$

for all $M \ge M_0$ and all $t \in [0,T], v \in V$. We have

$$\lim_{M\to\infty} \int_0^T \|Z^M_{\Psi,\Gamma}(t) - Z^{M_0}_{\Psi,\Gamma}(t)\|^2_V dt = 0$$

and

$$\lim_{M\to\infty} \|Z^M_{\Psi,\Gamma}(t) - Z^{M_0}_{\Psi,\Gamma}(t)\|^2 = 0 \qquad \text{for all} \quad t \in [0,T].$$

For each $t \in [0,T]$ we define

$$Z_{\Psi,\Gamma}(\omega,t) := Z^{M_0}_{\Psi,\Gamma}(\omega,t) = \lim_{M\to\infty} Z^{M}_{\Psi,\Gamma}(\omega,t).$$

This definition is correct because $\omega \notin S$. Then (1.33) implies

$$\begin{aligned}(1.34) \qquad (Z_{\Psi,\Gamma}(t),v) &+ \int_0^t (\mathcal{A}Z_{\Psi,\Gamma}(s),v)ds = (a_0,v) \\ &+ \int_0^t \langle \mathcal{B}(X(s),Z_{\Psi,\Gamma}(s)) + \mathcal{B}(Z_{\Psi,\Gamma}(s),Y(s)),v\rangle ds \\ &+ \int_0^t \langle \Psi(s),v\rangle ds + \int_0^t (\mathcal{G}(s,Z_{\Psi,\Gamma}(s)),v)dw(s) + \int_0^t (\Gamma(s),v)dw(s)\end{aligned}$$

for all $\omega \in \Big(\Omega \cap \Omega''\Big) \setminus S, t \in [0,T], v \in V$. The process $(Z_{\Psi,\Gamma}(t))_{t\in[0,T]}$ is V-valued, $\mathcal{F} \times \mathcal{B}_{[0,T]}$-measurable, adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and has almost surely continuous trajectories in H, because all $Z^M_{\Psi,\Gamma}$ have this properties. For $Z^M_{\Psi,\Gamma}$ we can prove an analogous inequality as (1.27). Thus we get

$$\begin{aligned}(1.35) \quad E\Delta_Y(T)\|Z_{\Psi,\Gamma}(T)\|^2 &+ E\int_0^T \Delta_Y(t)\|Z_{\Psi,\Gamma}(t)\|^2_V dt \\ &\le \liminf_{M\to\infty} \Big\{E\Delta_{Y^M}(T)\|Z^M_{\Psi,\Gamma}(T)\|^2 + E\int_0^T \Delta_{Y^M}(t)\|Z^M_{\Psi,\Gamma}(t)\|^2_V dt\Big\} \\ &\le c\Big[E\|a_0\|^2 + E\int_0^T \|\Psi(s)\|^2_{V^*}ds + E\int_0^T \|\Gamma(s)\|^2 ds\Big],\end{aligned}$$

where c is the same constant as in (1.27). We obtain the other estimate by using in $(P_{\Psi,\Gamma})$ the Ito formula and then Proposition B.2.

Now we prove that equation $(P_{\Psi,\Gamma})$ has an almost surely unique solution. Let

$$e_1(t) := \Delta_Y(t)\exp\{-\lambda t\}.$$

We assume that $\tilde{Z}$ and Z are two solutions of $(P_{\Psi,\Gamma})$ which have almost surely continuous trajectories in H. Then for all $t \in [0,T]$ and a.e. $\omega \in \Omega$ we have

$$
\begin{aligned}
& e_1(t)\|\tilde{Z}(t)-Z(t)\|^2 + 2\int_0^t e_1(s)\langle \mathcal{A}\tilde{Z}(s)-\mathcal{A}Z(s), \tilde{Z}(s)-Z(s)\rangle ds \\
= \; & 2\int_0^t e_1(s)\langle \mathcal{B}(X(s),\tilde{Z}(s)-Z(s)) + \mathcal{B}(\tilde{Z}(s)-Z(s),Y(s)), \tilde{Z}(s)-Z(s)\rangle ds \\
- \; & \int_0^t e_1(s)(\lambda + \frac{b}{\nu}\|Y(s)\|_V^2)\|\tilde{Z}(s)-Z(s)\|^2 ds + \int_0^t e_1(s)\|\mathcal{G}(s,\tilde{Z}(s))-\mathcal{G}(s,Z(s))\|^2 ds \\
+ \; & 2\int_0^t e_1(s)(\mathcal{G}(s,\tilde{Z}(s))-\mathcal{G}(s,Z(s)), \tilde{Z}(s)-Z(s))dw(s).
\end{aligned}
$$

Taking into account the properties of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{G}$, it follows that for each $t \in [0,T]$ and a.e. $\omega \in \Omega$

$$
\begin{aligned}
e_1(t)\|\tilde{Z}(t)-Z(t)\|^2 \; & + \; \nu\int_0^t e_1(s)\|\tilde{Z}(s)-Z(s)\|_V^2 ds \\
& \leq \; 2\int_0^t e_1(s)(\mathcal{G}(s,\tilde{Z}(s))-\mathcal{G}(s,Z(s)), \tilde{Z}(s)-Z(s))dw(s).
\end{aligned} \tag{1.36}
$$

This implies (we use the same ideas as in the prove of Proposition B.2)

$$
E\int_0^T \Delta_Y(s)\|\tilde{Z}(s)-Z(s)\|_V^2 ds = 0.
$$

Hence $\tilde{Z}(\omega,t) = Z(\omega,t)$ for $P\times\Lambda$ a.e. $(\omega,t)\in(\Omega\times[0,T])$. Using this result and (1.36) we deduce that

$$
E \sup_{t\in[0,T]} \Delta_Y(t)\|\tilde{Z}(t)-Z(t)\|^2 = 0,
$$

which means that $(P_{\Psi,\Gamma})$ has an almost surely unique solution.

(ii) The existence, estimation, and (almost surely) uniqueness of the solution $Z_{n,\psi,\gamma}$ of $(P_{n,\psi,\gamma})$ can be proved analogously to the proof of **(i)**. ■

Lemma 1.3.2
We assume that $E\Delta_Y^{-2}(T) < \infty$. *Let* (ψ_n), (γ_n) *be sequences in* $\mathcal{L}_V^2(\Omega\times[0,T])$ *and* $\mathcal{L}_H^2(\Omega\times[0,T])$, *respectively, such that* $\psi_n \in \mathcal{L}^2_{(H_n,\|\cdot\|_V)}(\Omega\times[0,T]), \gamma_n \in \mathcal{L}^2_{H_n}(\Omega\times[0,T])$ *for each* $n\in\mathbb{N}$. *If* $(\mathrm{J}\psi_n)$ *converges weakly to* $\Psi\in\mathcal{L}^2_{V^*}(\Omega\times[0,T])$ *and* (γ_n) *converges weakly to* $\Gamma\in\mathcal{L}_H^2(\Omega\times[0,T])$, *then for* $n\to\infty$ *we have*

$$
\Delta_Y Z_{n,\psi_n,\gamma_n} \rightharpoonup \Delta_Y Z_{\Psi,\Gamma} \quad in \quad \mathcal{L}_V^2(\Omega\times[0,T])
$$

and

$$
\Delta_Y(T) Z_{n,\psi_n,\gamma_n}(T) \rightharpoonup \Delta_Y(T) Z_{\Psi,\Gamma}(T) \quad in \quad \mathcal{L}_H^2(\Omega).
$$

Proof. Because $(\mathrm{J}\psi_n)$ converges weakly to $\Psi \in \mathcal{L}^2_{V^*}(\Omega \times [0,T])$ and (γ_n) converges weakly to $\Gamma \in \mathcal{L}^2_H(\Omega \times [0,T])$, it follows that there exists a constant $c_3 > 0$ such that for all $n \in \mathbb{N}$

$$E\int_0^T \|\psi_n(t)\|_V^2 dt + E\int_0^T \|\gamma_n(t)\|^2 dt \le c_3.$$

For simplicity we define $Z_n := Z_{n,\psi_n,\gamma_n}$ and $Z_n^M := Z^M_{n,\psi_n,\gamma_n}$. Applying Theorem 1.3.1 we obtain

$$\sup_{1\le n}\Big\{E\Delta_Y(T)\|Z_n(T) - Z_{\Psi,\Gamma}(T)\|^2 + E\int_0^T \Delta_Y(s)\|Z_n(s) - Z_{\Psi,\Gamma}(s)\|_V^2 ds\Big\} \le (c_1 + c_2)\Big(c_3 + E\|a_0\|^2 + E\int_0^T \|\Psi(s)\|^2_{V^*} ds + E\int_0^T \|\Gamma(s)\|^2 ds\Big). \tag{1.37}$$

Let $\xi \in \mathcal{D}_{V^*}(\Omega \times [0,T])$ be arbitrary, but fixed. We want to prove that

$$\lim_{n\to\infty} E\int_0^T \langle \xi(s), Z_n(s) - Z_{\Psi,\Gamma}(s)\rangle ds = 0.$$

Since $\lim\limits_{M\to\infty} \mathcal{T}_M = T$ (see (1.25)) for a.e. $\omega \in \Omega$, $E\Delta_Y^{-2}(T) < \infty$ and $\xi \in \mathcal{D}_{V^*}(\Omega \times [0,T])$, we get

$$\lim_{M\to\infty} E\int_{\mathcal{T}_M}^T \Delta_Y^{-1}(s)\|\xi(s)\|^2_{V^*} ds = 0. \tag{1.38}$$

Let $\varepsilon > 0$. There exists a natural number $K = K_\varepsilon$ such that

$$\sup_{1\le n}\Big\{E\int_0^T \Delta_Y(s)\|Z_n(s) - Z_{\Psi,\Gamma}(s)\|_V^2 ds\Big\} E\int_{\mathcal{T}_K}^T \Delta_Y^{-1}(s)\|\xi(s)\|^2_{V^*} ds < \frac{\varepsilon^2}{4}\ . \tag{1.39}$$

Relation (1.38) implies

$$E\int_{\mathcal{T}_K}^T \langle \xi(s), Z_n(s) - Z_{\Psi,\Gamma}(s)\rangle ds \le \left(\sup_{1\le n}\Big\{E\int_0^T \Delta_Y(s)\|Z_n(s) - Z_{\Psi,\Gamma}(s)\|_V^2 ds\Big\} E\int_{\mathcal{T}_K}^T \Delta_Y^{-1}(s)\|\xi(s)\|^2_{V^*} ds\right)^{1/2} < \frac{\varepsilon}{2}$$

for all $n \in \mathbb{N}$.

From the (almost surely) uniqueness of the solutions of $(P^K_{\Psi,\Gamma})$ and $(P^K_{n,\psi_n,\gamma_n})$, respectively, we conclude that

$$E\int_0^{\mathcal{T}_K} \|Z_n(s) - Z_n^K(s)\|_V^2 ds = E\int_0^{\mathcal{T}_K} \|Z^K_{\Psi,\Gamma}(s) - Z_{\Psi,\Gamma}(s)\|_V^2 ds = 0.$$

Then for all $n \in \mathbb{N}$ we have

$$
\begin{aligned}
(1.40)\qquad & \left|E\int_0^T \langle \xi(s), Z_n(s) - Z_{\Psi,\Gamma}(s)\rangle ds\right| \\
\leq\ & \left|E\int_0^{\mathcal{T}_K} \langle \xi(s), Z_n(s) - Z_{\Psi,\Gamma}(s)\rangle ds\right| + \left|E\int_{\mathcal{T}_K}^T \langle \xi(s), Z_n(s) - Z_{\Psi,\Gamma}(s)\rangle ds\right| \\
\leq\ & \left|E\int_0^{\mathcal{T}_K} \langle \xi(s), Z_n(s) - Z_n^K(s)\rangle ds\right| + \left|E\int_0^{\mathcal{T}_K} \langle \xi(s), Z_n^K(s) - Z_{\Psi,\Gamma}^K(s)\rangle ds\right| \\
+\ & \left|E\int_0^{\mathcal{T}_K} \langle \xi(s), Z_{\Psi,\Gamma}^K(s) - Z_{\Psi,\Gamma}(s)\rangle ds\right| + \frac{\varepsilon}{2} = \left|E\int_0^{\mathcal{T}_K} \langle \xi(s), Z_n^K(s) - Z_{\Psi,\Gamma}^K(s)\rangle ds\right| + \frac{\varepsilon}{2}\,.
\end{aligned}
$$

In the following we prove that there exists an $n_\varepsilon > 0$ such that

$$\left|E\int_0^{\mathcal{T}_K} \langle \xi(s), Z_n^K(s) - Z_{\Psi,\Gamma}^K(s)\rangle ds\right| < \frac{\varepsilon}{2}$$

for all $n \geq n_\varepsilon$. Analogous to (1.28) (see the proof of Theorem 1.3.1) we have

$$E\int_0^T \|Z_n^K(s)\|_V^2 ds \leq c \exp\left\{\frac{bK}{\nu}\right\}\Big[E\|a_0\|^2 + c_3\Big].$$

Hence (Z_n^K) is a bounded sequence from $\mathcal{L}_V^2(\Omega \times [0,T])$. Consequently, by sequential weak compactness there exists a subsequence (n') of (n) and $Z^K \in \mathcal{L}_V^2(\Omega \times [0,T])$ such that for $n' \to \infty$ we have

$$(1.41)\qquad Z_{n'}^K \rightharpoonup Z^K \quad \text{in } \mathcal{L}_V^2(\Omega \times [0,T]).$$

As in the proof of Theorem 1.3.1 we can show that $\mathcal{B}_{n'}(X_{n'}^K, Z_{n'}^K) \rightharpoonup \mathcal{B}(X^K, Z^K)$ and $\mathcal{B}_{n'}(Z_{n'}^K, Y_{n'}^K) \rightharpoonup \mathcal{B}(Z^K, Y^K)$ in $\mathcal{L}_{V^*}^2(\Omega \times [0,T])$ for $n' \to \infty$. We take the limit $n' \to \infty$ in equation $(P_{n',\psi_{n'},\gamma_{n'}}^K)$, use the weak convergences given in the hypothesis and in (1.41), then the strong convergences of $\left(X_{n'}^K\right)$ to X^K and of $\left(Y_{n'}^K\right)$ to Y^K in the space $\mathcal{L}_H^2(\Omega \times [0,T])$ and Proposition A.3. Then we obtain

$$
\begin{aligned}
(Z^K(t), v) + \int_0^t \langle \mathcal{A}Z^K(s), v\rangle ds &= (a_0, v) + \int_0^t \langle \mathcal{B}(X^K(s), Z^K(s)) + \mathcal{B}(Z^K(s), Y^K(s)), v\rangle ds \\
&+ \int_0^t \langle \Psi(s), v\rangle ds + \int_0^t (\mathcal{G}(s, Z^K(s)), v) dw(s) + \int_0^t (\Gamma(s), v) dw(s)
\end{aligned}
$$

for all $v \in V$ and $P \times \Lambda$ a.e. $(\omega, t) \in \Omega \times [0,T]$. The (almost surely) uniqueness of the solution of equation $(P_{\Psi,\Gamma}^K)$ implies that

$$Z^K(\omega, t) = Z_{\Psi,\Gamma}^K(\omega, t) \quad \text{for } P \times \Lambda \quad \text{a.e.} \quad (\omega, t) \in \Omega \times [0,T].$$

Hence
$$Z^K_{n'} \rightharpoonup Z^K_{\Psi,\Gamma} \quad \text{in} \quad \mathcal{L}^2_V(\Omega \times [0,T]).$$
We also see that each weakly convergent subsequence of $\left(Z^K_n\right)$ converges weakly to the same limit $Z^K_{\Psi,\Gamma}$. Therefore, the whole sequence $\left(Z^K_n\right)$ converges weakly to $Z^K_{\Psi,\Gamma}$ in $\mathcal{L}^2_V(\Omega \times [0,T])$ (see Proposition A.1).

Hence, there exists $n_\varepsilon > 0$ such that for all $n \geq n_\varepsilon$ we have
$$E\int_0^{\mathcal{T}_K} \langle \xi(s), Z^K_n(s) - Z^K_{\Psi,\Gamma}(s)\rangle ds < \frac{\varepsilon}{2}\ .$$
Using (1.40) we deduce that
$$\left|E\int_0^T \langle \xi(s), Z_n(s) - Z_{\Psi,\Gamma}(s)\rangle ds\right| < \varepsilon \quad \text{for all} \quad n \geq n_\varepsilon$$
and consequently,
$$\lim_{n\to\infty} E\int_0^T \langle \xi(s), Z_n(s) - Z_{\Psi,\Gamma}(s)\rangle ds = 0.$$
Because $\mathcal{D}_{V^*}(\Omega \times [0,T])$ is dense in $\mathcal{L}^2_{V^*}(\Omega \times [0,T])$ and $\Delta_Y Z_n, \Delta_Y Z_{\Psi,\Gamma} \in \mathcal{L}^2_V(\Omega \times [0,T])$ (we do not know whether $Z_n, Z_{\Psi,\Gamma} \in \mathcal{L}^2_V(\Omega \times [0,T])$) we conclude that for $n \to \infty$

$$\Delta_Y Z_n \rightharpoonup \Delta_Y Z_{\Psi,\Gamma} \quad \text{in} \quad \mathcal{L}^2_V(\Omega \times [0,T]). \tag{1.42}$$

We want to prove that for all $\xi \in \mathcal{D}_V(\Omega)$ we have

$$\lim_{n\to\infty} E(Z_{\Psi,\Gamma}(T) - Z_n(T), \xi) = 0. \tag{1.43}$$

Let $\xi = v\phi \in \mathcal{D}_V(\Omega)$, $\varepsilon > 0$. There exists an index $K \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ we get

$$\begin{aligned} |E(Z_{\Psi,\Gamma}(T) - Z_n(T), v)\phi| &\leq |E(Z_{\Psi,\Gamma}(T) - Z_n(T), \Pi_K v)\phi| \\ &+ \|v - \Pi_K v\| \Big\{ E\Big(\phi^2 \Delta_Y^{-1}(T)\Big) \sup_{1\leq n} \Big[E\Delta_Y(T)\|Z_{\Psi,\Gamma}(T) - Z_n(T)\|^2\Big]\Big\}^{1/2} \\ &\leq |E(Z_{\Psi,\Gamma}(T) - Z_n(T), \Pi_K v)\phi| + \frac{\varepsilon}{2}\ . \end{aligned} \tag{1.44}$$

In the second inequality we have used (1.37). From (P_{n,ψ_n}) and $(P_{\Psi,\Gamma})$ we conclude that

$$
\begin{aligned}
& E\phi(Z_{\Psi,\Gamma}(T) - Z_n(T), \Pi_K v) + E\phi\int_0^T \langle \mathcal{A}Z_{\Psi,\Gamma}(s) - \mathcal{A}Z_n(s), \Pi_K v\rangle ds \\
= \; & E\phi\int_0^T \langle \mathcal{B}(X(s), Z_{\Psi,\Gamma}(s)) - \mathcal{B}(X_n(s), Z_n(s)) + \mathcal{B}(Z_{\Psi,\Gamma}(s), Y(s)) - \mathcal{B}(Z_n(s), Y_n(s)), \Pi_K v\rangle ds \\
+ \; & E\phi\int_0^T \langle \Psi(s) - \mathrm{J}\psi_n(s), \Pi_K v\rangle ds + E\phi\int_0^T (\mathcal{G}(s, Z_{\Psi,\Gamma}(s) - Z_n(s)), \Pi_K v) dw(s) \\
+ \; & E\phi\int_0^T (\Gamma(s) - \gamma_n(s), \Pi_K v) dw(s).
\end{aligned}
$$

In the above equation we take the limit $n \to \infty$, use the weak convergence (1.42) and obtain that there exists an $n_\varepsilon > 0$ such that

$$|E(Z_{\Psi,\Gamma}(T) - Z_n(T), \Pi_K v)\phi| < \frac{\varepsilon}{2} \quad \text{for all } n \geq n_\varepsilon.$$

We use (1.44) to deduce that

$$|E(Z_{\Psi,\Gamma}(T) - Z_n(T), v)\phi| < \varepsilon \quad \text{for all } n \geq n_\varepsilon.$$

Hence (1.43) yields. Since $\mathcal{D}_V(\Omega)$ is dense in $\mathcal{L}^2_V(\Omega) \hookrightarrow \mathcal{L}^2_H(\Omega)$ and since we have $\Delta_Y(T)Z_n(T), \Delta_Y(T)Z_{\Psi,\Gamma}(T) \in \mathcal{L}^2_H(\Omega)$ (note, we do not know whether $Z_n(T), Z_{\Psi,\Gamma}(T) \in \mathcal{L}^2_H(\Omega)$) we conclude that

$$\Delta_Y(T)Z_n(T) \rightharpoonup \Delta_Y(T)Z_{\Psi,\Gamma}(T) \quad \text{in} \quad \mathcal{L}^2_H(\Omega). \qquad \blacksquare$$

Remark 1.3.3

Theorem 1.3.1 and Lemma 1.3.2 hold also if $\mathcal{G}$ is not a mapping satisfying hypothesis **(v)** from Section 1.1, but a stochastic process belonging to the space $\mathcal{L}^2_H(\Omega \times [0,T])$.

1.4 Linear approximation of the solution of the stochastic Navier-Stokes equation

In this section we approximate the solution of the Navier-Stokes equation by the solutions of a sequence of linear equations with additive noise and prove that the approximations $\left(u_n\right)$ converge in mean square to the solution of (1.1).

For each $n = 1, 2, 3, \ldots$ we consider the **linear** evolution equation with additive noise

$$(\hat{P}_n) \qquad (\mathbf{u_n}(t), v) + \int_0^t \langle \mathcal{A}\mathbf{u_n}(s), v\rangle ds = (x_0, v) + \int_0^t \langle \mathcal{B}(\mathbf{u_{n-1}}(s), \mathbf{u_n}(s)), v\rangle ds + \int_0^t (\Phi(s, \mathbf{u_{n-1}}(s)), v) ds + \int_0^t (\mathcal{C}(s, \mathbf{u_{n-1}}(s)), v) dw(s),$$

for all $v \in V$, $t \in [0,T]$, and a.e. $\omega \in \Omega$, where $\mathbf{u_0}(t) = 0$ for all $t \in [0,T]$ and a.e. $\omega \in \Omega$.

Remark 1.4.1
1) In equation $(\hat{P}_n)$, considering that u_{n-1} is known, the operators $\mathcal{A}$ and $\mathcal{B}$ depend linearly on u_n and the noise is additive with respect to u_n.
2) The approximation method given in this section holds also, if the sequence of approximations (u_n) starts with $u_0(t) := x_0$ for all $t \in [0,T]$ and a.e. $\omega \in \Omega$.

Theorem 1.4.2
For each $n \in \mathbb{N}$ equation $(\hat{P}_n)$ has an almost surely unique solution $u_n \in \mathcal{L}^2_V(\Omega \times [0,T])$ with almost surely continuous trajectories in H.

PROOF. We prove the statement by induction. We apply successively Theorem 1.3.1 on $Z_{\Psi,\Gamma} := u_n$, $a_0 := x_0$, $X := u_{n-1}, Y := 0, a_0 := x_0$, $\Psi(s) := \Phi(s, u_{n-1}(s))$, $\Gamma := 0$, $\mathcal{G}(s) := \mathcal{C}(s, u_{n-1}(s))$ (for $\mathcal{G}$ we also take into account Remark 1.3.3). ■

Now we establish some properties for the solutions of the equations $(\hat{P}_n)$, $n \in \mathbb{N}$.

Lemma 1.4.3
There exists a positive constant c_1 (depending only on λ, μ, ν, and T) such that each of the expressions

$$\sup_{t\in[0,T]} E\|u_n(t)\|^4, \quad E\Big(\int_0^T \|u_n(s)\|_V^2 ds\Big)^2$$

$(n = 1, 2, \ldots)$ *is less than or equal to* $c_1 E\|x_0\|^4$.

PROOF. Let $n \in \mathbb{N}$. We define $\tilde{z}(t) = e^{-(9\lambda+5\sqrt{\mu})t}$, $t \in [0,T]$. Using the Ito formula we have

$$\tilde{z}(t)\|u_n(t)\|^2 + 2\int_0^t \tilde{z}(s)\langle \mathcal{A}u_n(s), u_n(s)\rangle ds = \|x_0\|^2 + 2\int_0^t \tilde{z}(s)(\Phi(s, u_{n-1}(s)), u_n(s)) ds + \int_0^t \tilde{z}(s)\|\mathcal{C}(s, u_{n-1}(s))\|^2 ds - (9\lambda + 5\sqrt{\mu})\int_0^t \tilde{z}(s)\|u_n(s)\|^2 ds + 2\int_0^t \tilde{z}(s)(\mathcal{C}(s, u_{n-1}(s)), u_n(s)) dw(s)$$

and

$$\begin{aligned}\tilde{z}(t)\|u_n(t)\|^4 &+ 4\int_0^t \tilde{z}(s)\langle \mathcal{A}u_n(s), u_n(s)\rangle \|u_n(s)\|^2 ds \\ &= \|x_0\|^4 + 2\int_0^t \tilde{z}(s)\|\mathcal{C}(s, u_{n-1}(s))\|^2\|u_n(s)\|^2 ds - (9\lambda + 5\sqrt{\mu})\int_0^t \tilde{z}(s)\|u_n(s)\|^4 ds \\ &+ 4\int_0^t \tilde{z}(s)(\Phi(s, u_{n-1}(s)), u_n(s))\|u_n(s)\|^2 ds + 4\int_0^t \tilde{z}(s)(\mathcal{C}(s, u_{n-1}(s)), u_n(s))^2 ds \\ &+ 4\int_0^t \tilde{z}(s)(\mathcal{C}(s, u_{n-1}(s)), u_n(s))\|u_n(s)\|^2 dw(s).\end{aligned}$$

Using the properties of $\mathcal{A}$, Φ, and $\mathcal{C}$ and some elementary calculations, we obtain

$$\begin{aligned}2\nu\int_0^t \tilde{z}(s)\|u_n(s)\|_V^2 ds &\le \|x_0\|^2 + (\lambda + \sqrt{\mu})\int_0^t \tilde{z}(s)\|u_{n-1}(s)\|^2 ds \\ &+ 2\int_0^t \tilde{z}(s)(\mathcal{C}(s, u_{n-1}(s)), u_n(s))dw(s)\end{aligned} \tag{1.45}$$

and

$$\begin{aligned}\tilde{z}(t)\|u_n(t)\|^4 &+ 4\nu\int_0^t \tilde{z}(s)\|u_n(s)\|_V^2\|u_n(s)\|^2 ds + 2(3\lambda + \sqrt{\mu})\int_0^t \tilde{z}(s)\|u_n(s)\|^4 ds \\ &\le \|x_0\|^4 + (3\lambda + \sqrt{\mu})\int_0^t \tilde{z}(s)\|u_{n-1}(s)\|^4 ds \\ &+ 4\int_0^t \tilde{z}(s)(\mathcal{C}(s, u_{n-1}(s)), u_n(s))\|u_n(s)\|^2 dw(s).\end{aligned} \tag{1.46}$$

Using (1.46) and the ideas from the proof of Proposition B.2 we get

$$\begin{aligned}&E\tilde{z}(t)\|u_n(t)\|^4 + 2(3\lambda + \sqrt{\mu})E\int_0^t \tilde{z}(s)\|u_n(s)\|^4 ds \\ &\qquad\le E\|x_0\|^4 + (3\lambda + \sqrt{\mu})E\int_0^t \tilde{z}(s)\|u_{n-1}(s)\|^4 ds.\end{aligned} \tag{1.47}$$

By successive application of (1.47), we obtain

$$E\tilde{z}(t)\|u_n(t)\|^4 + 2(3\lambda + \sqrt{\mu})E\int_0^t \tilde{z}(s)\|u_n(s)\|^4 ds \le \Big(1 + \frac{1}{2} + \ldots + \frac{1}{2^{n-1}}\Big)E\|x_0\|^4.$$

Consequently, there exists a constant $c_2 > 0$ (independent of n) such that

$$\sup_{t\in[0,T]} E\|u_n(t)\|^4 + 2(3\lambda+\sqrt{\mu})E\int_0^T \|u_n(s)\|^4 ds \le c_2 E\|x_0\|^4. \tag{1.48}$$

In (1.45) we square both sides of the inequality. Then we use the properties of the stochastic integral and those of the Lebesgue integral to obtain

$$\begin{aligned} & 4\nu^2 E\Big(\int_0^T \tilde{z}(s)\|u_n(s)\|_V^2 ds\Big)^2 \le 3E\|x_0\|^4 + 3(\lambda+\sqrt{\mu})^2 E\Big(\int_0^T \tilde{z}(s)\|u_{n-1}(s)\|^2 ds\Big)^2 \\ + \quad & 12E\Big|\int_0^T \tilde{z}(s)(\mathcal{C}(s,u_{n-1}(s)),u_n(s))dw(s)\Big|^2 \le 3E\|x_0\|^4 + c_3 E\int_0^T \tilde{z}^2(s)\|u_{n-1}(s)\|^4 ds \\ + \quad & E\int_0^T \tilde{z}^2(s)\|u_n(s)\|^4 ds, \end{aligned}$$

where c_3 is a positive constant depending on λ, μ, and T. Taking into account (1.48) and the properties of $\tilde{z}$, it follows that there exists a constant c_4 depending on (λ, μ, ν and T) such that

$$E\Big(\int_0^T \|u_n(s)\|_V^2 ds\Big)^2 \le c_4 E\|x_0\|^4. \qquad \blacksquare$$

We define

$$\tilde{e}(t) = \Delta_U(t)\exp\{-(\lambda+\sqrt{\mu})t\}$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$ and introduce the following notations:

$$s_N(t) = \sum_{n=1}^N \tilde{e}(t)\|u_n(t) - U(t)\|^2, \tag{1.49}$$

$$S_N(t) = \sum_{n=1}^N \tilde{e}(t)\|u_n(t) - U(t)\|_V^2, \tag{1.50}$$

where N is a natural number, $t \in [0,T]$, $\omega \in \Omega$.

Lemma 1.4.4
The following convergences hold:

$$\lim_{n\to\infty} E\int_0^T \tilde{e}(s)\|u_n(s) - U(s)\|_V^2 ds = 0$$

and

$$\lim_{n\to\infty} E\tilde{e}(t)\|u_n(t) - U(t)\|^2 = 0 \quad \textit{for all} \quad t \in [0,T].$$

PROOF. Let $n \in \mathbb{N}$. Using (1.1), $(\hat{P}_n)$ and the Ito formula we obtain

$$\tilde{e}(t)\|u_n(t) - U(t)\|^2 + 2\int_0^t \tilde{e}(s)\langle \mathcal{A}u_n(s) - \mathcal{A}U(s), u_n(s) - U(s)\rangle ds \tag{1.51}$$

$$\begin{aligned}
&= 2\int_0^t \tilde{e}(s)\langle \mathcal{B}(u_{n-1}(s), u_n(s)) - \mathcal{B}(U(s), U(s)), u_n(s) - U(s)\rangle ds \\
&- \frac{b}{\nu}\int_0^t \tilde{e}(s)\|U(s)\|_V^2 \|u_n(s) - U(s)\|^2 ds - (\lambda + \sqrt{\mu})\int_0^t \tilde{e}(s)\|u_n(s) - U(s)\|^2 ds \\
&+ 2\int_0^t \tilde{e}(s)(\Phi(s, u_{n-1}(s)) - \Phi(s, U(s)), u_n(s) - U(s))ds \\
&+ \int_0^t \tilde{e}(s)\|\mathcal{C}(s, u_{n-1}(s)) - \mathcal{C}(s, U(s))\|^2 ds \\
&+ 2\int_0^t \tilde{e}(s)(\mathcal{C}(s, u_{n-1}(s)) - \mathcal{C}(s, U(s)), u_n(s) - U(s))dw(s),
\end{aligned}$$

for all $t \in [0, T]$ and a.e. $\omega \in \Omega$. From the properties of $\mathcal{B}$ we can derive the following estimate:

$$\begin{aligned}
&2\langle \mathcal{B}(u_{n-1}(s), u_n(s)) - \mathcal{B}(U(s), U(s)), u_n(s) - U(s)\rangle \\
&= -2\langle \mathcal{B}(u_{n-1}(s) - U(s), u_n(s) - U(s)), U(s)\rangle \\
&\le 2\sqrt{b}\|U(s)\|_V \|u_{n-1}(s) - U(s)\|_V^{\frac{1}{2}} \|u_{n-1}(s) - U(s)\|^{\frac{1}{2}} \|u_n(s) - U(s)\|_V^{\frac{1}{2}} \|u_n(s) - U(s)\|^{\frac{1}{2}} \\
&\le \frac{\nu}{2}\|u_{n-1}(s) - U(s)\|_V^2 + \frac{\nu}{2}\|u_n(s) - U(s)\|_V^2 \\
&+ \frac{b}{2\nu}\|U(s)\|_V^2 \|u_{n-1}(s) - U(s)\|^2 + \frac{b}{2\nu}\|U(s)\|_V^2 \|u_n(s) - U(s)\|^2
\end{aligned}$$

for all $s \in [0, T]$ and a.e. $\omega \in \Omega$. Using this estimation and the continuity of $\mathcal{C}$ in (1.51), we obtain

$$\begin{aligned}
&\tilde{e}(t)\|u_n(t) - U(t)\|^2 + \frac{3\nu}{2}\int_0^t \tilde{e}(s)\|u_n(s) - U(s)\|_V^2 ds + (\lambda + \sqrt{\mu})\int_0^t \tilde{e}(s)\|u_n(s) - U(s)\|^2 ds \\
&\le \frac{\nu}{2}\int_0^t \tilde{e}(s)\|u_{n-1}(s) - U(s)\|_V^2 ds + 2\int_0^t \tilde{e}(s)(\mathcal{C}(s, u_{n-1}(s)) - \mathcal{C}(s, U(s)), u_n(s) - U(s))dw(s) \\
&+ \frac{b}{2\nu}\int_0^t \tilde{e}(s)\|U(s)\|_V^2 \Big(\|u_{n-1}(s) - U(s)\|^2 - \|u_n(s) - U(s)\|^2\Big) ds
\end{aligned}$$

$$+ \quad (\lambda + \sqrt{\mu}) \int_0^t \tilde{e}(s) \|u_{n-1}(s) - U(s)\|^2 ds$$

for all $t \in [0, T]$ and a.e. $\omega \in \Omega$.

Summing up these relations for $n = 1$ to an arbitrary natural number N, we get

$$s_N(t) + \nu \int_0^t S_N(s) ds + \frac{b}{2\nu} \int_0^t \tilde{e}(s) \|U(s)\|_V^2 \|u_N(s) - U(s)\|^2 ds$$

$$+ \quad (\lambda + \sqrt{\mu}) \int_0^t \tilde{e}(s) \|u_N(s) - U(s)\|^2 ds \le \frac{\nu}{2} \int_0^t \tilde{e}(s) \|u_0(s) - U(s)\|_V^2 ds$$

$$+ \quad (\lambda + \sqrt{\mu}) \int_0^t \tilde{e}(s) \|u_0(s) - U(s)\|^2 ds + \frac{b}{2\nu} \int_0^t \tilde{e}(s) \|U(s)\|_V^2 \|u_0(s) - U(s)\|^2 ds$$

$$+ \quad 2 \sum_{n=1}^N \int_0^t \tilde{e}(s) (\mathcal{C}(s, u_{n-1}(s)) - \mathcal{C}(s, U(s)), u_n(s) - U(s)) dw(s)$$

for all $t \in [0, T]$ and a.e. $\omega \in \Omega$, where s_N and S_N are defined in (1.49) and (1.50). Taking the mathematical expectation we have

$$E s_N(t) + \nu E \int_0^t S_N(s) ds \le \frac{\nu}{2} E \int_0^t \tilde{e}(s) \|u_0(s) - U(s)\|_V^2 ds$$

$$+ \quad (\lambda + \sqrt{\mu}) E \int_0^t \tilde{e}(s) \|u_0(s) - U(s)\|^2 ds + \frac{b}{2\nu} E \int_0^t \tilde{e}(s) \|U(s)\|_V^2 \|u_0(s) - U(s)\|^2 ds$$

for all $t \in [0, T]$. But $0 < \tilde{e}(s) \le 1$ and $u_0(s) = 0$ for all $s \in [0, T]$, a.e. $\omega \in \Omega$. Applying Lemma 1.2.6 and Lemma 1.4.3 it follows that there exists a positive constant c, which does not depend on N, such that

$$E s_N(t) + \nu E \int_0^t S_N(s) ds \le E \int_0^T \Big(\frac{\nu}{2} \|U(s)\|_V^2 + (\lambda + \sqrt{\mu}) \|U(s)\|^2 + \frac{b}{2\nu} \|U(s)\|_V^2 \|U(s)\|^2 \Big) ds \le c$$

for all $t \in [0, T]$ and all natural numbers N. Consequently, for all $t \in [0, T]$ we have

$$\sum_{n=1}^\infty E \tilde{e}(t) \|u_n(t) - U(t)\|^2 + \nu \sum_{n=1}^\infty E \int_0^t \tilde{e}(s) \|u_n(s) - U(s)\|_V^2 ds \le c$$

Hence

$$\lim_{n \to \infty} E \int_0^T \tilde{e}(s) \|u_n(s) - U(s)\|_V^2 ds = 0$$

and for all $t \in [0,T]$ we have

$$\lim_{n\to\infty} E\tilde{e}(t)\|u_n(t) - U(t)\|^2 = 0. \qquad \blacksquare$$

The **main result** of this section is the following theorem.

Theorem 1.4.5
The following convergences hold:

$$\lim_{n\to\infty} E\int_0^T \|u_n(s) - U(s)\|_V^2 ds = 0$$

and

$$\lim_{n\to\infty} E\|u_n(t) - U(t)\|^2 = 0 \quad \textit{for all} \quad t \in [0,T].$$

PROOF. We take $\mathcal{T}_M := \mathcal{T}_M^U$. From Lemma 1.4.4 it follows that for each fixed natural number M we have

$$\lim_{n\to\infty} E\int_0^{\mathcal{T}_M} \|u_n(s) - U(s)\|_V^2 ds = 0 \quad \text{and} \quad \lim_{n\to\infty} E\|u_n(\mathcal{T}_M) - U(\mathcal{T}_M)\|^2 = 0.$$

First we apply Proposition B.3 for $\mathcal{T} := T$, $Q_n(\mathcal{T}) := \int_0^{\mathcal{T}} \|u_n(s) - U(s)\|^2 ds$, use Lemma 1.4.3 and Lemma 1.2.6 to obtain

$$\lim_{n\to\infty} E\int_0^T \|u_n(s) - U(s)\|_V^2 ds = 0.$$

Let $t \in [0,T]$. Now we apply Proposition B.3 for $\mathcal{T} := t$, $Q_n(\mathcal{T}) := \|u_n(\mathcal{T}) - U(\mathcal{T})\|$, use Lemma 1.4.3 and Lemma 1.2.6 to get

$$\lim_{n\to\infty} E\|u_n(t) - U(t)\|^2 = 0. \qquad \blacksquare$$