Chapter 2

Optimal Control

We consider the stochastic Navier-Stokes equation controlled by linear and continuous feedback controls, respectively, by bounded controls (which are not feedback controls). Since the considered equation is nonlinear, we are dealing with a nonconvex optimization problem. Our purpose is to prove in Section 2.2 , Section 2.3 and Section 2.4 the existence of optimal and ε-optimal controls. In Section 2.5 we investigate a special property for the solution of the stochastic Navier-Stokes equation, which we assume to be fulfilled in the following sections. In Section 2.6 we calculate the Gateaux derivative of the cost functional and in Section 2.7 we formulate a stochastic minimum principle. We complete the statement of the stochastic minimum principle by giving in Section 2.8 the equations for the adjoint processes. In the last three sections we use and adapt the ideas from A. Bensoussan [3] for the case of the stochastic Navier-Stokes equation.

2.1 Formulation of the control problem

First we consider the stochastic Navier-Stokes equation controlled by **linear and continuous feedback controls**. In this case we denote by $\mathcal{U}$ the set of all admissible controls, which we assume to be the set of all functions $\Phi : [0,T] \times H \to H$ satisfying the following conditions: for each $t \in [0,T]$ we have $\Phi(t,\cdot) \in \mathcal{L}(H)$ and

$$\|\Phi(t_1,x_1) - \Phi(t_2,x_2)\|^2 \leq \alpha|t_1 - t_2|^2 + \mu\|x_1 - x_2\|^2 \quad \text{for all } t_1,t_2 \in [0,T], x_1,x_2 \in H$$

where $\alpha, \mu > 0$ are given constants.

Our purpose is to control the solution U_Φ of the Navier-Stokes equation

$$(2.1)\qquad (U_\Phi(t),v)+\int_0^t\langle\mathcal{A}U_\Phi(s),v\rangle ds = (x_0,v)+\int_0^t\langle\mathcal{B}(U_\Phi(s),U_\Phi(s)),v\rangle ds + \int_0^t(\Phi(s,U_\Phi(s)),v)ds+\int_0^t(\mathcal{C}(s,U_\Phi(s)),v)dw(s)$$

for all $v\in V$, $t\in[0,T]$, a.e. $\omega\in\Omega$, by the feedback controls $\Phi\in\mathcal{U}$. We consider the following **cost functional**

$$(2.2)\qquad \mathcal{J}(\Phi)=E\int_0^T\mathcal{L}[s,U_\Phi(s),\Phi(s,U_\Phi(s))]ds+E\mathcal{K}[U_\Phi(T)],\qquad \Phi\in\mathcal{U},$$

where $\mathcal{L}:[0,T]\times H\times H\to\mathbb{R}_+$, $\mathcal{K}:H\to\mathbb{R}_+$ are mappings satisfying the conditions:

$(\mathbf{H_1})$
$$|\mathcal{L}(t,x_1,y_1)-\mathcal{L}(t,x_2,y_2)|\le c_{\mathcal{L}}\Big(\|x_1-x_2\|^2+\|y_1-y_2\|^2\Big),$$
$$|\mathcal{K}(x_1)-\mathcal{K}(x_2)|\le c_{\mathcal{K}}\|x_1-x_2\|^2$$

for all $t\in[0,T]$, $x_1,x_2,y_1,y_2\in H$, where $c_{\mathcal{L}},c_{\mathcal{K}}$ are positive constants;

$(\mathbf{H_2})$ for all $x,y\in H$ we assume that $\mathcal{L}(\cdot,x,y)\in\mathcal{L}^2_H[0,T]$.

We denote by $(\mathcal{P})$ the problem of minimizing $\mathcal{J}$ among the admissible controls.

Now we consider the stochastic Navier-Stokes equation controlled by **bounded controls**, which are not feedback controls. In this case we denote by $\mathcal{U}^b$ the set of all admissible controls, which we assume to be the set of all functions $\Phi\in\mathcal{L}^2_H(\Omega\times[0,T])$ satisfying the condition:

$$\|\Phi(\omega,t)\|\le\rho\quad\text{for } P\times\Lambda \text{ a.e. } (\omega,t)\in\Omega\times[0,T],$$

where $\rho>0$ is a given constant.

In this case the stochastic Navier-Stokes equation has the form

$$(2.3)\qquad (U_\Phi(t),v)+\int_0^t\langle\mathcal{A}U_\Phi(s),v\rangle ds = (x_0,v)+\int_0^t\langle\mathcal{B}(U_\Phi(s),U_\Phi(s)),v\rangle ds + \int_0^t(\Phi(s),v)ds+\int_0^t(\mathcal{C}(s,U_\Phi(s)),v)dw(s)$$

for all $v\in V$, $t\in[0,T]$, a.e. $\omega\in\Omega$, where $\Phi\in\mathcal{U}^b$. The cost functional is in this case given by

$$(2.4)\qquad \mathcal{J}(\Phi)=E\int_0^T\mathcal{L}[s,U_\Phi(s),\Phi(s)]ds+E\mathcal{K}[U_\Phi(T)],\qquad \Phi\in\mathcal{U}^b,$$

where $\mathcal{L}$ and $\mathcal{K}$ satisfy $(\mathbf{H_1})$ and $(\mathbf{H_2})$. We denote by $(\mathcal{P}^b)$ the problem of minimizing $\mathcal{J}$, given in equation (2.4), among the admissible controls of the set $\mathcal{U}^b$.

Remark 2.1.1
In their paper [1] F. Abergel and R. Temam investigate the deterministic Navier-Stokes equation by controlling turbulence inside the flow. They give a cost functional which involves the vorticity in the fluid. For our problem $(\mathcal{P}^b)$ this would be $\mathcal{L}(t, U_\Phi(t), \Phi(t)) := \|\nabla \times U_\Phi(t)\|^2 + \|\Phi(t)\|^2,\ \mathcal{K} := 0$.

2.2 Existence of optimal controls

First we prove some properties of the cost functional $\mathcal{J}$.

Lemma 2.2.1

(i) *Let (Φ_n) be a sequence in $\mathcal{U}$ and let $\Phi \in \mathcal{U}$ be such that*

$$\lim_{n\to\infty} \int_0^T \|\Phi_n(t,\cdot) - \Phi(t,\cdot)\|^2_{\mathcal{L}(H)} dt = 0.$$

Then $\lim_{n\to\infty} \mathcal{J}(\Phi_n) = \mathcal{J}(\Phi)$.

(ii) *Let (Φ_n) be a sequence in $\mathcal{U}^b$ and let $\Phi \in \mathcal{U}^b$ be such that*

$$\lim_{n\to\infty} \int_0^T \|\Phi_n(t) - \Phi(t)\|^2 dt = 0.$$

Then $\lim_{n\to\infty} \mathcal{J}(\Phi_n) = \mathcal{J}(\Phi)$.

Proof. **(i)** Let $U := U_\Phi$ and $e(t) = \Delta_U(t) \exp\{-(\lambda + 2\sqrt{\mu} + 1)t\}$. It follows by the Ito formula that

$$e(t)\|U(t) - U_{\Phi_n}(t)\|^2 + 2\int_0^t e(s)\langle \mathcal{A}U(s) - \mathcal{A}U_{\Phi_n}(s), U(s) - U_{\Phi_n}(s)\rangle ds$$

$$\begin{aligned}
= {} & 2\int_0^t e(s)\langle \mathcal{B}(U(s),U(s)) - \mathcal{B}(U_{\Phi_n}(s), U_{\Phi_n}(s)), U(s) - U_{\Phi_n}(s)\rangle ds \\
- {} & \frac{b}{\nu}\int_0^t e(s)\|U(s)\|_V^2 \|U(s) - U_{\Phi_n}(s)\|^2 ds - (\lambda + 2\sqrt{\mu} + 1)\int_0^t e(s)\|U(s) - U_{\Phi_n}(s)\|^2 ds \\
+ {} & 2\int_0^t e(s)(\Phi(s, U(s)) - \Phi_n(s, U_{\Phi_n}(s)), U(s) - U_{\Phi_n}(s)) ds \\
+ {} & \int_0^t e(s)\|\mathcal{C}(s, U(s)) - \mathcal{C}(s, U_{\Phi_n}(s))\|^2 ds \\
+ {} & 2\int_0^t e(s)(\mathcal{C}(s, U(s)) - \mathcal{C}(s, U_{\Phi_n}(s)), U(s) - U_{\Phi_n}(s)) dw(s).
\end{aligned}$$

In view of the properties of $\mathcal{B}$ we can write

$$\begin{aligned} & 2\langle \mathcal{B}(U(s), U(s)) - \mathcal{B}(U_{\Phi_n}(s), U_{\Phi_n}(s)), U(s) - U_{\Phi_n}(s)\rangle \\ & \qquad = 2\langle \mathcal{B}(U(s) - U_{\Phi_n}(s), U(s)), U(s) - U_{\Phi_n}(s)\rangle \\ & \qquad \leq \frac{b}{\nu}\|U(s)\|_V^2\|U(s) - U_{\Phi_n}(s)\|^2 + \nu\|U(s) - U_{\Phi_n}(s)\|_V^2. \end{aligned}$$

Now we use the properties of $\mathcal{A}$, Φ, $\mathcal{C}$, and those of the stochastic integral to obtain

$$\begin{aligned} & E\sup_{s\in[0,t]} e(s)\|U(s) - U_{\Phi_n}(s)\|^2 + \nu E\int_0^t e(s)\|U(s) - U_{\Phi_n}(s)\|_V^2 ds \\ \leq\ & 2E\int_0^t e(s)\|\Phi(s, U(s)) - \Phi_n(s, U(s))\|^2 ds \\ +\ & 4E\sup_{s\in[0,t]}\Big|\int_0^s e(r)(\mathcal{C}(r, U(r)) - \mathcal{C}(r, U_{\Phi_n}(r)), U(r) - U_{\Phi_n}(r))dw(r)\Big| \\ \leq\ & 2E\int_0^t e(s)\|\Phi(s, U(s)) - \Phi_n(s, U(s))\|^2 ds + k_1 E\int_0^t \sup_{r\in[0,s]}\Big\{e(r)\|U(r) - U_{\Phi_n}(r)\|^2\Big\}ds \\ +\ & \frac{1}{2}E\sup_{s\in[0,t]} e(s)\|U(s) - U_{\Phi_n}(s))\|^2, \end{aligned}$$

where k_1 is a positive constant and $t \in [0, T]$. By Gronwall's Lemma we get

$$\begin{aligned} E\sup_{s\in[0,t]} e(s)\|U(s) - U_{\Phi_n}(s)\|^2 &+ 2\nu E\int_0^t e(s)\|U(s) - U_{\Phi_n}(s)\|_V^2 ds \\ &\leq 4e^{2k_1 T}E\int_0^T \|\Phi(s, U(s)) - \Phi_n(s, U(s))\|^2 ds, \end{aligned}$$

for all $t \in [0, T]$.

We take $t := \mathcal{T}_M^U$. Using the hypothesis and the above inequality it follows that for each fixed $M \in \mathbb{N}$ we have

$$\lim_{n\to\infty} E\|U(\mathcal{T}_M^U) - U_{\Phi_n}(\mathcal{T}_M^U)\|^2 = 0, \qquad \lim_{n\to\infty} E\int_0^{\mathcal{T}_M^U} \|U(s) - U_{\Phi_n}(s)\|_V^2 ds = 0.$$

Applying Proposition B.3 for $\mathcal{T} := T$, $\mathcal{T}_M := \mathcal{T}_M^U$, $Q_n(\mathcal{T}) := \|U(\mathcal{T}) - U_{\Phi_n}(\mathcal{T})\|^2$, respectively, $Q_n(\mathcal{T}) := \int_0^{\mathcal{T}} \|U(s) - U_{\Phi_n}(s)\|_V^2 ds$, we obtain

$$\lim_{n\to\infty} E\|U(T) - U_{\Phi_n}(T)\|^2 = 0, \qquad \lim_{n\to\infty} E\int_0^T \|U(s) - U_{\Phi_n}(s)\|_V^2 ds = 0.$$

The continuity properties of $\mathcal{L}$ and $\mathcal{K}$ imply $\lim_{n\to\infty} \mathcal{J}(\Phi_n) = \mathcal{J}(\Phi)$.
(ii) We use the same method as in **(i)**. ■

Using Lemma 2.2.1 and the generalized *Weierstraß Theorem* we obtain the following theorem.

Theorem 2.2.2

(i) *If the admissible controls are in a compact subset of $\mathcal{U}$, then there exist optimal feedback controls for the minimization problem $(\mathcal{P})$.*

(ii) *If the admissible controls are in a compact subset of $\mathcal{U}^b$, then there exist optimal controls for the minimization problem $(\mathcal{P}^b)$.*

2.3 The existence of optimal feedback controls

It is difficult to guarantee the compactness of the set of admissible controls and therefore it is useful to derive the existence of optimal controls using other methods. In this section we prove the existence of optimal feedback controls.

Let (Φ_n) be a sequence in $\mathcal{U}$.

Lemma 2.3.1
There exist a subsequence (n') of (n) and a mapping $\Phi \in \mathcal{U}$ such that for all $t \in [0,T]$, $x, y \in H$ we have

$$\lim_{n'\to\infty} (\Phi_{n'}(t,x), y) = (\Phi(t,x), y). \tag{2.5}$$

Proof. Let $\{t_1, t_2, \ldots\}$ be a dense subset of $[0,T]$ and recall that $\{h_1, h_2, \ldots\}$ is an orthonormal base in H. The sequence $\Big(\Phi_n(t_1,h_1)\Big)_{n\in\mathbb{N}}$ is a bounded sequence in H. Hence there exists a subsequence $(n_k^{1,1})$ of (n) and an element $z_1^1 \in H$ such that for all $y \in H$ we have

$$\lim_{k\to\infty} (\Phi_{n_k^{1,1}}(t_1,h_1), y) = (z_1^1, y).$$

The sequence $\Big(\Phi_n(t_1,h_2)\Big)_{n\in\mathbb{N}}$ is a bounded sequence in H. Hence there exists a subsequence $(n_k^{1,2})$ of $(n_k^{1,1})$ and an element $z_2^1 \in H$ such that for all $y \in H$ we have

$$\lim_{k\to\infty} (\Phi_{n_k^{1,2}}(t_1,h_2), y) = (z_2^1, y).$$

This procedure we repeat for all $h_1, h_2, \dots$ and then we take the "diagonal sequence" $(n_k^{1,k})_{k\in\mathbb{N}}$, which has the property

$$\lim_{k\to\infty}(\Phi_{n_k^{1,k}}(t_1,h_i),y) = (z_i^1,y)$$

for all $y \in H$ and $i \in \mathbb{N}$. Of course, the subsequence $(n_k^{1,k})$ depends on t_1. Now we repeat the procedure from above for $t_2, t_3, \dots$ and take again the "diagonal sequence" $(n_k^{k,k})_{k\in\mathbb{N}}$, which we denote by (n'). For all $y \in H$ and $i, j \in \mathbb{N}$, we have

$$\lim_{n'\to\infty}(\Phi_{n'}(t_j,h_i),y) = (z_i^j,y), \tag{2.6}$$

where $z_i^j \in H$.

We want to prove that for each fixed $i, j \in \mathbb{N},\ x \in H$ the sequence $\Big(\Phi_{n'}(t_j,x),h_i)\Big)_{n'\in\mathbb{N}}$ is convergent.

Let $\varepsilon > 0$. There exists $p_\varepsilon \in \mathbb{N}$ such that

$$\|x - x_{p_\varepsilon}\|^2 < \frac{\varepsilon}{3\sqrt{\mu}},$$

where $x_{p_\varepsilon} = \Pi_{p_\varepsilon} x$. Equation (2.6) implies that there exists $n'_0 \in \mathbb{N}$ such that for all $n', m' \geq n'_0$

$$\Big|(\Phi_{n'}(t_j,x_{p_\varepsilon}),h_i) - (\Phi_{m'}(t_j,x_{p_\varepsilon}),h_i)\Big| < \frac{\varepsilon}{3}\ .$$

For all $n', m' \geq n'_0$ we have

$$\begin{aligned}\Big|(\Phi_{n'}(t_j,x),h_i) - (\Phi_{m'}(t_j,x),h_i)\Big| &< \Big|(\Phi_{n'}(t_j,x-x_{p_\varepsilon}),h_i)\Big| + \Big|(\Phi_{m'}(t_j,x-x_{p_\varepsilon}),h_i)\Big| \\ &+ \Big|(\Phi_{n'}(t_j,x_{p_\varepsilon}),h_i) - (\Phi_{m'}(t_j,x_{p_\varepsilon}),h_i)\Big| < \varepsilon.\end{aligned}$$

Hence $\Big(\Phi_{n'}(t_j,x),h_i)\Big)_{n'\in\mathbb{N}}$ is a Cauchy sequence, and we can define the function $f_{i,j} : H \to \mathbb{R}$ by

$$f_{i,j}(x) = \lim_{n'\to\infty}(\Phi_{n'}(t_j,x),h_i).$$

Obviously, $f_{i,j}$ is linear. Let $p \in \mathbb{N}$. We have

$$\sum_{i=1}^{p} f_{i,j}^2(x) = \lim_{n'\to\infty}\sum_{i=1}^{p}(\Phi_{n'}(t_j,x),h_i)^2 \leq \limsup_{n'\to\infty}\|\Phi_{n'}(t_j,x)\|^2 \leq \mu\|x\|^2 < \infty \tag{2.7}$$

and

$$\begin{aligned}\sum_{i=1}^{p}|f_{i,j}(x) - f_{i,k}(x)|^2 &= \lim_{n'\to\infty}\sum_{i=1}^{p}(\Phi_{n'}(t_j,x) - \Phi_{n'}(t_k,x),h_i)^2 \\ &\leq \limsup_{n'\to\infty}\|\Phi_{n'}(t_j,x) - \Phi_{n'}(t_k,x)\|^2 \leq \alpha|t_j - t_k|^2 < \infty\end{aligned} \tag{2.8}$$

for all $j, k \in \mathbb{N},\ x \in H$.

We define $\Phi : \{t_1, t_2, \ldots\} \times H \to H$ as follows:

$$\Phi(t_j, x) = \sum_{i=1}^{\infty} f_{i,j}(x) h_i, \quad j \in \mathbb{N},\ x \in H.$$

By (2.7) and (2.8) we see that for each $j \in \mathbb{N}$ the mapping $\Phi(t_j, \cdot)$ is linear and continuous with

$$\|\Phi(t_j, x)\|^2 = \sum_{i=1}^{\infty} f_{i,j}^2(x) \le \mu \|x\|^2 \quad \text{for all } x \in H,$$

as soon as for each $j, k \in \mathbb{N}$ we have

$$\|\Phi(t_j, x) - \Phi(t_k, x)\|^2 = \sum_{i=1}^{\infty} |f_{i,j}(x) - f_{i,k}(x)|^2 \le \alpha |t_j - t_k|^2 \quad \text{for all } x \in H. \tag{2.9}$$

Now we define $\Phi : [0, T] \times H \to H$. Let $t \in [0, T]$. There exists a sequence $(\tilde{t}_n)$ in $\{t_1, t_2, \ldots\}$ such that $\lim\limits_{n\to\infty} \tilde{t}_n = t$, and we put

$$\Phi(t, x) = \lim_{n\to\infty} \Phi(\tilde{t}_n, x).$$

Using (2.9) it can be proved that this definition is independent of the choice of $(\tilde{t}_n)$. Obviously, we have

$$\|\Phi(t_1, x) - \Phi(t_2, x)\|^2 \le \alpha |t_1 - t_2|^2$$

for all $t_1, t_2 \in [0, T]$ and all $x \in H$. Consequently, $\Phi \in \mathcal{U}$ and by the construction of Φ we deduce that it satisfies (2.5). ■

For convenience, in the following we will denote the subsequence of indices (n') obtained in Lemma 2.3.1 by (n).

For $n = 1, 2, \ldots$ we consider the evolution equation

$$(E_{\Phi_n}) \qquad \begin{aligned} (\tilde{\mathbf{U}}_{\mathbf{\Phi_n}}(t), v) + \int_0^t \langle \mathcal{A}\tilde{\mathbf{U}}_{\mathbf{\Phi_n}}(s), v\rangle ds &= (x_0, v) + \int_0^t \langle \mathcal{B}(U_\Phi(s), \tilde{\mathbf{U}}_{\mathbf{\Phi_n}}(s)), v\rangle ds \\ &+ \int_0^t (\Phi_n(s, U_\Phi(s)), v) ds + \int_0^t (\mathcal{C}(s, U_\Phi(s)), v) dw(s) \end{aligned}$$

for all $t \in [0, T], v \in V$ and a.e. $\omega \in \Omega$. By Theorem 1.3.1, applied for $Z_{\Psi,\Gamma} := \tilde{U}_{\Phi_n}, X := U_\Phi$, $Y := 0$, $a_0 := x_0$, $\Psi(s) := \Phi_n(s, U_\Phi(s))$, $\mathcal{G} := 0$, $\Gamma(s) := \mathcal{C}(s, U_\Phi(s))$ it follows that there exists an almost surely unique solution $\tilde{U}_{\Phi_n} \in \mathcal{L}_V^2(\Omega \times [0, T])$ of (E_{Φ_n}), which has almost surely continuous trajectories in H and

$$E \sup_{t\in[0,T]} \|\tilde{U}_{\Phi_n}(t)\|^4 + E\Big(\int_0^T \|\tilde{U}_{\Phi_n}(s)\|_V^2 ds\Big)^2 \le \tilde{c}_1 \Big[E\|x_0\|^4 + E\int_0^T \|U_\Phi(s)\|^4 ds\Big],$$

where $\tilde{c}_1 > 0$ is a constant (independent of n).

For $n = 1, 2, \ldots$ we consider the n-dimensional evolution equation

$$(E_{n,\Phi_n})\quad (\tilde{\mathbf{U}}_{\mathbf{n},\mathbf{\Phi_n}}(t), v) + \int_0^t \langle \mathcal{A}\tilde{\mathbf{U}}_{\mathbf{n},\mathbf{\Phi_n}}(s), v\rangle ds = (x_{0n}, v) + \int_0^t \langle \mathcal{B}_n(\Pi_n U_\Phi(s), \tilde{\mathbf{U}}_{\mathbf{n},\mathbf{\Phi_n}}(s)), v\rangle ds$$
$$+ \int_0^t \Big(\Pi_n \Phi_n(s, U_\Phi(s)), v\Big) ds + \int_0^t \Big(\Pi_n \mathcal{C}(s, U_\Phi(s)), v\Big) dw(s)$$

for all $t \in [0,T], v \in H_n$ and a.e. $\omega \in \Omega$. By Theorem 1.3.1, applied for $Z_{n,\psi,\gamma} := \tilde{U}_{n,\Phi_n}$, $X_n := \Pi_n U_\Phi$, $Y_n := 0$, $a_{0n} := x_{0n}$, $\psi(s) := \Pi_n \Phi_n(s, U_\Phi(s))$, $\mathcal{G} := 0$, $\gamma(s) := \Pi_n \mathcal{C}(s, U_\Phi(s))$ it follows that there exists an almost surely unique solution $\tilde{U}_{n,\Phi_n} \in \mathcal{L}^2_{(H_n,\|\cdot\|_V)}(\Omega \times [0,T])$ of (E_{n,Φ_n}), which has almost surely continuous trajectories in H and

$$E \sup_{t\in[0,T]} \|\tilde{U}_{n,\Phi_n}(t)\|^4 + E\Big(\int_0^T \|\tilde{U}_{n,\Phi_n}(s)\|_V^2 ds\Big)^2 \le \tilde{c}_2 \Big[E\|x_0\|^4 + E\int_0^T \|U_\Phi(s)\|^4 ds\Big],$$

where $\tilde{c}_2 > 0$ is a constant (independent of n).

Theorem 2.3.2
The following convergences hold:

$$\lim_{n\to\infty} E\int_0^T \|U_\Phi(s) - \tilde{U}_{\Phi_n}(s)\|_V^2 ds = 0, \quad \lim_{n\to\infty} E\|U_\Phi(T) - \tilde{U}_{\Phi_n}(T)\|^2 = 0,$$

$$\lim_{n\to\infty} E\int_0^T \|U_\Phi(s) - \tilde{U}_{n,\Phi_n}(s)\|_V^2 ds = 0 \quad \textit{and} \quad \lim_{n\to\infty} E\|U_\Phi(T) - \tilde{U}_{n,\Phi_n}(T)\|^2 = 0.$$

Proof. We consider the evolution equation

$$(\mathbf{z}(t), v) + \int_0^t \langle \mathcal{A}\mathbf{z}(s), v\rangle ds = (x_0, v) + \int_0^t (\mathcal{C}(s, U_\Phi(s)), v) dw(s) \tag{2.10}$$

for all $t \in [0,T], v \in V$ and a.e. $\omega \in \Omega$. There exists an almost surely unique solution $z \in \mathcal{L}^2_V(\Omega \times [0,T])$ of (2.10), which has almost surely continuous trajectories in H (see [14], Theorem 4.1, p. 105). By using the ideas from Proposition B.2 we can prove that the estimate

$$E \sup_{t\in[0,T]} \|z(t)\|^2 + 2\nu E\int_0^T \|z(s)\|_V^2 ds \le c\Big[E\|x_0\|^2 + E\int_0^T \|U_\Phi(s)\|^2 ds\Big]$$

holds, where c is a positive constant depending on λ. From Theorem 1.2.2 we have

$$E \sup_{t\in[0,T]} \|U_\Phi(t)\|^2 < \infty, \quad E\int_0^T \|U_\Phi(s)\|_V^2 ds < \infty.$$

Hence, there exists $k(\omega) > 0$, independent of n, such that for all $n \in \mathbb{N}$ and a.e. $\omega \in \Omega$

$$\sup_{t\in[0,T]} \|\Pi_n z(t)\|^2 \leq \sup_{t\in[0,T]} \|z(t)\|^2 < k(\omega), \int_0^T \|\Pi_n z(s)\|_V^2 ds \leq \int_0^T \|z(s)\|_V^2 ds < k(\omega), \tag{2.11}$$

$$\sup_{t\in[0,T]} \|\Pi_n U_\Phi(t)\|^2 \leq \sup_{t\in[0,T]} \|U_\Phi(t)\|^2 < k(\omega), \int_0^T \|\Pi_n U_\Phi(s)\|_V^2 ds \leq \int_0^T \|U_\Phi(s)\|_V^2 ds < k(\omega). \tag{2.12}$$

By the properties of the stochastic integral and by the properties of U_Φ (see Lemma 1.2.6) we see that for all $s, t \in [0,T]$

$$\begin{aligned} E\Big\| \int_s^t \mathcal{C}(r, U_\Phi(r))dw(r)\Big\|_{V^*}^4 &\leq k_1 E\Big(\int_s^t \|\mathcal{C}(r, U_\Phi(r))\|_{V^*}^2 dr\Big)^2 \\ &\leq k_2 (t-s)^2 E \sup_{r\in[0,T]} \|U_\Phi(r)\|^4 \leq c(t-s)^2 E\|x_0\|^4 \end{aligned}$$

and

$$\begin{aligned} E\Big\| \int_s^t \Pi_n \mathcal{C}(r, U_\Phi(r))dw(r)\Big\|_{V^*}^4 &\leq k_1 E\Big(\int_s^t \Big\|\Pi_n \mathcal{C}(r, U_\Phi(r))\Big\|_{V^*}^2 dr\Big)^2 \\ &\leq k_2 (t-s)^2 E \sup_{r\in[0,T]} \|U_\Phi(r)\|^4 \leq c(t-s)^2 E\|x_0\|^4, \end{aligned}$$

where k_1, k_2 are positive constants. In the above estimates we need the V^*-norm, because we will apply the *Dubinsky Theorem* (see [35], Theorem 4.1, p. 132).

By the *Theorem of Kolmogorov-Centsov* (see [18], Theorem 2.8, p. 53; applied for $\alpha := 4, \beta := 1$ and a process with values in a Hilbert space) it follows that there exist a random variable $\chi(\omega)$ and a positive constant δ such that

$$\Big\| \int_s^t \mathcal{C}(r, U_\Phi(r))dw(r)\Big\|_{V^*}^2 \leq \delta |t-s|^{2\gamma}, \tag{2.13}$$

$$\Big\| \int_s^t \Pi_n \mathcal{C}(r, U_\Phi(r))dw(r)\Big\|_{V^*}^2 \leq \delta |t-s|^{2\gamma}, \tag{2.14}$$

for $\gamma \in \Big(0, \dfrac{1}{4}\Big)$ and for every $t, s \in [0,T]$ with $|t-s| < \chi(\omega)$ and a.e. $\omega \in \Omega$.

Let $\tilde{\Omega} \subseteq \Omega$ with $P(\tilde{\Omega}) = 1$ be such that for all $\omega \in \tilde{\Omega}$ we have:

- equations (2.1) and (2.10) hold for all $t \in [0,T]$, $v \in V$;
- for each $n = 1, 2, \ldots$ equations (E_{Φ_n}) and (E_{n,Φ_n}) hold for all $t \in [0,T]$, $v \in V$, respectively $v \in H_n$;
- the inequalities in (2.11), (2.12), (2.13) and (2.14) hold.

From (E_{Φ_n}), (2.10), (2.11), (2.12), and the properties of $\mathcal{A}, \mathcal{B}, \Phi_n$, it follows that for all $\omega \in \tilde{\Omega}$ we have[1]

$$\begin{aligned}\sup_{t\in[0,T]} \|\tilde{U}_{\Phi_n}(t) - z(t)\|^2 &+ \nu\int_0^T \|\tilde{U}_{\Phi_n}(s) - z(s)\|_V^2 ds \le \frac{4}{\nu}\int_0^T \|\mathcal{B}(U_\Phi(s), z(s))\|_{V^*}^2 ds \\ &+ \frac{4\mu c_{HV}}{\nu}\int_0^T \|U_\Phi(s)\|^2 ds \le \frac{2b}{\nu}\sup_{t\in[0,T]}\|U_\Phi(t)\|^2 \int_0^T \|U_\Phi(s)\|_V^2 ds \\ &+ \frac{2b}{\nu}\sup_{t\in[0,T]}\|z(t)\|^2\int_0^t \|z(s)\|_V^2 ds + \frac{4\mu c_{HV}}{\nu}\int_0^T \|U_\Phi(s)\|^2 ds \le c_1(k^2(\omega) + k(\omega))\end{aligned}$$

where c_1 is a positive constant independent of n and ω. Analogously, using (E_{n,Φ_n}) and (2.10) we have

$$\begin{aligned}&\sup_{t\in[0,T]} \|\tilde{U}_{n,\Phi_n}(t) - \Pi_n z(t)\|^2 + \nu\int_0^T \|\tilde{U}_{n,\Phi_n}(s) - \Pi_n z(s)\|_V^2 ds \\ &\quad\le \frac{2b}{\nu}\sup_{t\in[0,T]}\|\Pi_n U_\Phi(t)\|^2\int_0^T \|\Pi_n U_\Phi(s)\|_V^2 ds + \frac{2b}{\nu}\sup_{t\in[0,T]}\|\Pi_n z(t)\|^2\int_0^t \|\Pi_n z(s)\|_V^2 ds \\ &\quad+ \frac{4\mu c_{HV}}{\nu}\int_0^T \|U_\Phi(s)\|^2 ds \le c_1(k^2(\omega) + k(\omega))\end{aligned}$$

where c_1 is the same constant as above.

Hence for all $n \in \mathbb{N}$ we have

$$\sup_{t\in[0,T]} \|\tilde{U}_{\Phi_n}(t)\|^2 + \nu\int_0^T \|\tilde{U}_{\Phi_n}(s)\|_V^2 ds < c_2(\omega) \quad \text{for all } \omega \in \tilde{\Omega} \tag{2.15}$$

and

$$\sup_{t\in[0,T]} \|\tilde{U}_{n,\Phi_n}(t)\|^2 + \nu\int_0^T \|\tilde{U}_{n,\Phi_n}(s)\|_V^2 ds < c_2(\omega) \quad \text{for all } \omega \in \tilde{\Omega}, \tag{2.16}$$

where $c_2(\omega)$ is positive, independent of n.

Let $\omega \in \tilde{\Omega}$. For this ω, we consider the sets

$$S = \left\{\tilde{U}_{\Phi_n}(\omega,\cdot)\,\middle|\, n = 1, 2, \dots\right\}, \quad \tilde{S} = \left\{\tilde{U}_{n,\Phi_n}(\omega,\cdot)\,\middle|\, n = 1, 2, \dots\right\}.$$

For each of these sets we want to apply the *Dubinsky Theorem*. By (2.15) and (2.16) we get that $S \subset \mathcal{L}_V^2[0,T]$ and $\tilde{S} \subset \mathcal{L}_V^2[0,T]$ are bounded. We have to verify that S, respectively $\tilde{S}$, are

[1]Since $V \hookrightarrow H$ we have $\|v\|^2 \le c_{HV}\|v\|_V^2$ for all $v \in V$.

equi-continuous in $C([0,T],V^*)$. From (E_{Φ_n}) and the Schwarz inequality we have

$$\begin{aligned}\|\tilde{U}_{\Phi_n}(t)-\tilde{U}_{\Phi_n}(s)\|^2_{V^*} &\leq (t-s)\int_s^t \Big(\|\mathcal{A}\tilde{U}_{\Phi_n}(r)\|^2_{V^*}+\|\mathcal{B}(U_\Phi(r),\tilde{U}_{\Phi_n}(r))\|^2_{V^*}+\|\Phi_n(r,U_\Phi(r))\|^2_{V^*}\Big)dr \\ &+ \Big\|\int_s^t \mathcal{C}(r,U_\Phi(r))dw(r)\Big\|^2_{V^*}\end{aligned}$$

for each $t,s\in[0,T], t>s$. By (2.13), (2.15), and the properties of $\mathcal{A},\mathcal{B},\Phi_n$ we obtain

$$\|\tilde{U}_{\Phi_n}(t)-\tilde{U}_{\Phi_n}(s)\|^2_{V^*}\leq c_3(\omega)(t-s)+\delta(t-s)^{2\gamma}$$

for $\gamma\in\Big(0,\dfrac{1}{4}\Big)$ and for every $t,s\in[0,T]$ with $|t-s|<\chi(\omega)$, where $c_3(\omega)>0$ is independent of n. Consequently, S is equi-continuous in $C([0,T],V^*)$. Analogously we can prove that $\tilde{S}$ is equi-continuous in $C([0,T],V^*)$. Now, using the *Dubinsky Theorem*, it follows that S and $\tilde{S}$ are relatively compact in $\mathcal{L}^2_H[0,T]$ and hence there exists a subsequence (n') of (n) and $\tilde{U},U^*\in\mathcal{L}^2_H[0,T]$ such that

$$\lim_{n'\to\infty}\int_0^T\|\tilde{U}_{\Phi_{n'}}(s)-\tilde{U}(s)\|^2ds=0 \tag{2.17}$$

and

$$\lim_{n'\to\infty}\int_0^T\|\tilde{U}_{n',\Phi_{n'}}(s)-U^*(s)\|^2ds=0. \tag{2.18}$$

We use $(E_{\Phi_{n'}})$ and (2.1), the generalized chain rule, the properties of $\mathcal{A}$ and $\mathcal{B}$ to obtain

$$\begin{aligned}\|\tilde{U}_{\Phi_{n'}}(T)-U_\Phi(T)\|^2 &+ 2\nu\int_0^T\|\tilde{U}_{\Phi_{n'}}(s)-U_\Phi(s)\|^2_V ds \\ &\leq 2\int_0^T\Big(\Phi_{n'}(s,U_\Phi(s))-\Phi(s,U_\Phi(s)),\tilde{U}_{\Phi_{n'}}(s)-\tilde{U}(s)\Big)ds \\ &+ 2\int_0^T\Big(\Phi_{n'}(s,U_\Phi(s))-\Phi(s,U_\Phi(s)),\tilde{U}(s)-U_\Phi(s)\Big)ds.\end{aligned}$$

According to Lemma 2.3.1, (2.17), and the properties of $\Phi_n,\Phi\in\mathcal{U}$ we get

$$\lim_{n'\to\infty}\|\tilde{U}_{\Phi_{n'}}(T)-U_\Phi(T)\|^2=0,\quad \lim_{n'\to\infty}\int_0^T\|\tilde{U}_{\Phi_{n'}}(s)-U_\Phi(s)\|^2_V ds=0. \tag{2.19}$$

Every subsequence of $\Big(\tilde{U}_{\Phi_n}(\omega,\cdot)\Big)$ has a further subsequence, which converges in the space $\mathcal{L}^2_V[0,T]$ to the same limit $U_\Phi(\omega,\cdot)$ (because we can repeat all arguments of above). Applying Proposition

A.1, it follows that the whole sequence $\left(\tilde{U}_{\Phi_n}(\omega,\cdot)\right)$ converges to $U_\Phi(\omega,\cdot)$ in the space $\mathcal{L}_V^2[0,T]$. Analogously we conclude that the whole sequence $\left(\tilde{U}_{\Phi_n}(\omega,T)\right)$ converges to $U_\Phi(\omega,T)$ in H.

Our arguments from above worked for an arbitrary fixed $\omega \in \tilde{\Omega}$. Hence (2.19) holds for a.e. $\omega \in \Omega$ and for the whole sequence (n). Taking into consideration that the processes $\left(\tilde{U}_{\Phi_n}(t)\right)_{t\in[0,T]}$ and $\left(U_\Phi(t)\right)_{t\in[0,T]}$ are uniformly integrable (see Theorem 1.3.1 and Lemma 1.2.6) it follows that

$$\lim_{n\to\infty} E\int_0^T \|U_\Phi(s) - \tilde{U}_{\Phi_n}(s)\|_V^2 ds = 0$$

and

$$\lim_{n\to\infty} E\|U_\Phi(T) - \tilde{U}_{\Phi_n}(T)\|^2 = 0.$$

Now we prove the convergences for the sequence $(\tilde{U}_{n,\Phi_n})$. We use $(E_{n',\Phi_{n'}})$ and (2.1), the generalized chain rule, the properties of $\mathcal{A}$ and $\mathcal{B}$ to obtain

$$\begin{aligned}(2.20)\quad \|\tilde{U}_{n,\Phi_n}(T) - \Pi_n U_\Phi(T)\|^2 &+ \nu\int_0^T \|\tilde{U}_{n,\Phi_n}(s) - \Pi_n \mathcal{U}_\Phi(s)\|_V^2 ds\\ &\le \int_0^T \|\mathcal{B}(\Pi_n U_\Phi(s), \Pi_n U_\Phi(s)) - \mathcal{B}(U_\Phi(s), U_\Phi(s))\|_{V^*}^2 ds\\ &+ 2\int_0^T \Big(\Pi_n \Phi_n(s, U_\Phi(s)) - \Phi(s, U_\Phi(s)), \tilde{U}_{n,\Phi_n}(s) - \Pi_n U_\Phi(s)\Big) ds.\end{aligned}$$

By the properties of $\mathcal{B}$ we have

$$\begin{aligned}&\int_0^T \|\mathcal{B}(\Pi_n U_\Phi(s), \Pi_n U_\Phi(s)) - \mathcal{B}(U_\Phi(s), U_\Phi(s))\|_{V^*}^2 ds\\ &\qquad\le 2b\Big(\sup_{t\in[0,T]} \|U_\Phi(t) - \Pi_n U_\Phi(t)\|^2 \int_0^T \|U_\Phi(s)\|_V^2 ds\Big)^{1/2}\\ &\qquad\times \Big(\sup_{t\in[0,T]} \|U_\Phi(t)\|^2 \int_0^T \|U_\Phi(s) - \Pi_n U_\Phi(s)\|_V^2 ds\Big)^{1/2}\end{aligned}$$

and by (1.4) it follows

$$(2.21)\qquad \lim_{n'\to\infty}\int_0^T \|\mathcal{B}(\Pi_{n'} U_\Phi(s), \Pi_{n'} U_\Phi(s)) - \mathcal{B}(U_\Phi(s), U_\Phi(s))\|_{V^*}^2 ds = 0.$$

We write

$$\int_0^T \Big(\Pi_n\Phi_n(s,U_\Phi(s)) - \Phi(s,U_\Phi(s)), \tilde{U}_{n,\Phi_n}(s) - \Pi_n U_\Phi(s)\Big)ds$$

$$= \int_0^T \Big(\Phi_n(s,U_\Phi(s)) - \Phi(s,U_\Phi(s)), \tilde{U}_{n,\Phi_n}(s) - U^*(s)\Big)ds$$

$$+ \int_0^T \Big(\Phi_n(s,U_\Phi(s)) - \Phi(s,U_\Phi(s)), U^*(s) - U_\Phi(s)\Big)ds$$

$$+ \int_0^T \Big(\Phi_n(s,U_\Phi(s)) - \Phi(s,U_\Phi(s)), U_\Phi(s) - \Pi_n U_\Phi(s)\Big)ds.$$

By using this equality for n', as soon as (1.4), (2.12), (2.16), (2.18), the properties of Φ_n, Φ and Lemma 2.3.1 we get

$$\lim_{n'\to\infty}\int_0^T \Big(\Pi_{n'}\Phi_{n'}(s,U_\Phi(s)) - \Phi(s,U_\Phi(s)), \tilde{U}_{n',\Phi_{n'}}(s) - \Pi_{n'} U_\Phi(s)\Big)ds = 0. \tag{2.22}$$

From (2.21) and (2.22) we obtain that the right side of the inequality in (2.20) tends to zero. Therefore

$$\lim_{n'\to\infty} \|\tilde{U}_{n',\Phi_{n'}}(T) - \Pi_{n'}U_\Phi(T)\|^2 = 0, \quad \lim_{n'\to\infty}\int_0^T \|\tilde{U}_{n',\Phi_{n'}}(s) - \Pi_{n'}U_\Phi(s)\|_V^2 ds = 0.$$

Hence by (1.4) and (1.6) we have

$$\lim_{n'\to\infty} \|\tilde{U}_{n',\Phi_{n'}}(T) - U_\Phi(T)\|^2 = 0, \quad \lim_{n'\to\infty}\int_0^T \|\tilde{U}_{n',\Phi_{n'}}(s) - U_\Phi(s)\|_V^2 ds = 0. \tag{2.23}$$

Every subsequence of $\Big(\tilde{U}_{n,\Phi_n}(\omega,\cdot)\Big)$ has a further subsequence, which converges in the space $\mathcal{L}_V^2[0,T]$ to the same limit $U_\Phi(\omega,\cdot)$ (because we can repeat all arguments of above). Applying Proposition A.1, it follows that the whole sequence $\Big(\tilde{U}_{n,\Phi_n}(\omega,\cdot)\Big)$ converges to $U_\Phi(\omega,\cdot)$ in the space $\mathcal{L}_V^2[0,T]$, respectively. Analogously we conclude that the whole sequence $\Big(\tilde{U}_{n,\Phi_n}(\omega,T)\Big)$ converges in H to $U_\Phi(\omega,T)$.

Our arguments from above worked for an arbitrary fixed $\omega\in\tilde{\Omega}$. Hence (2.23) holds for a.e. $\omega\in\Omega$ and for the whole sequence (n). Taking into consideration that the processes $\Big(\tilde{U}_{n,\Phi_n}(t)\Big)_{t\in[0,T]}$ and $\Big(U_\Phi(t)\Big)_{t\in[0,T]}$ are uniformly integrable (see Theorem 1.3.1 and Lemma 1.2.6) and using (1.5) it follows that the conclusions of this theorem hold. ■

Let $U_{n,\Phi}$ be the solution of (P_n) (see Section 1.2) using the feedback control $\Phi_n := \Pi_n\Phi$. Note that $U_{n,\Phi} = U_{n,\Pi_n\Phi}$ for $\Phi\in\mathcal{U}$ or $\Phi\in\mathcal{U}^b$.

Theorem 2.3.3
The following convergences hold:

$$\lim_{n\to\infty} E\int_0^T \|U_\Phi(s) - U_{\Phi_n}(s)\|_V^2 ds = 0, \quad \lim_{n\to\infty} E\|U_\Phi(T) - U_{\Phi_n}(T)\|^2 = 0,$$

$$\lim_{n\to\infty} E\int_0^T \|U_\Phi(s) - U_{n,\Phi_n}(s)\|_V^2 ds = 0 \quad \textit{and} \quad \lim_{n\to\infty} E\|U_\Phi(T) - U_{n,\Phi_n}(T)\|^2 = 0.$$

Proof. We write $U := U_\Phi$. Let $M \in \mathbb{N}$ and let $\mathcal{T}_M := \mathcal{T}_M^U$ be the stopping time of U. We write

$$e(t) = \Delta_U^2(t)\exp\{-(2\lambda + 2\sqrt{\mu} + 1)t\}.$$

It follows by the Ito formula that for a.e. $\omega \in \Omega$ we have

$$\begin{aligned}
(2.24)\quad & e(\mathcal{T}_M)\|\tilde{U}_{\Phi_n}(\mathcal{T}_M) - U_{\Phi_n}(\mathcal{T}_M)\|^2 + 2\int_0^{\mathcal{T}_M} e(s)\langle \mathcal{A}\tilde{U}_{\Phi_n}(s) - \mathcal{A}U_{\Phi_n}(s), \tilde{U}_{\Phi_n}(s) - U_{\Phi_n}(s)\rangle ds \\
= \;& 2\int_0^{\mathcal{T}_M} e(s)\langle \mathcal{B}(U(s), \tilde{U}_{\Phi_n}(s)) - \mathcal{B}(U_{\Phi_n}(s), U_{\Phi_n}(s)), \tilde{U}_{\Phi_n}(s) - U_{\Phi_n}(s)\rangle ds \\
- \;& \frac{2b}{\nu}\int_0^{\mathcal{T}_M} e(s)\|U(s)\|_V^2\|\tilde{U}_{\Phi_n}(s) - U_{\Phi_n}(s)\|^2 ds - (2\lambda + 2\sqrt{\mu} + 1)\int_0^{\mathcal{T}_M} e(s)\|\tilde{U}_{\Phi_n}(s) - U_{\Phi_n}(s)\|^2 ds \\
+ \;& 2\int_0^{\mathcal{T}_M} e(s)(\Phi_n(s, U(s) - U_{\Phi_n}(s)), \tilde{U}_{\Phi_n}(s) - U_{\Phi_n}(s))ds \\
+ \;& 2\int_0^{\mathcal{T}_M} e(s)(\mathcal{C}(s, U(s)) - \mathcal{C}(s, U_{\Phi_n}(s)), \tilde{U}_{\Phi_n}(s) - U_{\Phi_n}(s))dw(s) \\
+ \;& \int_0^{\mathcal{T}_M} e(s)\|\mathcal{C}(s, U(s)) - \mathcal{C}(s, U_{\Phi_n}(s))\|^2 ds.
\end{aligned}$$

In view of the properties of $\mathcal{B}$ we can write

$$\begin{aligned}
& 2\langle \mathcal{B}(U(s), \tilde{U}_{\Phi_n}(s)) - \mathcal{B}(U_{\Phi_n}(s), U_{\Phi_n}(s)), \tilde{U}_{\Phi_n}(s) - U_{\Phi_n}(s)\rangle \\
= \;& 2\langle \mathcal{B}(U(s) - U_{\Phi_n}(s), \tilde{U}_{\Phi_n}(s) - U(s)), \tilde{U}_{\Phi_n}(s) - U_{\Phi_n}(s)\rangle \\
+ \;& 2\langle \mathcal{B}(U(s) - \tilde{U}_{\Phi_n}(s), U(s)), \tilde{U}_{\Phi_n}(s) - U_{\Phi_n}(s)\rangle \\
+ \;& 2\langle \mathcal{B}(\tilde{U}_{\Phi_n}(s) - U_{\Phi_n}(s), U(s)), \tilde{U}_{\Phi_n}(s) - U_{\Phi_n}(s)\rangle
\end{aligned}$$

$$
\begin{aligned}
\leq\ & 2\sqrt{b}\|\tilde{U}_{\Phi_n}(s)-U(s)\|_V\|U(s)-U_{\Phi_n}(s)\|_V^{\frac{1}{2}}\|U(s)-U_{\Phi_n}(s)\|^{\frac{1}{2}} \\
\times\ & \|\tilde{U}_{\Phi_n}(s)-U_{\Phi_n}(s)\|_V^{\frac{1}{2}}\|\tilde{U}_{\Phi_n}(s)-U_{\Phi_n}(s)\|^{\frac{1}{2}} \\
+\ & \frac{2b}{\nu}\|U(s)\|\|U(s)\|_V\|U(s)-\tilde{U}_{\Phi_n}(s)\|_V\|U(s)-\tilde{U}_{\Phi_n}(s)\|+\nu\|\tilde{U}_{\Phi_n}(s)-U_{\Phi_n}(s)\|_V^2 \\
+\ & \frac{2b}{\nu}\|U(s)\|_V^2\|\tilde{U}_{\Phi_n}(s)-U_{\Phi_n}(s)\|^2.
\end{aligned}
$$

Using this estimates in (2.24) and after some elementary calculations, we obtain

$$
\begin{aligned}
& Ee(\mathcal{T}_M)\|\tilde{U}_{\Phi_n}(\mathcal{T}_M)-U_{\Phi_n}(\mathcal{T}_M)\|^2+\nu E\int_0^{\mathcal{T}_M} e(s)\|\tilde{U}_{\Phi_n}(s)-U_{\Phi_n}(s)\|_V^2 ds \\
\leq\ & 2\sqrt{b}\Big(E\int_0^T\|\tilde{U}_{\Phi_n}(s)-U(s)\|_V^2 ds\Big)^{\frac{1}{2}} \\
\times\ & \Big(E\int_0^T\Big(\|U(s)-U_{\Phi_n}(s)\|_V^2\|U(s)-U_{\Phi_n}(s)\|^2+\|\tilde{U}_{\Phi_n}(s)-U_{\Phi_n}(s)\|_V^2\|\tilde{U}_{\Phi_n}(s)-U_{\Phi_n}(s)\|^2\Big)ds\Big)^{\frac{1}{2}} \\
+\ & \frac{2bM}{\nu}\Big(E\int_0^T\|U(s)-\tilde{U}_{\Phi_n}(s)\|_V^2 ds\Big)^{1/2}\Big(E\sup_{t\in[0,T]}\|U(t)-\tilde{U}_{\Phi_n}(t)\|^2\Big)^{1/2} \\
+\ & (\mu+2\lambda)E\int_0^T\|U(s)-\tilde{U}_{\Phi_n}(s)\|^2 ds.
\end{aligned}
$$

Using the above inequalitiy, Theorem 2.3.2 and Lemma 1.2.6 we have

$$
\lim_{n\to\infty} E\|U(\mathcal{T}_M)-U_{\Phi_n}(\mathcal{T}_M)\|^2=0, \qquad \lim_{n\to\infty} E\int_0^{\mathcal{T}_M}\|U(s)-U_{\Phi_n}(s)\|^2 ds=0.
$$

By Proposition B.3, applied on $\mathcal{T} := T$, $Q_n(\mathcal{T}) := \|U(\mathcal{T})-U_{\Phi_n}(\mathcal{T})\|^2$, respectively $Q_n(\mathcal{T}) := \int_0^{\mathcal{T}}\|U(s)-U_{\Phi_n}(s)\|_V^2 ds$, we get

$$
\lim_{n\to\infty} E\int_0^T\|U(s)-U_{\Phi_n}(s)\|_V^2 ds=0 \quad \text{and} \quad \lim_{n\to\infty} E\|U(T)-U_{\Phi_n}(T)\|^2=0.
$$

Now we prove the convergences for the sequence $(\tilde{U}_{n,\Phi_n})$. It follows by the Ito formula and the properties for $\mathcal{A}$ that for a.e. $\omega \in \Omega$ we have

$$
\begin{aligned}
(2.25) \qquad & e(\mathcal{T}_M)\|\tilde{U}_{n,\Phi_n}(\mathcal{T}_M) - U_{n,\Phi_n}(\mathcal{T}_M)\|^2 + 2\nu \int_0^{\mathcal{T}_M} e(s)\|\tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\|_V^2 ds \\
= \;& 2\int_0^{\mathcal{T}_M} e(s)\langle \mathcal{B}(\Pi_n U(s), \tilde{U}_{n,\Phi_n}(s)) - \mathcal{B}(U_{n,\Phi_n}(s), U_{n,\Phi_n}(s)), \tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\rangle ds \\
- \;& \frac{2b}{\nu}\int_0^{\mathcal{T}_M} e(s)\|U(s)\|_V^2 \|\tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\|^2 ds \\
- \;& (2\lambda + 2\sqrt{\mu} + 1)\int_0^{\mathcal{T}_M} e(s)\|\tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\|^2 ds \\
+ \;& 2\int_0^{\mathcal{T}_M} e(s)\Big(\Pi_n \Phi_n(s, U(s) - U_{n,\Phi_n}(s)), \tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\Big) ds \\
+ \;& 2\int_0^{\mathcal{T}_M} e(s)\Big(\Pi_n \mathcal{C}(s, U(s)) - \Pi_n \mathcal{C}(s, U_{n,\Phi_n}(s)), \tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\Big) dw(s) \\
+ \;& \int_0^{\mathcal{T}_M} e(s)\Big\|\Pi_n \mathcal{C}(s, U(s)) - \Pi_n \mathcal{C}(s, U_{n,\Phi_n}(s))\Big\|^2 ds.
\end{aligned}
$$

In view of the properties of $\mathcal{B}$ and (1.2) we can write

$$
\begin{aligned}
& 2\langle \mathcal{B}(\Pi_n U(s), \tilde{U}_{n,\Phi_n}(s)) - \mathcal{B}(U_{n,\Phi_n}(s), U_{n,\Phi_n}(s)), \tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\rangle \\
= \;& 2\langle \mathcal{B}(\Pi_n U(s) - U_{n,\Phi_n}(s), \tilde{U}_{n,\Phi_n}(s) - \Pi_n U(s)), \tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\rangle \\
+ \;& 2\langle \mathcal{B}(\Pi_n U(s) - \tilde{U}_{n,\Phi_n}(s), \Pi_n U(s)), \tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\rangle \\
+ \;& 2\langle \mathcal{B}(\tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s), \Pi_n U(s)), \tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\rangle \\
\le \;& 2\sqrt{b}\|\tilde{U}_{n,\Phi_n}(s) - \Pi_n U(s)\|_V \|\Pi_n U(s) - U_{n,\Phi_n}(s)\|_V^{\frac{1}{2}} \|\Pi_n U(s) - U_{n,\Phi_n}(s)\|^{\frac{1}{2}} \\
\times \;& \|\tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\|_V^{\frac{1}{2}} \|\tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\|^{\frac{1}{2}} \\
+ \;& \frac{2b}{\nu}\|U(s)\|\|U(s)\|_V \|\Pi_n U(s) - \tilde{U}_{n,\Phi_n}(s)\|_V \|\Pi_n U(s) - \tilde{U}_{n,\Phi_n}(s)\| \\
+ \;& \nu\|\tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\|_V^2 + \frac{2b}{\nu}\|U(s)\|_V^2 \|\tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\|^2.
\end{aligned}
$$

Using this estimates in (2.25) and after some elementary calculations, we obtain

$$
\begin{aligned}
& Ee(\mathcal{T}_M)\|\tilde{U}_{n,\Phi_n}(\mathcal{T}_M) - U_{n,\Phi_n}(\mathcal{T}_M)\|^2 + \nu E\int_0^{\mathcal{T}_M} e(s)\|\tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\|_V^2 ds \\
\leq \; & 2\sqrt{b}\Big(E\int_0^T \|\tilde{U}_{n,\Phi_n}(s) - \Pi_n U(s)\|_V^2 ds\Big)^{\frac{1}{2}} \\
\times \; & \Bigg(E\int_0^T \Big(\|\Pi_n U(s) - U_{n,\Phi_n}(s)\|_V^2 \|\Pi_n U(s) - U_{n,\Phi_n}(s)\|^2 \\
+ \; & \|\tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\|_V^2 \|\tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\|^2\Big) ds\Bigg)^{\frac{1}{2}} \\
+ \; & \frac{2bM}{\nu}\Big(E\int_0^T \|\Pi_n U(s) - \tilde{U}_{n,\Phi_n}(s)\|_V^2 ds\Big)^{1/2}\Big(E\sup_{t\in[0,T]}\|\Pi_n U(t) - \tilde{U}_{n,\Phi_n}(t)\|^2\Big)^{1/2} \\
+ \; & (\mu + 2\lambda)E\int_0^T e(s)\|U(s) - \tilde{U}_{n,\Phi_n}(s)\|^2 ds.
\end{aligned}
$$

Using Theorem 2.3.2 and Lemma 1.2.6 we have

$$
\lim_{n'\to\infty} E\|\tilde{U}_{n,\Phi_n}(\mathcal{T}_M) - U_{n,\Phi_n}(\mathcal{T}_M)\|^2 = 0, \qquad \lim_{n'\to\infty} E\int_0^{\mathcal{T}_M} \|\tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\|^2 ds = 0.
$$

By Proposition B.3, applied for $\mathcal{T} := T$, $Q_n(\mathcal{T}) := \|\tilde{U}_{n,\Phi_n}(\mathcal{T}) - U_{n,\Phi_n}(\mathcal{T})\|^2$, respectively $Q_n(\mathcal{T}) := \int_0^{\mathcal{T}} \|\tilde{U}_{n',\Phi_{n'}}(s) - U_{n,\Phi_n}(s)\|_V^2 ds$, we get

$$
\lim_{n\to\infty} E\int_0^T \|\tilde{U}_{n,\Phi_n}(s) - U_{n,\Phi_n}(s)\|_V^2 ds = 0 \quad \text{and} \quad \lim_{n\to\infty} E\|\tilde{U}_{n,\Phi_n}(T) - U_{n,\Phi_n}(T)\|^2 = 0.
$$

Now we use Theorem 2.3.2 to obtain the conclusion of this theorem. ■

The **main result** of this section is the following theorem, in which we prove that there exists at least one optimal feedback control for problem $(\mathcal{P})$.

Theorem 2.3.4
Assume for all $t \in [0,T], x \in H$ that the mappings $\mathcal{L}(t,x,\cdot), \mathcal{K}(\cdot)$ are weakly lower semicontinuous. Then there exists an optimal feedback control for problem $(\mathcal{P})$.

PROOF. Let (Φ_n) be a minimizing sequence for problem $(\mathcal{P})$. We apply Lemma 2.3.1 and Theorem 2.3.3 for this sequence. Therefore there exists a subsequence (n') of (n) and $\Phi \in \mathcal{U}$ such that for all $t \in [0,T]$, $x, y \in H$ and a.e. $\omega \in \Omega$ it holds

$$\lim_{n' \to \infty} (\Phi_{n'}(t, U_{\Phi_{n'}}(t)), y) = (\Phi(t, U_\Phi(t)), y).$$

From $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, and Theorem 2.3.3 we have

$$\begin{aligned} E\int_0^T \mathcal{L}[t, U_\Phi(t), \Phi(t, U_\Phi(t))]dt &\leq \liminf_{n' \to \infty} E\int_0^T \mathcal{L}[t, U_\Phi(t), \Phi_{n'}(t, U_{\Phi_{n'}}(t))]dt \\ &\leq \liminf_{n' \to \infty} \left(E\int_0^T \mathcal{L}[t, U_{\Phi_{n'}}(t), \Phi_{n'}(t, U_{\Phi_{n'}}(t))]dt + c_{\mathcal{L}} E\int_0^T \|U_\Phi(t) - U_{\Phi_{n'}}(t)\|_V^2 dt \right) \\ &\leq \liminf_{n' \to \infty} E\int_0^T \mathcal{L}[t, U_{\Phi_{n'}}(t), \Phi_{n'}(t, U_{\Phi_{n'}}(t))]dt \end{aligned}$$

and

$$E\mathcal{K}[U_\Phi(T)] \leq \liminf_{n' \to \infty} E\mathcal{K}[U_{\Phi_{n'}}(T)].$$

Consequently,

$$\mathcal{J}(\Phi) \leq \liminf_{n' \to \infty} \mathcal{J}(\Phi_{n'}).$$

But (Φ_n) is a minimizing sequence for problem $(\mathcal{P})$. Hence

$$\mathcal{J}(\Phi) = \min_{\Psi \in \mathcal{U}} \mathcal{J}(\Psi)$$

and therefore $\Phi \in \mathcal{U}$ is an optimal feedback control for problem $(\mathcal{P})$. ■

Remark 2.3.5

We can not use this method in the case of problem $(\mathcal{P}^b)$, because the minimizing sequence (Φ_n) then belongs to the space $\mathcal{L}_H^2(\Omega \times [0,T])$ and we can not find (as in Lemma 2.3.1) a subsequence (n') of (n) *independent of* ω, t such that $(\Phi_{n'}(\omega, t))$ would converge in H to a process $\Phi \in \mathcal{L}_H^2(\Omega \times [0,T])$ for $P \times \Lambda$ a.e. $(\omega, t) \in \Omega \times [0,T]$. The *independence with respect to* ω is essential in the proof of Theorem 2.3.2.

2.4 Existence of ε-optimal feedback controls

For $(\mathcal{P})$ we formulate the corresponding n-dimensional control problem

$$(\mathcal{P}_n) \qquad \begin{cases} \mathcal{J}_n(\Phi_n) \to \min \\ \Phi_n \in \mathcal{U}_n \end{cases}$$

where

$$\mathcal{J}_n(\Phi_n) = E\int_0^T \mathcal{L}[s, U_{n,\Phi_n}(s), \Phi_n(s, U_{n,\Phi_n}(s))]ds + E\mathcal{K}[U_{n,\Phi_n}(T)]$$

and

$$\mathcal{U}_n := \Big\{\Phi_n : [0,T] \times H_n \to H_n | \Phi_n = \Pi_n\Phi, \Phi \in \mathcal{U}\Big\}.$$

Here U_{n,Φ_n} is the solution of (P_n) using the feedback control $\Phi_n \in \mathcal{U}_n$.

Analogously we define the n-dimensional control problem corresponding to $(\mathcal{P}^b)$. We denote this problem by $(\mathcal{P}^b_n)$.

Theorem 2.4.1
Let (Φ_n) be a sequence in $\mathcal{U}$ such that $\Phi_n \in \mathcal{U}_n$ for each $n \in \mathbb{N}$. There exists a subsequence (n') of (n) such that

$$\lim_{n'\to\infty} E\int_0^T \|U_{\Phi_{n'}}(s) - U_{n',\Phi_{n'}}(s)\|_V^2 ds = 0 \quad \textit{and} \quad \lim_{n'\to\infty} E\|U_{\Phi_{n'}}(T) - U_{n',\Phi_{n'}}(T)\|^2 = 0,$$

where $U_{\Phi_{n'}}$ and $U_{n',\Phi_{n'}}$ are the solutions of (2.1), respectively $(P_{n'})$, using the feedback control $\Phi_{n'}$.

Proof. First we apply Lemma 2.3.1 on the sequence (Φ_n). Consequently, there exist a subsequence (n') of (n) and $\Phi \in \mathcal{U}$ such that for all $t \in [0,T]$, $x, y \in H$

$$\lim_{n'\to\infty} (\Phi_{n'}(t,x), y) = (\Phi(t,x), y).$$

By Theorem 2.3.3 it follows that

$$\lim_{n'\to\infty} E\int_0^T \|U_{\Phi_{n'}}(s) - U_\Phi(s)\|_V^2 ds = 0, \quad \lim_{n'\to\infty} E\|U_{\Phi_{n'}}(T) - U_\Phi(T)\|^2 = 0 \tag{2.26}$$

and

$$\lim_{n'\to\infty} E\int_0^T \|U_{n',\Phi_{n'}}(s) - U_\Phi(s)\|_V^2 ds = 0, \quad \lim_{n'\to\infty} E\|U_{n',\Phi_{n'}}(T) - U_\Phi(T)\|^2 = 0. \tag{2.27}$$

We see that

$$E\int_0^T \|U_{\Phi_{n'}}(s) - U_{n',\Phi_{n'}}(s)\|_V^2 ds \le 2E\int_0^T \|U_{\Phi_{n'}}(s) - U_\Phi(s)\|_V^2 ds + 2E\int_0^T \|U_\Phi(s) - U_{n',\Phi_{n'}}(s)\|_V^2 ds.$$

By using (2.26) and (2.27) we have

$$\lim_{n'\to\infty} E\int_0^T \|U_{\Phi_{n'}}(s) - U_{n',\Phi_{n'}}(s)\|_V^2 ds = 0.$$

Analogously we deduce

$$\lim_{n'\to\infty} E\|U_{\Phi_{n'}}(T) - U_{n',\Phi_{n'}}(T)\|^2 = 0. \qquad \blacksquare$$

Theorem 2.4.2
Assume that for sufficiently large n there exists optimal controls for problem $(\mathcal{P}_n)$. If $\Phi^* \in \mathcal{U}$ is an optimal control for problem $(\mathcal{P})$ and $\varepsilon > 0$ is arbitrary fixed, then there exists $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$ $\Phi^*_n \in \mathcal{U}_n$ is an optimal control for the n-dimensional control problem $(\mathcal{P}_n)$ and

$$|\mathcal{J}_n(\Phi^*_n) - \mathcal{J}(\Phi^*)| < \varepsilon, \quad \mathcal{J}(\Phi^*_n) - \mathcal{J}(\Phi^*) < \varepsilon,$$

hence ${\Phi^*}_n$ is an ε-optimal control for problem $(\mathcal{P})$.

Proof. Let $\varepsilon > 0$ and take

$$\varepsilon^* := \frac{\varepsilon}{2(c_\mathcal{L} + c_\mathcal{K})},$$

where $c_\mathcal{L}, c_\mathcal{K}$ are that constants that occure in $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ from Section 2.1.

For each $m \in \mathbb{N}$ let $\tilde{\Phi}_m := \Pi_m \Phi^*$. From Theorem 1.2.7 and the properties (1.5) (from Section 1.1) it follows that there exists an $m_\varepsilon > 0$ such that for all $m \geq m_\varepsilon$ it holds

$$E\int_0^T \|U_{m,\Phi^*}(s) - U_{\Phi^*}(s)\|^2 ds + E\|U_{m,\Phi^*}(T) - U_{\Phi^*}(T)\|^2 < \varepsilon^*$$

and

$$E\int_0^T \|\tilde{\Phi}_m(s, U_{m,\Phi^*}(s)) - \Phi^*(s, U_{\Phi^*}(s))\|^2 ds < \varepsilon^*.$$

Let $n \geq m_\varepsilon$ and let Φ^*_n be an optimal control for the n-dimensional control problem $(\mathcal{P}_n)$. By Theorem 2.4.1, applied on (Φ^*_n), there exists a subsequence (n') of (n) and $n'_\varepsilon \geq m_\varepsilon$ such that for all $n' \geq n'_\varepsilon$ we have

$$E\int_0^T \|U_{\Phi^*_{n'}}(s) - U_{n',\Phi^*_{n'}}(s)\|^2 ds + E\|U_{\Phi^*_{n'}}(T) - U_{n',\Phi^*_{n'}}(T)\|^2 < \varepsilon^*. \tag{2.28}$$

First case: $\mathcal{J}_{n'}(\Phi^*_{n'}) - \mathcal{J}(\Phi^*) \geq 0$. Then by using the properties of $\mathcal{L}$ and $\mathcal{K}$ (given in Section 2.1), we have for all $n \geq n'_\varepsilon$

$$\begin{aligned} 0 &\leq \mathcal{J}_{n'}(\Phi^*_{n'}) - \mathcal{J}(\Phi^*) \leq \mathcal{J}_{n'}(\tilde{\Phi}_{n'}) - \mathcal{J}(\Phi^*) \\ &\leq c_\mathcal{L}\Big(E\int_0^T \|U_{n',\Phi^*}(s) - U_{\Phi^*}(s)\|^2 ds + E\int_0^T \|\tilde{\Phi}_{n'}(s, U_{n',\Phi^*}(s)) - \Phi^*(s, U_{\Phi^*}(s))\|^2 ds\Big) \\ &+ c_\mathcal{K} E\|U_{n',\Phi^*}(T) - U_{\Phi^*}(T)\|^2 < (c_\mathcal{L} + c_\mathcal{K})\varepsilon^* < \frac{\varepsilon}{2}. \end{aligned}$$

Second case: $\mathcal{J}_{n'}(\Phi^*_{n'}) - \mathcal{J}(\Phi^*) < 0$. We write

$$\begin{aligned} 0 &< \mathcal{J}(\Phi^*) - \mathcal{J}_{n'}(\Phi^*_{n'}) \leq \mathcal{J}(\Phi^*_{n'}) - \mathcal{J}_{n'}(\Phi^*_{n'}) \\ &\leq c_{\mathcal{L}} E \int_0^T \|U_{n',\Phi^*_{n'}}(s) - U_{\Phi^*_{n'}}(s)\|^2 ds + c_{\mathcal{K}} E \|U_{n',\Phi^*_{n'}}(T) - U_{\Phi^*_{n'}}(T)\|^2 \\ &< (c_{\mathcal{L}} + c_{\mathcal{K}})\varepsilon^* < \frac{\varepsilon}{2}. \end{aligned}$$

Hence for all $n' \geq n'_\varepsilon$ we have for all $n' \geq n'_\varepsilon$

$$|\mathcal{J}_{n'}(\Phi^*_{n'}) - \mathcal{J}(\Phi^*)| < \frac{\varepsilon}{2} < \varepsilon.$$

Using this inequality and (2.28) we get

$$\begin{aligned} 0 &\leq \mathcal{J}(\Phi^*_{n'}) - \mathcal{J}(\Phi^*) \leq |\mathcal{J}(\Phi^*_{n'}) - \mathcal{J}_n(\Phi^*_{n'})| + |\mathcal{J}_{n'}(\Phi^*_{n'}) - \mathcal{J}(\Phi^*)| \\ &\leq c_{\mathcal{L}} E \int_0^T \|U_{\Phi^*_{n'}}(s) - U_{n',\Phi^*_{n'}}(s)\|^2 ds + c_{\mathcal{K}} E \|U_{\Phi^*_{n'}}(T) - U_{n,\Phi^*_{n'}}(T)\|^2 + \frac{\varepsilon}{2} \\ &< (c_{\mathcal{L}} + c_{\mathcal{K}})\varepsilon^* + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

The sequence (n'), $n'_\varepsilon \in \mathbb{N}$ and ${\Phi^*}_{n'}$ obtained above satisfy the conclusion of the theorem. ■

2.5 A special property

We will prove a special property for the solution U_Φ of the Navier-Stokes equation (2.1) (respectively (2.3) in the case of bounded controls):

$$E \exp\Big\{\beta \int_0^T \|U_\Phi(s)\|_V^2 ds\Big\} < K < \infty \tag{2.29}$$

where $\beta > 0$ satisfies certain conditions and K is a positive constant independent of Φ. We need this property because of the special structure of the Navier-Stokes equation. If (2.29) is not satisfied, we have to consider in the cost functional $\mathcal{J}$ together with the expression of the state U_Φ the expression Δ_{U_Φ} too (as a discount factor). The computations are in this case more complicated. If $\beta := \dfrac{b}{\nu}$ then (2.29) becomes

$$E\Delta^{-1}_{U_\Phi}(T) < K < \infty.$$

We want to show that there exist situations for which (2.29) is satisfied. We formulate some conditions which assure that (2.29) holds. Of course, these conditions are not the only possible ones.

For the stochastic Navier-Stokes equation we assume that the supplementary assumptions hold **(v')** $\mathcal{C}$ satisfies assumption **(v)** from Section 1.1 and

$$\gamma := \sup_{\substack{x\in H\\ t\in[0,T]}} \|\mathcal{C}(t,x)\|^2 < \infty;$$

(vii') $x_0 \in H$ (it does not depend on ω).

Let $\Phi \in \mathcal{U}$ or $\Phi \in \mathcal{U}^b$ (we recall the definition of $\mathcal{U}$ and $\mathcal{U}^b$ from Section 2.1). We make the convention: if $\Phi \in \mathcal{U}$ then we take $\rho := 0$ and if $\Phi \in \mathcal{U}^b$ then we take $\mu := 0$.

We consider the conditions:

(C$_1$) $\dfrac{(\nu - \sqrt{\mu}c_{HV})^2}{2\gamma c_{HV}} > \beta;$

(C$_2$) $1 - 2\sqrt{\mu}T > 0, \quad \dfrac{2\nu(\nu - \sqrt{\mu}c_{HV})}{\gamma c_{HV}} > \beta, \quad \dfrac{\nu(1-2\sqrt{\mu}\gamma T)^2}{4\gamma T} > \beta;$

(C$_3$) $\dfrac{\nu^2 e^{-2\sqrt{\mu}T}}{\gamma c_{HV}} > \beta.$

A possible interpretation for this conditions in the case $\beta := \frac{4b}{\nu}$ (which will be used in the following sections) is given at the end of this section in Remark 2.5.2.

Theorem 2.5.1
Assume that hypotheses **(i)-(iv)**, **(v')**, **(vii')** *are fulfilled and* $\Phi \in \mathcal{U}$ *or* $\Phi \in \mathcal{U}^b$. *If one of the conditions* (**C$_1$**), (**C$_2$**) *or* (**C$_3$**) *holds, then there exists a positive constant* K *independent of* Φ *such that inequality* (2.29) *is satisfied.*

Proof. Applying the Ito formula for $U := U_\Phi$ and using the properties of $\mathcal{A}, \mathcal{B}$ and Φ we obtain

$$\begin{aligned}\|U(t)\|^2 &+ 2\nu\int_0^t \|U(s)\|_V^2 ds \le \|x_0\|^2 + \frac{\rho^2}{2\varepsilon} + 2(\sqrt{\mu}+\varepsilon)\int_0^t \|U(s)\|^2 ds \\ &+ 2\int_0^t (\mathcal{C}(s,U(s)),U(s))dw(s) + \int_0^t \|\mathcal{C}(s,U(s))\|^2 ds\end{aligned} \tag{2.30}$$

with $\varepsilon > 0$.

We assume that (**C$_1$**) is fulfilled: There exists a sufficiently small $\varepsilon > 0$ such that

$$\frac{(\nu - \sqrt{\mu}c_{HV} - \varepsilon c_{HV})^2}{2\gamma c_{HV}} \ge \beta. \tag{2.31}$$

We find an $\eta > 0$ such that

$$2\eta(\nu - \sqrt{\mu}c_{HV} - \varepsilon c_{HV} - \gamma c_{HV}\eta) = \beta. \tag{2.32}$$

By (2.30) and assumption **(v')** we get

$$\eta\|U(t)\|^2 + 2\eta\nu\int_0^t \|U(s)\|_V^2 ds \leq \eta\|x_0\|^2 + \eta\gamma T + \frac{\eta\rho^2}{2\varepsilon} + 2(\sqrt{\mu}+\varepsilon)\eta\int_0^t \|U(s)\|^2 ds \tag{2.33}$$

$$+ \quad 2\eta^2\int_0^t (\mathcal{C}(s,U(s)),U(s))^2 ds + 2\eta\int_0^t (\mathcal{C}(s,U(s)),U(s))dw(s) - 2\eta^2\int_0^t (\mathcal{C}(s,U(s)),U(s))^2 ds,$$

which implies that

$$\begin{aligned}\eta\|U(t)\|^2 \quad &+ \quad 2\eta(\nu-\sqrt{\mu}c_{HV}-\varepsilon c_{HV}-\gamma\eta c_{HV})\int_0^t \|U(s)\|_V^2 ds \leq \eta\|x_0\|^2 + \eta\gamma T + \frac{\eta\rho^2}{2\varepsilon}\\ &+ \quad 2\eta\int_0^t (\mathcal{C}(s,U(s)),U(s))dw(s) - 2\eta^2\int_0^t (\mathcal{C}(s,U(s)),U(s))^2 ds.\end{aligned}$$

Hence

$$\begin{aligned}&E\exp\Big\{2\eta(\nu-\sqrt{\mu}c_{HV}-\varepsilon c_{HV}-\gamma\eta c_{HV})\int_0^t \|U(s)\|_V^2 ds\Big\}\\ \leq\ &\exp\{\eta\|x_0\|^2+\eta\gamma T+\frac{\eta\rho^2}{2\varepsilon}\}\\ \times\ &E\exp\Big\{2\eta\int_0^t(\mathcal{C}(s,U(s)),U(s))dw(s) - 2\eta^2\int_0^t (\mathcal{C}(s,U(s)),U(s))^2 ds\Big\}.\end{aligned}$$

By Levi's inequality (see [12], p. 331) it then follows that

$$E\exp\Big\{2\eta(\nu-\sqrt{\mu}c_{HV}-\varepsilon c_{HV}-\gamma\eta c_{HV})\int_0^t \|U(s)\|_V^2 ds\Big\} \leq \exp\Big\{\eta\|x_0\|^2+\eta\gamma T+\frac{\eta\rho^2}{2\varepsilon}\Big\}.$$

By using $(\mathbf{C}_1)$, (2.31) and (2.32), we can find a positive constant K independent of Φ such that

$$E\exp\Big\{\beta\int_0^t \|U(s)\|_V^2 ds\Big\} \leq \exp\Big\{\eta\|x_0\|^2+\eta\gamma T+\frac{\eta\rho^2}{2\varepsilon}\Big\} < K < \infty.$$

Now we assume that $(\mathbf{C}_2)$ is fulfilled: There exists a sufficiently small $\varepsilon > 0$ such that

$$1-2\sqrt{\mu}T-2\varepsilon T > 0, \qquad \frac{\nu(1-2\sqrt{\mu}T-2\varepsilon T)^2}{4\gamma T} \geq \beta. \tag{2.34}$$

By the Ito formula and the property of $\mathcal{B}$ we have

$$\exp\{\eta\|U(t)\|^2\} + 2\eta\int_0^t \langle \mathcal{A}U(s),U(s)\rangle \exp\{\eta\|U(s)\|^2\}ds \leq \exp\{\eta\|x_0\|^2\}$$

$$+ \quad \eta\int_0^t \Big(2(\Phi(s,U(s)),U(s)) + \|\mathcal{C}(s,U(s))\|^2\Big)\exp\{\eta\|U(s)\|^2\}ds$$

$$+ \quad 2\eta\int_0^t (\mathcal{C}(s,U(s)),U(s))\exp\{\eta\|U(s)\|^2\}dw(s) + 2\eta^2\int_0^t (\mathcal{C}(s,U(s)),U(s))^2\exp\{\eta\|U(s)\|^2\}ds,$$

with $\eta > 0$. Now we use the properties of $\mathcal{A},\Phi,\mathcal{C}$ to obtain

$$\exp\{\eta\|U(t)\|^2\} + 2\eta\Big(\frac{\nu}{c_{HV}} - \sqrt{\mu} - \varepsilon - \eta\gamma\Big)\int_0^t \|U(s)\|^2\exp\{\eta\|U(s)\|^2\}ds \le \exp\{\eta\|x_0\|^2\}$$

$$+ \quad \eta\Big(\gamma + \frac{\rho^2}{2\varepsilon}\Big)\int_0^t \exp\{\eta\|U(s)\|^2\}ds + 2\eta\int_0^t (\mathcal{C}(s,U(s)),U(s))\exp\{\eta\|U(s)\|^2\}dw(s)$$

for all $t \in [0,T]$. We chose η such that $\nu - \sqrt{\mu}c_{HV} - \varepsilon c_{HV} - \eta\gamma c_{HV} > 0$ (see (2.38)). Then by Proposition B.2 (applied for real valued processes) we get

$$E\exp\{\eta\|U(t)\|^2\} \le c\exp\{\eta\|x_0\|^2\} \quad \text{for all } t\in[0,T], \tag{2.35}$$

where $c > 0$ is a constant depending on $\eta,\gamma,T,\rho,\varepsilon$. Taking into account the Hölder and the Levi inequality in (2.33), we have

$$\begin{aligned} E\exp\Big\{2\eta\nu\int_0^T \|U(s)\|_V^2 ds\Big\} &\le \exp\Big\{\eta\|x_0\|^2 + \eta\gamma T + \frac{\eta\rho^2}{2\varepsilon}\Big\} \\ &\times \Bigg(E\exp\Big\{2\eta p\int_0^T (\mathcal{C}(s,U(s)),U(s))dw(s) - 2\eta^2p^2\int_0^T (\mathcal{C}(s,U(s)),U(s))^2 ds\Big\}\Bigg)^{\frac{1}{p}} \\ &\times \Bigg(E\exp\Big\{2\eta q(\sqrt{\mu}+\varepsilon+\eta\gamma p)\int_0^T \|U(s)\|^2 ds\Big\}\Bigg)^{\frac{1}{q}} \\ &\le \exp\Big\{\eta\|x_0\|^2 + \eta\gamma T + \frac{\eta\rho^2}{2\varepsilon}\Big\}\Bigg(E\exp\Big\{2\eta q(\sqrt{\mu}+\varepsilon+\eta\gamma p)\int_0^T \|U(s)\|^2 ds\Big\}\Bigg)^{\frac{1}{q}} \end{aligned} \tag{2.36}$$

where $p,q > 1, \dfrac{1}{p} + \dfrac{1}{q} = 1$ (the exact value of p is given in (2.38)). By Jensen's inequality we obtain

$$E\exp\Big\{2\eta q(\sqrt{\mu}+\varepsilon+\eta\gamma p)\int_0^T \|U(s)\|^2 ds\Big\} \le \frac{1}{T}E\int_0^T \exp\Big\{2\eta q T(\sqrt{\mu}+\varepsilon+\eta\gamma p)\|U(s)\|^2\Big\}ds. \tag{2.37}$$

We set

$$\eta := \frac{\beta}{2\nu} \quad \text{and} \quad p = \frac{2}{1 - 2\sqrt{\mu}T - 2\varepsilon T}\;. \tag{2.38}$$

Using (2.34) and (2.38) we write the condition for β as follows:

$$\beta \le \nu \frac{(1 - 2\sqrt{\mu}T - 2\varepsilon T)p - 1}{\gamma T p^2}.$$

This implies

$$2qT(\sqrt{\mu} + \varepsilon + \eta\gamma p) \le 1.$$

By applying these estimates in (2.35), (2.36), and (2.37) we get

$$E \exp\Big\{2\eta\nu \int_0^T \|U(s)\|_V^2 ds\Big\} \le T^{-\frac{1}{q}} \exp\Big\{\eta\|x_0\|^2 + \eta\gamma T + \frac{\eta\rho^2}{2\varepsilon}\Big\}\Big(\int_0^T E\exp\Big\{\eta\|U(s)\|^2\Big\}ds\Big)^{\frac{1}{q}} < \infty.$$

Hence there exists a positive constant K independent of Φ such that (2.29) holds.

We assume that $(\mathbf{C}_3)$ is fulfilled: There exists a sufficiently small $\varepsilon > 0$ such that

$$\frac{\nu^2 e^{-2(\sqrt{\mu}+\varepsilon)T}}{2\gamma c_{HV}} \ge \beta \tag{2.39}$$

and $\eta > 0$ such that

$$2\eta e^{-2(\sqrt{\mu}+\varepsilon)T}(\nu - \gamma c_{HV}\eta) = \beta. \tag{2.40}$$

By the Ito formula and the properties of $\mathcal{A}, \mathcal{B}, \Phi$ we obtain

$$\begin{aligned} e^{-2(\sqrt{\mu}+\varepsilon)t}\|U(t)\|^2 \quad &+ \quad 2\nu\int_0^t e^{-2(\sqrt{\mu}+\varepsilon)s}\|U(s)\|_V^2 ds \le \|x_0\|^2 + \frac{\rho^2}{2\varepsilon} \\ &+ \quad 2\int_0^t e^{-2(\sqrt{\mu}+\varepsilon)s}(\mathcal{C}(s,U(s)),U(s))dw(s) + \int_0^t e^{-2(\sqrt{\mu}+\varepsilon)s}\|\mathcal{C}(s,U(s))\|^2 ds. \end{aligned}$$

For an arbitrary fixed $\eta > 0$ we write

$$\begin{aligned} & 2\eta(\nu - \gamma\eta c_{HV})\int_0^t e^{-2(\sqrt{\mu}+\varepsilon)s}\|U(s)\|_V^2 ds \le \eta\|x_0\|^2 + \eta\gamma T + \frac{\eta\rho^2}{2\varepsilon} \\ + \quad & 2\eta\int_0^t e^{-2(\sqrt{\mu}+\varepsilon)s}(\mathcal{C}(s,U(s)),U(s))dw(s) - 2\eta^2\int_0^t e^{-4(\sqrt{\mu}+\varepsilon)s}(\mathcal{C}(s,U(s)),U(s))^2 ds. \end{aligned}$$

Hence

$$\begin{aligned} & E\exp\Big\{2\eta e^{-2(\sqrt{\mu}+\varepsilon)T}(\nu - \gamma\eta c_{HV})\int_0^T \|U(s)\|_V^2 ds\Big\} \le \exp\Big\{\eta\|x_0\|^2 + \eta\gamma T + \frac{\eta\rho^2}{2\varepsilon}\Big\} \\ \times \quad & E\exp\Big\{2\eta\int_0^T e^{-2(\sqrt{\mu}+\varepsilon)s}(\mathcal{C}(s,U(s)),U(s))dw(s) - 2\eta^2\int_0^T e^{-4(\sqrt{\mu}+\varepsilon)s}(\mathcal{C}(s,U(s)),U(s))^2 ds\Big\}. \end{aligned}$$

Using Levi's inequality we have

$$E\exp\Big\{2\eta e^{-2(\sqrt{\mu}+\varepsilon)T}(\nu-\gamma\eta c_{HV})\int\limits_0^t \|U(s)\|_V^2 ds\Big\} \le \exp\Big\{\eta\|x_0\|^2+\eta\gamma T+\frac{\eta\rho^2}{2\varepsilon}\Big\}.$$

By (2.40) it follows that there exists a positive constant K independent of Φ such that

$$E\exp\Big\{\beta\int\limits_0^t \|U(s)\|_V^2 ds\Big\} \le \exp\Big\{\eta\|x_0\|^2+\eta\gamma T+\eta\frac{\rho^2}{2\varepsilon}\Big\} < K < \infty. \qquad \blacksquare$$

Remark 2.5.2

1) In the following sections of this chapter we need the condition

$$E\Delta_{U_\Phi}^{-4}(T) = E\exp\Big\{4\frac{b}{\nu}\int\limits_0^T \|U_\Phi(s)\|_V^2 ds\Big\} < K < \infty. \tag{2.41}$$

By taking $\beta := \frac{4b}{\nu}$ the conditions mentioned at the beginning of this section become

(C$_1$) $\dfrac{\nu(\nu-\sqrt{\mu}c_{HV})^2}{8\gamma c_{HV}} > b$;

(C$_2$) $1-2\sqrt{\mu}T > 0, \quad \dfrac{2\nu^2(\nu-\sqrt{\mu}c_{HV})}{4\gamma c_{HV}} > b, \quad \dfrac{\nu^2(1-2\sqrt{\mu}T)^2}{16\gamma T} > b$;

(C$_3$) $\dfrac{\nu^3 e^{-2\sqrt{\mu}T}}{4\gamma c_{HV}} > b$.

If one of these conditions is fulfilled then (2.41) holds.

2) If $\Phi \in \mathcal{U}^b$, then by the convention from the beginning of this section we have $\mu := 0$ and the three conditions from above can be written as follows:

(C$_1^b$) $\dfrac{\nu^3}{8\gamma c_{HV}} > b$;

(C$_2^b$) $\dfrac{\nu^3}{2\gamma c_{HV}} > b, \quad \dfrac{\nu^2}{16\gamma T} > b$;

(C$_3^b$) $\dfrac{\nu^3}{4\gamma c_{HV}} > b$.

If one of these conditions is fulfilled then (2.41) holds.

3) These conditions seem to be very complicate, but they can be interpreted as follows: if ν, involving the viscosity, is large (we have a "very viscous fluid") then we can act with large external forces (μ can be chosen large) and we can have a "strong" influence of the Brownian motion (γ can be chosen large). The inequality $1-2\sqrt{\mu}T > 0$ is satisfied if we have large external forces and a small interval $[0,T]$ or conversely, a large interval and small external forces.

2.6 The Gateaux derivative of the cost functional

For the mappings that occur in the expression of the cost functional $\mathcal{J}$ (see (2.2)) we will assume further some supplementary conditions:

$(\mathbf{H_3})$ the mappings $\mathcal{L}(t,\cdot,\cdot),\mathcal{K}(\cdot)$ are Fréchet differentiable for each fixed $t\in[0,T]$;

$(\mathbf{H_4})$ the mappings $\mathcal{L}_x(t,\cdot,\cdot),\ \mathcal{L}_y(t,\cdot,\cdot),\mathcal{K}'(\cdot)$ are Lipschitz continuous and

$$\|\mathcal{L}_x(t,x,y)\|+\|\mathcal{L}_y(t,x,y)\|\le k_{\mathcal{L}}(1+\|x\|+\|y\|)\qquad\text{and}\qquad\|\mathcal{K}'(x)\|\le k_{\mathcal{K}}(1+\|x\|)$$

for all $t\in[0,T], x,y\in H$, where $k_{\mathcal{L}},k_{\mathcal{K}}$ are positive constants;

$(\mathbf{H_5})$ $\mathcal{L}_x(\cdot,x,y),\mathcal{L}_y(\cdot,x,y)\in\mathcal{L}^2_H[0,T]$ for all $x,y\in H$.

For the stochastic Navier-Stokes equation we assume that the supplementary condition holds:

(v") $\mathcal{C}$ satisfies assumption **(v)** from Section 1.1 and for each $t\in[0,T]$ the mapping $\mathcal{C}(t,\cdot)$ is Fréchet differentiable and $\mathcal{C}'(t,x)\in\mathcal{L}^2(H,H)$, the Fréchet derivative of $\mathcal{C}(t,\cdot)$ at the point x, satisfies

$$\|\mathcal{C}'(t,x)(y)\|\le k_{\mathcal{C}'}\|y\|\quad\text{for all } t\in[0,T],x,y\in H$$

where $k_{\mathcal{C}'}$ is a positive constant independent of t and x.

Using the properties of $\mathcal{B}$ it can be proved that the mapping $x\in V\mapsto\mathcal{B}(x,x)\in V^*$ is Fréchet differentiable and

$$\mathcal{B}'(x)(y)=\mathcal{B}(x,y)+\mathcal{B}(y,x)\quad\text{for all } x,y\in V.$$

We consider the case of **bounded controls**. Let $\Phi,\Upsilon\in\mathcal{U}^b$ such that for sufficiently small $\theta>0$ we have $\Phi+\theta\Upsilon\in\mathcal{U}^b$. We denote by

$$X_\theta:=\frac{U_{\Phi+\theta\Upsilon}-U_\Phi-\theta Z_\Upsilon}{\theta}. \tag{2.42}$$

Throughout this section we assume that β,ν,γ,ρ,T are chosen in such a way that

$$E\Delta^{-4}_{U_\Phi}(T)<K<\infty.$$

We recall here the results mentioned in Remark 2.5.2.

Let $\Upsilon\in\mathcal{L}^2_H(\Omega\times[0,T])$. We consider the stochastic evolution equation

$$\begin{aligned}(Z_\Upsilon(t),v)\ &+\ \int_0^t\langle\mathcal{A}Z_\Upsilon(s),v\rangle ds=\int_0^t\langle\mathcal{B}'(U_\Phi(s))(Z_\Upsilon(s)),v\rangle ds\\ &+\ \int_0^t(\Upsilon(s),v)ds+\int_0^t(\mathcal{C}'(s,U_\Phi(s))(Z_\Upsilon(s)),v)dw(s)\end{aligned} \tag{2.43}$$

for all $v\in V$, $t\in[0,T]$ and a.e. $\omega\in\Omega$.

Lemma 2.6.1
There exists a V-valued, $\mathcal{F} \times \mathcal{B}_{[0,T]}$-measurable process $\left(Z_\Upsilon(t)\right)_{t\in[0,T]}$ adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$, satisfying (2.43) and which has almost surely continuous trajectories in H. The solution is almost surely unique and there exists a constant $c > 0$ (independent of Υ) such that

$$E\Delta_{U_\Phi}(T)\|Z_\Upsilon(T)\|^2 + E\int_0^T \Delta_{U_\Phi}(t)\|Z_\Upsilon(t)\|_V^2 dt \leq cE\int_0^T \|\Upsilon(t)\|^2 dt$$

and

$$E\Delta^2_{U_\Phi}(T)\|Z_\Upsilon(T)\|^4 + E\Big(\int_0^T \Delta_{U_\Phi}(t)\|Z_\Upsilon(t)\|_V^2 dt\Big)^2 \leq cE\int_0^T \|\Upsilon(t)\|^4 dt.$$

Proof. We apply Theorem 1.3.1 on $X = Y := U_\Phi, a_0 := 0, \Psi := \Upsilon, \Gamma := 0, \mathcal{G}(s,h) := \mathcal{C}'(s, U_\Phi(s))(h)$, $Z_{\Psi,\Gamma} := Z_\Upsilon$. ■

Lemma 2.6.2

(i) *There exists a positive constant c independent of θ such that*

$$E\Delta^2_{U_\Phi}(T)\Big\|\frac{U_{\Phi+\theta\Upsilon}(T) - U_\Phi(T)}{\theta}\Big\|^4 + E\Big(\int_0^T \Delta_{U_\Phi}(s)\Big\|\frac{U_{\Phi+\theta\Upsilon}(s) - U_\Phi(s)}{\theta}\Big\|_V^2 ds\Big)^2 \leq cE\int_0^T \|\Upsilon(s)\|^4 ds$$

and

$$\lim_{\theta\searrow 0} \frac{1}{\theta^2} E\int_0^T \Delta^2_{U_\Phi}(s)\|U_{\Phi+\theta\Upsilon}(s) - U_\Phi(s)\|_V^2 \|U_{\Phi+\theta\Upsilon}(s) - U_\Phi(s)\|^2 ds = 0.$$

(ii) *The following convergences hold*

$$\lim_{\theta\searrow 0} E\|X_\theta(T)\|^2 = 0 \quad \textit{and} \quad \lim_{\theta\searrow 0} E\int_0^T \|X_\theta(s)\|_V^2 ds = 0,$$

where X_θ is defined in (2.42).

Proof. For all $t \in [0,T]$ and a.e. $\omega \in \Omega$ let

$$e_1(t) = \Delta_{U_\Phi}(t)\exp\{-(2\lambda+1)t\}.$$

(i) We use the Ito formula and the properties of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ to get

$$e_1(t)\|U_{\Phi+\theta\Upsilon}(t) - U_\Phi(t)\|^2 + \nu\int_0^t e_1(s)\|U_{\Phi+\theta\Upsilon}(s) - U_\Phi(s)\|_V^2 ds \tag{2.44}$$

$$\leq \theta^2\int_0^t e_1(s)\|\Upsilon(s)\|^2 ds + 2\int_0^t e_1(s)(\mathcal{C}(s, U_{\Phi+\theta\Upsilon}(s)) - \mathcal{C}(s, U_\Phi(s)), U_{\Phi+\theta\Upsilon}(s) - U_\Phi(s))dw(s)$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$. Using Proposition B.2 we obatin

$$E\Big\{\sup_{t\in[0,T]} \Delta^2_{U_\Phi}(t)\Big\|\frac{U_{\Phi+\theta\Upsilon}(t)-U_\Phi(t)}{\theta}\Big\|^4\Big\} \quad + \quad \nu^2 E\Big(\int_0^T \Delta_{U_\Phi}(s)\Big\|\frac{U_{\Phi+\theta\Upsilon}(s)-U_\Phi(s)}{\theta}\Big\|_V^2 ds\Big)^2 \leq cE\int_0^T \|\Upsilon(s)\|^4 ds$$

where c is a positive constant independent of θ. We write

$$\begin{aligned} & E\int_0^T \Delta^2_{U_\Phi}(s)\|U_{\Phi+\theta\Upsilon}(s)-U_\Phi(s)\|_V^2\|U_{\Phi+\theta\Upsilon}(s)-U_\Phi(s)\|^2 ds \\ \leq\ & \theta^4\Big[E\Big\{\sup_{t\in[0,T]} \Delta^2_{U_\Phi}(t)\Big\|\frac{U_{\Phi+\theta\Upsilon}(t)-U_\Phi(t)}{\theta}\Big\|^4\Big\}E\Big(\int_0^T \Delta_{U_\Phi}(s)\Big\|\frac{U_{\Phi+\theta\Upsilon}(s)-U_\Phi(s)}{\theta}\Big\|_V^2 ds\Big)^2\Big]^{\frac{1}{2}}. \end{aligned}$$

Consequently,

$$\lim_{\theta\searrow 0}\frac{1}{\theta^2}E\int_0^T \Delta^2_{U_\Phi}(s)\|U_{\Phi+\theta\Upsilon}(s)-U_\Phi(s)\|_V^2\|U_{\Phi+\theta\Upsilon}(s)-U_\Phi(s)\|^2 ds = 0.$$

(ii) By the Ito formula and the properties of $\mathcal{A}, \Phi, \Upsilon$ we get

$$\begin{aligned} & e_1(T)\|X_\theta(T)\|^2 + 2\nu\int_0^T e_1(s)\|X_\theta(s)\|_V^2 ds \\ \leq\ & 2\int_0^T e_1(s)\langle \mathcal{B}(X_\theta(s), U_\Phi(s)) + \frac{1}{\theta}\mathcal{B}(U_{\Phi+\theta\Upsilon}(s)-U_\Phi(s), U_{\Phi+\theta\Upsilon}(s)-U_\Phi(s)), X_\theta(s)\rangle ds \\ -\ & (2\lambda+1)\int_0^T e_1(s)\|X_\theta(s)\|^2 ds - \frac{b}{\nu}\int_0^T e_1(s)\|U_\Phi(s)\|^2\|X_\theta(s)\|^2 ds \\ +\ & 2\int_0^T \frac{e_1(s)}{\theta}\Big(\mathcal{C}(s, U_{\Phi+\theta\Upsilon}(s)) - \mathcal{C}(s, U_\Phi(s)) - \theta\mathcal{C}'(s, U_\Phi(s))(Z_\Upsilon(s)), X_\theta(s)\Big) dw(s) \\ +\ & \int_0^T \frac{e_1(s)}{\theta^2}\Big\|\mathcal{C}(s, U_{\Phi+\theta\Upsilon}(s)) - \mathcal{C}(s, U_\Phi(s)) - \theta\mathcal{C}'(s, U_\Phi(s))(Z_\Upsilon(s))\Big\|^2 ds. \end{aligned}$$

Using the properties of $\mathcal{B}, \Phi, \Upsilon, \mathcal{C}$, it follows that

$$Ee_1(T)\|X_\theta(T)\|^2 + \frac{3\nu}{4}E\int_0^T e_1(s)\|X_\theta(s)\|_V^2 ds$$

$$\leq \frac{4b}{\nu\theta^2} E\int_0^T e_1(s)\|U_{\Phi+\theta\Upsilon}(s) - U_\Phi(s)\|^2 \|U_{\Phi+\theta\Upsilon}(s) - U_\Phi(s))\|_V^2 ds$$

$$+ \frac{2}{\theta^2} E\int_0^T e_1(s)\|\mathcal{C}(s, U_\Phi(s) + \theta Z_\Upsilon(s)) - \mathcal{C}(s, U_\Phi(s)) - \theta\mathcal{C}'(s, U_\Phi(s))(Z_\Upsilon(s))\|^2 ds.$$

Applying **(i)**, the properties of $\mathcal{C}$ and the Lebesgue Theorem we conclude

$$\lim_{\theta\searrow 0} E\Delta_{U_\Phi}(T)\|X_\theta(T)\|^2 = 0, \quad \lim_{\theta\searrow 0} E\int_0^T \Delta_{U_\Phi}(s)\|X_\theta(s)\|_V^2 ds = 0. \tag{2.45}$$

We see that

$$E\Delta_{U_\Phi}^2(T)\|X_\theta(T)\|^4 + E\Big(\int_0^T \Delta_{U_\Phi}(s)\|X_\theta(s)\|_V^2 ds\Big)^2 \tag{2.46}$$

$$\leq 8\Bigg[E\Delta_{U_\Phi}^2(T)\Bigg\|\frac{U_{\Phi+\theta\Upsilon}(T) - U_\Phi(T)}{\theta}\Bigg\|^4 + E\Delta_{U_\Phi}^2(T)\|Z_\Upsilon(T)\|^4$$

$$+ E\Bigg(\int_0^T \Delta_{U_\Phi}(s)\Bigg\|\frac{U_{\Phi+\theta\Upsilon}(s) - U_\Phi(s)}{\theta}\Bigg\|_V^2 ds\Bigg)^2 + E\Big(\int_0^T \Delta_{U_\Phi}(s)\|Z_\Upsilon(s)\|_V^2 ds\Big)^2\Bigg].$$

By using the Schwarz inequality we obtain

$$E\|X_\theta(T)\|^2 \leq \Big(E\Delta_{U_\Phi}(T)\|X_\theta(T)\|^2\Big)^{\frac{1}{2}} \Big(E\Delta_{U_\Phi}^{-4}(T)\Big)^{\frac{1}{4}} \Big(E\Delta_{U_\Phi}^2(T)\|X_\theta(T)\|^4\Big)^{\frac{1}{4}}.$$

Taking into account (2.45), (2.46), **(i)**, Theorem 1.3.1 and the condition $E\Delta_{U_\Phi}^{-4}(T) < K < \infty$ it follows that

$$\lim_{\theta\searrow 0} E\|X_\theta(T)\|^2 = 0.$$

Analogously we can prove that

$$\lim_{\theta\searrow 0} E\int_0^T \|X_\theta(s)\|_V^2 ds = 0. \qquad \blacksquare$$

Remark 2.6.3

For the proof of **(i)** in Lemma 2.6.2 we do not need the condition $E\Delta_{U_\Phi}^{-4}(T) < \infty$.

Theorem 2.6.4
The cost functional $\mathcal{J}$ is Gateaux differentiable with

$$\frac{\mathrm{d}\mathcal{J}(\Phi+\theta\Upsilon)}{\mathrm{d}\theta}\Big|_{\theta=0} = E\int_0^T\Big(\mathcal{L}_x[t,U_\Phi(t),\Phi(t)],Z_\Upsilon(t)\Big)dt + E\int_0^T\Big(\mathcal{L}_y[t,U_\Phi(t),\Phi(t)],\Upsilon(t)\Big)dt + E\Big(\mathcal{K}'[U_\Phi(T)],Z_\Upsilon(T)\Big). \tag{2.47}$$

Proof. We have

$$\mathcal{K}(x)-\mathcal{K}(\tilde{x}) = \int_0^1\Big(\mathcal{K}'[\tilde{x}+r(x-\tilde{x}))],x-\tilde{x}\Big)dr \tag{2.48}$$

and

$$\mathcal{L}(t,x,y)-\mathcal{L}(t,\tilde{x},\tilde{y}) = \int_0^1\Big(\mathcal{L}_x[t,\tilde{x}+r(x-\tilde{x}),\tilde{y}+r(y-\tilde{y})],x-\tilde{x}\Big)dr + \int_0^1\Big(\mathcal{L}_y[t,\tilde{x}+r(x-\tilde{x}),\tilde{y}+r(y-\tilde{y})],y-\tilde{y}\Big)dr \tag{2.49}$$

for all $x,\tilde{x},y,\tilde{y}\in H,\ t\in[0,T]$. Equation (2.48) implies

$$\mathcal{K}[U_{\Phi+\theta\Upsilon}(T)]-\mathcal{K}[U_\Phi(T)] = \int_0^1\theta\Big(\mathcal{K}'[U_\Phi(T)+r\theta(X_\theta(T)+Z_\Upsilon(T))],X_\theta(T)+Z_\Upsilon(T)\Big)dr.$$

Using (2.49) we obtain

$$\begin{aligned}
&\mathcal{L}[t,U_{\Phi+\theta\Upsilon}(t),(\Phi+\theta\Upsilon)(t)]-\mathcal{L}[t,U_\Phi(t),\Phi(t)]\\
&= \int_0^1\Big\{\theta\Big(\mathcal{L}_x[t,U_\Phi(t)+r\theta(X_\theta(t)+Z_\Upsilon(t)),\Phi(t)+r\theta\Upsilon(t)],X_\theta(t)+Z_\Upsilon(t)\Big)\\
&+ \Big(\mathcal{L}_y[t,U_\Phi(t)+r\theta(X_\theta(t)+Z_\Upsilon(t)),\Phi(t)+r\theta\Upsilon(t)],\theta\Upsilon(t)\Big)dr\\
&= \int_0^1\theta\Big\{\Big(\mathcal{L}_x[t,U_\Phi(t)+r\theta(X_\theta(t)+Z_\Upsilon(t)),\Phi(t)+r\theta\Upsilon(t)],X_\theta(t)+Z_\Upsilon(t)\Big)\\
&+ \Big(\mathcal{L}_y[t,U_\Phi(t)+r\theta(X_\theta(t)+Z_\Upsilon(t)),\Phi(t)+r\theta\Upsilon(t)],\Upsilon(t)\Big)\Big\}dr,
\end{aligned}$$

for all $t\in[0,T]$. Using the properties of $\mathcal{L},\mathcal{K}$, and Lemma 2.6.2 it follows that relation (2.47) holds. ■

Remark 2.6.5

Now we consider the case of feedback controls. Let $\Phi, \Upsilon \in \mathcal{U}$ such that for sufficiently small $\theta > 0$ we have $\Phi + \theta\Upsilon \in \mathcal{U}$.

We assume that $\beta, \nu, \gamma, \rho, T$ are chosen in such a way that

$$E\Delta_{U_\Phi}^{-4}(T) < K < \infty.$$

We recall here the results mentioned in Remark 2.5.2.

Analogously to Theorem 2.6.4 it can be proved that the cost functional $\mathcal{J}$ is Gateaux differentiable with

$$\left.\frac{\mathrm{d}\mathcal{J}(\Phi+\theta\Upsilon)}{\mathrm{d}\theta}\right|_{\theta=0} = E\int_0^T \Big(\mathcal{L}_x[t, U_\Phi(t), \Phi(t, U_\Phi(t)], Z_\Upsilon(t)\Big)dt + \tag{2.50}$$

$$+E\int_0^T \Big(\mathcal{L}_y[t, U_\Phi(t), \Phi(t, U_\Phi(t)], \Upsilon(t, U_\Phi(t)) + \Phi(t, Z_\Upsilon(t))\Big)dt + E\Big(\mathcal{K}'[U_\Phi(T)], Z_\Upsilon(T)\Big),$$

where Z_Υ is the solution of the evolution equation

$$\begin{aligned}(Z_\Upsilon(t), v) + \int_0^t \langle \mathcal{A}Z_\Upsilon(s), v\rangle ds &= \int_0^t \langle \mathcal{B}'(U_\Phi)(Z_\Upsilon(s)), v\rangle ds + \int_0^t (\Upsilon(s, U_\Phi(s)) \\ &+ \Phi(s, Z_\Upsilon(s)), v)ds + \int_0^t (\mathcal{C}'(s, U_\Phi(s))(Z_\Upsilon(s)), v)dw(s)\end{aligned}$$

for all $v \in V$, $t \in [0,T]$ and a.e. $\omega \in \Omega$. To establish the existence and almost surely uniqueness of the solution of this equation we use the same methods as in Theorem 1.3.1.

2.7 A stochastic minimum principle

We will state a stochastic minimum principle in the case of problem $(\mathcal{P}^b)$. Let $\Phi^* \in \mathcal{U}^b$ be an optimal control with $E\Delta_{U_{\Phi^*}}^{-4}(T) < \infty$, $\Psi \in \mathcal{L}^2_{V^*}(\Omega\times[0,T]), \Gamma \in \mathcal{L}^2_H(\Omega\times[0,T])$ and let $Z_{\Psi,\Gamma}$ be the solution of

$$\begin{aligned}(Z_{\Psi,\Gamma}(t), v) &+ \int_0^t \langle \mathcal{A}Z_{\Psi,\Gamma}(s), v\rangle ds = \int_0^t \langle \mathcal{B}'(U_{\Phi^*}(s))(Z_{\Psi,\Gamma}(s)), v\rangle ds \\ &+ \int_0^t \langle \Psi(s), v\rangle ds + \int_0^t (\mathcal{C}'(s, U_{\Phi^*}(s))(Z_{\Psi,\Gamma}(s)), v)dw(s) + \int_0^t (\Gamma(s), v)dw(s)\end{aligned} \tag{2.51}$$

for all $v \in V$, $t \in [0,T]$ and a.e. $\omega \in \Omega$. This equation is $(P_{\Psi,\Gamma})$ from Section 1.3 applied for $X = Y := U_{\Phi^*}, a_0 := 0, \mathcal{G}(s,h) := \mathcal{C}'(s, U_{\Phi^*}(s))(h)$.

The mapping

$$
\begin{aligned}
(\Psi,\Gamma) &\in \mathcal{L}^2_{V^*}(\Omega\times[0,T])\times\mathcal{L}^2_H(\Omega\times[0,T])\mapsto \\
&\mapsto E\int_0^T\Big(\mathcal{L}_x[t,U_{\Phi^*}(t),\Phi^*(t)],Z_{\Psi,\Gamma}(t)\Big)dt+E\Big(\mathcal{K}'[U_{\Phi^*}(T)],Z_{\Psi,\Gamma}(T)\Big)\in\mathbb{R}
\end{aligned}
$$

is linear and continuous, because

$$(\Psi,\Gamma)\in\mathcal{L}^2_{V^*}(\Omega\times[0,T])\times\mathcal{L}^2_H(\Omega\times[0,T])\mapsto Z_{\Psi,\Gamma}\in\mathcal{L}^2_V(\Omega\times[0,T])$$

and

$$(\Psi,\Gamma)\in\mathcal{L}^2_{V^*}(\Omega\times[0,T])\times\mathcal{L}^2_H(\Omega\times[0,T])\mapsto Z_{\Psi,\Gamma}(T)\in\mathcal{L}^2_H(\Omega)$$

are linear. By using the properties for $\mathcal{L},\mathcal{K},U_{\Phi^*},\Phi^*$ and Theorem 1.3.1 we get

$$
\begin{aligned}
&\Big|E\int_0^T\Big(\mathcal{L}_x[t,U_{\Phi^*}(t),\Phi^*(t)],Z_{\Psi,\Gamma}(t)\Big)dt+E\Big(\mathcal{K}'[U_{\Phi^*}(T)],Z_{\Psi,\Gamma}(T)\Big)\Big|\le\Big(E\Delta^{-2}_{U_{\Phi^*}}(T)\Big)^{1/4}\\
\times\ &\Big\{\Big[E\Big(\int_0^T\|\mathcal{L}_x[t,U_{\Phi^*}(t),\Phi^*(t)]\|^2dt\Big)^2\Big]^{1/4}+\Big(E\|\mathcal{K}'[U_{\Phi^*}(T)]\|^4\Big)^{1/4}\Big\}\\
\times\ &\Big\{\Big(E\int_0^T\Delta_{U_{\Phi^*}}(t)\|Z_{\Psi,\Gamma}(t)\|^2dt\Big)^{1/2}+\Big(E\Delta_{U_{\Phi^*}}(T)\|Z_{\Psi,\Gamma}(T)\|^2\Big)^{1/2}\Big\}\\
\le\ &\tilde c\Big(E\Delta^{-2}_{U_{\Phi^*}}(T)\Big)^{1/4}\Big(E\int_0^T\|\Psi(t)\|^2_{V^*}dt+E\int_0^T\|\Gamma(t)\|^2dt\Big)^{1/2}
\end{aligned}
$$

where $\tilde c$ is a positive constant independent of Ψ and Γ. By the Riesz Theorem it follows that there exist in a unique way processes

$$p\in\mathcal{L}^2_V(\Omega\times[0,T]),\quad q\in\mathcal{L}^2_H(\Omega\times[0,T])$$

such that

$$
\begin{aligned}
E\int_0^T\langle\Psi(t),p(t)\rangle dt &+ E\int_0^T\Big(\Gamma(t),q(t)\Big)dt\\
&= E\int_0^T\Big(\mathcal{L}_x[t,U_{\Phi^*}(t),\Phi^*(t)],Z_{\Psi,\Gamma}(t)\Big)dt+E\Big(\mathcal{K}'[U_{\Phi^*}(T)],Z_{\Psi,\Gamma}(T)\Big)
\end{aligned}
\tag{2.52}
$$

for all $\Psi\in\mathcal{L}^2_{V^*}(\Omega\times[0,T]),\Gamma\in\mathcal{L}^2_H(\Omega\times[0,T])$.

Let

$$\mathcal{H}(t,v,\tilde v,x,y):=\mathcal{L}(t,x,y)+\langle-\mathcal{A}x+\mathcal{B}(x,x),v\rangle+(\mathcal{C}(t,x),\tilde v)+(y,v)$$

for $v,x\in V,v,\tilde v,y\in H$.

Lemma 2.7.1
For all $\Upsilon \in \mathcal{U}^b$ *we have*

$$\left.\frac{\mathrm{d}\mathcal{J}(\Phi^* + \theta(\Upsilon - \Phi^*))}{\mathrm{d}\theta}\right|_{\theta=0} = E\int_0^T \Big(\mathcal{H}_y[t, p(t), q(t), U_{\Phi^*}(t), \Phi^*(t)], \Upsilon(t) - \Phi^*(t)\Big)dt \geq 0. \tag{2.53}$$

Proof. Since $\mathcal{U}^b$ is convex it follows that $\Phi^* + \theta(\Upsilon - \Phi^*) \in \mathcal{U}^b$ for all $\theta \in [0,1]$. Equation (2.52) and Theorem 2.6.4 implies

$$\left.\frac{\mathrm{d}\mathcal{J}(\Phi^* + \theta(\Upsilon - \Phi^*))}{\mathrm{d}\theta}\right|_{\theta=0} = E\int_0^T \Big(\Upsilon(t) - \Phi^*(t), p(t)\Big)dt + E\int_0^T \Big(\mathcal{L}_y[t, U_{\Phi^*}(t), \Phi^*(t)], \Upsilon(t) - \Phi^*(t)\Big)dt.$$

Since Φ^* is an optimal control, we have

$$\left.\frac{\mathrm{d}\mathcal{J}(\Phi^* + \theta(\Upsilon - \Phi^*))}{\mathrm{d}\theta}\right|_{\theta=0} \geq 0.$$

Taking into account the definition of $\mathcal{H}$, it follows that (2.53) holds. ■

The statement of **the stochastic minimum principle** is contained in the following theorem.

Theorem 2.7.2
If $\Phi^* \in \mathcal{U}^b$ *is an optimal control, then for all* $h \in H$ *with* $\|h\| \leq \rho$ *the inequality*

$$\Big(\mathcal{L}_y[t, U_{\Phi^*}(t), \Phi^*(t)] + p(t), h - \Phi^*(t)\Big) \geq 0 \tag{2.54}$$

holds for $P \times \Lambda$ *a.e.* $(\omega, t) \in \Omega \times [0,T]$.

Proof. Let $h \in H$ with $\|h\| \leq \rho$. We denote by

$$\xi(\omega, t) := \Big(\mathcal{H}_y[t, p(t), q(t), U_{\Phi^*}(t), \Phi^*(t)], h - \Phi^*(t)\Big), \quad \mathcal{S} := \{(\omega, t) \in \Omega \times [0,T] | \xi(\omega, t) < 0\},$$

and $\mathcal{S}_t := \{\omega \in \Omega | \xi(\omega, t) < 0\}$ for each $t \in [0,T]$. Obviously, for each $t \in [0,T]$ the set $\mathcal{S}_t$ is $\mathcal{F}_t$-measurable. We take

$$\Upsilon(\omega, t) = \begin{cases} h & , \quad \omega \in \mathcal{S}_t \\ \Phi^*(\omega, t) & , \quad \omega \notin \mathcal{S}_t. \end{cases}$$

We see that $\Upsilon \in \mathcal{U}^b$ and

$$\Big(\mathcal{H}_y[t, p(t), q(t), U_{\Phi^*}(t), \Phi^*(t)], \Upsilon(t) - \Phi^*(t)\Big) = I_{\mathcal{S}_t}(\omega)\xi(\omega, t) \leq 0. \tag{2.55}$$

From Lemma 2.7.1 and (2.55) it follows

$$\begin{aligned} 0 &\leq E\int_0^T \Big(\mathcal{H}_y[t, p(t), q(t), U_{\Phi^*}(t), \Phi^*(t)], \Upsilon(t) - \Phi^*(t)\Big)dt \\ &= \int_{\mathcal{S}} \Big(\mathcal{H}_y[t, p(t), q(t), U_{\Phi^*}(t), \Phi^*(t)], h - \Phi^*(t)\Big)d(P \times \Lambda) = \int_{\mathcal{S}} \xi(\omega, t)d(P \times \Lambda) \leq 0. \end{aligned}$$

Consequently, $(P \times \Lambda)(\mathcal{S}) = 0$ and therefore

$$\Big(\mathcal{H}_y[t, p(t), q(t), U_{\Phi^*}(t), \Phi^*(t)], h - \Phi^*(t)\Big) = \Big(\mathcal{L}_y[t, U_{\Phi^*}(t), \Phi^*(t)] + p(t), h - \Phi^*(t)\Big) \geq 0$$

for $P \times \Lambda$ a.e. $(\omega, t) \in \Omega \times [0,T]$ and all $h \in H$ with $\|h\| \leq \rho$. ■

2.8 Equation of the adjoint processes

To complete the statement of the stochastic minimum principle, we need to derive the equation for the processes $\Big(p(t)\Big)_{t\in[0,T]}$ and $\Big(q(t)\Big)_{t\in[0,T]}$, called **adjoint equation**. We will use an approximation procedure and we will derive the equation for the approximation processes $\Big(p_n(t)\Big)_{t\in[0,T]}$, $\Big(q_n(t)\Big)_{t\in[0,T]}$ $(n \in \mathbb{N})$.

In this section we specialize the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ namely by $\Big(\mathcal{F}_{[w(r):r\leq t]}\Big)_{t\in[0,T]}$, which is the filtration generated by the Wiener process $\Big(w(t)\Big)_{t\in[0,T]}$.

Let $n \in \mathbb{N}$, $\psi \in \mathcal{L}^2_{(H_n,\|\cdot\|_V)}(\Omega \times [0,T]), \Gamma \in \mathcal{L}^2_{H_n}(\Omega \times [0,T])$ and let $\Phi^* \in \mathcal{U}^b$ be an optimal control with $E\Delta^{-4}_{U_{\Phi^*}}(T) < \infty$ (Remark 2.5.2 contains sufficient conditions for this inequality). We consider $Z_{n,\psi,\gamma}$ to be the solution of

$$\begin{aligned}(2.56)\qquad (Z_{n,\psi,\gamma}(t), v) &+ \int_0^t (\mathcal{A}_n Z_{n,\psi,\gamma}(s), v)ds = \int_0^t (\mathcal{B}'_n(U^n_{\Phi^*}(s))(Z_{n,\psi,\gamma}(s)), v)ds \\ &+ \int_0^t (\psi(s), v)ds + \int_0^t (\mathcal{C}'_n(s, U_{\Phi^*}(s))(Z_{n,\psi,\gamma}(s)), v)dw(s) + \int_0^t (\gamma(s), v)dw(s)\end{aligned}$$

for all $v \in H_n$, $t \in [0,T]$ and a.e. $\omega \in \Omega$, where $\mathcal{B}'_n(x)(y) := \sum_{i=1}^n \langle \mathcal{B}'(x)(y), h_i\rangle h_i$ for all $x, y \in V$, $U^n_{\Phi^*} := \Pi_n U_{\Phi^*}$, $\mathcal{C}'_n := \Pi_n \mathcal{C}'$. This equation is $(P_{n,\psi,\gamma})$ from Section 1.3 applied on $a_0 := 0$, $X = Y := U_{\Phi^*}, \mathcal{G}_n(s,h) := \mathcal{C}'_n(s, U_{\Phi^*}(s))(h)$.

The mapping

$$\begin{aligned}(\psi, \gamma) &\in \mathcal{L}^2_{(H_n,\|\cdot\|_V)}(\Omega \times [0,T]) \times \mathcal{L}^2_{H_n}(\Omega \times [0,T]) \mapsto \\ &\mapsto E\int_0^T \Big(\mathcal{L}_x[t, U_{\Phi^*}(t), \Phi^*(t)], Z_{n,\psi,\gamma}(t)\Big)dt + E\Big(\mathcal{K}'[U_{\Phi^*}(T)], Z_{n,\psi,\gamma}(T)\Big) \in \mathbb{R}\end{aligned}$$

is linear and continuous (by the same arguments as in the infinite dimensional case from Section 2.7).

By the Riesz Theorem it follows that there exist in a unique way processes

$$p_n \in \mathcal{L}^2_{(H_n,\|\cdot\|_V)}(\Omega \times [0,T]), q_n \in \mathcal{L}^2_{H_n}(\Omega \times [0,T])$$

such that

$$\begin{aligned}(2.57)\quad E\int_0^T \Big(\psi(t), p_n(t)\Big)dt \;&+\; E\int_0^T \Big(\gamma(t), q_n(t)\Big)dt \\ &= E\int_0^T \Big(\mathcal{L}_x[t, U_{\Phi^*}(t), \Phi^*(t)], Z_{n,\psi,\gamma}(t)\Big)dt + E\Big(\mathcal{K}'[U_{\Phi^*}(T)], Z_{n,\psi,\gamma}(T)\Big)\end{aligned}$$

for all $\psi \in \mathcal{L}^2_{(H_n,\|\cdot\|_V)}(\Omega \times [0,T]), \gamma \in \mathcal{L}^2_{H_n}(\Omega \times [0,T])$.

Let $\Psi \in \mathcal{L}^2_{V^*}(\Omega \times [0,T]), \Gamma \in \mathcal{L}^2_H(\Omega \times [0,T])$ and set

$$\Psi_n := \sum_{i=1}^n \langle \Psi, h_i \rangle h_i, \qquad \Gamma_n := \Pi_n \Gamma.$$

We have

$$\begin{aligned}(2.58)\qquad E\int_0^T \langle \Psi(t), p_n(t)\rangle dt \;&+\; E\int_0^T \Big(\Gamma(t), q_n(t)\Big)dt \\ &= E\int_0^T \Big(\Psi_n(t), p_n(t)\Big)dt + E\int_0^T \Big(\Gamma_n(t), q_n(t)\Big)dt \\ &= E\int_0^T \Big(\Delta^{-1}_{U_{\Phi^*}}(t)\mathcal{L}_x[t, U_{\Phi^*}(t), \Phi^*(t)], \Delta_{U_{\Phi^*}}(t) Z_{n,\Psi_n,\Gamma_n}(t)\Big)dt \\ &+ E\Big(\Delta^{-1}_{U_{\Phi^*}}(T)\mathcal{K}'[U_{\Phi^*}(T)], \Delta_{U_{\Phi^*}}(T) Z_{n,\Psi_n,\Gamma_n}(T)\Big).\end{aligned}$$

From the properties of the solution of the Navier-Stokes equation (see Lemma 1.2.6) and from the hypothesis on $\mathcal{L}$ and $\mathcal{K}$ we can deduce that

$$\Delta^{-1}_{U_{\Phi^*}}(t)\mathcal{L}_x[t, U_{\Phi^*}(t), \Phi^*(t)] \in \mathcal{L}^2_H(\Omega \times [0,T]), \quad \Delta^{-1}_{U_{\Phi^*}}(T)\mathcal{K}'[U_{\Phi^*}(T)] \in \mathcal{L}^2_H(\Omega).$$

We have $\Psi = \lim_{n\to\infty} \Psi_n$ in the space $\mathcal{L}^2_{V^*}(\Omega \times [0,T])$ and $\Gamma = \lim_{n\to\infty} \Gamma_n$ in the space $\mathcal{L}^2_H(\Omega \times [0,T])$. Now we use Lemma 1.3.2 and (2.52) in (2.58) to obtain

$$\lim_{n\to\infty}\left\{E\int_0^T\langle\Psi(t),p_n(t)\rangle dt+E\int_0^T\Big(\Gamma(t),q_n(t)\Big)dt\right\}$$

$$= E\int_0^T\Big(\mathcal{L}_x[t,U_{\Phi^*}(t),\Phi^*(t)],Z_{\Psi,\Gamma}(t)\Big)dt++E\Big(\mathcal{K}'[U_{\Phi^*}(T)],Z_{\Psi,\Gamma}(T)\Big)$$

$$= E\int_0^T\langle\Psi(t),p(t)\rangle dt+E\int_0^T\Big(\Gamma(t),q(t)\Big)dt,$$

for all $\Psi\in\mathcal{L}^2_{V^*}(\Omega\times[0,T]),\Gamma\in\mathcal{L}^2_H(\Omega\times[0,T])$. Hence, for $n\to\infty$ we have

$$p_n\rightharpoonup p\quad\text{in}\quad\mathcal{L}^2_V(\Omega\times[0,T])\quad\text{and}\quad q_n\rightharpoonup q\quad\text{in}\quad\mathcal{L}^2_H(\Omega\times[0,T]). \tag{2.59}$$

In (2.57) we take $\psi:=p_n,\gamma:=q_n$, use the weak convergence from above and Lemma 1.3.2. Then

$$\lim_{n\to\infty}\Big\{E\int_0^T\|p_n(t)\|^2dt\quad+\quad E\int_0^T\|q_n(t)\|^2dt\Big\}=E\int_0^T\Big(\mathcal{L}_x[t,U_{\Phi^*}(t),\Phi^*(t)],Z_{\mathrm{J}p,q}(t)\Big)dt \tag{2.60}$$

$$+\quad E\Big(\mathcal{K}'[U_{\Phi^*}(T)],Z_{\mathrm{J}p,q}(T)\Big)=E\int_0^T\|p(t)\|^2_Vdt+E\int_0^T\|q(t)\|^2dt.$$

From (2.59) and (2.60) it follows that the following strong convergences hold:

$$\lim_{n\to\infty}p_n=p\quad\text{in }\mathcal{L}^2_V(\Omega\times[0,T])\quad\text{and}\quad\lim_{n\to\infty}q_n=q\quad\text{in }\mathcal{L}^2_H(\Omega\times[0,T]). \tag{2.61}$$

Now we derive the equations for $\Big(p_n(t)\Big)_{t\in[0,T]}$ and $\Big(q_n(t)\Big)_{t\in[0,T]}$ $(n\in\mathbb{N})$ and then by passing to the limit obtain the equation for $\Big(p(t)\Big)_{t\in[0,T]}$ and $\Big(q(t)\Big)_{t\in[0,T]}$.

We consider the following matrices:

$$\tilde{\mathcal{A}}_n:=\Big(\langle\mathcal{A}h_j,h_i\rangle\Big)_{i,j=1,n},\ I_n:=\Big(\delta_{i,j}\Big)_{i,j=1,n},$$

$$\tilde{\mathcal{B}}_n(s):=\Big(\langle\mathcal{B}'(U^n_{\Phi^*}(s))(h_j),h_i\rangle\Big)_{i,j=1,n},\ \tilde{\mathcal{C}}_n(s):=\Big((\mathcal{C}'(s,U_{\Phi^*}(s))(h_j),h_i)\Big)_{i,j=1,n}.$$

The last two matrices depend on s and ω and are $\mathcal{F}_s$-measurable.

For each natural number n we introduce the $n\times n$ matrix processes

$$\Big(X_n(t)\Big)_{t\in[0,T]}=\Big(\Big(X^{i,j}_n(t)\Big)_{i,j=1,n}\Big)_{t\in[0,T]},\qquad\Big(Y_n(t)\Big)_{t\in[0,T]}=\Big(\Big(Y^{i,j}_n(t)\Big)_{i,j=1,n}\Big)_{t\in[0,T]}$$

as the solutions of the stochastic matrix equations

$$X_n(t)+\int_0^t\tilde{\mathcal{A}}_nX_n(s)ds=I_n+\int_0^t\tilde{\mathcal{B}}_n(s)X_n(s)ds+\int_0^t\tilde{\mathcal{C}}_n(s)X_n(s)dw(s) \tag{2.62}$$

and

$$(2.63)\quad Y_n(t) - \int_0^t Y_n(s)\tilde{\mathcal{A}}_n(s)ds = I_n - \int_0^t Y_n(s)\tilde{\mathcal{B}}_n(s)ds + \int_0^t Y_n(s)\tilde{\mathcal{C}}_n(s)\tilde{\mathcal{C}}_n(s)ds - \int_0^t Y_n(s)\tilde{\mathcal{C}}_n(s)dw(s)$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$. To prove the existence and (almost surely) uniqueness of the solution of (2.62) and (2.63), respectively, we consider the above equations as linear evolution equations with the unknown variable X_n, respectively Y_n. Their coeficients may depend on ω and s (see $\tilde{\mathcal{B}}_n, \tilde{\mathcal{C}}_n$). We use the same techniques as in the investigation of equations $(P_{\Psi,\Gamma})$, $(P_{n,\psi,\gamma})$ in Section 1.3. For each $i,j \in \{1,\ldots,n\}$ the process $\left(Y_n^{i,j}(t)\right)_{t\in[0,T]}$ has continuous trajectories in $\mathbb{R}$.

Using the Ito formula we obtain

$$(2.64)\qquad Y_n(t)X_n(t) = I_n \quad \text{for all } t\in[0,T],\ \text{a.e. } \omega\in\Omega$$

and hence

$$(2.65)\qquad X_n(t)Y_n(t) = I_n \quad \text{for all } t\in[0,T],\ \text{a.e. } \omega\in\Omega.$$

If $M := \left(M_{i,j}\right)_{i,j=1,n}$ is a matrix of real numbers and $h \in H_n$, then we write

$$Mh := \sum_{i,j=1}^{n} M_{i,j}(h,h_j)h_i.$$

We write $\widehat{M}$ for the transposed matrix of M.

Theorem 2.8.1
The processes $\left(p(t)\right)_{t\in[0,T]}$ and $\left(q(t)\right)_{t\in[0,T]}$ satisfy the adjoint equation

$$\begin{aligned}(\mathcal{K}'[U_{\Phi^*}(T)] - p(t), v) \quad &- \int_t^T \langle \mathcal{A}v, p(s)\rangle ds \\ &= -\int_t^T \langle \mathcal{B}(U_{\Phi^*}(s),v) + \mathcal{B}(v,U_{\Phi^*}(s)), p(s)\rangle ds - \int_t^T (\mathcal{L}_x[s,U_{\Phi^*}(s),\Phi^*(s)],v)ds \\ &- \int_t^T (\mathcal{C}'(s,U_{\Phi^*}(s))(v), q(s))ds + \int_t^T (q(s),v)dw(s),\end{aligned}$$

for all $t \in [0,T], v \in V$ and a.e. $\omega\in\Omega$. The processes $\left(p(t)\right)_{t\in[0,T]}$ and $\left(q(t)\right)_{t\in[0,T]}$ are uniquely characterized by this equation.

Proof. Let $\psi \in \mathcal{D}_V(\Omega\times[0,T])$, $\gamma\in\mathcal{D}_H(\Omega\times[0,T])$ and we define for each $K\in\mathbb{N}$ the stopping time

$$\mathcal{T}_K^n := \min\{\mathcal{T}_K^{Y_n^{i,j}} \ :\ 1 \le i,j \le n\}$$

and we take

$$\psi_K := I_{[0,\mathcal{T}_K^n]}\psi,\ \gamma_K := I_{[0,\mathcal{T}_K^n]}\gamma.$$

We consider the H_n-valued process

$$W_n^K(t) := Y_n(t)Z_{n,\psi_K,\gamma_K}(t)$$

where the process $\Big(Z_{n,\psi_K,\gamma_K}(t)\Big)_{t\in[0,T]}$ is the solution of (2.56) (with ψ_K and γ_K instead of ψ and γ). Using (2.65) we obtain for all $t \in [0,T]$ and a.e. $\omega \in \Omega$ that

$$Z_{n,\psi_K,\gamma_K}(t) = X_n(t)W_n^K(t). \tag{2.66}$$

Using (2.56), (2.63), and the Ito formula it follows that the process $\Big(W_n^K(t)\Big)_{t\in[0,T]}$ satisfies

$$(W_n^K(t),h) = \int_0^t (Y_n(s)\psi_K(s),h)ds - \int_0^t (Y_n(s)\tilde{\mathcal{C}}_n(s)\gamma_K(s),h)ds + \int_0^t (Y_n(s)\gamma_K(s),h)dw(s) \tag{2.67}$$

for all $t \in [0,T]$, $h \in H_n$ and a.e. $\omega \in \Omega$.

We use (2.57) and (2.66) to obtain

$$\begin{aligned} & E\int_0^T \Big(\psi_K(t),p_n(t)\Big)dt + E\int_0^T \Big(\gamma_K(t),q_n(t)\Big)dt \\ = \;& E\int_0^T \Big(\mathcal{L}_x[t,U_{\Phi^*}(t),\Phi^*(t)],Z_{n,\psi_K,\gamma_K}(t)\Big)dt + E\Big(\mathcal{K}'[U_{\Phi^*}(T)],Z_{n,\psi_K,\gamma_K}(T)\Big) \\ = \;& E\int_0^T \Big(\widehat{X}_n(t)\mathcal{L}_x^n[t,U_{\Phi^*}(t),\Phi^*(t)],W_n^K(t)\Big)dt + E\Big(\widehat{X}_n(T)\mathcal{K}_n'[U_{\Phi^*}(T)],W_n^K(T)\Big) \end{aligned} \tag{2.68}$$

where $\mathcal{L}_x^n(t,x,y) := \Pi_n\mathcal{L}_x(t,x,y), \mathcal{K}_n'(x) := \Pi_n\mathcal{K}'(x),\ t \in [0,T], x,y \in H$. Let us define the H_n-valued random variable

$$\xi_n = \widehat{X}_n(T)\mathcal{K}_n'[U_{\Phi^*}(T)] + \int_0^T \widehat{X}_n(t)\mathcal{L}_x^n[t,U_{\Phi^*}(t),\Phi^*(t)]dt, \tag{2.69}$$

and the H_n-valued process

$$\zeta_n(t) = -\int_0^t \widehat{X}_n(t)\mathcal{L}_x^n[t,U_{\Phi^*}(t),\Phi^*(t)]dt + E(\xi_n|\mathcal{F}_t) \tag{2.70}$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$. By the representation theorem of Levy (see [18], Theorem 4.15, p. 182 and Problem 4.17, p. 184) we have

$$E(\xi_n|\mathcal{F}_t) = E\xi_n + \int_0^t G_n(s)dw(s) \tag{2.71}$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$ where $G_n \in \mathcal{L}^2_{H_n}(\Omega \times [0,T])$. Without loss of generality we can assume that the process $\Big(\zeta_n(t)\Big)_{t\in[0,T]}$ has continuous trajectories in H. We see that $\zeta_n(T) = \widehat{X}_n(T)\mathcal{K}'_n[U_{\Phi^*}(T)]$ for a.e. $\omega \in \Omega$.

By using (2.69), (2.70), and (2.71) we deduce by Ito's calculus that

$$E\Big(\zeta_n(T), W_n^K(T)\Big) = -E\int_0^T \Big(\widehat{X}_n(t)\mathcal{L}_x^n[t, U_{\Phi^*}(t), \Phi^*(t)], W_n^K(t)\Big)dt$$

$$+E\int_0^T \Big\{(\zeta_n(t), Y_n(t)\psi_K(t) - Y_n(t)\tilde{\mathcal{C}}_n(t)\gamma_K(t)) + (G_n(t), Y_n(t)\gamma_K(t))\Big\}dt.$$

Here we have omitted to write explicitly an intermediate step: To consider stopping times for G_n. After taking the mathematical expectation in the above relation (with $\mathcal{T}_M^{G_n}$ instead of T) we let these stopping times to tend to T and use the almost surely continuity of the trajectories of ζ_n and W_n^K. Then we obtain the above equality.

Hence,

$$E\Big(\widehat{X}_n(T)\mathcal{K}'_n[U_{\Phi^*}(T)], W_n^K(T)\Big) + E\int_0^T \Big(\widehat{X}_n(t)\mathcal{L}_x^n[t, U_{\Phi^*}(t), \Phi^*(t)], W_n^K(t)\Big)dt$$

$$= E\int_0^T \Big(\widehat{Y}_n(t)\zeta_n(t), \psi_K(t)\Big)dt + E\int_0^T \Big(\widehat{Y}_n(t)G_n(t) - \widehat{\tilde{\mathcal{C}}_n}(t)\widehat{Y}_n(t)\zeta_n(t), \gamma_K(t)\Big)dt.$$

The processes ψ, γ were arbitrary fixed, and by (2.57) and (2.66) it follows that

$$I_{[0,\mathcal{T}_K^n]}(t)p_n(t) = I_{[0,\mathcal{T}_K^n]}(t)\widehat{Y}_n(t)\zeta_n(t) \quad \text{for } P \times \Lambda \text{ a.e. } (\omega,t) \in \Omega \times [0,T] \tag{2.72}$$

and

$$I_{[0,\mathcal{T}_K^n]}(t)q_n(t) = I_{[0,\mathcal{T}_K^n]}(t)\Big(\widehat{Y}_n(t)G_n(t) - \widehat{\tilde{\mathcal{C}}_n}(t)\widehat{Y}_n(t)\zeta_n(t)\Big) = I_{[0,\mathcal{T}_K^n]}(t)\Big(\widehat{Y}_n(t)G_n(t) - \widehat{\tilde{\mathcal{C}}_n}(t)p_n(t)\Big)$$

for $P \times \Lambda$ a.e. $(\omega,t) \in \Omega \times [0,T]$. Since $\lim_{K\to\infty} \mathcal{T}_K^n = T$ for a.e. $\omega \in \Omega$ (see Proposition B.1) and by using (2.72) we have

$$0 = \lim_{K\to\infty} E\int_0^{\mathcal{T}_K^n} \|p_n(t) - \widehat{Y}_n(t)\zeta_n(t)\|dt = E\int_0^T \|p_n(t) - \widehat{Y}_n(t)\zeta_n(t)\|dt.$$

This implies

$$p_n(t) = \widehat{Y}_n(t)\zeta_n(t) \quad \text{for } P \times \Lambda \text{ a.e. } (\omega,t) \in \Omega \times [0,T].$$

Analogously we obtain

$$q_n(t) = \widehat{Y}_n(t)G_n(t) - \widehat{\tilde{\mathcal{C}}_n}(t)p_n(t)\Big) \quad \text{for } P \times \Lambda \text{ a.e. } (\omega,t) \in \Omega \times [0,T].$$

We can identify $\Big(p_n(t)\Big)_{t\in[0,T]}$ with a process which has continuous trajectories in H. Then for all $t\in[0,T]$ we have

$$p_n(t)=\widehat{Y}_n(t)\zeta_n(t)\quad\text{for all } t\in[0,T]\quad\text{and}\quad p_n(T)=\mathcal{K}'_n[U_{\Phi^*}(T)]\quad\text{for a.e}\quad \omega\in\Omega.$$

By using the equations for $\Big(\widehat{Y}_n(t)\Big)_{t\in[0,T]}$ and $\Big(\zeta_n(t)\Big)_{t\in[0,T]}$ it follows by the Ito calculus that $\Big(p_n(t)\Big)_{t\in[0,T]}$ satisfies for all $t\in[0,T]$ and a.e. $\omega\in\Omega$ the n-dimensional evolution equation

$$\begin{aligned}(2.73)\qquad p_n(T)-p_n(t)-\int_t^T\widehat{\mathcal{A}}_np_n(s)ds &= -\int_t^T\Big\{\widehat{\widetilde{\mathcal{B}}}_n(s)p_n(s)+\mathcal{L}_x^n[s,U_{\Phi^*}(s),\Phi^*(s)]\Big\}ds\\ &- \int_t^T\widehat{\widetilde{\mathcal{C}}}_n(s)q_n(s)ds+\int_t^T q_n(s)dw(s)\end{aligned}$$

with $p_n(T)=\mathcal{K}'_n[U_{\Phi^*}(T)]$. Equation (2.73) can be written equivalently as

$$\begin{aligned}(p_n(T)-p_n(t),v) &- \int_t^T\langle\mathcal{A}v,p_n(s)\rangle ds\\ &= -\int_t^T\langle\mathcal{B}(U^n_{\Phi^*}(s),v)+\mathcal{B}(v,U^n_{\Phi^*}(s)),p_n(s)\rangle ds-\int_t^T(\mathcal{L}_x^n[s,U_{\Phi^*}(s),\Phi^*(s)],v)ds\\ &- \int_t^T\Big(\mathcal{C}'(s,U_{\Phi^*}(s))(v),q_n(s)\Big)ds+\int_t^T(q_n(s),v)dw(s),\end{aligned}$$

for all $t\in[0,T], v\in H_n$ and a.e. $\omega\in\Omega$. In this equation we take the limit for $n\to\infty$, use (2.61), and obtain

$$\begin{aligned}(2.74)\qquad (\mathcal{K}'[U_{\Phi^*}(T)]-p(t),v)-\int_t^T\langle\mathcal{A}v,p(s)\rangle ds&\\ = -\int_t^T\langle\mathcal{B}(U_{\Phi^*}(s),v)+\mathcal{B}(v,U_{\Phi^*}(s)),p(s)\rangle ds&-\int_t^T(\mathcal{L}_x[s,U_{\Phi^*}(s),\Phi^*(s)],h)ds\\ - \int_t^T\Big(\mathcal{C}'(s,U_{\Phi^*}(s))(v),q(s)\Big)ds+\int_t^T(q(s),v)dw(s),&\end{aligned}$$

for $P\times\Lambda$ a.e. $(\omega,t)\in\Omega\times[0,T]$ and all $v\in V$. We can identify $\Big(p(t)\Big)_{t\in[0,T]}$ with a process which has continuous trajectories in H and satisfies (2.74) for all $t\in[0,T]$ and a.e. $\omega\in\Omega$.

In order to show that (2.74) characterize in a unique way the adjoint processes $\Big(p(t)\Big)_{t\in[0,T]}$ and $\Big(q(t)\Big)_{t\in[0,T]}$, let us take any processes $\Big(p(t)\Big)_{t\in[0,T]}$ and $\Big(q(t)\Big)_{t\in[0,T]}$ which satisfy (2.74). Let

$\Psi \in \mathcal{L}^2_{V^*}(\Omega \times [0,T])$, $\Gamma \in \mathcal{L}^2_H(\Omega \times [0,T])$ and let $Z_{\Psi,\Gamma}$ be the solution of (2.51). Then we have

$$\begin{aligned}
E\Big(p(T), Z_{\Psi,\Gamma}(T)\Big) &= E\int_0^T \Big\{\langle \mathcal{A}Z_{\Psi,\Gamma}(t), p(t)\rangle - \langle \mathcal{B}'(U_{\Phi^*}(t))(Z_{\Psi,\Gamma}(t)), p(t)\rangle \\
&- \Big(\mathcal{L}_x[t, U_{\Phi^*}(t), \Phi^*(t)], Z_{\Psi,\Gamma}(t)\Big) - \Big(\mathcal{C}'(t, U_{\Phi^*}(t))(Z_{\Psi,\Gamma}(t)), q(t)\Big) - \langle \mathcal{A}Z_{\Psi,\Gamma}(t), p(t)\rangle \\
&+ \langle \mathcal{B}'(U_{\Phi^*}(t))(Z_{\Psi,\Gamma}(t)), p(t)\rangle + \langle \Psi(t), p(t)\rangle + \Big(\mathcal{C}'(t, U_{\Phi^*}(t))(Z_{\Psi,\Gamma}(t)), q(t)\Big) + (\Gamma(t), q(t))\Big\}dt.
\end{aligned}$$

Hence for all $\Psi \in \mathcal{L}^2_{V^*}(\Omega \times [0,T]), \Gamma \in \mathcal{L}^2_H(\Omega \times [0,T])$ we get

$$\begin{aligned}
E\Big(\mathcal{K}'[U_{\Phi^*}(T)], Z_{\Psi,\Gamma}(T)\Big) &+ E\int_0^T \Big(\mathcal{L}_x[t, U_{\Phi^*}(t), \Phi^*(t)], Z_{\Psi,\Gamma}(t)\Big)dt \\
&= E\int_0^T \langle \Psi(t), p(t)\rangle dt + E\int_0^T (\Gamma(t), q(t))dt.
\end{aligned}$$

Therefore, $\Big(p(t)\Big)_{t\in[0,T]}$ and $\Big(q(t)\Big)_{t\in[0,T]}$ must be the processes that are uniquely defined in (2.52). ■