Chapter 3

About the Dynamic Programming Equation

In Section 3.1 of this chapter we prove that the solution of the stochastic Navier-Stokes equation is a Markov process (see Theorem 3.1.1). In Section 3.2 we illustrate the dynamic programming approach (called also Bellman's principle) and we give a formal derivation of Bellman's equation. Bellman's principle turns the stochastic control problem into a deterministic control problem of a nonlinear partial differential equation of second order (see equation (3.11)) involving the infinitesimal generator. To round off the results of Chapter 2 we give a sufficient condition for an optimal control (Theorem 3.2.3 and Theorem 3.2.4). This condition requires a suitably behaved solution of the Bellman equation and an admissible control satisfying a certain equation. In this section we consider the finite dimensional stochastic Navier-Stokes equation, i.e., the equations (P_n) used in the Galerkin method in Section 1.2. The approach would be very complicate for the infinite dimensional case, because in this case it is difficult to obtain the infinitesimal generator. M.J. Vishik and A.V. Fursikov investigated in Chapter 11 of [35] the inverse Kolmogorov equations, which give the inifinitesimal generator of the process being solution of the considered equation, only for the case of $n = 2$ for (0.1). We take into account ideas and results on optimal control of Markov diffusion processes from the book of W.H. Fleming and R.W. Rishel [9] and adapt them for our problem.

3.1 The Markov property

An important property used in the dynamic programming approach is the Markov property of the solution of the Navier-Stokes equation. We will prove this property in this section.

Let us introduce the following σ-algebras

$$\sigma_{[U(s)]} := \sigma\{U(s)\}, \quad \sigma_{[U(r):r\leq s]} := \sigma\{U(r) : r \leq s\}$$

and the event

$$\sigma_{[U(s)=y]} := \{\omega : U(s) = y\}.$$

We define for the solution $U := U_\Phi$ of the Navier-Stokes equation (2.1), where $\Phi \in \mathcal{U}$, the **transition function**

$$\bar{P}(s,x,t,A) := P(U(t) \in A | \sigma_{[U(s)=x]})$$

with $s,t \in [0,T], s < t, x \in H, A \in B(H)$. In the following theorem we prove that **the solution of the Navier-Stokes equation is a Markov process**. This means that the state $U(s)$ at time s must contain all probabilistic information relevant to the evolution of the process for times $t > s$.

Theorem 3.1.1

(i) *For fixed $s,t \in [0,T], s < t, A \in B(H)$ the mapping*

$$y \in H \mapsto \bar{P}(s,y,t,A) \in \mathbb{R}$$

is measurable.

(ii) *The following equalities hold*

$$P(U(t) \in A|\mathcal{F}_s) = P(U(t) \in A|\sigma_{[U(s)]})$$

and

$$P\Big(U(t) \in A \Big| \sigma_{[U(r):r \le s]}\Big) = P\Big(U(t) \in A \Big| \sigma_{[U(s)]}\Big)$$

for all $s,t \in [0,T], s < t, y \in H, A \in B(H)$.

Proof. **(i)** Let $s,t \in [0,T], s < t, y \in H$. We denote by $\Big(\tilde{U}(t,s,y)\Big)_{t\in[s,T]}$ the solution of the Navier-Stokes equation starting in s with the initial value y, i.e. $\tilde{U}(s,s,y) = y$ for a.e. $\omega \in \Omega$.

Let $A \in B(H)$. Without loss of generality we can consider the set A to be closed. Let (a_n) be a sequence of continuous and uniformly bounded functions $a_n : H \to \mathbb{R},\ n \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \|a_n(y) - I_A(y)\| = 0 \quad \text{for all } y \in H. \tag{3.1}$$

By the uniqueness of the solution of the Navier-Stokes equation and from the definition of the transition function we have

$$\bar{P}(s,y,t,A) = E\Big(I_A(U(t))\Big|\sigma_{[U(s)=y]}\Big) = E\Big(I_A(\tilde{U}(t,s,y))\Big).$$

We consider an arbitrary sequence (y_n) in H such that $\lim_{n\to\infty} \|y_n - y\| = 0$. Using the same method as in the proof of Lemma 2.2.1 we can prove that

$$\lim_{n\to\infty} E\|\tilde{U}(t,s,y_n) - \tilde{U}(t,s,y)\|^2 = 0. \tag{3.2}$$

Therefore $\Big(\tilde{U}(t,s,y_n)\Big)$ converges in probability to $\tilde{U}(t,s,y)$. Using (3.2) and the Lebesgue Theorem it follows that for all $k \in \mathbb{N}$

$$\lim_{n\to\infty} Ea_k\Big(\tilde{U}(t,s,y_n)\Big) = Ea_k\Big(\tilde{U}(t,s,y)\Big).$$

We conclude that for each $k \in \mathbb{N}$ the mapping

$$y \in H \mapsto Ea_k\Big(\tilde{U}(t,s,y)\Big) \in \mathbb{R}$$

is continuous. Hence it is measurable. By the Lebesgue Theorem and (3.1) we deduce that for all $y \in H$

$$\lim_{k\to\infty} Ea_k\Big(\tilde{U}(t,s,y)\Big) = EI_A\Big(\tilde{U}(t,s,y)\Big).$$

Consequently, $\bar{P}(s,\cdot,t,A) = EI_A\Big(\tilde{U}(t,s,\cdot)\Big)$ is measurable, because it is the pointwise limit of measurable functions.

(ii) First we prove that for each fixed $s,t \in [0,T], s < t, y \in H$ the random variable $\tilde{U}(t,s,y)$ (considered as a H-valued random variable) is independent of $\mathcal{F}_s$. By relation (1.12) from Section 1.2 we have

$$\lim_{M\to\infty} \|\tilde{U}_n^M(t,s,y) - \tilde{U}_n(t,s,y)\| = 0 \quad \text{for each } n \in \mathbb{N} \text{ and a.e. } \omega \in \Omega, \tag{3.3}$$

and by Theorem 1.2.7 it follows that there exists a subsequence (n') of (n) such that

$$\lim_{n'\to\infty} \|\tilde{U}_{n'}(t,s,y) - \tilde{U}(t,s,y)\| = 0 \quad \text{for a.e. } \omega \in \Omega \tag{3.4}$$

where $\Big(\tilde{U}_n^M(t,s,y)\Big)_{t\in[s,T]}$ and $\Big(\tilde{U}_n(t,s,y)\Big)_{t\in[s,T]}$ are the solutions of (P_n^M) and (P_n), respectively, if we start in s with the initial value y (see Section 1.2). Since for fixed n, M the random variable $\tilde{U}_n^M(t,s,y)$ is approximated by Picard-iteration and each Picard-approximation is independent of $\mathcal{F}_s$ (as a H-valued random variable), it follows by Proposition B.4 that $\tilde{U}_n(t,s,y)$ is independent of $\mathcal{F}_s$. Using (3.3), (3.4), and Proposition B.4 we conclude that $\tilde{U}(t,s,y)$ is independent of $\mathcal{F}_s$.

Let $A \in B(H)$. Now we apply Proposition B.5 for $\widehat{\mathcal{F}} := \mathcal{F}_s, f(y,\omega) := I_A\Big(\tilde{U}(t,s,y)\Big)$, $\xi(\omega) := U(s)$. Hence

$$E\Big(I_A\Big(\tilde{U}(t,s,U(s))\Big)\Big|\mathcal{F}_s\Big) = E\Big(I_A\Big(\tilde{U}(t,s,U(s))\Big)\Big|\sigma_{[U(s)]}\Big). \tag{3.5}$$

Since the solution of the Navier-Stokes equation is (almost surely) unique it follows that

$$\tilde{U}(t,s,U(s)) = U(t) \quad \text{for all } t \in [s,T] \text{ and a.e. } \omega \in \Omega.$$

Then relation (3.5) becomes

$$E\Big(I_A\Big(U(t)\Big)\Big|\mathcal{F}_s\Big) = E\Big(I_A\Big(U(t)\Big)\Big|\sigma_{[U(s)]}\Big).$$

Consequently,

$$P\Big(U(t) \in A\Big|\mathcal{F}_s\Big) = P\Big(U(t) \in A\Big|\sigma_{[U(s)]}\Big). \tag{3.6}$$

We know

$$\sigma_{[U(s)]} \subseteq \sigma_{[U(r):r\leq s]} \subseteq \mathcal{F}_s.$$

Taking into account the properties of the conditional expectation and (3.6) we deduce that

$$\begin{aligned} P\Big(U(t)\in A\Big|\sigma_{[U(r):r\le s]}\Big) &= E\Big(E\Big(U(t)\in A\Big|\mathcal{F}_s\Big)\Big|\sigma_{[U(r):r\le s]}\Big) \\ &= E\Big(E\Big(U(t)\in A\Big|\sigma_{[U(s)]}\Big)\Big|\sigma_{[U(r):r\le s]}\Big) = P\Big(U(t)\in A\Big|\sigma_{[U(s)]}\Big). \end{aligned}$$ ■

Corollary 3.1.2 ([11], Chapter 3, Section 9, pp. 59)

(i) *For fixed* $s,t\in[0,T], s<t, y\in H$ *the mapping*

$$A\in B(H)\mapsto \bar P(s,y,t,\cdot)\in \mathbb{R}$$

is a probability measure.

(ii) *The Chapman-Kolmogorov equation*

$$\bar P(s,y,t,A)=\int_H \bar P(r,x,t,A)\bar P(s,y,r,dx)$$

holds for any $r,s,t\in[0,T], s<r<t, y\in H, A\in B(H)$.

Remark 3.1.3

1) We have the **autonomous version** of the stochastic Navier-Stokes equation if for $t\in[0,T]$, $h\in H$ we have $\mathcal{C}(t,h)=\mathcal{C}(h)$ and $\Phi(t,h)=\Phi(h)$ for $\Phi\in\mathcal{U}$. In this case $\Big(U_\Phi(t)\Big)_{t\in[0,T]}$ is a **homogeneous Markov process**, i.e., we have

$$\bar P(0,y,t-s,A)=\bar P(s,y,t,A) \tag{3.7}$$

for all $s,t\in[0,T], s<t, y\in H, A\in B(H)$.

We prove the above property for $\Phi\in\mathcal{U}^a$, where $\mathcal{U}^a$ is the set of all autonomous feedback controls. Let $s,t\in[0,T], s<t, y\in H$. The solution U_Φ of the Navier-Stokes equation, which starts in s with the initial value y satisfies

$$\begin{aligned} (U_\Phi(t),v)+\int_s^t\langle\mathcal{A}U_\Phi(r),v\rangle dr &= (y,v)+\int_s^t\langle\mathcal{B}(U_\Phi(r),U_\Phi(r)),v\rangle dr \\ &+ \int_s^t(\Phi(U_\Phi(r)),v)dr+\int_s^t(\mathcal{C}(U_\Phi(r)),v)dw(r) \end{aligned}$$

for all $v\in V$ and a.e. $\omega\in\Omega$. We take $\tilde U(r)=U_\Phi(s+r), \tilde w(r):=w(s+r)-w(s)$ for $r\in[0,t-s]$. Then for $\tilde U(t-s)$ we have

$$\begin{aligned} (\tilde U(t-s),v)+\int_0^{t-s}\langle\mathcal{A}\tilde U(r),v\rangle dr &= (y,v)+\int_0^{t-s}\langle\mathcal{B}(\tilde U(r),\tilde U(r)),v\rangle dr \\ &+ \int_0^{t-s}(\Phi(\tilde U(r)),v)dr+\int_0^{t-s}(\mathcal{C}(\tilde U(r)),v)d\tilde w(r) \end{aligned}$$

for all $v \in V$ and a.e. $\omega \in \Omega$. Since $\Big(\tilde{w}(r)\Big)_{r\in[0,t-s]}$ and $\Big(w(r)\Big)_{r\in[s,t]}$ have the same distribution and because of the uniqueness of the solution of the Navier-Stokes equation, it follows that $\tilde{U}(t-s)$ and $U_\Phi(t)$ have the same distribution. Hence (3.7) holds.

2) The Galerkin approximations (the solutions of the equations (P_n) from Section 1.2) of the Navier-Stokes equation are also Markov processes.

3.2 Bellman's principle and Bellman's equation for the finite dimensional stochastic Navier-Stokes equation

Before we illustrate the dynamic programming approach (also called Bellman's principle) for our control problem, we need the definition of the infinitesimal generator associated to a process. This infinitesimal generator is a partial differential operator of second order (see Lemma 3.2.2) and it occurs in Bellman's equation.

Definition 3.2.1

Let $\Big(X(t)\Big)_{t\in[0,T]}$ be a process in the space $\mathcal{L}^2_H(\Omega\times[0,T])$ and let $t\in[0,T]$. The function $F: H \to \mathbb{R}$ is said to belong to the domain $\mathcal{D}_{\mathbf{A}_X(t)}$ of the **infinitesimal generator $\mathbf{A}_X$** of $\Big(X(t)\Big)_{t\in[0,T]}$ if the limit

$$\mathbf{A}_X(t)F(y) := \lim_{\theta\searrow 0}\frac{1}{\theta}\Big[E\Big(F(X(t+\theta))\Big|\sigma_{[X(t)=y]}\Big) - F(y)\Big], \tag{3.8}$$

exists and is finite for all $y \in H$.

We define $C^2(H)$ to be the set of all mappings $F: H \to \mathbb{R}$ which are twice continuously Fréchet differentiable in each point of H and which satisfy the conditions:
(i) F, F', F'' are locally bounded;
(ii) for each $h \in H$

$$\|F'(h)\| \le c_F(1+\|h\|), \quad \Big|\Big(F''(h)h_1, h_2\Big)\Big| \le c_F\|h_1\|\|h_2\|(1+\|h\|),$$

where c_F is a positive constant.

We define $C^{1,2}([0,T]\times H)$ to be the set of all mappings $G: [0,T]\times H \to \mathbb{R}$ such that
(i) for each fixed $t\in[0,T]$ we have $G(t,\cdot)\in C^2(H)$;
(ii) there exists the partial derivative G_t which is assumed to be continuous on $[0,T]$ and

$$|G_t(t,x)| \le c_G\|x\|$$

for all $t \in [0,T]$ and $x \in H$.

In this section we consider the n-dimensional stochastic Navier-Stokes equation

$$(P_n) \qquad (U_{n,\Phi}(t),v) + \int_0^t (\mathcal{A}_n U_{n,\Phi}(s),v)ds = (x_0,v) + \int_0^t (\mathcal{B}_n(U_{n,\Phi}(s),U_{n,\Phi}(s)),v)ds + \int_0^t (\Phi(s,U_{n,\Phi}(s)),v)ds + \int_0^t (\mathcal{C}_n(s,U_{n,\Phi}(s)),v)dw(s),$$

for all $v \in H_n$, $t \in [0,T]$ and a.e. $\omega \in \Omega$, controlled by *feedback controls* $\Phi \in \mathcal{U}_n$ (we proceed analogously in the case $\Phi \in \mathcal{U}_n^b$), where the set $\mathcal{U}_n$ (respectively $\mathcal{U}_n^b$) is defined in Section 2.4. We denote by

$$E_{t,y}(\cdot) := E\Big(\cdot \,\Big|\, \sigma_{[U_{n,\Phi}(t)=y]}\Big)$$

where $t \in [0,T], y \in H$.

We assume that the mappings $\mathcal{C}(\cdot,x), \mathcal{L}(\cdot,x,y)$ are continuous on $[0,T]$ for each $x,y \in H$. The formula of the infinitesimal generator for the process $\Big(U_{n,\Phi}(t)\Big)_{t\in[0,T]}$ is given in the following lemma.

Lemma 3.2.2
The infinitesimal generator of $\Big(U_{n,\Phi}(t)\Big)_{t\in[0,T]}$ *satisfies*

$$\mathbf{A}_{U_{n,\Phi}}(s)G(s,y) = G_t(s,y) + \Big(G_x(s,y), -\mathcal{A}_n y + \mathcal{B}(y,y) + \Phi(s,y)\Big) + \frac{1}{2}\Big(G_{xx}(s,y)\mathcal{C}_n(s,y), \mathcal{C}_n(s,y)\Big)$$

for all $s \in [0,T], y \in H_n, G \in C^{1,2}([0,T]\times H_n), \Phi \in \mathcal{U}_n$.
In the autonomous version of problem (P_n) *the infinitesimal generator of* $\Big(U_{n,\Phi}(t)\Big)_{t\in[0,T]}$ *satisfies*

$$\mathbf{A}_{U_{n,\Phi}}F(y) = \Big(F_x(y), -\mathcal{A}_n y + \mathcal{B}(y,y) + \Phi(y)\Big) + \frac{1}{2}\Big(F_{xx}(y)\mathcal{C}_n(y), \mathcal{C}_n(y)\Big)$$

for all $y \in H_n, F \in C^2(H_n), \Phi \in \mathcal{U}_n^a$.

Proof. Let $G \in C^{1,2}([0,T]\times H_n)$. We write $\Phi(r)$ instead of $\Phi(r,U_{n,\Phi}(r))$. By the Ito formula it follows that

$$\begin{aligned} & G(s+h,U_{n,\Phi}(s+h)) - G(s,U_{n,\Phi}(s)) \\ &= \int_s^{s+h} G_t(r,U_{n,\Phi}(r)) + \Big(G_x(r,U_{n,\Phi}(r)), -\mathcal{A}_n U_{n,\Phi}(r) + \mathcal{B}_n(U_{n,\Phi}(r),U_{n,\Phi}(r)) + \Phi(r)\Big)dr \\ &+ \frac{1}{2}\int_s^{s+h} \Big(G_{xx}(r,U_{n,\Phi}(r))\mathcal{C}_n(r,U_{n,\Phi}(r)), \mathcal{C}_n(r,U_{n,\Phi}(r))\Big)dr \\ &+ \int_s^{s+h} \Big(G_x(r,U_{n,\Phi}(r)), \mathcal{C}_n(r,U_{n,\Phi}(r))\Big)dw(r), \end{aligned}$$

for each $h, s \in [0, T]$ with $s + h \leq T$. In the above relation we take the conditional expectation $E_{s,y}$. We obtain

$$\frac{1}{h}\Big[E_{s,y}\Big(G(s+h, U_{n,\Phi}(s+h))\Big) - G(s,y)\Big]$$

$$= \quad E_{s,y}\Big\{\frac{1}{h}\int\limits_s^{s+h} G_t(r, U_{n,\Phi}(r)) + \Big(G_x(r, U_{n,\Phi}(r)), -\mathcal{A}_n U_{n,\Phi}(r) + \mathcal{B}_n(U_{n,\Phi}(r), U_{n,\Phi}(r)) + \Phi(r)\Big)dr\Big\}$$

$$+ \quad \frac{1}{2}E_{s,y}\Big\{\frac{1}{h}\int\limits_s^{s+h} \Big(G_{xx}(r, U_{n,\Phi}(r))\mathcal{C}_n(r, U_{n,\Phi}(r)), \mathcal{C}_n(r, U_{n,\Phi}(r))\Big)dr\Big\}.$$

We take $h \searrow 0$, use the properties of the process $\Big(U_{n,\Phi}(t)\Big)_{t\in[0,T]}$ (see Theorem 1.2.1 and Lemma 1.2.3) and those of $G, \Phi, \mathcal{C}_n$. Then, for each $t \in [0, T], y \in H_n$ we have

$$\mathbf{A}_{U_{n,\Phi}}(s)G(s,y) = G_t(s,y) + \Big(G_x(s,y), -\mathcal{A}_n y + \mathcal{B}(y,y) + \Phi(s,y)\Big) + \frac{1}{2}\Big(G_{xx}(s,y)\mathcal{C}_n(s,y), \mathcal{C}_n(s,y)\Big).$$

We proceed similarly in the autonomous case. ■

We consider the cost functional

$$\mathcal{J}(s,y,\Phi) := E_{s,y}\Big\{\int\limits_s^T \mathcal{L}[r, U_{n,\Phi}(r), \Phi(r, U_{n,\Phi}(r))]dr + \mathcal{K}[U_{n,\Phi}(T)]\Big\}$$

where $s \in [0, T], y \in H_n$ and the feedback control $\Phi \in \mathcal{U}_n$.

To illustrate the dynamic programming approach we give a *formal derivation of Bellman's equation*, our arguments are of heuristic nature. *Bellman's principle* turns the stochastic control problem $(\mathcal{P}_n)$ into a problem about a nonlinear differential equation of second order (see equation (3.11)).

In dynamic programming the optimal expected system performance is considered as a function of the initial data

$$W(s,y) = \inf_{\Phi\in\mathcal{U}_n} \mathcal{J}(s,y,\Phi).$$

If $W \in C^{1,2}([0,T] \times H_n)$, then by using (P_n), the Ito formula, and Lemma 3.2.2, it follows that

$$\text{(3.9)} \quad E_{s,y}W(t, U_{n,\Phi}(t)) - W(s,y) = E_{s,y}\int\limits_s^t \Big(W_t(r, U_{n,\Phi}(r)) + \mathbf{A}_{U_{n,\Phi}}(r)W(r, U_{n,\Phi}(r))\Big)dr.$$

Suppose that the controller uses Φ for times $s \leq r \leq t$ and uses an optimal control Φ^* after time t. His expectet performance cannot be less than $W(s, y)$. Thus for all $y \in H_n$ let

$$\tilde{\Phi}(r,y) = \begin{cases} \Phi(r,y) & \text{for} \quad s \leq r \leq t \\ \Phi^*(r,y) & \text{for} \quad t < r \leq T. \end{cases}$$

By the Chapman-Kolmogorov equation (see Corollary 3.1.2) and the properties of the conditional expectation we have

$$\mathcal{J}(s,y,\tilde{\Phi}) = E_{s,y}\int_s^t \mathcal{L}[r,U_{n,\Phi}(r),\Phi(r,U_{n,\Phi}(r))]dr + E_{s,y}\mathcal{J}(t,U_{n,\Phi^*}(t),\Phi^*).$$

Because $\Phi^* \in \mathcal{U}_n$ is an optimal control, then for all $t\in[0,T], y\in H_n$ we have

$$W(t,U_{n,\Phi^*}(t)) = \mathcal{J}(t,U_{n,\Phi^*}(t),\Phi^*), \quad W(s,y)\le \mathcal{J}(s,y,\Phi)$$

and

$$W(s,y) \le E_{s,y}\int_s^t \mathcal{L}[r,U_{n,\Phi}(r),\Phi(r,U_{n,\Phi}(r))]dr + E_{s,y}W(t,U_{n,\Phi^*}(t)) \tag{3.10}$$

In (3.10) we have equality if an optimal control $\Phi := \Phi^*$ is used during $[s,t]$. By (3.9) and (3.10) we obtain

$$0 \le E_{s,y}\int_s^t \Big\{\mathcal{L}[r,U_{n,\Phi}(r),\Phi(r,U_{n,\Phi}(r))] + W_t(r,U_{n,\Phi}(r)) + \mathbf{A}_{U_{n,\Phi}}(r)W(r,U_{n,\Phi}(r))\Big\}dr.$$

In the above inequality we divide by $t-s$, take $t\searrow s$, use the continuity properties of $\Big(U_{n,\Phi}(t)\Big)_{t\in[0,T]}$ (see Theorem 1.2.1 and Lemma 1.2.3) and those of $W, \mathbf{A}_{U_{n,\Phi}}, \mathcal{L}$. Thus

$$0 \le \mathcal{L}[s,y,\Phi(s,y)] + W_t(s,y) + \mathbf{A}_{U_{n,\Phi}}(s)W(s,y).$$

Equality holds above, if $\Phi = \Phi^*$. For W we have derived formally the **continuous-time dynamic programming equation of optimal stochastic control theory**, also called **Bellman's equation**

$$0 = W_t(s,y) + \min_{\Phi\in\mathcal{U}_n}\Big\{\mathcal{L}[s,y,\Phi(s,y)] + \mathbf{A}_{U_{n,\Phi}}(s)W(s,y)\Big\} \quad s\in[0,T],\ y\in H_n \tag{3.11}$$

with the boundary condition

$$W(T,y) = \mathcal{K}(y), \quad y\in H_n.$$

The **main result** of this section is a sufficient condition for a minimum (Theorem 3.2.3 and for the autonomous case Theorem 3.2.4). The sufficient condition requires a suitably behaved solution W of the Bellman equation (3.11) and an admissible control Φ^* satisfying (3.14). Such a result is called *verification theorem.*

Theorem 3.2.3
Let W be the solution of Bellman's equation

$$0 = W_t(s,y) + \inf_{\Phi\in\mathcal{U}_n}\Big\{\mathcal{L}[s,y,\Phi(s,y)] + \mathbf{A}_{U_{n,\Phi}}(s)W(s,y)\Big\} \tag{3.12}$$

for all $(s,y)\in[0,T]\times H_n$, satisfying the boundary condition

$$W(T,U_{n,\Phi}(T)) = \mathcal{K}(U_{n,\Phi}(T)) \quad for\ all\ \Phi\in\mathcal{U}. \tag{3.13}$$

If $W\in C^{1,2}([0,T]\times H_n)$, then:

(i) *$W(s,y) \le \mathcal{J}(s,y,\Phi)$ for any $\Phi \in \mathcal{U}_n$, $s \in [0,T], y \in H_n$.*

(ii) *If $\Phi^* \in \mathcal{U}_n$ is a feedback control such that*

$$\mathcal{L}[s,y,\Phi^*(s,y)] + \mathbf{A}_{U_{n,\Phi^*}}(s)W(s,y) = \min_{\Phi\in\mathcal{U}_n}\Big\{\mathcal{L}[s,y,\Phi(s,y)] + \mathbf{A}_{U_{n,\Phi}}(s)W(s,y)\Big\} \tag{3.14}$$

for all $s \in [0,T]$ and $y \in H_n$, then $W(s,y) = \mathcal{J}(s,y,\Phi^*)$ for all $s \in [0,T], y \in H_n$. Thus Φ^* is an optimal feedback control.

Proof. **(i)** Let $\Phi \in \mathcal{U}_n, s \in [0,T], y \in H_n$. From (3.12) it follows that

$$0 \le W_t(r, U_{n,\Phi}(r)) + \mathcal{L}[r, U_{n,\Phi}(r), \Phi(r, U_{n,\Phi}(r))] + \mathbf{A}_{U_{n,\Phi}}(r)W(r, U_{n,\Phi}(r)), \quad r \in [0,T].$$

We integrate from s to T, use (3.9), take the conditional expectation $E_{s,y}$ and have

$$W(s,y) \le E_{s,y}W(T, U_{n,\Phi}(T)) + E_{s,y}\int_s^T \mathcal{L}[r, U_{n,\Phi}(r), \Phi(r, U_{n,\Phi}(r))]dr.$$

Now we use the boundary condition (3.13) and hence

$$W(s,y) \le \mathcal{J}(s,y,\Phi).$$

(ii) We use the same arguments as above. Instead of Φ we take Φ^*, and instead of $\le$ we take $=$. ■

Let us state a corresponding verification theorem for the **autonomous version** of the problem, formulated at the end of Section 3.1. The cost functional is given by

$$\mathcal{J}(y,\Phi) = E_y\Big\{\int_0^T \mathcal{L}[U_{n,\Phi}(r), \Phi(U_{n,\Phi}(r))]dr + \mathcal{K}[U_{n,\Phi}(T)]\Big\},$$

with $y \in H_n, \Phi \in \mathcal{U}_n^a$ (see Remark 3.1.3) and $E_y(\cdot) = E\Big(\cdot|\sigma_{[U_{n,\Phi}(0)=y]}\Big)$. The mapping $\mathcal{L}$ that occurs in the expression of the cost functional does not depend on $r \in [0,T]$ and satisfies the conditions $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ from Section 2.1.

Analogously to Theorem 3.2.3 we can prove the following verification theorem.

Theorem 3.2.4
Let W be the solution of Bellman's equation

$$0 = \inf_{\Phi\in\mathcal{U}_n^a}\Big\{\mathcal{L}[y,\Phi(y)] + \mathbf{A}_{U_{n,\Phi}}W(y)\Big\} \quad \text{for all} \quad y \in H_n$$

with the boundary condition

$$W(U_{n,\Phi}(T)) = \mathcal{K}(U_{n,\Phi}(T)) \quad \text{for all } \Phi \in \mathcal{U}.$$

If $W \in C^2(H_n)$, then:

(i) $W(y) \leq \mathcal{J}(y, \Phi)$ *for any* $\Phi \in \mathcal{U}_n^a$ *and any initial data* $y \in H_n$.

(ii) *If* $\Phi^* \in \mathcal{U}_n^a$ *is a feedback control such that*

$$\mathcal{L}[y, \Phi^*(y)] + \mathbf{A}_{U_{n,\Phi^*}} W(y) = \min_{\Phi \in \mathcal{U}_n^a} \left\{ \mathcal{L}[y, \Phi(y)] + \mathbf{A}_{U_{n,\Phi}} W(y) \right\} \quad \text{for all } y \in H_n,$$

then $W(y) = \mathcal{J}(y, \Phi^*)$ *for all* $y \in H_n$. *Thus* Φ^* *is an optimal feedback control.*