Appendix A

Basic Convergence Results

For the convenience of the reader we recall some basic convergence results.

Proposition A.1 ([36], Proposition 10.13, p. 480).
Let (x_n) be a sequence in a Banach space S. Then the following assertions hold:

(i) *If S is reflexive and (x_n) is bounded, then (x_n) has a weakly convergent subsequence. If, in addition, every weakly convergent subsequence of (x_n) has the same limit $x \in S$, then (x_n) converges weakly to x.*

(ii) *If every subsequence of (x_n) has a subsequence which converges strongly to the same limit $x \in S$, then $x_n \to x$.*

Proposition A.2 ([37], Proposition 21.27, p.261).
Let S_1 and S_2 be Banach spaces and let $L : S_1 \to S_2$ be a continuous linear operator. If (x_n) is a sequence in S_1 such that $x_n \rightharpoonup x$ (where $x \in S_1$), then $L(x_n) \rightharpoonup L(x)$.

Proposition A.3
If S is a Banach space and if (x_n) is a sequence from $\mathcal{L}_S^2(\Omega \times [0,T])$ which converges weakly to $x \in \mathcal{L}_S^2(\Omega \times [0,T])$, then for $n \to \infty$ the following assertions are true:

(i) $$\int_0^t x_n(s)dw(s) \rightharpoonup \int_0^t x(s)dw(s) \quad \textit{and} \quad \int_0^t x_n(s)ds \rightharpoonup \int_0^t x(s)ds \quad \textit{in} \quad \mathcal{L}_S^2(\Omega \times [0,T]);$$

(ii) $$\int_0^T x_n(s)dw(s) \rightharpoonup \int_0^T x(s)dw(s) \quad \textit{and} \quad \int_0^T x_n(s)ds \rightharpoonup \int_0^T x(s)ds \quad \textit{in} \quad \mathcal{L}_S^2(\Omega).$$

Proof. We apply Proposition A.2 on $S_1 = S_2 := \mathcal{L}_S^2(\Omega\times[0,T])$, $L : \mathcal{L}_S^2(\Omega\times[0,T]) \to \mathcal{L}_S^2(\Omega\times[0,T])$, where

$$L(x) := \int_0^t x(s)dw(s).$$

Obviously, is L a linear mapping. By the properties of the stochastic integral we have

$$\begin{aligned}\|L(x)\|^2_{\mathcal{L}^2_S(\Omega\times[0,T])} &= E\int_0^T \Big\|\int_0^t x(s)dw(s)\Big\|^2_S dt \le TE\sup_{t\in[0,T]}\Big\|\int_0^t x(s)dw(s)\Big\|^2_S \\ &\le 4TE\int_0^T \|x(t)\|^2_S dt = 4T\|x\|^2_{\mathcal{L}^2_S(\Omega\times[0,T])}.\end{aligned}$$

Hence L is continuous and we can apply Proposition A.2. The other convergences are proved analogously. ■

Appendix B

Stopping Times

Let $\Big(Q(t)\Big)_{t\in[0,T]}$ be a V-valued process with

$$\int_0^T \|Q(s)\|_V^2 ds < \infty \quad \text{and} \quad \sup_{t\in[0,T]} \|Q(t)\|^2 < \infty$$

for a.e. $\omega \in \Omega$. For each $M \in \mathbb{N}$ we introduce the following **stopping times**

$$\widetilde{\mathcal{T}}_M^Q = \begin{cases} T\ , \quad \text{if} \quad \sup_{t\in[0,T]} \|Q(t)\|^2 < M \\ \inf\Big\{t \in [0,T] : \|Q(t)\|^2 \geq M\Big\}, \quad \text{otherwise,} \end{cases}$$

$$\widehat{\mathcal{T}}_M^Q = \begin{cases} T\ , \quad \text{if} \quad \int_0^T \|Q(s)\|_V^2 ds < M \\ \inf\Big\{t \in [0,T] : \int_0^t \|Q(s)\|_V^2 ds \geq M\Big\}, \quad \text{otherwise.} \end{cases}$$

We define

$$\mathcal{T}_M^Q := \min\{\widetilde{\mathcal{T}}_M^Q, \widehat{\mathcal{T}}_M^Q\}.$$

We see that for all $t \in [0,T]$ and a.e. $\omega \in \Omega$ we have

$$\|Q(t \wedge \mathcal{T}_M^Q)\|^2 \leq M, \qquad \int_0^{t\wedge\mathcal{T}_M^Q} \|Q(s)\|_V^2 ds \leq M.$$

Proposition B.1
The following convergences hold:

$$\lim_{M\to\infty} P(\mathcal{T}_M^Q < T) = 0$$

and

$$\lim_{M\to\infty} \mathcal{T}_M^Q = T \quad for\ a.e.\ \omega \in \Omega.$$

Proof. Using some elementary inequalities we obtain

$$\begin{aligned}
&\lim_{M\to\infty} P(\mathcal{T}_M^Q < T) \le \lim_{M\to\infty} P(\widetilde{\mathcal{T}}_M^Q < T) + \lim_{M\to\infty} P(\widehat{\mathcal{T}}_M^Q < T) \\
\le\ & \lim_{M\to\infty} P\Big(\sup_{t\in[0,T]} \|Q(t)\|^2 \ge M\Big) + \lim_{M\to\infty} P\Big(\int_0^T \|Q(s)\|_V^2 ds\Big) \ge M\Big) \\
\le\ & P\Big(\bigcap_{M=1}^{\infty} \Big\{\sup_{t\in[0,T]} \|Q(t)\|^2 \ge M\Big\}\Big) + P\Big(\bigcap_{M=1}^{\infty} \Big\{\int_0^T \|Q(s)\|_V^2 \ge M\Big\}\Big) = 0.
\end{aligned}$$

The sequence $\Big(T - \mathcal{T}_M^Q\Big)$ is monotone decreasing (for a.e. $\omega \in \Omega$). We have proved above that it converges in probability to zero. Therefore it converges to zero for almost every $\omega \in \Omega$. ■

Proposition B.2
We assume that the following assumptions are fulfilled:
(1) $k_1, k_2 > 0$ *are real numbers;*
(2) a_0 *is a* H*-valued* $\mathcal{F}_0$*-measurable random variable with* $E\|a_0\|^4 < \infty$*;*
(3) $F_1 \in \mathcal{L}^1_{\mathbb{R}}(\Omega \times [0,T]), F_2 \in \mathcal{L}^2_H(\Omega \times [0,T])$.
(4) $F_3 : [0,T] \times H \to H$ *is a mapping such that for all* $t \in [0,T], x \in H$ *we have* $\|F_3(t,x)\| \le k_{F_3}\|x\|$ *with* k_{F_3} *a positive constant and* $F_3(\cdot, x) \in \mathcal{L}^2_H[0,T]$ *for all* $x \in H$*;*
(5) $\Big(Q(t)\Big)_{t\in[0,T]}$ *is a* V*-valued process with*

$$\int_0^T \|Q(s)\|_V^2 ds < \infty \quad and \quad \sup_{t\in[0,T]} \|Q(t)\|^2 < \infty \quad for\ a.e.\ \omega \in \Omega,$$

satisfying the inequality

$$\begin{aligned}
\|Q(t)\|^2 + k_1\int_0^t \|Q(s)\|_V^2 ds \ &\le\ \|a_0\|^2 + k_2\int_0^t \|Q(s)\|^2 ds \\
&+\ \int_0^t |F_1(s)| ds + \int_0^t (F_2(s) + F_3(s, Q(s)), Q(s)) dw(s)
\end{aligned}$$

for all $t \in [0,T]$ *and a.e.* $\omega \in \Omega$. *Then there exists a positive constant* c *(depending on* k_1, k_2, k_{F_3}, T*) such that*

$$E \sup_{t\in[0,T]} \|Q(t)\|^2 + E\int_0^T \|Q(s)\|_V^2 ds \leq c\Big[E\|a_0\|^2 + E\int_0^T |F_1(s)| ds + E\int_0^T \|F_2(s)\|^2 ds\Big] \tag{B.1}$$

and if $E\int_0^T |F_1(s)|^2 ds < \infty,\ E\int_0^T \|F_2(s)\|^4 ds < \infty$ *then*

$$E \sup_{t\in[0,T]} \|Q(t)\|^4 + E\Big(\int_0^T \|Q(s)\|_V^2 ds\Big)^2 \leq c\Big[E\|a_0\|^4 + E\int_0^T |F_1(s)|^2 ds + E\int_0^T \|F_2(s)\|^4 ds\Big]. \tag{B.2}$$

Proof. We consider the stopping times $\mathcal{T}_M := \mathcal{T}_M^Q$, $M \in \mathbb{N}$. Using **(5)** it follows that for all $t \in [0,T]$

$$\begin{aligned}\sup_{s\in[0,t\wedge\mathcal{T}_M]} \|Q(s)\|^2 &+ k_1 \int_0^{t\wedge\mathcal{T}_M} \|Q(s)\|_V^2 ds \leq 2\|a_0\|^2 + 2k_2 \int_0^{t\wedge\mathcal{T}_M} \|Q(s)\|^2 ds \\ &+ 2\int_0^{t\wedge\mathcal{T}_M} |F_1(s)| ds + 2 \sup_{s\in[0,t\wedge\mathcal{T}_M]} \Big|\int_0^s (F_2(r) + F_3(r,Q(r)), Q(r)) dw(r)\Big|.\end{aligned}$$

and

$$\begin{aligned}\sup_{s\in[0,t\wedge\mathcal{T}_M]} \|Q(s)\|^4 &+ k_1^2\Big(\int_0^{t\wedge\mathcal{T}_M} \|Q(s)\|_V^2 ds\Big)^2 \leq 16\|a_0\|^4 + 16k_2^2\Big(\int_0^{t\wedge\mathcal{T}_M} \|Q(s)\|^2 ds\Big)^2 \\ &+ 16\Big(\int_0^{t\wedge\mathcal{T}_M} |F_1(s)| ds\Big)^2 + 16 \sup_{s\in[0,t\wedge\mathcal{T}_M]} \Big|\int_0^s (F_2(r) + F_3(r,Q(r)), Q(r)) dw(r)\Big|^2.\end{aligned}$$

Now we use the Burkholder inequality (see [18], p. 166) and the Schwarz inequality to obtain

$$\begin{aligned}E \sup_{s\in[0,t\wedge\mathcal{T}_M]} \|Q(s)\|^2 + k_1 E\int_0^{t\wedge\mathcal{T}_M} \|Q(s)\|_V^2 ds &\leq 2E\|a_0\|^2 + 2k_2 E\int_0^{t\wedge\mathcal{T}_M} \|Q(s)\|^2 ds \\ + 2E\int_0^{t\wedge\mathcal{T}_M} |F_1(s)| ds + \frac{1}{2}E \sup_{s\in[0,t\wedge\mathcal{T}_M]} \|Q(s)\|^2 &+ c_1 E\int_0^{t\wedge\mathcal{T}_M} \|F_2(s) + F_3(s,Q(s))\|^2 ds\end{aligned}$$

and

$$\begin{aligned}E \sup_{s\in[0,t\wedge\mathcal{T}_M]} \|Q(s)\|^4 + k_1^2 E\Big(\int_0^{t\wedge\mathcal{T}_M} \|Q(s)\|_V^2 ds\Big)^2 &\leq 16E\|a_0\|^4 + 16k_2^2 TE\int_0^{t\wedge\mathcal{T}_M} \|Q(s)\|^4 ds \\ + 16TE\int_0^{t\wedge\mathcal{T}_M} |F_1(s)|^2 ds + \frac{1}{2}E \sup_{s\in[0,t\wedge\mathcal{T}_M]} \|Q(s)\|^4 &+ c_2 E\int_0^{t\wedge\mathcal{T}_M} \|F_2(s) + F_3(s,Q(s))\|^4 ds\end{aligned}$$

for all $t \in [0,T]$, where c_1, c_2 are positive constants. Consequently, for all $t \in [0,T]$, we have

$$E \sup_{s\in[0,t]} I_{[0,\mathcal{T}_M]}(s)\|Q(s)\|^2 + 2k_1 E\int_0^t I_{[0,\mathcal{T}_M]}(s)\|Q(s)\|_V^2 ds \leq 4E\|a_0\|^2$$

$$+ \quad 4(k_2 + c_1 k_{F_3})E\int_0^t \sup_{r\in[0,s]} I_{[0,\mathcal{T}_M]}(r)\|Q(r)\|^2 dr + 4E\int_0^T |F_1(s)|ds + 4c_1 E\int_0^T \|F_2(s)\|^2 ds$$

and

$$E \sup_{s\in[0,t]} I_{[0,\mathcal{T}_M]}(s)\|Q(s)\|^4 + 2k_1^2 E\Big(\int_0^t I_{[0,\mathcal{T}_M]}(s)\|Q(s)\|_V^2 ds\Big)^2 \leq 32E\|a_0\|^4$$

$$+ \quad (32k_2^2 T + 16c_2 k_{F_3})E\int_0^t \sup_{r\in[0,s]} I_{[0,\mathcal{T}_M]}(r)\|Q(r)\|^4 dr + 32TE\int_0^T |F_1(s)|^2 ds + 16c_2 E\int_0^T \|F_2(s)\|^4 ds.$$

By Gronwall's Lemma it follows that there exists a positive constant c^* (independent of M) such that

$$E \sup_{s\in[0,T\wedge\mathcal{T}_M]} \|Q(s)\|^2 + 2k_1 E \int_0^{T\wedge\mathcal{T}_M} \|Q(s)\|_V^2 ds \leq c^*\Big[E\|a_0\|^2 + E\int_0^T |F_1(s)|ds + E\int_0^T \|F_2(s)\|^2 ds\Big]$$

and

$$E \sup_{s\in[0,T\wedge\mathcal{T}_M]} \|Q(s)\|^4 + 2k_1^2 E\Big(E \int_0^{T\wedge\mathcal{T}_M} \|Q(s)\|_V^2 ds\Big)^2 \leq c^*\Big[E\|a_0\|^4 + E\int_0^T |F_1(s)|^2 ds + E\int_0^T \|F_2(s)\|^4 ds\Big].$$

Now we use Proposition B.1, take the limit $M \to \infty$ in the above inequalities to obtain (B.1) and (B.2). ■

Proposition B.3
Let $(\mathcal{T}_M)$ and $\mathcal{T}$ be stopping times, such that

$$\lim_{M\to\infty} P(\mathcal{T}_M < \mathcal{T}) = 0.$$

Let (Q_n) be a sequence of processes from the space $\mathcal{L}^2_{\mathbb{R}}([0,T]\times\Omega)$ such that for each fixed M we have

$$\lim_{n\to\infty} E|Q_n(\mathcal{T}_M)| = 0$$

and there exists a positive constant c independent of n such that

$$E|Q_n(\mathcal{T})|^2 < c \quad for\ all \quad n \in \mathbb{N}.$$

Then

$$\lim_{n\to\infty} E|Q_n(\mathcal{T})| = 0.$$

Proof. Let $\varepsilon, \delta > 0$. There exists $M_0 \in \mathbb{N}$ such that

$$P(\mathcal{T}_{M_0} < \mathcal{T}) \leq \frac{\varepsilon}{2}.$$

By the hypothesis it follows that for this M_0 we have $\lim_{n\to\infty} E|Q_n(\mathcal{T}_{M_0})| = 0$. Consequently, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{\delta} E|Q_n(\mathcal{T}_{M_0})| \leq \frac{\varepsilon}{2}$$

for all $n \geq n_0$. We write

$$\begin{aligned} P\Big(|Q_n(\mathcal{T})| \geq \delta\Big) &\leq P\Big(\mathcal{T}_{M_0} < \mathcal{T}\Big) + P\Big(\{\mathcal{T}_{M_0} = \mathcal{T}\} \wedge \Big\{|Q_n(\mathcal{T})| \geq \delta\Big\}\Big) \\ &\leq \frac{\varepsilon}{2} + P\Big(|Q_n(\mathcal{T}_{M_0})| \geq \delta\Big) \leq \frac{\varepsilon}{2} + \frac{1}{\delta} E|Q_n(\mathcal{T}_{M_0})| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for all $n \geq n_0$. Hence for all $\delta > 0$ we get $\lim_{n\to\infty} P\Big(|Q_n(\mathcal{T})| \geq \delta\Big) = 0$. Therefore, the sequence $\Big(|Q_n(\mathcal{T})|\Big)$ converges in probability to zero. From the hypothesis it follows that this sequence is uniformly integrable (with respect to $\omega \in \Omega$). Hence it converges also in mean to zero

$$\lim_{n\to\infty} E|Q_n(\mathcal{T})| = 0.$$ ■

Proposition B.4

Let $\widehat{\mathcal{F}} \subseteq \mathcal{F}$ be a σ-algebra, (Q_n) be a sequence of H-valued random variables which converges for a.e. $\omega \in \Omega$ to Q. If each random variable Q_n is independent of $\widehat{\mathcal{F}}$, then Q is independent of $\widehat{\mathcal{F}}$.

Proof. The random variable Q is independent of $\widehat{\mathcal{F}}$ if

(B.3) $$P(\{\|Q\| < x\} \cap A) = P(\|Q\| < x)P(A)$$

for all $x \in \mathbb{R}, A \in \widehat{\mathcal{F}}$. The hypothesis implies that the sequence $\Big(\|Q_n\|\Big)$ converge in probability to $\|Q\|$. Therefore, the sequence of their distribution functions is convergent

(B.4) $$\lim_{n\to\infty} F_{\|Q_n\|}(x) = F_{\|Q\|}(x)$$

for each $x \in \mathbb{R}$ which is continuity point of $F_{\|Q\|}$.

Let $x \in \mathbb{R}, A \in \widehat{\mathcal{F}}, \delta > 0$. First we consider that $F_{\|Q\|}$ is continuous in x. Then using the hypothesis and (B.4) we get

(B.5) $$\lim_{n\to\infty} P(\{\|Q_n\| < x\} \cap A) = \lim_{n\to\infty} P(\|Q_n\| < x)P(A) = P(\|Q\| < x)P(A).$$

We write

$$\begin{aligned} P(\{\|Q\| < x - \delta\} \cap A) &\leq P(\{\|Q\| < x - \delta\} \cap \{\|Q_n\| < x\} \cap A) \\ &+ P(\{\|Q\| < x - \delta\} \cap \{\|Q_n\| \geq x\} \cap A) \\ &\leq P(\{\|Q_n\| < x\} \cap A) + P\Big(\Big|\|Q\| - \|Q_n\|\Big| > \delta\Big). \end{aligned}$$

Analogously we have

$$P(\{\|Q_n\| < x\} \cap A) \leq P(\{\|Q\| < x + \delta\} \cap A) + P\Big(\Big|\|Q\| - \|Q_n\|\Big| > \delta\Big).$$

Consequently,

$$\begin{aligned} P(\{\|Q\| < x - \delta\} \cap A) - P\Big(\Big|\|Q\| - \|Q_n\|\Big| > \delta\Big) &\leq P(\|Q_n\| < x)P(A) \\ &\leq \quad P(\{\|Q\| < x + \delta\} \cap A) + P\Big(\Big|\|Q\| - \|Q_n\|\Big| > \delta\Big). \end{aligned}$$

In the inequalities above we take the limit $n \to \infty$ and use (B.5) to obtain

$$P(\{\|Q\| < x - \delta\} \cap A) \leq P(\|Q\| < x)P(A) \leq P(\{\|Q\| < x + \delta\} \cap A).$$

Let $\delta \searrow 0$ in the inequalities above. Then

$$P(\{\|Q\| \leq x\} \cap A) \leq P(\|Q\| < x)P(A) \leq P(\{\|Q\| \leq x\} \cap A).$$

Since x is a point of continuity for $F_{\|Q\|}$ we have

$$P(\{\|Q\| \leq x\} \cap A) = P(\{\|Q\| < x\} \cap A).$$

Consequently, (B.3) holds and Q is independent of $\widehat{\mathcal{F}}$.

Now we consider that x is not a point of continuity of $F_{\|Q\|}$. Let (x_n) be a monotone increasing sequence of continuity points of $F_{\|Q\|}$ which converges to x. Then

$$\lim_{n\to\infty} F_{\|Q\|}(x_n) = F_{\|Q\|}(x),$$

and because x_n is a point of continuity for $F_{\|Q\|}$, we have

$$P(\{\|Q\| < x_n\} \cap A) = P(\|Q\| < x_n)P(A).$$

Now we take the limit $n \to \infty$ and conclude that (B.3) holds. Hence Q is independent of $\widehat{\mathcal{F}}$. ■

Proposition B.5
Let $\widehat{\mathcal{F}} \subseteq \mathcal{F}$ be a σ-algebra, $f : H \times \Omega \to H$ be a mapping such that for each $x \in H$ the random variable $f(x,\cdot)$ is bounded, measurable and independent of $\widehat{\mathcal{F}}$. Let ξ be a H-valued $\widehat{\mathcal{F}}$-measurable random variable. Then

$$E(f(\xi,\omega)|\widehat{\mathcal{F}}) = E(f(\xi,\omega)|\sigma_{[\xi]}),$$

where $\sigma_{[\xi]}$ is the σ-algebra generated by the random variable ξ.

This Proposition can be proved analogously to Lemma 1, p. 63 in [11]. ■