A Counterexample to the Modularity of Unicity of Normal Forms for Join Conditional Term Rewriting Systems

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Abstract
In [1], Middeldorp proved that unicity of normal forms is a modular property for semi-equational conditional term rewriting systems. It remained an open problem, however, whether this is also the case for join conditional term rewriting systems. This short paper answers the question in the negative by giving a counterexample.

1 Preliminaries

1.1 Abstract reduction systems
A pair \( A = (A, \rightarrow) \) is called an abstract reduction system if \( A \) is a set and \( \rightarrow \), the reduction relation of \( A \), is a binary relation on \( A \).

Let \( \xrightarrow{*} \) be the reflexive-transitive closure of \( \rightarrow \) and let \( \leftrightarrow \) be the reflexive-symmetric-transitive closure of \( \rightarrow \).

For \( a, b \in A \) let \( a \downarrow b \) if there is \( c \in A \) with \( a \xrightarrow{*} c \) and \( b \xrightarrow{*} c \).

\( a, b \in A \) are called convertible if \( a \leftrightarrow b \) and they are called joinable if \( a \downarrow b \).

\( a \in A \) is called a normal form of \( A \) if \( \neg \exists b \in A \ a \rightarrow b \).

\( a \in A \) is called isolated in \( A \) if \( \neg \exists b \in A \ b \rightarrow a \).

Lemma 1.1 Let \( a, b \in A \) with \( a \leftrightarrow b \) and let \( a \) be an isolated normal form of \( A \). Then \( a = b \).

Proof. Since \( a \) is isolated, there is no \( d \in A \) with \( d \rightarrow a \).

Since \( a \) is a normal form, there is no \( d \in A \) with \( a \rightarrow d \).

So, from \( a \leftrightarrow b \) it follows that \( a = b \). \( \square \)

\( A \) is called terminating if there is no \( (a_k)_{k \in \mathbb{N}} \in A^\mathbb{N} \) with \( \forall k \in \mathbb{N} \ a_k \rightarrow a_{k+1} \).

\( A \) is called confluent if \( \forall a, b \in A \ (a \leftrightarrow b \Rightarrow a \downarrow b) \).

\( A \) has the unique normal forms property, or \( A \) is UN for short, if for all normal forms \( a, b \) of \( A \), \( a \leftrightarrow b \) implies \( a = b \).
Lemma 1.2  If $A$ is confluent, then $A$ is UN.

Proof. This is an easy standard result. □

Lemma 1.3  The following two assertions are equivalent:

(i) $A$ is UN.

(ii) for all non-isolated normal forms $a, b$ of $A$, $a \leftrightarrow b$ implies $a = b$.

Proof. The direction “(i) $\Rightarrow$ (ii)” is trivial.
For the other direction, assume (ii) and take any two normal forms $a, b$ with $a \leftrightarrow b$.
If both $a, b$ are non-isolated, then (ii) gives $a = b$;
otherwise, at least one of them is isolated, and so, by Lemma 1.1, $a = b$. □

1.2 Signatures, terms and substitutions

We always use the same set of variables $V$, consisting of the variables $x_1, x_2, \ldots$.
A pair $F = (F^*, \rho)$ is called a signature, if $F^*$, the set of function symbols of $F$, has the property $F^* \cap V = \emptyset$ and $\rho$, the arity function of $F$, maps $F^*$ to $\mathbb{N}$.
The $f \in F^*$ with $\rho(f) = 0$ are called constants.

$T(F)$, the set of terms of $F$, is defined inductively by

(i) $V \subseteq T(F)$

(ii) If $f \in F^*$, $n := \rho(f)$ and $t_1, \ldots, t_n \in T(F)$ then $f(t_1, \ldots, t_n) \in T(F)$.

If $\rho(f) = 0$, then $f$ is written instead of $f()$.

For $t \in T(F)$ let $FS(t)$ denote the set of all function symbols of $F$ appearing in $t$ and let $\text{Var}(t)$ denote the set of all variables appearing in $t$.

We define the length $|t|$ of $t \in T(F)$ inductively by

(i) If $v \in V$ then $|v| := 1$.

(ii) If $f \in F^*$, $n := \rho(f)$ and $t_1, \ldots, t_n \in T(F)$ then

$$|f(t_1, \ldots, t_n)| := 1 + |t_1| + \ldots + |t_n|.$$  

A function $\sigma : T(F) \rightarrow T(F)$ is called a substitution on $F$ if

$$\forall f \in F^* \forall t_1, \ldots, \tau(t) \in T(F) \quad \sigma(f(t_1, \ldots, t_\tau(f))) = f(\sigma(t_1), \ldots, \sigma(t_\tau(f)))$$

and there are only finitely many $v \in V$ with $\sigma(v) \neq v$.

As usual, $t^\sigma$ is written instead of $\sigma(t)$.

An easy induction on the complexity of the argument proves that a substitution is uniquely determined by its values on $V$.  

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Lemma 1.4 Let $\sigma$ be a substitution on $\mathcal{F}$ and $t \in \mathcal{T}(\mathcal{F})$. Then we have $\text{FS}(t) \subset \text{FS}(t^\sigma)$.

Proof. This is an easy induction on the complexity of $t$. □

Up to Lemma 1.5, let $s, t \in \mathcal{T}(\mathcal{F})$ with $\text{Var}(s) \cap \text{Var}(t) = \emptyset$. Then $s$ and $t$ are called unifiable if there is a substitution $\sigma$ on $\mathcal{F}$ with $s^\sigma = t^\sigma$.

A substitution $\sigma$ on $\mathcal{F}$ is called a most general unifier (mgu) of $s$ and $t$ if

(i) $s^\sigma = t^\sigma$

(ii) for all substitutions $\tau$ on $\mathcal{F}$ with $s^\tau = t^\tau$ there is a substitution $\gamma$ on $\mathcal{F}$ with $\tau = \gamma \circ \sigma$.

Lemma 1.5 Let $s$ and $t$ be unifiable. Then we have

1. There is an mgu of $s$ and $t$.

2. Let $\sigma, \tau$ be two mgu’s of $s$ and $t$. Then there is a substitution $\gamma$ on $\mathcal{F}$ such that $\tau = \gamma \circ \sigma$ and $\gamma$ is a bijection on $\mathcal{V}$.

Proof. See [2] Theorem 4.5.8 and Lemma 4.5.3. □

1.3 Join conditional rewriting systems

Some of the definitions in this subsection are a little bit “rough”. For details, see for instance [1] and [3].

A pair $\mathcal{S} = (\mathcal{F}, \mathcal{R})$ is called a join conditional term rewriting system (jCTRS), if $\mathcal{F}$ is a signature and $\mathcal{R}$ is a set of rules

$$(R) \quad s \rightarrow t \quad \Leftarrow \quad p_1 \downarrow q_1 \land \ldots \land p_n \downarrow q_n,$$

where

(i) $(R)$ is the name of the rule

(ii) $s \in \mathcal{T}(\mathcal{F})$ with $s \notin \mathcal{V}$

(iii) $t \in \mathcal{T}(\mathcal{F})$ with $\text{Var}(t) \subset \text{Var}(s)$

(iv) $n \in \mathbb{N}$ and $p_1, \ldots, p_n, q_1, \ldots, q_n \in \mathcal{T}(\mathcal{F})$.

The subformula $p_1 \downarrow q_1 \land \ldots \land p_n \downarrow q_n$ is called the condition of the rule $(R)$. Of course, if $n = 0$, everything concerning the condition has always to be dropped. In this case the rule is called unconditioned and has the form

$$(R) \quad s \rightarrow t.$$
Now, let □ be a fresh constant symbol, called a hole and let \( F_\square \) be the extension of \( F \) with this new constant.

\( C \) is called a (unary) context on \( F \), if \( C \in \mathcal{T}(F_\square) \) and there is exactly one hole in \( C \). If \( t \in \mathcal{T}(F) \) then \( C[t] \) denotes the result of replacing this hole by \( t \).

For \( l, r \in \mathcal{T}(F) \) and \((R) \in \mathcal{R} \), we simultaneously define \( l \rightarrow_{(R)} r \) and \( l \rightarrow_{R} r \) inductively by

(i) Consider a rule

\[
(R) \quad s \rightarrow t \iff p_1 \downarrow q_1 \land \ldots \land p_n \downarrow q_n
\]

of \( \mathcal{R} \). Then \( l \rightarrow_{(R)} r \) if there is a substitution \( \sigma \) and a context \( C \) with

\[
l = C[s^\sigma] \land r = C[t^\sigma] \\
\land \forall j \in \{1, \ldots, n\} \quad p_j^\sigma \downarrow_R q_j^\sigma.
\]

(ii) \( l \rightarrow_{R} r \) if there is a \((R) \in \mathcal{R} \) with \( l \rightarrow_{(R)} r \).

The index \( \mathcal{R} \) often is suppressed, if it is clear which set of rules is considered.

**Lemma 1.6** Let

\[
(R) \quad s \rightarrow t \iff p_1 \downarrow q_1 \land \ldots \land p_n \downarrow q_n
\]

be in \( \mathcal{R} \) and let \( l, r \in \mathcal{T}(F) \) with \( l \rightarrow_{(R)} r \). Then we have

1. \( \text{FS}(s) \subset \text{FS}(l) \)
2. \( \text{FS}(t) \subset \text{FS}(r) \).

**Proof.** This follows easily from Lemma 1.4. \qed

Up to Theorem 1.1 we assume that the left-hand-sides of distinct rules in \( \mathcal{R} \) have disjoint sets of variables. (This can always be achieved by renumbering the variables.)

Let there be two rules

\[
(R) \quad s \rightarrow t \iff p_1 \downarrow q_1 \land \ldots \land p_n \downarrow q_n \\
(R') \quad s' \rightarrow t' \iff p'_1 \downarrow q'_1 \land \ldots \land p'_m \downarrow q'_m
\]

in \( \mathcal{R} \) and \( u \in \mathcal{T}(F) \setminus \mathcal{V} \) and let \( C \) be a context with

\[
s = C[u]
\]

and \( u \) and \( s' \) are unifiable with the mgu \( \sigma \).

Then the formula

\[
t^\sigma = (C[t'])^\sigma \iff p_1^\sigma \downarrow q_1^\sigma \land \ldots \land p_n^\sigma \downarrow q_n^\sigma \land p'_1^\sigma \downarrow q'_1^\sigma \land \ldots \land p'_m^\sigma \downarrow q'_m^\sigma
\]

is called a critical pair of \( \mathcal{S} \).
The pair is called an overlay if \( s = u \).

A critical pair

\[
l = r \iff \exists_{p_1 \downarrow q_1 \land \ldots \land p_n \downarrow q_n}
\]

is called joinable if for all substitutions \( \tau \)

\[
( p_1^\tau \downarrow_R q_1^\tau \land \ldots \land p_n^\tau \downarrow_R q_n^\tau \Rightarrow l^\tau \downarrow_R r^\tau )
\]

and it is called trivial if \( l = r \).

**Theorem 1.1** Let \( \mathcal{S} \) be terminating and all its critical pairs are joinable overlays. Then \( \mathcal{S} \) is confluent.

**Proof.** See [4], Theorem 4. \( \square \)

Now, for \( j \in \{1, 2\} \), let \( \mathcal{S}_j = (\mathcal{F}_j, \mathcal{R}_j) \) be a jCTRS and \( \mathcal{F}_j = (\mathcal{F}_j^s, \rho_j) \).

Let \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) be disjoint, which means that \( \mathcal{F}_1^s \cap \mathcal{F}_2^s = \emptyset \).

Then \( \mathcal{F}_\oplus := (\mathcal{F}_1^s \cup \mathcal{F}_2^s, \rho_1 \cup \rho_2) \) is a signature

and so \( \mathcal{S}_1 \oplus \mathcal{S}_2 := (\mathcal{F}_\oplus, \mathcal{R}_1 \cup \mathcal{R}_2) \) is a jCTRS, called the direct sum of \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \).

A property is called modular for jCTRS if for all disjoint jCTRS’s \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) the direct sum \( \mathcal{S}_1 \oplus \mathcal{S}_2 \) satisfies the property if and only if both \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) do.

## 2 The counterexample

The idea behind the following counterexample is to take a condition, which forces a certain term at the position of a variable.

If for a rule two of these conditions are taken, which force two different terms \( b_1 \) resp. \( b_2 \) for the same variable \( x \), then these conditions contradict each other. So the rule cannot be applied in the system, it is “sleeping”.

On the other hand, in the direct sum a “forking” can be substituted for \( x \), namely a term \( F(b_1, b_2) \), where \( F \) is a function symbol of the other system, so that the substituted term can be reduced to \( b_1 \) as well as to \( b_2 \). Now the sleeping rule is applicable, which gives the desired contradiction to UN (see Figure 1).

![Figure 1: “Waking up” the sleeping rule makes the direct sum not to be UN.](image-url)
All what is left to do is to ensure, that each of the two single systems is UN:

• In the system that gives the forking, the condition of the rule is used to reach confluence. (This will be proved using Theorem 1.1.)

• In the system with the “sleeping” rule, rules of the form “\( t \rightarrow t \)” are introduced to avoid normal forms, with the exception of only two constants, that are used for the contradiction. One of these constants can only be reached using the sleeping rule; so it is isolated in the single system. Now, Lemma 1.3 implies that the system is UN.

\textbf{Def. 2.1} Let the jCTRS \( S_1 = (\mathcal{F}_1, \mathcal{R}_1) \) be defined by

- the signature \( \mathcal{F}_1 = (\mathcal{F}_1^s, \rho_1) \) with
  \( \mathcal{F}_1^s := \{ F \} \) and \( \rho_1(F) := 2 \)

- the rules
  \[
  \mathcal{R}_1 := \begin{cases}
  (R1) & F(x_1, x_2) \rightarrow x_1 \quad \text{“forking, left side”} \\
  (R2) & F(x_3, x_4) \rightarrow x_4 \quad \quad \iff \quad x_3 \downarrow x_4 \quad \text{“forking, right side”}
  \end{cases}
  \]

\textbf{Def. 2.2} Let the jCTRS \( S_2 = (\mathcal{F}_2, \mathcal{R}_2) \) be defined by

- the signature \( \mathcal{F}_2 = (\mathcal{F}_2^s, \rho_2) \) with
  \( \mathcal{F}_2^s := \{ f, g_1, g_2, a_1, a_2, b_1, b_2, c_1, c_2, d \} \)
  and \( \rho_2(f) := \rho_2(g_1) := \rho_2(g_2) := 1 \) \( \land \rho_2(a_1) := \rho_2(a_2) := \rho_2(b_1) := \rho_2(b_2) := \rho_2(c_1) := \rho_2(c_2) := \rho_2(d) := 0 \)

- the rules
  \[
  \mathcal{R}_2 := \begin{cases}
  (Ra) & f(x) \rightarrow a_1 \quad \quad \text{“active rule”} \\
  (Rs) & f(x) \rightarrow a_2 \quad \iff \quad g_1(x) \downarrow c_1 \quad \land \quad g_2(x) \downarrow c_2 \quad \text{“sleeping rule”} \\
  (S1) & g_1(b_1) \rightarrow c_1 \\
  (S2) & g_2(b_2) \rightarrow c_2 \end{cases}
  \]
  \[
  \begin{cases}
  (F1) & b_1 \rightarrow d \\
  (F2) & b_2 \rightarrow d \end{cases} \quad \begin{cases}
  (Ng_1) & g_1(x) \rightarrow g_1(x) \\
  (Ng_2) & g_2(x) \rightarrow g_2(x) \\
  (Nc_1) & c_1 \rightarrow c_1 \\
  (Nc_2) & c_2 \rightarrow c_2 \\
  (Nd) & d \rightarrow d \end{cases} \quad \text{“conditions for forking”}
  \]
  \[
  \begin{cases}
  (Ng_1) & g_1(x) \rightarrow g_1(x) \\
  (Ng_2) & g_2(x) \rightarrow g_2(x) \\
  (Nc_1) & c_1 \rightarrow c_1 \\
  (Nc_2) & c_2 \rightarrow c_2 \\
  (Nd) & d \rightarrow d \end{cases} \quad \text{“prevention of normal forms”}
  \]

(\text{where for readability the index “1” of the variable } x \text{ is suppressed}).
Lemma 2.1 \( S_1 \) is confluent.

Proof. This will be proved using Theorem 1.1:

\( S_1 \) is terminating because for all \( a, b \in T(\mathcal{F}_1), \ a \rightarrow_{\mathcal{R}_1} b \) implies \( |a| > |b| \).

All the rules of the system have the left-hand-side \( F(x, y) \), where \( x, y \) are different variables. Since the only non-variable subterm of this term is the term itself, all critical pairs must be overlays.

An mgu \( \sigma \) of \( f(x_1, x_2) \) and \( f(x_3, x_4) \) is given by

\[
\sigma(x_1) := x_3 \quad \land \quad \sigma(x_2) := x_4 \quad \land \quad \forall j > 2 \ \sigma(x_j) := x_j,
\]

so essentially, the only non-trivial critical pair of \( S_1 \) is given by

\[
x_3 = x_4 \iff x_3 \downarrow x_4,
\]

which, of course, is joinable.

The trivial critical pairs (generally) are joinable. \( \Box \)

Lemma 2.2 \( S_1 \) is UN.

Proof. This follows from Lemma 2.1 using Lemma 1.2. \( \Box \)

Up to Lemma 2.5 we suppress the index “\( \mathcal{R}_2 \)”.

Lemma 2.3 Let \( j \in \{1, 2\} \) and \( t \in T(\mathcal{F}_2) \). Then we have

1. \( (c_j \rightarrow t) \iff t = c_j \)
2. \( (c_j \rightsquigarrow t) \iff t = c_j \)
3. \( (t \rightarrow c_j) \iff t = c_j \lor t = g_j(b_j) \)
4. \( (t \rightarrow g_j(b_j) \iff t = g_j(b_j) \)
5. \( (t \rightsquigarrow c_j \iff t = c_j \lor t = g_j(b_j) \)
6. \( (t \downarrow c_j \land t \neq c_j \iff t = g_j(b_j) \)

Proof. All implications “\( \Leftarrow \)” trivially follow from the obvious rules.

Now, we prove the other direction:

\( ad 1 \). Let \( c_j \rightarrow_{(R)} t \) with \( (R) \in \mathcal{R}_2 \).

Then, by Lemma 1.6.1, we have \( (R) = (Nc_j) \), so \( t = c_j \).

\( ad 2 \). This follows from 1. by induction on the number of reduction steps in \( c_j \rightsquigarrow t \).
ad 3. Let \( t \rightarrow_{(R)} c_j \) with \((R) \in \mathcal{R}_2\).
Then, by Lemma 1.6.2, we have \((R) \in \{(S_j), (Nc_j)\}\).
If \((R) = (S_j)\) then \( t = g_j(b_j)\); otherwise \( t = c_j\).

ad 4. Let \( t \rightarrow_{(R)} g_j(b_j) \) with \((R) \in \mathcal{R}_2\).
Then, by Lemma 1.6.2, we have \((R) = (Ng_j)\), so \( t = g_j(b_j)\).

ad 5. This follows from 3. and 4. .

ad 6. Let \( t \downarrow c_j \) and \( t \neq c_j \).
From the first assumption, there is an \( s \in \mathcal{T}(\mathcal{F}_2) \) with \( t \rightarrow s \) and \( c_j \rightarrow s \).
Since \( c_j \rightarrow s \), it follows by 2. that \( s = c_j \).
So, with \( t \rightarrow s \) and \( t \neq c_j \) it follows by 5. that \( t = g_j(b_j)\). \(\square\)

Lemma 2.4 \( \forall l, r \in \mathcal{T}(\mathcal{F}_2) \neg (l \rightarrow_{(Rs)} r) \)

Proof. Assume there are \( l, r \in \mathcal{T}(\mathcal{F}_2) \) with \( l \rightarrow_{(Rs)} r \).
Then there is a context \( C \) and a substitution \( \sigma \) on \( \mathcal{F}_2 \) with
\[
    l = C[f(x)]^{\sigma} \quad \land \quad r = C[a_2^{\sigma}]
\]
and
\[
    (g_1(x_1))^{\sigma} \downarrow c_1^{\sigma} \quad \land \quad (g_2(x_1))^{\sigma} \downarrow c_2^{\sigma}. \quad (1)
\]
Since \( c_1, c_2 \) are constants, \( c_1^{\sigma} = c_1 \) \( \land \quad c_2^{\sigma} = c_2 \). \(\quad (2)\)
With \( s := x_1^{\sigma} \in \mathcal{T}(\mathcal{F}_2) \),
it follows from (1) and (2) that
\[
    g_1(s) \downarrow c_1 \quad \land \quad g_2(s) \downarrow c_2,
\]
so, by Lemma 2.3.6
\[
    b_1 = s = b_2, \quad \text{contradiction.} \quad \square
\]

Lemma 2.5 \( \mathcal{S}_2 \) is UN.

Proof. Because of \((Ra), (Ng_j), (Fj), (Nc_j)\) and \((Nd) \) (for \( j \in \{1, 2\} \))
none of the function signs \( f, g_j, b_j, c_j, d \) can appear in a normal form of \( \mathcal{S}_2 \).
So, the only possible normal forms of \( \mathcal{S}_2 \) are \( a_1, a_2 \) and the \( v \in \mathcal{V} \).
Because of Lemma 1.6.2, each \( v \in \mathcal{V} \) is isolated.
Because of Lemma 1.6.2 and Lemma 2.4, \( a_2 \) is isolated.
So, only \( a_1 \) can be a non-isolated normal form of \( \mathcal{S}_2 \).
Thus, Lemma 1.3 implies that \( \mathcal{S}_2 \) is UN. \(\square\)
Lemma 2.6 \( S_1 \oplus S_2 \) is not UN.

Proof. In this proof, we suppress the index “\( \mathcal{R}_1 \cup \mathcal{R}_2 \)”. Because of (F1) and (F2) 
\[ b_1 \downarrow b_2, \]
so, by (R1) and (R2) 
\[ \forall j \in \{1, 2\} \quad F(b_1, b_2) \rightarrow b_j, \]
thus, with (Sj), 
\[ \forall j \in \{1, 2\} \quad g_j \left( F(b_1, b_2) \right) \rightarrow g_j(b_j) \rightarrow c_j, \]
especially 
\[ \forall j \in \{1, 2\} \quad g_j \left( F(b_1, b_2) \right) \downarrow c_j. \]
So, by (Rs) 
\[ f \left( F(b_1, b_2) \right) \rightarrow a_2. \]
Since furthermore by (Ra) 
\[ f \left( F(b_1, b_2) \right) \rightarrow a_1, \]
it follows that 
\[ a_1 \leftrightarrow a_2. \]
On the other hand, by Lemma 1.6.1, \( a_1, a_2 \) are normal forms of \( S_1 \oplus S_2 \). So, \( S_1 \oplus S_2 \) cannot be UN. \( \square \)

Theorem 2.1 UN is not a modular property for join conditional rewriting systems.

Proof. Since \( S_1 \) and \( S_2 \) are disjoint, Lemmata 2.2, 2.5 and 2.6 show that these two jCTRS’s are a counterexample.

3 Concluding remarks

1. The idea of a sleeping rule might be useful for other questions about modularity for jCTRS.

2. Of course, another example can be constructed by deleting the \( g_j \):
   - \( g_1, g_2 \) are left out in (Rs), (S1) and (S2).
   - The rules (Ng1) and (Ng2) can be dropped.
   - Now, only some straightforward changes have to be done at Lemma 2.3 and the proofs of Lemmata 2.4, 2.5 and 2.6.

Despite of the obvious advantage of reducing the number of rules and function signs, I think that the original counterexample is easier to see:
If \( g_1, g_2 \) are left out, the conditions of (Rs) are fulfilled, if and only if \( c_j \) or \( b_j \) are substituted for \( x_1 \). So for each condition, there is not only one but a pair of possible insertions for \( x_1 \), which makes the example a little more complicated to understand.
References


