Preface

> *'The attractiveness of multifractals may to some extent be due to mystery. All mystery is eliminated, however, when one understands the nature of the formal manipulations.'*
>
> *B. Mandelbrot*[1]

In November 1996, the German national science foundation DFG[2] sponsored a workshop at the University of Göttingen, entitled 'Fractals and Dynamics'. This workshop concentrated on the great breadth of the theory of fractals. Special emphasis was given to the newly developing theory of multifractals. Several other related topics were represented as well; such as fractional calculus, harmonic analysis for random walks and certain aspects of complex dynamics.
The basic goal was to bring together researchers from diverse areas of the theory of fractals and in particular multifractals, to inform about new developments, and to encourage further interaction.
The proceedings give elaborated abstracts of nearly all of the talks held at the workshop, ordered according to the timetable of the workshop. These notes are not refereed.
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Göttingen, January 15th, 1997.

[1] 'An Introduction to Multifractal Distribution Functions, in *Fluctuations and Pattern Formation*, ed. H.E. Stanley and N. Ostrowsky, Kluwer Acad., Dordrecht, 1988, page 279-291.

[2] SPP 'Ergodentheorie, Analysis und effiziente Simulation dynamischer Systeme' (Universität Göttingen).

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Chapter 1

Sierpiński Gasket as a Martin Boundary

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In this note we explain and describe a method how to define a harmonic structure on p.c.f. fractals (and more general fractals) using the concept of harmonicity for random walks and Markov chains. This is done by representing the Sierpiński gasket as the boundary of a certain random walk and Markov chain. The motivation for this representation originates in a description of the gasket as the sphere at infinity of a discrete semigroup action on its interior. The main results are in a forthcoming paper by Denker and Sato [1].

1.1 Limit sets.

The Sierpiński gasket in $N-1$ dimensional Euclidean space $\mathbb{R}^{N-1}$ is determined by N points $p_1, ...p_N$ and the non-degenerate regular simplex $\Delta = \Delta(p_1, ..., p_N)$ generated by them. For every fixed $i_0 \in \{1, .., N\}$, the midpoints $p_{j,i_{i_0}} = \frac{p_{i_0}+p_j}{2}$ $(j = 1, ..., N)$ define a corresponding simplex $\Delta(i_0) = \Delta(p_{1,i_0}, ..., p_{N,i_0}) \subset \Delta$ and an affine map

$$f_{i_0} : \Delta(p_1, ..., p_N) \to \Delta(p_{1,i_0}, ..., p_{N,i_0})$$

satisfying $f_{i_0}(p_i) = p_{i,i_0}$.
Let $\mathcal{A} = \{1, ..., N\}$ and $\mathcal{W} = \mathcal{A}^{\mathbb{N}}$ denote the space of all finite words over the alphabet $\mathcal{A}$, including the empty word $\emptyset$. $\mathcal{W}$ has a natural semigroup structure defined by the product of two words

$$\mathbf{wv} = w_1...w_n v_1...v_m \quad \forall \mathbf{w} = w_1...w_n, \mathbf{v} = v_1...v_m \in \mathcal{W}.$$

We use the convention $f \circ g(x) = f(g(x))$ for the iterated application of two maps. For a word $\mathbf{w} \in \mathcal{W}$ there are two possibilities to define $f_{\mathbf{w}}$. One choice is

$$f_{\mathbf{w}} = \begin{cases} f_{w_1} \circ f_{w_2} \circ ... \circ f_{w_n} & \text{if } \mathbf{w} = w_1 w_2 ... w_n \\ \text{id. if } \mathbf{w} = \emptyset. \end{cases}$$

Let $\Delta(\mathbf{w}) = f_{\mathbf{w}}\Delta$. Note that $f_{\mathbf{w}} \circ f_{\mathbf{v}} = f_{\mathbf{wv}}$ and hence $f_{\mathbf{w}}(\Delta(\mathbf{v})) = \Delta(\mathbf{wv}) \subset \Delta(\mathbf{w})$ for $\mathbf{w}, \mathbf{v} \in \mathcal{W}$.
The other choice is obtained by setting $\tilde{f}_{\mathbf{w}} = f_{\mathbf{w}'}$ where $\mathbf{w}' = w_n...w_2w_1$ for $\mathbf{w} = w_1w_2...w_n$. In this case we get $\tilde{f}_{\mathbf{w}} \circ \tilde{f}_{\mathbf{v}} = \tilde{f}_{\mathbf{vw}}$ and hence $\tilde{f}_{\mathbf{w}}(\Delta(\mathbf{v})) = \Delta(\mathbf{w}'\mathbf{v}) \subset \Delta(\mathbf{w}')$ for $\mathbf{w}, \mathbf{v} \in \mathcal{W}$. No matter how $f_{\mathbf{w}}$ is defined (usually the first choice is taken in the literature), the Sierpiński gasket associated to $p_1, ..., p_N$ is

$$\mathcal{S} = \bigcap_{m=1}^{\infty} \bigcup_{\mathbf{w} \in \mathcal{W}_m} \Delta(\mathbf{w}).$$

Let Σ denote the space of infinite words over the alphabet $\mathcal{A} = \{1, ..., N\}$, which is the natural limit set of the semigroup $\mathcal{W}$ in the sense that for $x = x_1x_2... \in \Sigma$

$$\lim_{n\to\infty} \mathbf{w}_n = x$$

where $\mathbf{w}_n = x_1x_2...x_n \in \mathcal{W}$. Note that $f_{\mathbf{w}_n}$ converges (necessarily to some constant function $\xi \in \mathcal{S}$). However, this action is not compatible with the semigroup action on $\mathcal{W}$; instead, the family $f_{\mathbf{w}_n}$ forms an iterated function system. On the other hand, $\mathbf{w} \to \tilde{f}_{\mathbf{w}}$ is an action of the semigroup, but $\tilde{f}_{\mathbf{w}_n}$ does not converge. The problem we are concerned with is to define an action of $\mathcal{W}$ on a set $D \subset \Delta$ which has $\mathcal{S}$ as its limit set and is represented by Σ.
The representation part is easy and well known. Define an equivalence relation $\sim$ on Σ by $x \sim y$ if $x = y$ or if $x = x_1x_2,,,x_nbaaa...$ for some $n \geq 0$, $a \neq b \in \mathcal{A}$ and $y = x_1...x_nabbb....$ Introducing the metric $d(x, y) = \sum_{n=1}^{\infty} 2^{-n} 1_{\{x_n \neq y_n\}}$ on Σ, it follows that $\Sigma/\sim$ is Lipschitz equivalent to the Sierpiński gasket $\mathcal{S}$ via the map $x \to \xi(x) = \lim_{n\to\infty} f_{x_1x_2...x_n}\Delta$.
We will now define the action of $\mathcal{W}$. It is motivated by the action of a discrete group acting on the hyperbolic space. The fundamental domain of the action is the set $D(\emptyset) = \Delta \setminus \bigcup_{i=1}^{N} \Delta(i)$, the open simplex generated by all points $p_{i,j}$ with $1 \leq i \neq j \leq N$. For $\mathbf{w} \in \mathcal{W}_n$ define $D(\mathbf{w}) = f_{\mathbf{w}}(D(\emptyset))$. The sets $D(\mathbf{w})$ are pairwise disjoint and satisfy $D(\mathbf{w}) \subset \Delta(\mathbf{w})$. The action $\mathcal{W}$ on

$$D = \bigcup_{n=0}^{\infty} \bigcup_{\mathbf{w} \in \mathcal{W}_n} D(\mathbf{w})$$

is defined by

$$g_i : D(\mathbf{w}) \to D(\mathbf{w}i) \quad \forall \mathbf{w} \in \mathcal{W}$$

$$g_i(\xi) = f_{\mathbf{w}} \circ f_i \circ f_{\mathbf{w}}^{-1}(\xi), \ (\xi \in D(\mathbf{w})).$$

Then $g_{\mathbf{v}} \circ g_{\mathbf{w}} = g_{\mathbf{wv}}$ and the limit set is $\mathcal{S}$ in the sense that for $x \in \Sigma \quad \lim_{n\to\infty} g_{x_1x_2...x_n} D(\emptyset) = \xi(x)$. The equivalence relation $\sim$ can be represented by the map

$$S : D(\mathbf{w}) \to D(\mathbf{w}^*) \quad \forall \mathbf{w} \in \mathcal{W}$$

$$S(\xi) = f_{\mathbf{w}^*} \circ f_{\mathbf{w}}^{-1}(\xi) \ (\xi \in D(\mathbf{w})),$$

where $\mathbf{w}^* = \mathbf{v}ba^l$ for $\mathbf{w} = \mathbf{v}ab^l$ with $\mathbf{v} \in \mathcal{W}$ and $a \neq b \in \mathcal{A}$ and $\mathbf{w}^* = \mathbf{w}$ else. Note that S is the identity on all sets $D(\mathbf{w})$ with $\mathbf{w} = x^q$ for some $q \geq 0$ and $1 \leq x \leq N$.
Let G denote the semigroup generated by these $N + 1$ maps. Note that

(1) $\forall \mathbf{w} \in \mathcal{W}_n$ such that $\mathbf{w} \neq \mathbf{w}^*$

$$Sg_{\mathbf{w}} = g_{\mathbf{w}^*}.$$

(2) S^2 is the identity on D.

(3) The semigroup G consists of all elements of the form

$$g = g_{w_n} g_{w_{n-1}} ... g_{w_L} S g_{w_{L-1}}^{t_{L-1}} ... S g_{w_2}^{t_2} S g_{w_1}^{t_1} S^{\epsilon_0}$$

where $\mathbf{w} = w_1 ... w_n \in \mathcal{W}_n$ is a word of length $n \geq 0$, $1 \leq L \leq n$, $t_j \geq 1$ $(1 \leq j < L)$, and where $\epsilon_0 = 0$ or $= 1$. Hence the limit set of G agrees with $\mathcal{S}$.

1.2 Random walks.

Define a random walk $Z_1, Z_2, ...$ on G by

$$\mathbf{P}(Z_1 = g_i \circ S^k) = \mu(g_i \circ S^k) = \frac{1}{2N} \qquad (i = 1, ..., N, k = 0, 1).$$

Let Y_i be independent with distribution μ so that $Z_n = Y_1 \circ Y_2 \circ ... \circ Y_n$.
Since the word length of Z_n is a.s. at least n, the random walk Z_n is transitive. The random walk acts on D and Z_n converges a.s. to a point in $\mathcal{S}$.
Let ν denote the Bernoulli measure on Σ with 1-dimensional marginal $\nu([i]) = N^{-1}$. Since the projection map $\Sigma \to \Sigma/\sim$ is a measure isomorphism, we may consider ν as a measure on $\Sigma/\sim$ and on $\mathcal{S}$ as well.
Each $g \in G$ defines a map $g : \mathcal{S} \to \mathcal{S}$ setting

$$g\xi = \lim_{n \to \infty} g_{x_1 ... x_n} \circ gD(\emptyset),$$

whenever $\xi = \xi(x)$ for $x \in \Sigma$. Note that the involution S acts as the identity on $\mathcal{S}$. G also acts on functions $\phi : \mathcal{S} \to \mathbb{R}$ by the operators P_g defined by

$$[P_g \phi](\xi) = \int_G \phi(g' \circ g(\xi)) \mu(dg'),$$

and in case $g = g_{\mathbf{w}} = g_{w_n ... w_1}$ for some $\mathbf{w} = w_1 ... w_n \in \mathcal{W}$

$$[P_g \phi](\xi) = \sum_{\mathbf{v} \in \mathcal{W}} p(\mathbf{w}, \mathbf{v}) \phi(\mathbf{v}\xi).$$

Theorem 1.2.1
(1) $P_g^* \nu = \frac{g + S \circ g}{2} * \nu$ *for every* $g \in G$. *In particular,* $P_{g_{\mathbf{w}}}^* \nu = \frac{g_{\mathbf{w}} + g_{\mathbf{w}^*}}{2} * \nu$ *for* $\mathbf{w} \in \mathcal{W}$, $\mathbf{w} \neq \mathbf{w}^*$.
(2) $(\mathcal{S}, \nu)$ *is a boundary of* P. *This means (cf. [3]):*
(a) G *acts on* $\mathcal{S}$.
(b) ν *is* μ*-stationary, i.e.* $\mu * \nu = \nu$.

1.3 Markov chains.

There is a natural projection $G^{\mathbb{N}} \to \mathcal{W}^{\mathbb{N}}$. The image of the measure $\mu^{\mathbb{N}}$ is a Markov measure with transition probabilities $p(\mathbf{v}, \mathbf{w})$ $(\mathbf{v}, \mathbf{w} \in \mathcal{W})$ defined as follows:

(a) For $\mathbf{v} = a^k$ where $a \in \mathcal{A}$ and $k \geq 0$

$$p(a^k, \mathbf{w}) = \begin{cases} \frac{1}{N} & \text{if } \exists \gamma \in \mathcal{A} \text{ such that } \mathbf{w} = a^k\gamma \\ 0 & \text{otherwise.} \end{cases}$$

(b) For $\mathbf{v} = \mathbf{u}ab^k$ where $\mathbf{u} \in \mathcal{W}$, $a, b \in \mathcal{A}$, $(a \neq b)$, and $k \geq 1$

$$p(\mathbf{u}ab^k, \mathbf{w}) = \begin{cases} \frac{1}{2N} & \text{if } \exists \gamma \in \mathcal{A} \text{ such that } \mathbf{w} = \mathbf{v}\gamma \text{ or } \mathbf{v}^*\gamma \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $(X_n)_{n\geq 1}$ the Markov chain defined by p and the state space $\mathcal{W}$. The n-step transition probabilities are defined recursively by

$$p(n; \mathbf{v}, \mathbf{w}) = \sum_{\mathbf{u}\in\mathcal{W}} p(n-1, \mathbf{v}, \mathbf{u})p(\mathbf{u}, \mathbf{w}), \qquad \mathbf{v}, \mathbf{w} \in \mathcal{W},\ n \geq 1,$$

where $p(0; \mathbf{v}, \mathbf{w}) = \delta_{\mathbf{v}}(\mathbf{w})$ ($\delta_{\mathbf{v}}$ is the Dirac function at $\mathbf{v}$). Evidently we have $p(n; \mathbf{v}, \mathbf{w}) > 0$ only if $d(\mathbf{v}, \mathbf{w}) = n$, hence the Green function $g : \mathcal{W} \times \mathcal{W} \to \mathbb{R}_+$,

$$g(\mathbf{v}, \mathbf{w}) = \sum_{n=0}^{\infty} p(n; \mathbf{v}, \mathbf{w}) = p(d(\mathbf{v}, \mathbf{w}); \mathbf{v}, \mathbf{w}), \quad \mathbf{v}, \mathbf{w} \in \mathcal{W},$$

is well defined. It is easy to see that $g(\emptyset, \mathbf{w}) = \frac{1}{N^{d(\mathbf{w})}}$ for any $\mathbf{w} \in \mathcal{W}$, where $d(\mathbf{w})$ denotes the word length of $\mathbf{w}$. The Martin kernel is defined by

$$k(\mathbf{v}, \mathbf{w}) = \frac{g(\mathbf{v}, \mathbf{w})}{g(\emptyset, \mathbf{w})}, \qquad \mathbf{v}, \mathbf{w} \in \mathcal{W}.$$

Define a metric ρ on $\mathcal{W}$ by

$$\rho(\mathbf{v}, \mathbf{w}) = \left|2^{-d(\mathbf{v})} - 2^{-d(\mathbf{w})}\right| + \sum_{\mathbf{u}\in\mathcal{W}} a(\mathbf{u}) \frac{|k(\mathbf{u}, \mathbf{w}) - k(\mathbf{u}, \mathbf{v})|}{1 + |k(\mathbf{u}, \mathbf{w}) - k(\mathbf{u}, \mathbf{v})|}$$

where $\{a(\mathbf{u});\ \mathbf{u} \in \mathcal{W}\}$ is some fixed sequence of strictly positive numbers such that $\sum_{\mathbf{u}\in\mathcal{W}} a(\mathbf{u}) = 1$. The collection $\overline{\mathcal{W}}$ of all equivalence classes of Cauchy sequences in $\mathcal{W}$ is the ρ-completion of $\mathcal{W}$ and is called the Martin space. This is a compact metric space with the extension of ρ, and $\mathcal{W}$ is an open dense subset of $\overline{\mathcal{W}}$. The Martin boundary

$$M = \partial\overline{\mathcal{W}} = \overline{\mathcal{W}} \setminus \mathcal{W}$$

is also a compact metric space. Moreover, for every fixed $\mathbf{v} \in \mathcal{W}$ the function $\mathbf{w} \to k(\mathbf{v}, \mathbf{w})$ has an extension to a continuous function on $\overline{\mathcal{W}}$. The extension is also denoted by $k(\mathbf{v}, \xi)$, $\xi \in \overline{\mathcal{W}}$.
Define the function θ of a word $\mathbf{v}$ by

$$\theta(\mathbf{v}) = \begin{cases} 1 & \text{if } \mathbf{v} \text{ includes at least two letters} \\ 2 & \text{otherwise.} \end{cases}$$

Theorem 1.3.1 *Let* $\mathbf{v}a(\neq \emptyset)$ *and* $\mathbf{w}$ *be words. If* $k(\mathbf{v}a, \mathbf{w}) > 0$, *then* $\mathbf{w}$ *has the form*

$$\mathbf{w} = \mathbf{v}w_0w_1w_2w_3\cdots w_n \quad \text{or} \quad (vw_0)^*w_1w_2w_3\cdots w_n$$

for some $w_0, w_1, w_2, \cdots, w_n \in \mathcal{A}$. *More precisely,*

(1) if $n = 0$ *then* $\mathbf{v}a = \mathbf{w}$ *and* $k(\mathbf{v}a, \mathbf{w}) = N^{d(\mathbf{v}a)}$.

(2) if $n = 1$ *then* $\mathbf{w} = \mathbf{v}ac$ *or* $\mathbf{w} = (va)^*c$ *for some* $c \in \mathcal{A}$ *and* $k(\mathbf{v}a, \mathbf{w}) = k(\mathbf{v}a, \mathbf{w}^*) = \frac{\theta}{2}N^{d(\mathbf{v}a)}$.

(3) if $n \geq 2$ *then*

$$k(\mathbf{v}a, \mathbf{w}) = k((\mathbf{v}a)^*, \mathbf{w}) = \frac{\theta(\mathbf{v}a)}{4}N^{d(\mathbf{v}a)}\left(\sum_{k=0}^{n}\frac{1_a(w_k)}{2^k} + \frac{1_a(w_n)}{2^n}\right).$$

Corollary 1.3.2 $\mathcal{S}$ *is homeomorphic to the Martin boundary of* $(X_n)_{n\in\mathbb{N}}$.

1.4 Harmonic functions.

The Markov operator P for the Markov chain X_n is defined by

$$(Pf)(\mathbf{v}) = \sum_{\mathbf{w}\in\mathcal{W}} p(\mathbf{v}, \mathbf{w})f(\mathbf{w}), \quad \mathbf{v} \in \mathcal{W}$$

for a non-negative function f on $\mathcal{W}$. A non-negative function $f : \mathcal{W} \to \mathbb{R}$ is called P-harmonic if

$$(Pf)(\mathbf{v}) = f(\mathbf{v}), \quad \mathbf{v} \in \mathcal{W}.$$

The boundary representations of $\mathcal{S}$ permits to describe P-harmonic functions Dynkin's Theorem ([2]).

Theorem 1.4.1

(1) For every bounded P*-harmonic function* h, *there exists a bounded measurable function* $\phi : \mathcal{S} \to \mathbb{R}$ *such that*

$$h(\mathbf{w}) = \int_{\mathcal{S}} k(\mathbf{w}, y)\phi(y)\nu(dy).$$

(2) If $\phi : \mathcal{S} \to \mathbb{R}$ *is measurable and bounded, then*

$$h(\mathbf{w}) = \int_{\mathcal{S}} k(\mathbf{w}, y)\phi(y)\nu(dy)$$

is P*-harmonic on* $\mathcal{W}$.

We say that a function $h : G \to \mathbb{R}$ is μ-harmonic for the random walk if

$$\int h(g' \circ g)\mu(dg') = h(g)$$

for all $g \in G$.
Let $\mathcal{W}' = \{\mathbf{w} \in \mathcal{W} : \mathbf{w} \neq \mathbf{w}^*\}$ and $G' = \{g_{\mathbf{w}} \in G : \ \mathbf{w} \in \mathcal{W}'\}$. The connection between P- and μ-harmonic function is described in the following way:

Theorem 1.4.2

(1) For every bounded measurable function $\phi : \mathcal{S} \to \mathbb{R}$,

$$h(g) = \int \phi(g\xi)\nu(d\xi) \qquad (g \in G)$$

is μ-harmonic.

(2) Every bounded μ-harmonic function h, restricted to G', is P-harmonic.

(3) Every bounded P-harmonic function h has a representation as

$$h(\mathbf{w}) = \int \phi(g_{\mathbf{w}}\xi)\nu(d\xi) \qquad \forall \mathbf{w} \in \mathcal{W}'$$

for some measurable $\phi : \mathcal{S} \to \mathbb{R}$.

We will say that the functions h as in the last theorem are harmonic. Let $\mathcal{H}$ denote the class of harmonic functions, i.e. $h \in \mathcal{H}$ is a function on $\mathcal{W}'$ which has extensions to a P- and a μ-harmonic function.
It is easy to see that

(1) If h is P-harmonic and $\mathbf{w} \in \mathcal{W}$, then $h(\mathbf{w}) = h(\mathbf{w}^*)$.

(2) If h is μ-harmonic and $g \in G$, then $h(g) = h(S \circ g)$.

Hence harmonic functions are constant on $\sim$-equivalence classes, and we may consider them on $\mathcal{W}/\!\sim$ and $\mathcal{W}'/\!\sim$. Since the latter space is embedded into $\mathcal{S}$ via the bi-Lipschitz map

$$\{\mathbf{w}ab^k, \mathbf{w}ba^k\} \to \xi(\mathbf{w}ab^\infty) = \xi(\mathbf{w}ba^\infty),$$

we obtain a natural class of harmonic functions on $\mathcal{S}$. These are continuous functions in $\mathcal{H}$ (thus extend to continuous functions on $\mathcal{S}$) and are constant on sets of the form $\{\mathbf{w}a^k : k \geq 1\}$ where $\mathbf{w}a \in \mathcal{W}'$. This class of functions includes the harmonic functions on $\mathcal{S}$ introduced by Kigami in [4] and [5].

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Chapter 2

Multifractal Geometry

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2.1 Introduction

Within the last 3 or 4 years the multifractal spectra of various classes of measures exhibiting some degree of self-similarity have been computed rigorously (cf. eg. [1], [3], [4], [13], [16], [17], [18], [24], [25], [26], [27], [32], [35], [36], [38]), and attempts to construct a general multifractal geometry for arbitrary measures (analogous to the existing fractal geometry for sets) have been initiated by Olsen [25], Pesin[33], [34] and Peyriére [35]. This survey intends to show that the multifractal formalism introduced in [25], [33], [34], [35] leads to a multifractal geometry for product measures, for slices of measures (i.e. intersections of measures with lower dimensional subspaces), and for general intersections of measures, which is completely analogous to the fractal geometry for product sets, for slices of sets (i.e. intersections of sets with lower dimensional subspaces), and for general intersections of sets, respectively.
The main problem in multifractal analysis is as follows. For a metric space X we denote the family of regular Radon measures on X by $\mathcal{M}(X)$ and we denote the family of Borel probability measures on X by $\mathcal{P}(X)$. For $\mu \in \mathcal{M}(X)$ and $x \in X$ we define the upper and lower local dimension of μ at x by

$$\overline{\alpha}_\mu(x) = \limsup_{r \searrow 0} \frac{\log \mu B(x,r)}{\log r}$$

resp.

$$\underline{\alpha}_\mu(x) = \liminf_{r \searrow 0} \frac{\log \mu B(x,r)}{\log r}$$

where $B(x,r)$ denotes the closed ball with centre x and radius r. If $\overline{\alpha}_\mu(x)$ and $\underline{\alpha}_\mu(x)$ agree we refer to the common value as the local dimension of μ at x and denote it by $\alpha_\mu(x)$. For each $\alpha \geq 0$ define $\Delta_\mu(\alpha)$ by

$$\Delta_\mu(\alpha) = \{x \in \operatorname{supp} \mu \mid \alpha_\mu(x) = \alpha\}$$

where $\operatorname{supp}\mu$ denotes the topological support of μ. The main problem in multifractal analysis is to estimate the size of the sets $\Delta_\mu(\alpha)$; this is done by introducing the functions $f_\mu, F_\mu : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$f_\mu(\alpha) = \dim \Delta_\mu(\alpha)\,, \qquad F_\mu(\alpha) = \operatorname{Dim} \Delta_\mu(\alpha)$$

where dim and Dim denote Hausdorff dimension and packing dimension respectively. These and similar functions are generically known as "the multifractal spectrum of μ", "the singularity spectrum of μ", "the spectrum of scaling indices" or simply "the $f(\alpha)$-spectrum". The function $f(\alpha) = f_\mu(\alpha)$ was first explicitly defined by the physicists Halsey et al. in 1986 in their seminal paper [9]. The reader is referred to Cawley & Mauldin [3], Lau [15] or Olsen [25], [31] for a more detailed historical account of multifractality, and an extensive list of references. Recently Olsen [25], Pesin [33], [34] and Peyriére [35] suggested that certain multifractal generalisations of the Hausdorff measure and packing measure might be useful in multifractal analysis. In Section 2 we recall the multifractal formalism in [25], [33], [34], [35]. In Section 3 we discuss the multifractal geometry of product measures. In Section 4 we investigate the multifractal structure of slices of measures (i.e. the intersection of a measure in Euclidean spaces with a lower dimensional subspace), and in Section 5 we discuss the multifractal geometry of general intersection measures.

2.2 The multifractal formalism of Olsen, Pesin and Peyriére

This section gives a brief summary of the main results in [25]. We first recall the definition of the Hausdorff measure, the centred Hausdorff measure and the packing measure. Let X be a metric space, $E \subseteq X$ and $\delta > 0$. A countable family $\mathcal{B} = (B(x_i, r_i))_i$ of closed balls in X is called a centred δ-covering of E if $E \subseteq \cup_i B(x_i, r_i)$, $x_i \in E$ and $0 < r_i < \delta$ for all i. The family $\mathcal{B}$ is called a centred δ-packing of E if $x_i \in E$, $0 < r_i < \delta$ and $B(x_i, r_i) \cap B(x_j, r_j) = \emptyset$ for all $i \neq j$. Let $E \subseteq X$, $t \geq 0$ and $\delta > 0$. Now put

$$\mathcal{H}^t_\delta(E) = \inf\{\sum_i \operatorname{diam}(E_i)^t \mid E \subseteq \bigcup_{i=1}^\infty E_i\,,\ \operatorname{diam} E_i < \delta\}\,.$$

The t-dimensional Hausdorff measure $\mathcal{H}^t(E)$ of E is defined by

$$\mathcal{H}^t(E) = \sup_{\delta>0} \mathcal{H}^t_\delta(E)\,.$$

The reader is referred to [6] for more information on $\mathcal{H}^t$. Next we define the centred Hausdorff measure introduced by Raymond & Tricot in [37]. Put

$$\overline{\mathcal{C}}^t_\delta(E) = \inf\left\{\sum_{i=1}^\infty (2r_i)^t \mid (B(x_i, r_i))_i \text{ is a centred } \delta\text{-covering of } E\right\}\,.$$

The t-dimensional centred pre-Hausdorff measure $\overline{\mathcal{C}}^t(E)$ of E is defined by

$$\overline{\mathcal{C}}^t(E) = \sup_{\delta>0} \overline{\mathcal{C}}^t_\delta(E)\,.$$

The set function $\overline{\mathcal{C}}^t$ is not necessarily monotone, and hence not necessarily an outer measure, c.f. [37], pp. 137-138. But $\overline{\mathcal{C}}^t$ give rise to a Borel measure, called the t-dimensional centred Hausdorff measure $\mathcal{C}^t(E)$ of E, as follows

$$\mathcal{C}^t(E) = \sup_{F \subseteq E} \overline{\mathcal{C}}^t(F)\,.$$

It is easily seen (c.f. [37], Lemma 3.3) that $2^{-t}\mathcal{C}^t \leq \mathcal{H}^t \leq \mathcal{C}^t$. We will now define the packing measure. Write

$$\overline{\mathcal{P}}^t_\delta(E) = \sup\left\{\sum_{i=1}^{\infty}(2r_i)^t \mid (B(x_i,r_i))_i \text{ is a centered } \delta\text{-packing of } E\right\}.$$

The t-dimensional prepacking measure $\overline{\mathcal{P}}^t(E)$ of E is defined by

$$\overline{\mathcal{P}}^t(E) = \inf_{\delta>0} \overline{\mathcal{P}}^t_\delta(E)\,.$$

The set function $\overline{\mathcal{P}}^t$ is not necessarily countable sub-additive, and hence not necessarily an outer measure, c.f. [39] or [6]. But $\overline{\mathcal{P}}^t$ give rise to a Borel measure, namely the t-dimensional packing measure $\mathcal{P}^t(E)$ of E, as follows

$$\mathcal{P}^t(E) = \inf_{E \subseteq \bigcup_{i=1}^{\infty} E_i} \sum_{i=1}^{\infty} \overline{\mathcal{P}}^t(E_i)\,.$$

The packing measure was introduced by Taylor and Tricot in [39] using centred δ-packings of open balls, and by Raymond and Tricot in [37] using centred δ-packings of closed balls. Also recall that the Hausdorff dimension $\dim(E)$, the packing dimension $\mathrm{Dim}(E)$ and the logarithmic index $\Delta(E)$ of E is defined by

$$\begin{aligned}
\dim(E) &= \sup\{t \geq 0 \mid \mathcal{H}^t(E) = \infty\}\\
\mathrm{Dim}(E) &= \sup\{t \geq 0 \mid \mathcal{P}^t(E) = \infty\}\\
\Delta(E) &= \sup\{t \geq 0 \mid \overline{\mathcal{P}}^t(E) = \infty\}\,.
\end{aligned}$$

We refer the reader to [40] and [37] for more information on the centred Hausdorff measure, the packing measure and the packing dimension.
Olsen [25] suggested that some multifractal generalisations of the centred Hausdorff measure and the packing measure might be useful in multifractal analysis. For $q \in \mathbb{R}$ define $\varphi_q : [0,\infty[\to \overline{\mathbb{R}}_+ = [0,\infty]$ by

$$\begin{aligned}
\varphi_q(x) &= \begin{cases} \infty & \text{for } x = 0 \\ x^q & \text{for } 0 < x \end{cases} && \text{for } q < 0\\
\varphi_q(x) &= 1 && \text{for } q = 0\\
\varphi_q(x) &= \begin{cases} 0 & \text{for } x = 0 \\ x^q & \text{for } 0 < x \end{cases} && \text{for } 0 < q
\end{aligned}$$

For $\mu \in \mathcal{M}(X)$, $E \subseteq X$, $q, t \in \mathbb{R}$ and $\delta > 0$ write

$$\begin{aligned}
\overline{\mathcal{H}}^{q,t}_{\mu,\delta}(E) &= \inf\{\sum_i \varphi_q(\mu(B(x_i,r_i)))(2r_i)^t \mid (B(x_i,r_i))_i \text{ centred } \delta\text{-covering of } E\},\\
\overline{\mathcal{H}}^{q,t}_{\mu,\delta}(\emptyset) &= 0\\
\overline{\mathcal{H}}^{q,t}_{\mu}(E) &= \sup_{\delta>0} \overline{\mathcal{H}}^{q,t}_{\mu,\delta}(E)\\
\mathcal{H}^{q,t}_{\mu}(E) &= \sup_{F\subseteq E} \overline{\mathcal{H}}^{q,t}_{\mu}(F)\,.
\end{aligned}$$

We also make the dual definitions

$$\begin{aligned}
\overline{\mathcal{P}}^{q,t}_{\mu,\delta}(E) &= \sup\{\sum_i \varphi_q(\mu(B(x_i,r_i)))(2r_i)^t \mid (B(x_i,r_i))_i \text{ centred } \delta\text{-packing of } E\},\\
\overline{\mathcal{P}}^{q,t}_{\mu,\delta}(\emptyset) &= 0\\
\overline{\mathcal{P}}^{q,t}_{\mu}(E) &= \inf_{\delta>0} \overline{\mathcal{P}}^{q,t}_{\mu,\delta}(E)\\
\mathcal{P}^{q,t}_{\mu}(E) &= \inf_{E\subseteq\cup_i E_i} \sum_i \overline{\mathcal{P}}^{q,t}_{\mu}(E_i)\,.
\end{aligned}$$

It is proven in [25] that $\mathcal{H}^{q,t}_\mu$ and $\mathcal{P}^{q,t}_\mu$ are measures on the family of Borel subsets of X. The measure $\mathcal{H}^{q,t}_\mu$ is of course a multifractal generalisation of the centred Hausdorff measure, whereas $\mathcal{P}^{q,t}_\mu$ is a multifractal generalisation of the packing measure. In fact, it is easily seen that the following holds for $t \geq 0$,

$$2^{-t}\mathcal{H}^{0,t}_\mu \leq \mathcal{H}^t \leq \mathcal{H}^{0,t}_\mu\,,\ \mathcal{P}^t = \mathcal{P}^{0,t}_\mu\,,\ \overline{\mathcal{P}}^t = \overline{\mathcal{P}}^{0,t}_\mu\,.$$

The next result shows that the measures $\mathcal{H}^{q,t}_\mu$, $\mathcal{P}^{q,t}_\mu$ and the pre-measure $\overline{\mathcal{P}}^{q,t}_\mu$ in the usual way assign a dimension to each subset E of X.

Proposition 2.2.1 *There exist unique extended real valued numbers* $\Delta^q_\mu(E) \in [-\infty,\infty]$, $\mathrm{Dim}^q_\mu(E) \in [-\infty,\infty]$ *and* $\dim^q_\mu(E) \in [-\infty,\infty]$ *such that*

$$\overline{\mathcal{P}}^{q,t}_\mu(E) = \begin{cases} \infty & \text{for } t < \Delta^q_\mu(E)\\ 0 & \text{for } \Delta^q_\mu(E) < t \end{cases}$$

$$\mathcal{P}^{q,t}_\mu(E) = \begin{cases} \infty & \text{for } t < \mathrm{Dim}^q_\mu(E)\\ 0 & \text{for } \mathrm{Dim}^q_\mu(E) < t \end{cases}$$

$$\mathcal{H}^{q,t}_\mu(E) = \begin{cases} \infty & \text{for } t < \dim^q_\mu(E)\\ 0 & \text{for } \dim^q_\mu(E) < t \end{cases}$$

Proof: See [25], Proposition 1.1]. □

The number $\dim^q_\mu(E)$ is an obvious multifractal analogue of the Hausdorff dimension $\dim(E)$ of E whereas $\mathrm{Dim}^q_\mu(E)$ and $\Delta^q_\mu(E)$ are obvious multifractal analogues of the packing dimension $\mathrm{Dim}(E)$ and the logarithmic index $\Delta(E)$ of E respectively. In fact, it follows immediately from the definitions that

$$\dim(E) = \dim^0_\mu(E)\,,\ \mathrm{Dim}(E) = \mathrm{Dim}^0_\mu(E)\,,\ \Delta(E) = \Delta^0_\mu(E)\,.$$

Next we define multifractal dimension functions $b_\mu, B_\mu, \Lambda_\mu : \mathbb{R} \to [-\infty, \infty]$ by

$$b_\mu(q) = \dim_\mu^q(\operatorname{supp}\mu)\,,\ B_\mu(q) = \operatorname{Dim}_\mu^q(\operatorname{supp}\mu)\,,\ \Lambda_\mu(q) = \Delta_\mu^q(\operatorname{supp}\mu)\,.$$

We will now give a very brief list of some of the most important properties of the measures $\mathcal{H}_\mu^{q,t}$ and $\mathcal{P}_\mu^{q,t}$, and the corresponding dimension functions. We remark that we will not state all theorems in their most general form. The reader is referred to Olsen [25] for a detailed study of the measure $\mathcal{H}_\mu^{q,t}$ and $\mathcal{P}_\mu^{q,t}$, and the dimension functions b_μ, B_μ and Λ_μ. For $\mu \in \mathcal{M}(X)$ and $a > 1$ write $T_a(\mu) = \limsup_{r \searrow 0} \left(\sup_{x \in \operatorname{supp}\mu} \frac{\mu B(x,ar)}{\mu B(x,r)} \right)$ and define the family $\mathcal{M}_0(X)$ of doubling measures on X by $\mathcal{M}_0(X) = \{\mu \in \mathcal{M}(X) \mid T_a(\mu) < \infty \quad \text{for some } a > 1\}$. It follows from [25] that the definition of $\mathcal{M}_0(X)$ is independent of the number $a > 1$, i.e. $T_a(\mu) < \infty$ for all $a > 1$ if and only if $T_a(\mu) < \infty$ for some $a > 1$. Furthermore, write $\mathcal{P}_0(X) = \mathcal{P}(X) \cap \mathcal{M}_0(X)$.

Proposition 2.2.2 *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ and $q, t \in \mathbb{R}$. Then*

i) $\mathcal{H}_\mu^{q,t} \le \mathcal{P}_\mu^{q,t}$ *for* $\mu \in \mathcal{M}_0(\mathbb{R}^n)$, *and* $\mathcal{P}_\mu^{q,t} \le \overline{\mathcal{P}}_\mu^{q,t}$ *for* $\mu \in \mathcal{M}(\mathbb{R}^n)$.

ii) $\dim_\mu^q \le \operatorname{Dim}_\mu^q \le \Delta_\mu^q$, *in particular* $b_\mu \le B_\mu \le \Lambda_\mu$.

iii) b_μ *is decreasing, and* B_μ *and* Λ_μ *are convex and decreasing.*

Proof: See [25]. □

Write

$$\underline{a}_\mu = \sup_{0<q} -\frac{b_\mu(q)}{q}\,,\ \ \overline{a}_\mu = \inf_{q<0} -\frac{b_\mu(q)}{q}\,.$$

For a real valued function $f : \mathbb{R} \to \mathbb{R}$ we define the Legendre transform $f^* : \mathbb{R} \to [-\infty, \infty]$ of f by

$$f^*(x) = \inf_y (xy + f(y))\,.$$

Theorem 2.2.3 *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$. Then*

i) $\Delta_\mu(\alpha) = \emptyset$ *for* $\alpha \in \mathbb{R}_+ \setminus [\underline{a}_\mu, \overline{a}_\mu]$.

ii) $f_\mu(\alpha) \le b_\mu^*(\alpha)$ *for* $\alpha \in]\underline{a}_\mu, \overline{a}_\mu[$.

iii) $F_\mu(\alpha) \le B_\mu^*(\alpha)$ *for* $\alpha \in]\underline{a}_\mu, \overline{a}_\mu[$.

Proof: See [25]. □

2.3 Multifractal geometry of product measures

For $H \subseteq \mathbb{R}^{k+l}$ and $y \in \mathbb{R}^l$, let H^y denote the "section" defined by

$$H^y = \{x \in \mathbb{R}^k \mid (x, y) \in H\}\,.$$

The classical result on the fractal geometry of product sets is as follows.

Theorem 2.3.1 (Fractal Product Theorem [2], [7], [8], [10], [19], [40]])
Let $s,t \in \mathbb{R}_+$. Then there exists a number $c>0$ such that

$$\begin{aligned}\int \mathcal{H}^s(H^y)\,d\mathcal{H}^t(y) &\le c\mathcal{H}^{s+t}(H)\\ \mathcal{H}^{s+t}(E\times F) &\le c\mathcal{H}^s(E)\mathcal{P}^t(F)\\ \int \mathcal{H}^s(H^y)\,d\mathcal{P}^t(y) &\le c\mathcal{P}^{s+t}(H)\\ \mathcal{P}^{s+t}(E\times F) &\le c\mathcal{P}^s(E)\mathcal{P}^t(F)\end{aligned}$$

for $E\subseteq\mathbb{R}^k$, $F\subseteq\mathbb{R}^l$ and $H\subseteq\mathbb{R}^{k+l}$. In particular

$$\begin{aligned}\dim(E)+\dim(F) &\le \dim(E\times F)\\ &\le \dim(E)+\mathrm{Dim}(F)\\ &\le \mathrm{Dim}(E\times F)\le \mathrm{Dim}(E)+\mathrm{Dim}(F)\,.\end{aligned}$$

for $E\subseteq\mathbb{R}^k$ and $F\subseteq\mathbb{R}^l$.

The corresponding multifractal product measure inequalities are as follows.

Theorem 2.3.2 (Multifractal Product Theorem [28])
Let $\mu\in\mathcal{P}_0(\mathbb{R}^k)$, $\nu\in\mathcal{P}_0(\mathbb{R}^l)$ and $q,s,t\in\mathbb{R}$. Then there exists a number $c>0$ such that

$$\begin{aligned}\int \mathcal{H}_\mu^{q,s}(H^y)\,d\mathcal{H}_\nu^{q,t}(y) &\le c\mathcal{H}_{\mu\times\nu}^{q,s+t}(H)\\ \mathcal{H}_{\mu\times\nu}^{q,s+t}(E\times F) &\le c\mathcal{H}_\mu^{q,s}(E)\mathcal{P}_\nu^{q,t}(F)\\ \int \mathcal{H}_\mu^{q,s}(H^y)\,d\mathcal{P}_\nu^{q,t}(y) &\le c\mathcal{P}_{\mu\times\nu}^{q,s+t}(H)\\ \mathcal{P}_{\mu\times\nu}^{q,s+t}(E\times F) &\le c\mathcal{P}_\mu^{q,s}(E)\mathcal{P}_\nu^{q,t}(F)\end{aligned}$$

for $E\subseteq\mathbb{R}^k$, $F\subseteq\mathbb{R}^l$ and $H\subseteq\mathbb{R}^{k+l}$. In particular

$$\begin{aligned}\dim_\mu^q(E)+\dim_\nu^q(F) &\le \dim_{\mu\times\nu}^q(E\times F)\\ &\le \dim_\mu^q(E)+\mathrm{Dim}_\nu^q(F)\\ &\le \mathrm{Dim}_{\mu\times\nu}^q(E\times F)\le \mathrm{Dim}_\mu^q(E)+\mathrm{Dim}_\nu^q(F)\end{aligned}$$

for $E\subseteq\mathbb{R}^k$ and $F\subseteq\mathbb{R}^l$.

We observe that since $b_\mu(0)=\dim\operatorname{supp}\mu$ and $2^{-t}\mathcal{H}_\mu^{0,t}\le\mathcal{H}^t\le\mathcal{H}_\mu^{0,t}$ and $\mathcal{P}_\mu^{0,t}=\mathcal{P}^t$ for any measure μ and all positive t, the Fractal Product Theorem is obtained as a special case of the Multifractal Product Theorems by setting $q=0$.
We also observe that by taking Legendre transform of the inequalities in the Multifractal Product Theorem one can easily obtain upper and lower bounds for the multifractal spectrum of $\mu\times\nu$ (cf. [28]).

2.4 Multifractal geometry of slices of measures

For integers n and m with $0 \leq m \leq n$, let $G(n,m)$ denote the Grassmannian manifold of m-dimensional linear subspaces of $\mathbb{R}^n$. For $\Pi \in G(n,m)$ and $x \in \mathbb{R}^n$ write $\Pi + x = \{z + x \mid z \in \Pi\}$, and let $\Pi^\perp$ denote the orthogonal complement of Π. Let $\gamma_{n,m}$ be the unique Radon probability measure on $G(n,m)$ which is invariant under orthogonal transformations. Furthermore, we define the Hausdorff co-dimension $\operatorname{codim} E$ of a subset E of $\mathbb{R}^n$ by

$$\operatorname{codim} E = n - \dim E = \dim \mathbb{R}^n - \dim E\,.$$

Finally, if $(X, \mathcal{E}, \mu)$ is a measure space and $E \in \mathcal{E}$, $\mu \llcorner E$ denotes the restriction of μ to E, i.e. $(\mu \llcorner E)(F) = \mu(E \cap F)$ for all $F \in \mathcal{E}$. The Hausdorff dimension of slices of sets in $\mathbb{R}^n$ is now given by the following classical results.

Theorem 2.4.1 (Fractal Slice Theorem – Version 1 [20], [21])
Let $E \subseteq \mathbb{R}^n$ be a Borel set and write $t = \dim E$. Assume that $0 < \mathcal{H}^t(E) < \infty$.
If $n - m < \dim E$, then $(\mathcal{H}^t \llcorner E) \times \gamma_{n,m}$-a.a. $(x, \Pi) \in \mathbb{R}^n \times G(n,m)$ satisfy

$$\operatorname{codim}(E \cap (\Pi + x)) = \operatorname{codim}(E) + \operatorname{codim}(\Pi + x)\,.$$

If $\dim E < n - m$, then $(\mathcal{H}^t \llcorner E) \times \gamma_{n,m}$-a.a. $(x, \Pi) \in \mathbb{R}^n \times G(n,m)$ satisfy

$$\operatorname{codim}(E \cap (\Pi + x)) = n\,.$$

Theorem 2.4.2 (Fractal Slice Theorem – Version 2 [20], [21])
Let $E \subseteq \mathbb{R}^n$ be a Borel set. Assume that $0 < \mathcal{H}^{\dim E}(E) < \infty$. For all $\Pi \in G(n,m)$,

$$\mathcal{H}^{\dim E - (n-m)}(E \cap (\Pi + x)) < \infty \quad \text{for } \mathcal{H}^{n-m} \llcorner \Pi^\perp\text{-a.a. } x \in \Pi^\perp,$$

in particular

$$\operatorname{codim}(E \cap (\Pi + x)) \geq \operatorname{codim}(E) + \operatorname{codim}(\Pi + x) \quad \text{for } \mathcal{H}^{n-m} \llcorner \Pi^\perp\text{-a.a. } x \in \Pi^\perp.$$

The above theorems were first proved by Marstrand [20] for $n = 2$ and $m = 1$, and later extended to arbitrary integers n and m with $0 \leq m \leq n$ by Mattila [21]. The results state the the Hausdorff co-dimension of a planar intersection equals the sum of the Hausdorff co-dimensions of the sets involved. We will see that the multifractal formalism in [25] leads to a similar "principle of adding co-dimensions" in the multifractal case.
We will now define how to slice a Radon measure in $\mathbb{R}^n$ with a lower dimensional subspace. For $\Pi \in G(n,m)$, $x \in \mathbb{R}^n$ and $\delta > 0$ let π_Π denote the projection onto Π and write $B(\Pi + x, \delta) = \{z \in \mathbb{R}^n \mid \operatorname{dist}(z, \Pi + x) \leq \delta\}$. Fix $\mu \in \mathcal{M}(\mathbb{R}^n)$ and $\Pi \in G(n,m)$. For $\mathcal{H}^{n-m}$ almost all $x \in \Pi^\perp$ there following weak limit exists

$$\mu \cap (\mathcal{H}^m \llcorner (\Pi + x)) = \text{weak-lim}_{\delta \searrow 0}\, (2\delta)^{-(n-m)}\ \mu \llcorner B(\Pi + x, \delta)\,,$$

cf. [23], Chapter 10. If $\Pi \in G(n,m)$ and $x \in \mathbb{R}^n$ are such that $\mu \cap (\mathcal{H}^m \llcorner (\Pi + y))$ exists for $y = \pi_{\Pi^\perp}(x)$, then we write $\mu \cap (\mathcal{H}^m \llcorner (\Pi + x)) = \mu \cap (\mathcal{H}^m \llcorner (\Pi + y))$. The measure $\mu \cap (\mathcal{H}^m \llcorner (\Pi + x))$ is called the slice of μ by the plane $\Pi + x$. The measure

$\mu \cap (\mathcal{H}^m \llcorner (\Pi + x))$ has support in $\operatorname{supp}\mu \cap (\Pi + x)$, and if $\dim \mu > n - m$ then the family $\left(\mu \cap (\mathcal{H}^m \llcorner (\Pi + x))\right)_{x \in \Pi^\perp}$ disintegrates μ,

$$\mu = \int_{\Pi^\perp} \mu \cap (\mathcal{H}^m \llcorner (\Pi + x))\, d\mathcal{H}^{n-m}(x)\,,$$

cf. [23]. The intersection measure $\mu \cap (\mathcal{H}^m \llcorner (\Pi + x))$ can thus be thought of as a natural slice of μ.

We divide the multifractal analysis of the slices $\mu \cap (\mathcal{H}^m \llcorner (\Pi + x))$ into the following 3 cases determined by the relative size of μ and $\mathcal{H}^{n-m} \llcorner \Pi^\perp$.

The non-trivial case: This case is defined by

$$n - m < \dim \mu\,.$$

Since $\dim(\mathcal{H}^{n-m} \llcorner \Pi^\perp) = n - m$ for $\Pi \in G(n,m)$, we see that the non-trivial case asserts that the measure μ is "bigger" than the Hausdorff measure $\mathcal{H}^{n-m} \llcorner \Pi^\perp$ restricted to $\Pi^\perp$. Since the natural slices $\mu \cap (\mathcal{H}^m \llcorner (\Pi + x))$ are obtained by disintegrating μ with respect to $\mathcal{H}^{n-m} \llcorner \Pi^\perp$, we therefore expect that most slices $\mu \cap (\mathcal{H}^m \llcorner (\Pi + x))$ are "big" and have a non-trivial multifractal structure in the non-trivial case (provided the original measure has a non-trivial multifractal structure).

The critical case: This case is defined by

$$\dim \mu \leq n - m \leq \dim \operatorname{supp} \mu\,.$$

Since $\dim(\mathcal{H}^{n-m} \llcorner \Pi^\perp) = n - m$ for $\Pi \in G(n,m)$, we see that the critical case asserts that the "size" of the measure μ is "comparable" to the "size" of the Hausdorff measure $\mathcal{H}^{n-m} \llcorner \Pi^\perp$ restricted to $\Pi^\perp$ Since the natural slices $\mu \cap (\mathcal{H}^m \llcorner (\Pi + x))$ are obtained by disintegrating μ with respect to $\mathcal{H}^{n-m} \llcorner \Pi^\perp$, we therefore expect a rather subtle and intricate interplay between the multifractal structure of μ and the multifractal structure of the slices $\mu \cap (\mathcal{H}^m \llcorner (\Pi + x))$. We do not analyse the critical case in this paper.

The trivial case: This case is defined by

$$\dim \operatorname{supp} \mu < n - m\,.$$

Since $\dim(\mathcal{H}^{n-m} \llcorner \Pi^\perp) = n - m$ for $\Pi \in G(n,m)$, we see that the trivial case asserts that the measure μ is "smaller" than the Hausdorff measure $\mathcal{H}^{n-m} \llcorner \Pi^\perp$ restricted to $\Pi^\perp$ Since the natural slices $\mu \cap (\mathcal{H}^m \llcorner (\Pi + x))$ are obtained by disintegrating μ with respect to $\mathcal{H}^{n-m} \llcorner \Pi^\perp$, we therefore expect that most slices $\mu \cap (\mathcal{H}^m \llcorner (\Pi + x))$ are "small" and have a trivial multifractal structure in the trivial case. Indeed, it can be proved (cf. [29]) that for all $\Pi \in G(n,m)$, the slice $\mu \cap (\mathcal{H}^m \llcorner (\Pi + x))$ equals the zero measure for $\mathcal{H}^{n-m} \llcorner \Pi^\perp$-a.a. $x \in \Pi^\perp$.

In order to state our main results we need the following notation. For $\mu \in \mathcal{M}(\mathbb{R}^n)$, $m \in \{1, \ldots, n\}$ and $\Pi \in G(n,m)$ write

$$\begin{aligned} \hat{b}_\mu(q) &= b_\mu(q) - b_{\mathcal{H}^{n-m} \llcorner \Pi^\perp}(q) \\ &= b_\mu(q) - (n-m)(1-q). \end{aligned}$$

and

$$\begin{aligned} \alpha_{\min} &= \sup_{0<q} -\tfrac{\hat{b}_\mu(q)}{q}, \\ \alpha_{\max} &= \inf_{q<0} -\tfrac{\hat{b}_\mu(q)}{q}, \end{aligned}$$

$$q_{\min} = \{q < 0 \mid -\tfrac{\hat{b}_\mu(q)}{q} = \alpha_{\max}\},$$
$$q_{\max} = \{0 < q \mid -\tfrac{\hat{b}_\mu(q)}{q} = \alpha_{\min}\}.$$

Finally, in analogy with (4.1) we define the multifractal Hausdorff co-dimension function $\mathrm{co}b_\mu : \mathbb{R} \to [-\infty, \infty]$ of a locally finite Borel measure μ on $\mathbb{R}^n$ by

$$\mathrm{co}b_\mu = b_{\mathcal{H}^n} - b_\mu\,,$$

i.e. $\mathrm{co}b_\mu$ is the difference between the multifractal Hausdorff dimension function $b_{\mathcal{H}^n}$ of the n-dimensional Hausdorff measure $\mathcal{H}^n$ on $\mathbb{R}^n$ and the multifractal Hausdorff dimension function b_μ of the measure μ. We can now state a special case of the main results in [29].

Theorem 2.4.3 (Multifractal Slice Theorem – Version 1 [29])
Let $\mu \in \mathcal{P}_0(\mathbb{R}^n)$ be a Borel probability measure on $\mathbb{R}^n$ satisfying the technical conditions (**), (I) and (II) in [29]. Let $q_{\min} < q < q_{\max}$ and write $t = b_\mu(q)$. Assume that $0 < \mathcal{H}^{q,t}_\mu(\operatorname{supp}\mu) < \infty$.
If $n - m < \dim\mu$, then $(\mathcal{H}^{q,t}_\mu \llcorner \operatorname{supp}\mu) \times \gamma_{n,m}$-a.a. $(x, \Pi) \in \mathbb{R}^n \times G(n,m)$ satisfy

$$\mathrm{co}b_{\mu\cap(\mathcal{H}^m\llcorner(\Pi+x))}(q) = \mathrm{co}b_\mu(q) + \mathrm{co}b_{\mathcal{H}^m\llcorner(\Pi+x)}(q)\,.$$

If $\dim\operatorname{supp}\mu < n - m$, then $(\mathcal{H}^{q,t}_\mu \llcorner \operatorname{supp}\mu) \times \gamma_{n,m}$-a.a. $(x, \Pi) \in \mathbb{R}^n \times G(n,m)$ satisfy

$$\mathrm{co}b_{\mu\cap(\mathcal{H}^m\llcorner(\Pi+x))}(q) = \begin{cases} -\infty & q < 0 \\ n & q = 0 \\ \infty & 0 < q. \end{cases}$$

Theorem 2.4.4 (Multifractal Slice Theorem – Version 2 [29])
Let $\mu \in \mathcal{P}(\mathbb{R}^n)$ be a Borel probability measure on $\mathbb{R}^n$ satisfying the technical conditions (**), (I) and (II) in [29]. Assume that $0 < \mathcal{H}^{q,b_\mu(q)}_\mu(\operatorname{supp}\mu) < \infty$ for all $q_{\min} < q < q_{\max}$. For all $\Pi \in G(n,m)$ and all $q_{\min} < q < q_{\max}$,

$$\mathcal{H}^{q,b_\mu(q)-(n-m)(1-q)}_{\mu\cap(\mathcal{H}^m\llcorner(\Pi+x))}(\operatorname{supp}(\mu\cap(\mathcal{H}^m\llcorner(\Pi+x)))) < \infty \quad \text{for } \mathcal{H}^{n-m}\llcorner\Pi^\perp\text{-a.a. } x \in \Pi^\perp,$$

in particular

$$\mathrm{co}b_{\mu\cap(\mathcal{H}^m\llcorner(\Pi+x))}(q) \geq \mathrm{co}b_\mu(q) + \mathrm{co}b_{\mathcal{H}^m\llcorner(\Pi+x)}(q) \quad \text{for } \mathcal{H}^{n-m}\llcorner\Pi^\perp\text{-a.a. } x \in \Pi^\perp.$$

The Multifractal Slice Theorems state that the multifractal Hausdorff codimension function $b_{\mu\cap(\mathcal{H}^m\llcorner(\Pi+x))}$ of the planar intersection measure $\mu\cap(\mathcal{H}^m\llcorner(\Pi+x))$ equals the sum of multifractal Hausdorff co-dimension functions b_μ and $b_{\mathcal{H}^m\llcorner(\Pi+x)}$ of the measures involved. Hence, the formalism in [25] leads to a multifractal geometry for planar intersections $\mu\cap(\mathcal{H}^m\llcorner(\Pi+x))$ of measures μ which is completely analogue to the classical fractal geometry for planar intersections $E\cap(\Pi+x)$ of sets E: in both cases the Hausdorff co-dimension of the planar intersection (i.e. $\mu\cap(\mathcal{H}^m\llcorner(\Pi+x))$ or $E\cap(\Pi+x)$) is obtained by adding the Hausdorff co-dimension of the object which is being intersected (i.e. μ or E) and the Hausdorff co-dimension of the planar object which is performing the intersection (i.e. $\mathcal{H}^m\llcorner(\Pi+x)$ or $\Pi+x$). The Multifractal Slice Theorems can thus be viewed as multifractal generalisations of the Fractal Slice Theorems; in fact, since $b_\mu(0) = \dim\operatorname{supp}\mu$ and $2^{-t}\mathcal{H}^{0,t}_\mu \leq \mathcal{H}^t \leq \mathcal{H}^{0,t}_\mu$ for any measure μ and all $t \geq 0$, the Fractal Slice Theorems are obtained as special cases of the Multifractal Slice Theorems by setting $q = 0$.
We also observe that by taking Legendre transform of the inequalities in the Multifractal Slice Theorems and using Theorem 2.3 one can obtain an upper bound for the multifractal spectrum of the slice $\mu\cap(\mathcal{H}^m\llcorner(\Pi+x))$, cf. [29].

2.5 Multifractal geometry of intersections of measures

The "principle of adding co-dimensions" of slices described in the Fractal Slice Theorem and in the Multifractal Slice Theorem is just a special case of a more general "principle of adding co-dimensions" for general intersections of sets (due to Mattila [22] and Kahane [12]) and for general intersections of measures (due to Olsen [30]).
For $a \in \mathbb{R}^n$ and $r > 0$ let $T_a : \mathbb{R}^n \to \mathbb{R}^n$ denote the translation

$$T_a(x) = x + a\,,$$

and let $H_r : \mathbb{R}^n \to \mathbb{R}^n$ denote the homotety

$$H_r(x) = rx\,,$$

Let θ_n denote the Haar measure on the orthogonal group $\mathcal{O}(n)$ of $\mathbb{R}^n$, and let $\mathcal{L}^1$ denote Lebesgue measure on $\mathbb{R}$. The classical result on the fractal geometry of intersections of sets is as follows.

Theorem 2.5.1 (Fractal Intersection Theorem [23], [12])
Let $E, F \subseteq \mathbb{R}^n$ be Borel sets and write $t = \dim E$ and $s = \dim F$. Assume that

(1) $0 < \mathcal{H}^t(E) < \infty$,

(2) $\liminf_{r \searrow 0} \frac{1}{r^s}\mathcal{H}^s(F \cap B(x,r)) > 0$ *for all* $x \in F$.

If $0 < \dim E < n$, $0 < \dim F < n$ *and* $\dim E + \dim F - n > 0$, *then* $(\mathcal{H}^t \llcorner E) \times (\mathcal{H}^s \llcorner F) \times \theta_n \times \mathcal{L}^1$*-a.a.* $(x, y, O, r) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{O}(n) \times \mathbb{R}_+$ *satisfy*

$$\operatorname{codim}(E \cap (T_x \circ O \circ H_r \circ T_{-y})F) = \operatorname{codim}(E) + \operatorname{codim}((T_x \circ O \circ H_r \circ T_{-y})F)\,.$$

The above result states that the Hausdorff co-dimension of an intersection equals the sum of the Hausdorff co-dimensions of the sets involved. We will see that the multifractal formalism in [25] leads to a similar "principle of adding co-dimensions" in the multifractal case.
We will first need to define the intersection $\mu \cap \nu$ of two Radon measures on $\mathbb{R}^n$. If $E, F \subseteq \mathbb{R}^n$ and $\Delta = \{(x,x) | x \in \mathbb{R}^n\}$ denotes the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$, then clearly

$$E \cap F = \pi_1((E \times F) \cap \Delta)$$

where $\pi_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ denotes the projection

$$\pi_1(x, y) = x\,.$$

In analogy with (5.1) we now define the intersection $\mu \cap \nu$ of two Radon measures on $\mathbb{R}^n$ as follows,

$$\mu \cap \nu = ((\mu \times \nu) \cap (\mathcal{H}^{2n} \llcorner \Delta)) \circ \pi_1^{-1}\,;$$

the reader is referred to [23] for a discussion of intersection measures. It follows from Olsen [30] that if $\mu, \nu \in \mathcal{P}_0(\mathbb{R}^n)$ satisfy various regularity conditions (similar to the conditions

in the Multifractal Slice Theorems and to the conditions in the Fractal Intersection Theorem), then there exist two numbers $-\infty \leq q_{\min} < 0 < 1 < q_{\max} \leq \infty$ such that if $q_{\min} < q < q_{\max}$ and we put $t = b_\mu(q)$ and $s = b_\nu(q)$, then $(\mathcal{H}^{q,t}_\mu \llcorner \operatorname{supp}\mu) \times (\mathcal{H}^{q,s}_\mu \llcorner \operatorname{supp}\nu) \times \theta_n \times \mathcal{L}^1$-a.a. $(x, y, O, r) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{O}(n) \times \mathbb{R}_+$ satisfy

$$\operatorname{cob}_{\mu \cap (T_x \circ O \circ H_r \circ T_{-y})\nu}(q) = \operatorname{cob}_\mu(q) + \operatorname{cob}_{(T_x \circ O \circ H_r \circ T_{-y})\nu}(q)\,,$$

where $(T_x \circ O \circ H_r \circ T_{-y})\nu$ denotes the image measure

$$(T_x \circ O \circ H_r \circ T_{-y})\nu = \nu \circ (T_x \circ O \circ H_r \circ T_{-y})^{-1}\,.$$

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Chapter 3

Discontinuous Measures and a Failure of the Multifractal Formalism

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3.1 Introduction

Multifractal analysis is concerned with describing the complicated geometry of the singularities of distributions [11]. It has proved powerful in various applications such as image processing and compression [10]. While in praxis one prefers approaches using partitions of space, thus, the coarse graining multifractal spectrum f_{G}, a theoretical study preferably works with measures and, hence, with the notion of Hausdorff dimension and the corresponding spectrum f_{H}. Consequently, it is of considerable importance to know when the two approaches yield the same result ($f_{\mathrm{H}} = f_{\mathrm{G}}$). If they do in a particular situation, one says that the 'multifractal formalism' holds. By now, a variety of cases are known [1], [2], [3], [4], [6],[8], [9], [10], [14], [16], [17], [19].

As a particular toy model, *inverse measures* were introduced by Benoit Mandelbrot. In [13] it is argued that the multifractal spectra of the inverse measure is simply $\alpha \cdot f(1/\alpha)$. In [20],[21], the present author showed that this formula holds indeed for f_{H}. In this context, it was natural to introduce a broader class of self-similar measures where discontinuities are allowed [20], [21]. While the multifractal formalism holds in the case of continuous self-similar measures the author showed that it fails for discontinuous ones. As a first consequence, the formula $\alpha \cdot f(1/\alpha)$ can not hold for f_{G} in general. Secondly, it follows that a multiplicative structure implies neither the multifractal formalism nor concavity of the spectra. Finally, we mention that it becomes natural to include degenerated Hölder exponents, i.e. 0 and ∞, which appear also in different context [12],[19].

3.2 Multifractal analysis of inverse measures

Let μ be a probability measure on $[0,1]$ with its integral function $M(x) = \mu([0,x])$. Then, M is increasing and right-continuous. The differential of the inverse function M^* of M, defined as follows, is a probability measure denoted by μ^*, called the *inverse measure of* μ:

$$\mu^*([0,\theta]) := M^*(\theta) := \begin{cases} \inf\{t \,:\, M(t) > \theta\} & \text{if } \theta < 1 \\ 1 & \text{if } \theta = 1. \end{cases}$$

This procedure is involutive, i.e. $\mu^{**} = \mu$. Also, M^* is increasing and right-continuous, thus, μ^* is indeed a measure.
We are interested in the relation between the multifractal spectra $f_{\rm G}$ and $f_{\rm H}$ of μ and μ^* and possible implications of such a connection.

3.2.1 Coarse graining

First, note that μ-Hölder exponents and μ^*-Hölder exponents are reciprocal to each other unless a discontinuity lies on the boundary of an interval: Let a denote the left boundary point of an interval I. Then $|M(I)| = \mu(I\backslash\{a\})$ since M is right continuous.
We are introducing the *coarse grained spectrum* $f_{\rm G}(\alpha)$ now. Let G_δ be the set of all intervals $B = [l\delta, (l+1)\delta[$ with integer l and with $\mu(B) \neq 0$. In order to capture the local behaviour of μ one considers the neighbourhoods $B_1 := [(l-1)\delta, (l+2)\delta[$ of such intervals [18],[15]. Then,

$$N_\delta(\alpha,\varepsilon) = \#\{B \in G_\delta \,:\, |B_1|^{\alpha+\varepsilon} \le \mu(B_1) < |B_1|^{\alpha-\varepsilon}\},$$

and

$$f_{\rm G}(\alpha) := \lim_{\varepsilon\to 0} \limsup_{\delta\to 0} \frac{\log N_\delta(\alpha,\varepsilon)}{\log 1/\delta}$$

The coarse grained spectrum is related [18] to the *partition function* $\tau(q)$

$$\tau(q) := \liminf_{\delta\to 0} \frac{\log \sum_{B\in G_\delta} \mu(B_1)^q}{\log \delta}$$

through the Legendre transform

$$\tau(q) = \inf_{\alpha\in\mathbb{R}} \Big(q\alpha - f_{\rm G}(\alpha)\Big).$$

Consequently, the Legendre transform $f_{\rm L}$ of $\tau(q)$ is the concave hull of $f_{\rm G}$ and, thus, an upper bound to $f_{\rm G}$. It is natural to introduce $f_{\rm L}$ as the *Legendre spectrum* of μ. Equality $f_{\rm G} = f_{\rm L}$ may fail since $f_{\rm G}$ is not necessary concave. However, $f_{\rm G} = f_{\rm L}$ holds for discontinuous self-similar measures [21].
For a simple argument relating the spectra $f_{\rm G}$ and $f^*_{\rm G}$ of μ and μ^* use the approximation $N(\varepsilon,\alpha) \simeq \varepsilon^{-f(\alpha)}$. The μ measure of each of these ε-intervals is approximately ε^α. The function $M(t)$ maps these ε-intervals to $N(\varepsilon,\alpha)$ intervals, each of which has length approximately $\delta := \varepsilon^\alpha$. With $N(\varepsilon,\alpha) \simeq \delta^{-f(\alpha)/\alpha}$:

$$f^*_{\rm G}(1/\alpha) = \frac{f_{\rm G}(\alpha)}{\alpha}.$$

A more rigorous proof simply observes that computing $f_{\rm G}$ of the inverse measure amounts to computing the fixed mass spectrum of μ itself, provided both measures are *continuous*. The claim is then a corollary of a more general result.

3.2.2 Hausdorff spectrum

Now, we recall the definition of γ-*dimensional Hausdorff measure*

$$\eta^\gamma(E) = \sup_{\delta\to 0} \eta^\gamma_\delta(E), \qquad \eta^\gamma_\delta(E) = \inf\Big\{ \sum_k |I_k|^\gamma \ : \ E \subset \cup_k I_k \text{ and } |I_k| \le \delta \Big\}.$$

The *Hausdorff dimension* is then defined as

$$\dim(E) = \inf\{\gamma \ge 0 \ : \ \eta^\gamma(E) = 0\} = \sup\{\gamma \ge 0 \ : \ \eta^\gamma(E) = \infty\}.$$

Given a Hölder exponent α, the set K_α is defined by

$$K_\alpha := \left\{ t \in [0,1] \ : \ \lim_{I\to\{t\}} \frac{\log \mu(I)}{\log |I|} = \alpha \right\}.$$

Here, $I \to \{t\}$ will mean that I is an interval containing x, and that the length of I tends to zero.
The third multifractal spectrum, the *Hausdorff spectrum*, is defined as

$$f_{\rm H}(\alpha) = \dim(K_\alpha).$$

For continuous measures, the inversion formula for $f_{\rm H}$ follows directly from Proposition 1 below when applied to μ and μ^*, noting that $\mu^{**} = \mu$. In the case of discontinuous measures problems arise from the fact that the Hölder exponents of t and $M(t)$ are not necessarily reciprocal. A 'smoothing' procedure shows that these 'irregular' points do not contribute to the dimension of the sets K_α [21].

Proposition 1 *For any μ and any subset A of K_α one has*

$$\dim(A) \ge \alpha \cdot \dim(M(A)),$$

provided $0 < \alpha < \infty$.

For a proof fix $\alpha' < \alpha$ and let

$$A_m = \left\{ x \in A \ : \ \mu(I) \le |I|^{\alpha'} \text{ if } x \in I \text{ and } |I| \le 1/m \right\}.$$

Since A is a subset of K_α we have $A = \cup_{m\ge 1} A_m$. Let $\{I_j\}_j$ be a covering of A_m by intervals of length less than $1/n$ $(n > m)$. We have $|M(I_j)| \le \mu(I_j) \le |I_j|^{\alpha'} \le (1/n)^{\alpha'} := \delta_n \to 0$. Hence,

$$\eta^{\gamma/\alpha'}_{\delta_n}\Big(M(A_m)\Big) \le \sum |M(I_j)|^{\gamma/\alpha'} \le \sum |I_j|^\gamma.$$

Since $\{I_j\}_j$ was arbitrary we get $\dim(M(A)) = \sup_m \dim(M(A_m)) \le \dim(A)/\alpha'$.

3.3 Self-similar measures

Consider a set of similarity maps $w_0, \dots, w_{u-1}$ with *contraction ratios* $r_i \in]0,1[$ $(i = 0, \dots, u-1)$ and a *probability vector* $p_0, \dots, p_{u-1}$ ($p_i > 0$ for all i and $\sum p_i = 1$). Hutchinson [7] showed the existence of a unique probability measure with compact support such that

$$\mu(E) = \sum_{i=0}^{u-1} p_i \mu(w_i^{-1}(E))$$

For a simple example consider the random variable $X = \sum_n \varepsilon_n 2^{-n}$ where the ε_n are i.i.d. Bernoulli 0-1 variables. The distribution of X is then invariant under $w_i = x/2 + i/2$ with $p_i = \Pr[\varepsilon_n = i]$ $(i = 0, 1)$.
In the remainder we will assume that $]0,1[$ satisfies the open set condition which means that $w_i(]0,1[)$ are mutually disjoint subsets of $]0,1[$. Under this assumption it follows [1],[19], [3],[6],[14] that all the multifractal spectra coincide: $f_{\mathrm{H}} = f_{\mathrm{G}} = f_{\mathrm{L}}$ with $\tau(q) = \beta(q)$ and

$$\sum_{i=0}^{u-1} {p_i}^q {r_i}^{-\beta(q)} = 1. \tag{3.1}$$

For a proof consider the self-similar measure μ_q with the probabilities ${r_i}^{-\beta} {p_i}^q$ $(i = 0, \dots, u-1)$. By the LLN μ_q-almost every point has μ–Hölder exponent equal to $\alpha(q) = \beta'(q)$ and μ_q–Hölder exponent equal to $f(\alpha(q)) = q\alpha(q) - \beta(q)$. In other words, μ_q is supported on K_α and is there equivalent to $f(\alpha)$-dimensional Hausdorff measure. This gives (at least) the more difficult lower bound.

3.3.1 Discontinuous case

The *inverse* of a self-similar measure μ is readily obtained if $\sum r_i = 1$: simply interchange the roles of r_i and p_i. In other words, the inverse measure μ^* is in this case again a self-similar measure, invariant under w_i^* which have the same orientation as w_i and for which $w_i^*([0,1]) = [p_0 + \ldots + p_{i-1}, p_0 + \ldots + p_i]$. In particular, the respective partition functions of μ and μ^* are basically inverse to each other: $\tau^* = -q$.
Self-similar measures with $\sum r_i < 1$ are supported on Cantor sets. In other words, M is constant on a set of (Lebesgue) measure 1. Consequently, μ^* is purely atomic, hence, discontinuous in this case. Nevertheless, μ^* is still self-similar in the sense that it is invariant under a finite set of contraction. To see this, add (at the most $u+1$) maps w_j $(j = u, \dots v-1)$ to the family $w_0, \dots, w_{u-1}$ such that $]0,1[$ is still an open set and $r_0 + \ldots + r_{v-1} = 1$. The maps w_i^* for μ^* are defined as above. For $i = u, \dots v-1$ they are degenerate, though, and collapse the whole line into one point. They produce the atoms of μ^*.
With this it becomes natural to extend the scope of 'self-similarity' and to allow vanishing p_i and r_i. Applying the inversion formula shows then that the spectrum f_{H} can still be obtained as the Legendre transform of $\beta(q)$ where $0^x := 0$ for all x. However, the point $(0,0)$ has to be included if vanishing ratios, and hence atoms, are present. Consequently, one includes the point $(\infty, 1)$ in the spectrum if vanishing probabilities are involved reflecting the fact that intervals with zero measure have Hölder exponent ∞.

Also, we still have $f_{\mathrm{G}} = f_{\mathrm{L}}$ [21]. But $\tau(q) = \beta(q)$ holds only for $q \leq D^*$ where D^* is the (unique) zero of $\beta(q)$ and, thus, the dimension of the support of the inverse measure. For $q > D^*$ we have $\tau(q) = 0$. This follows immediately from the estimate

$$\begin{aligned}\sum_{B\in G_n} \mu(B_1)^q &\simeq \sum_{k=0}^{n-1} \sum_{r_{\varepsilon_l}\neq 0} \left(p_{\varepsilon_1}\cdot\ldots\cdot p_{\varepsilon_k}\right)^q = \sum_{k=0}^{n-1} \left({p_0}^q + \ldots + {p_{u-1}}^q\right)^k \\ &= c_q\Big(1 - \left({p_0}^q + \ldots + {p_{u-1}}^q\right)^n\Big).\end{aligned}$$

In other words, a coarse graining will find a whole hierarchy of atomic weights. The smallest ones ($k = n-1$) dominate the largest one ($k = 0$ with constant contribution) if and only if $q \leq D^*$. To have a picture note that the maps with $r_i = 0$ produce atoms which are then 'distributed' by the non-degenerate maps.
The equality $f_{\mathrm{G}} = f_{\mathrm{L}}$ can be established in this context using large deviations [5]. Note that this relation may fail if the measure is not purely multiplicative. Here, f_{G} and f_{L} equal the concave hull of the graph of the non-concave f_{H} (recall that it includes the origin). In particular, we have $f_{\mathrm{H}} < f_{\mathrm{G}}$ in a non-trivial interval. This is the first example of a failure of the multifractal formalism in the presence of a multiplicative process. Moreover, it supports our point of view that degenerate Hölder exponents have to be included in the spectrum. Otherwise, a convincing connection between f_{G} and f_{H} would not exist.

3.3.2 Equilibrium measures

A natural generalisation of the notion of self-similar measures are the equilibrium measures which appear in the theory of dynamical systems. In a typical situation on the line, one will consider a conformal mapping g which maps some disjoint intervals $I_i \subset [0,1]$ onto $[0,1]$ such that $-\log|g'|$ is negative and Hölder continuous. The invariant measure in question will then live on the repeller of g, more precisely, it will be the equilibrium measure of another Hölder continuous function ϕ. This scheme reduces to the self-similar case if g is such that the w_i are its inverse branches and if ϕ takes the constant value $\log p_i$ on I_i.
The multifractal formalism $f_{\mathrm{H}} = f_{\mathrm{L}}$ has been established for Cookie-cutters by Rand [17], and for equilibrium measures of certain Moran constructions by Pesin and Weiss [16]. Set $\psi = \exp(\phi - P\{\phi\})$ with P denoting the pressure function, and let β be (uniquely) defined through

$$P\{q\log\psi - \beta(-log|g'|)\} = 0.$$

Then, τ equals β, and the spectra of μ collapse with the Legendre transform β^*. Note, that the definition reduces to the usual one.
Reciprocal equilibrium measures: It is tempting to produce new measures analogously to self-similar measures, i.e. to exchange the roles of 'geometry' $-\log|g'|$ and 'mass' ϕ. Assume, therefore, that $\phi = -\log|h'|$ for some function h with properties analogous to g. Denote the h-invariant equilibrium measure for $\overline{\phi} := -\log|g'|$ by $\overline{\mu}$.
First, f_{H} of μ^* can be obtained by the inversion formula $f^*_{\mathrm{G}}(1/\alpha) = (1/\alpha) f_{\mathrm{G}}(\alpha)$ as usual [21], hence, by taking the Legendre transform of the inverse β^{-1}. In analogy with discontinuous self-similar measures, especially since gaps are present, we conjecture that the partition function of μ^* equals $\min\{\beta^{-1}, 0\}$.

Secondly, being an equilibrium measure, $\overline{\mu}$ has its f_{H} equal to $\overline{\beta}^*$ where, as before, $P\{t\log\overline{\psi} - \overline{\beta}(-\log|h'|)) = 0$ with $\overline{\psi} = \exp(\overline{\phi} - P\{\overline{\phi}\})$. Though very closely related, the spectra of μ^* and $\overline{\mu}$ are very well distinguished unless $P\{\phi\}$ and $P\{\overline{\phi}\}$ vanish. But this is the degenerate case when $\overline{\mu}$ and μ are supported on all of $[0,1]$.

Special feature of the spectra: One particular difference between the spectra of μ^* and $\overline{\mu}$ is the *slope of their tangent through the origin.* With the continuous $\overline{\mu}$, this slope is 1: The spectra must touch the bisector since $\overline{\tau}(1) = \overline{\beta}(1) = 0$. As they are strictly concave this can happen only in a point $\alpha > 0$. For μ^*, on the other hand, the touching point with the bisector is in 0, and the slope of the tangent through the origin is strictly less than 1 since $\beta^*(D) = 0$, $D = -\beta(0)$ being the dimension of the support of μ.

This fact reflects the fundamentally different way of dealing with the fact of 'loosing mass' when approximating the measure iteratively. With μ^*, loss of mass in the generating process is compensated by producing atoms. To the contrary with $\overline{\mu}$ which is 'renormalised' in each step by a factor e^{-P} in order to prevent it from dying out or exploding (compare [17]). This renormalisation by e^{-P} brings a shift in the Hölder exponents which causes the distinct yet closely related shape of the spectra of μ^* and $\overline{\mu}$.

It is this different way of compensating mass which causes the failure of the multifractal formalism for the inverse measure μ^*.

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Chapter 4

Fractals and Dynamics in $\mathbb{C}^n$

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4.1 Introduction

When dealing with Fractal Formalisms one could stick to the following programme.

- *Compare* different measures, e.g. Packing-, Hausdorff, Box-Counting-, SBR-, ...
- Find the *right* one (whatever this means ...)
- Investigate parameters, for example dimensions, mass, entropy, pressure, *characteristics*, resp.

One might want to ask the following questions.

- What *kind of* measures are supported on certain sets of interest?
- What does a measure tell us about the *topology* of its support?

We shall present some concepts from Functional Analysis and Holomorphic Dynamics which may be of interest for 'Fractal Formalists'.
Let us recall the well-known *Krejn-Mil'man-Theorem.*

Theorem 4.1.1 *In a linear topological space E, provided E^* is total, any compact convex set $Q \subset E$ coincides with the closed convex hull of the set extr(Q) of its extreme points*

$$z \in extr(Q) \ :\Leftrightarrow \ (z = (u+v)/2 \text{ for some } u, v \in Q \text{ implies } u = v = z)$$

The tight connection with measure theory and fractal formalism becomes clear with the following definitions.

Definition 4.1.2 *The total variation of a vector measure m on some measure space S (with values in E) is defined as*

$$\|m\| := |m|(S),$$

where, for $A \subseteq S$,

$$|m|(A) := \sup_{\{A_k\}_1^n} \sum_{k=1}^{n} \|m(A_k)\| \leq \infty$$

the supremum ranges over all finite measurable partitions A_k of A.

Definition 4.1.3 *A vector measure is called absolutely continuous if $\|m\|$ is finite and, for some $x \in L^1(S, |m|; E)$,*

$$m(A) = \int_A x(s) d|m|.$$

We say that a real Banach space E has the *Radon-Nikodym-property* if every vector measure with finite norm is absolutely continuous. It is interesting that this measure-theoretic property can be expressed in topological terms.

Definition 4.1.4 *For $C \subset\subset E$, a point $z^* \in C$ is called strongly exposed if there is $f \in E^*$ such that, for all $z \in C \setminus \{z^*\}$,*

$$f(z) < f(z^*),$$

and

$$\lim_{\varepsilon \to 0} diam\{z : f(z) > f(z^*) - \varepsilon\} = 0.$$

We call the set $extr(C)$ of all these points Choquet-Boundary $\partial_{CH} C$ of C.

The connection between Radon-Nikodym-property and topology is then given by the *Phelps-Rieffel-Theorem.*

Theorem 4.1.5 *A necessary and sufficient condition for a Banach space to have the Radon-Nikodym-property is that each of its closed bounded convex subsets be the closed convex hull of its strongly exposed points.*

Krejn-Mil'man-Theorem 'revisited' now reads:

Theorem 4.1.6 *Let*

$$x(s) := s,$$

where, for some compact $Q \subset\subset E$,

$$s \in S := \overline{extr(Q)}.$$

For a normalised measure μ on the compact set S, the centre of mass is given by

$$x_\mu := \int_S s d\mu.$$

Clearly, it belongs to Q, by the Krejn-Mil'man-Theorem any $x^* \in Q$ is centre of mass for some normalised μ_{x^*} on S.

It is remarkable that one can even restrict to measures which are supported only on extr(Q). This is the famous *Theorem of Choquet.*

Theorem 4.1.7 *μ_{x^*} can be chosen such that it is concentrated on extr(Q) and not on $S \setminus$ extr(Q) (Note that extr(Q) is G_δ in S).*

Clearly, the concept of Choquet-Boundary can be generalised to other convexity relations (pseudo-, harmonic, polynomial ... convexity). In general the following inclusions hold.

$$\partial_(CHO) \subset \partial_{SHI}(Q) := \overline{\partial_{CHI}(Q)} \subset \partial_{TOP}(Q).$$

We use ∂_{TOP} as abbreviation for the *topological boundary* and ∂_{SHI} for the *Shilov-Boundary* which can be characterised as follows

Definition 4.1.8 *The Shilov-Boundary is the set of points $z \in Q$ which are 'weakly exposed', i.e. for arbitrarily small neighbourhoods U of z there is a peak function f for the neighbourhood taking its maximal modulus in U but only smaller values outside of U.*

4.2 Julia sets and Choquet boundary

4.2.1 A bad example: Julia sets in $\mathbb{C}^1$

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree at least 2. Then we define its *Julia set* $J(f)$ as the set of points where the sequence (f^n) of the iterates of f is not *normal convergent.* In the following we shall investigate this set and some of its functional-analytic properties. First let us state a trivial but highly useful lemma.

Lemma 4.2.1 *An entire mapping $f : \mathbb{C} \to \mathbb{C}$ is a polynomial if an d only if f fulfils the following growth condition. For some $d \in \mathbb{N}$ and all $z \in \mathbb{C}$, $|z| > R_f$, where R_f is some real number depending on f, $k_1, k_2 > 0$*

$$k_1 \cdot |z|^d \leq |f(z)| \leq k_2 \cdot |z|^d.$$

The lemma allows us to give a topological characterisation of $J(f)$. First note that we can restrict our interest to a compact set. The existence of R_f implies that there is some finite constant r_f such that $|z| > r_f$ implies that an open neighbourhood of z converges to infinity under iteration of f. Hence the Julia set must be in the remaining set

$$K(f) := \bigcap_{n=0}^{\infty} \{z : |f(z)^n| \leq r_f\}.$$

Clearly, $K(f)$ is compact and (polynomially) convex. Furthermore, it is easy to see that, on the interior of $K(f)$, the sequence of iterates of f is normal. Since there cannot be normal convergence on any open set which contains (topological) boundary-points of $K(f)$ we have shown.

Theorem 4.2.2 *The Julia set of a polynomial $f : \mathbb{C} \to \mathbb{C}$ equals the topological boundary of the set $K(f)$ of points of bounded forward orbit under iteration of f. The latter set is polynomially convex.*

For any point $z^* \in \partial_{TOP}(K)$ in the topological boundary of $K(f)$ we can construct a peak function by choosing $z_0 \in \mathbb{C} \setminus K(f)$ and setting $\varphi(z) := \left|\frac{1}{z}\right|^\rho$ for some $\rho > 1$. This implies

Theorem 4.2.3 *For the Julia set of polynomial $f : \mathbb{C} \to \mathbb{C}$ the following equality holds:*

$$J(f) = \partial_{SHI}(K) = \partial_{CHO}(K) = \partial_{TOP}.$$

In this case, the concept of Choquet-Boundary does not give any additional piece of information about the topology of $K(f)$ which could not be obtained by application of simpler tools (e.g. by taking the topological boundary). What is remarkable is that it coincides with other sets which are defined in dynamical terms.

- $J(f)$ equals the closure of the set of repelling periodic points of f.
- $J(f)$ coincides with the support of the *harmonic measure* μ which is represented as

  $$\mu = \Delta G,$$

  where G is the *Green-Function* of $\mathbb{C} \setminus K$ which can be calculated as

  $$G(z) := \lim_{n\to\infty} \frac{1}{d^n} \log_+ |f^n(z)|;$$

  μ has *maximal entropy* $\log(d)$ as

  $$G \circ f(z) = d \cdot G(z)$$

  which implies that μ is a d-balanced measure.

4.2.2 A nice example: Julia sets in $\mathbb{C}^n$, $n \geq 2$

In order to perform iteration theory in $\mathbb{C}^n$, $n \geq 2$, one has to choose suitable mappings first. An interesting class of generic polynomial maps $f : \mathbb{C}^n \to \mathbb{C}^n$ consists of the so-called *strict polynomials*. These are entire maps $f : \mathbb{C}^n \to \mathbb{C}^n$ which fulfil a growth condition similar to that of polynomials in $\mathbb{C}^1$, namely for some $d \in \mathbb{N}$ and all $z \in \mathbb{C}^n$, $\|z\| > R_f$, where R_f is some real number depending on f, $k_1, k_2 > 0$

$$k_1 \cdot \|z\|^d \leq \|f(z)\| \leq k_2 \cdot \|z\|^d.$$

In complete analogy to the one-dimensional setting we see that for $\|z\| > r_f$ we get convergence to infinity, hence define

$$K(f) := \bigcap_{n=0}^{\infty} \{z : \|f(z)^n\| \leq r_f\}.$$

Again, $K(f)$ is polynomially convex and we may ask for the Choquet-Boundary of $K(f)$. It turns out that, in general the Choquet-Boundary is a proper subset of the topological boundary of $K(f)$, hence it makes to investigate it separately, moreover it coincides with the Julia set of f.
For the reader non-familiar with Julia sets in higher-dimensional spaces we give a brief account on their definition. The essential tool is so-called *weak normal convergence*.

Definition 4.2.4 *A point* $z \in \mathbb{C}^n$ *is called weakly normal w.r.t. to the sequence* f^n *if there is*

- *an open neighbourhood* $U \ni z$,
- *a family* $\mathcal{C}_x$ *of complex analytic sets with* $\dim(\mathcal{C}_x) \geq 1$ *indexed by* $x \in U$

such that

- *each* $x \in U$ *is contained in* $\mathcal{C}_x$,
- $f^n|_{\mathcal{C}_x}$ *is normal.*

The definition of the Julia set of f is obvious.

Definition 4.2.5 *The Julia set of a strict polynomial is given by the set of points where* f^n *is not weakly normal.*

Let us give an easy example. For the map

$$\sigma : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}$$

we get

$$K(\sigma) = \overline{\mathbb{B}} \times \overline{\mathbb{B}}.$$

Clearly, the topological boundary is given by the disjoint union

$$\partial_{TOP}(K(\sigma)) = S^1 \times S^1 \cup \mathbb{B} \times S^1 \cup S^1 \times \mathbb{B},$$

whereas the Julia set consists of

$$J(\sigma) = S^1 \times S^1.$$

Obviously,

$$J(\sigma) = \partial_{SHI}(K(\sigma)) = \overline{\partial_{CHO}(K(\sigma))} \subset \partial_{TOP}(K(\sigma)).$$

Since this also holds for arbitrary strict polynomials, we have found a dynamical characterisation of the Choquet-Boundary. Moreover, the dynamics also yields a suitable measure on $\partial_{CHO}(K(f))$. We can calculate the *generalised Green-Function* of $K(f)$ as

$$G(z) := \lim_{n\to\infty} \frac{1}{d^n} \log_+ \|f^n(z)\|;$$

application of dd^c yields a current whose n-th product gives a d-balanced measure

$$\mu := (dd^c)^n.$$

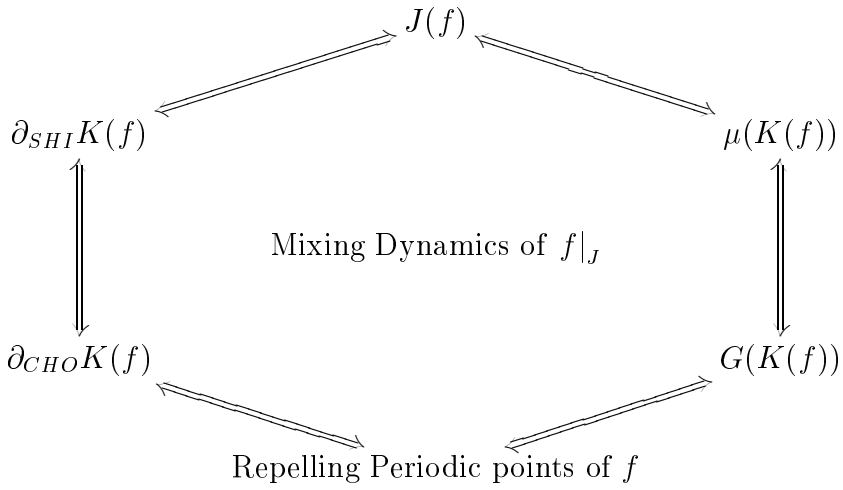
The support is exactly the Shilov-Boundary, hence

$$\operatorname{supp}(\mu) = \partial_{SHI} = \overline{\partial_{CHO}(K(\sigma))} \subset \partial_{TOP}(K(\sigma)).$$

It can also be shown that the Julia set equals the closure of the set of (purely) repelling periodic points of f which implies that the action of f on the Choquet-Boundary is *topologically mixing*, i.e. for all open sets $U \cap J(f) \neq \emptyset$ there exists $n_U < \infty$ such that

$$f^{n_U}(U) \supset J(f).$$

We obtain the following relations.



The following pictures show three-dimensional cross-cuts through Julia sets in $\mathbb{C}^2$ (see also picture on the front page).

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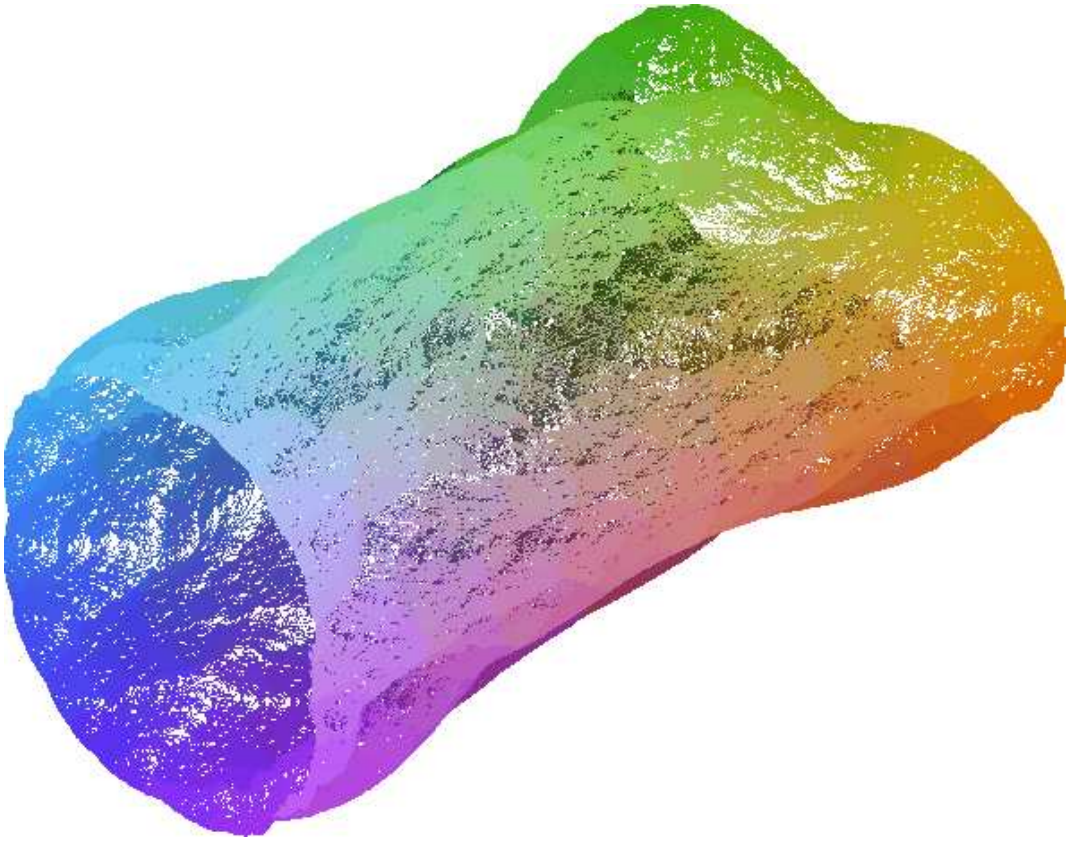

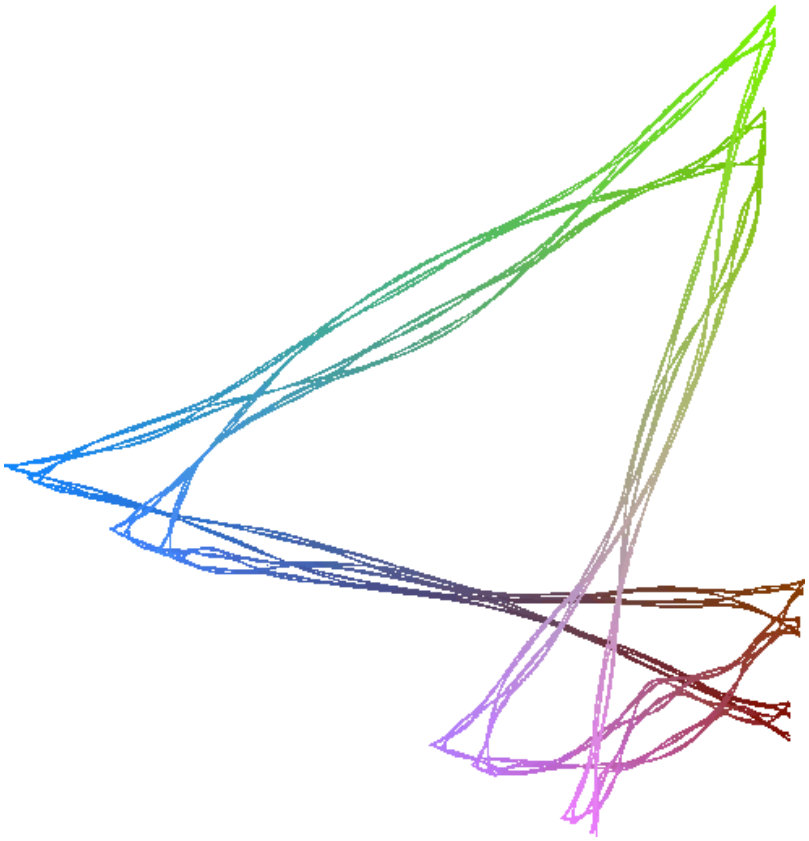

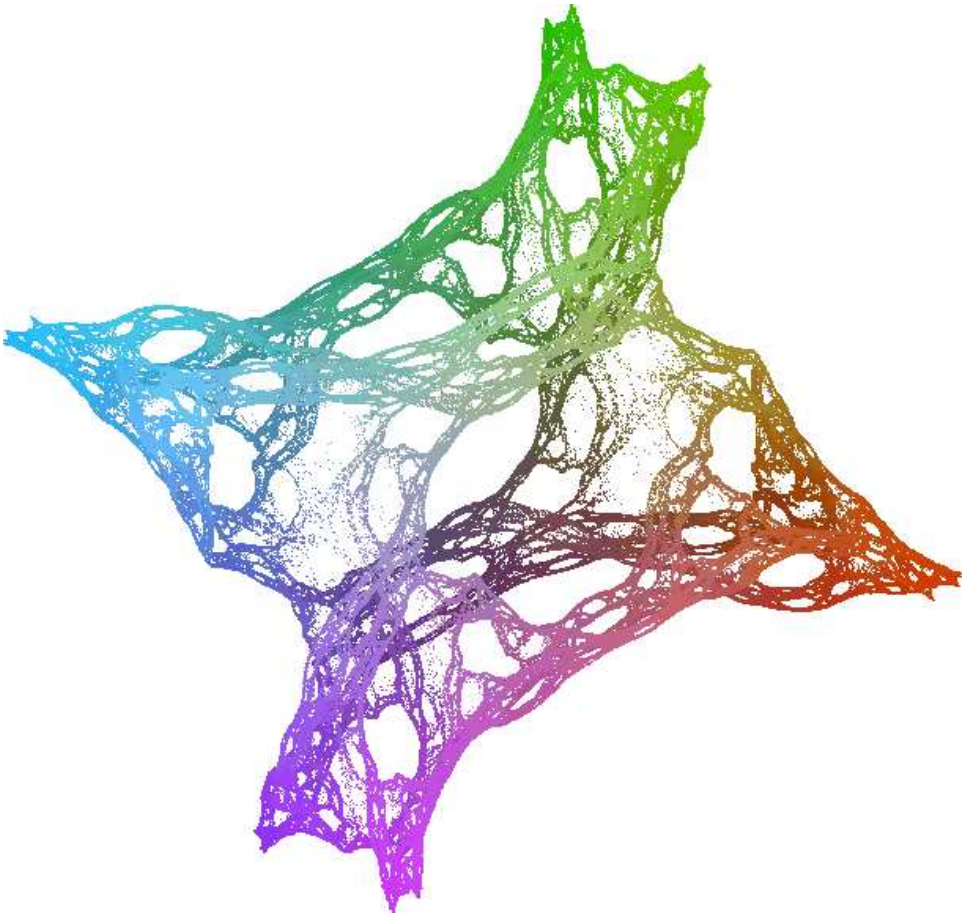

Chapter 5

Fractal Integrals and Differential Equations

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5.1 Introduction

The classical *ordinary differential equation*

$$\dot{x}(t) = a(x(t), t)$$

for $a \in C^1(\mathbb{R}^2, \mathbb{R})$ may be rewritten in integral form as

$$x(t) = x(0) + \int_0^t a(x(s), s)\, ds.$$

In many applications a time evolution of this kind is disturbed by a deterministic or random *"noise"*. The corresponding extension of the integral equation would read as follows:

$$x(t) = x(0) + \int_0^t a(x(s), s)\, ds + \int_0^t b(x(s), s)\, dZ(s) \tag{5.1}$$

where $dZ(s)$ represents the noise. Our first aim is to make the right–hand side precise for some classes of fractal mappings $Z(t)$, at least for Hölder continuous functions of order $\lambda > 1/2$. Secondly, we are interested in existence, uniqueness and representation of the solution of (1) with initial value $x(0) = x_\circ$ under appropriate conditions on the integrands a and b.
The following *example from finance mathematics* motivates our approach.
On the market two types of assets are traded continuously in time: The price P_1 of a safe asset, the so–called bond, satisfies the linear ODE

$$dP_1(t) = r_1(t)\, P_1(t)\, dt.$$

The risky assets are called stocks and the simplest model for their price P_2 is a stochastic differential equation in the Itô sense:

$$dP_2(t) = r_2(t)\,P_2(t)\,dt + \sigma(t)\,P_2(t)\,dW(t).$$

The interest rates $r_1(t)$ and $r_2(t)$ as well as the dispersion $\sigma(t)$ may assumed to be measurable random processes adapted to the filtration given by the Wiener process $W(t)$, e.g. in order to derive related option pricing procedures. Usually, r_1, r_2 and σ are chosen deterministic. The last equation makes also sense if the Wiener process $W(t)$ (or the "white noise" $dW(t)$) is replaced by arbitrary semimartingales. However, in real stock market processes long–range dependencies have been obtained which has led to search for more appropriate noise processes $dZ(t)$. *Fractional Brownian motion* $B^H(t)$ with Hurst exponent $H > 1/2$ provides one of them. The (stochastic) differential equation

$$d(P_2(t) = r_2(t)\,P_2(t)\,dt + \sigma_2(t)\,P_2(t)\,dB^H(t)$$

i.e.,

$$P_2(t) = P_2(0) + \int_0^t r_2(s)\,ds + \int_0^t \sigma_2(s)\,P_2(s)\,dB^H(s) \tag{5.2}$$

has already been considered in the literature, but without giving it a precise sense in the general situation. It turns out from our new approach that the known path-wise solution

$$P_2(t) = P_2(0)\,\exp\left\{\int_0^t r_2(s)\,ds + \int_0^t \sigma_2(s)\,dB^H(s)\right\}$$

is unique in the class of Hölder continuous function of order less than H. (In this formula $\int_0^s \sigma_2(s)\,dB^H(s)$ may be interpreted as Riemann–Stieltjes integral.)

5.2 Construction of the fractal integral

The second integrals on the right–hand sides of (1) and (2) will be obtained from the following more general construction.
Suppose that f and g are Borel measurable functions on a finite interval $[a, b]$. If g has bounded variation then it may be represented as $g = g_1 - g_2$ for monotone functions g_1 and g_2 with related measures μ_1 and μ_2, respectively. Recall, that the *Lebesgue–Stieltjes integral* of f with respect to g may be defined as

$$\text{(L.–S.)} \quad \int_a^b f\,dg = \int f\,d\mu_1 - \int f\,d\mu_2$$

provided that f is Lebesgue integrable with respect to the variation measure $\mu_1 + \mu_2$. If f is continuous then it agrees with the *Riemann–Stieltjes integral*

$$\text{(R.–S.)} \quad \int_a^b f\,dg = \lim_{\Delta \to 0} \sum_i f(x_i^*)\,(g(x_{i+1}) - g(x_i))$$

where convergence of the Riemann–Stieltjes sums holds uniformly in all finite partitions $a = x_0 \le x_0^* \le x_1 \le x_1^* \le \ldots \le x_n \le x_n^* \le x_{n+1} = b$ with $\Delta = \max_i (x_{i+1} - x_i)$.
The main idea for extending these integrals to fractal f and g relies on the following calculation rules in the smooth case:

$$\text{(L.–S.)} \quad \int_a^b f\, dg = \int_a^b f\, g'\, d\lambda$$

or

$$\text{(L.–S.)} \quad \int_a^b f\, dg = f(b)\, g(b) - f(a)\, g(a) - \int_a^b f'\, g\, d\lambda$$

where λ denotes Lebesgue measure. We simply replace the pairs (f, g') or (f', g) by the *fractional Weyl derivatives* $\left(D^\alpha_{\text{left}}\, f, D^{1-\alpha}_{\text{right}}\, g\right)$ for some $0 \le \alpha \le 1$. This method works if $f(a+) = \lim_{\epsilon \searrow 0} f(a+\epsilon) = 0$ and $g(b-) = \lim_{\epsilon \searrow 0} g(b-\epsilon) = 0$. Therefore we consider in the general case the auxiliary functions

$$f_{a+} := f - f(a+)\ , \quad g_{b-} := g - g(b-)$$

and define the *fractal integral*

$$\int_a^b f\, dg := (-1)^\alpha \int_a^b D^\alpha_{\text{left}} f_{a+}\ D^{1-\alpha}_{\text{right}} g_{b-}\ d\lambda + f(a+)(g(b-) - g(a+)) \tag{5.3}$$

if $f_{a+} \in I^\alpha_{\text{left}}\,(L_p)$, $g_{b-} \in I^{1-\alpha}_{\text{right}}\,(L_q)$ for some $0 \le \alpha \le 1$, $\frac{1}{p} + \frac{1}{q} \le 1$.
Here $I^\alpha_{\text{left}}\,(L_p)$ (or $I^\alpha_{\text{right}}\,(L_p)$) denotes the class of functions f which are representable as *fractional Riemann–Liouville integral* of order α of some L_p–function φ, i.e.,

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1}\, \varphi(y)\, dy$$

$$\left(f(x) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1}\, \varphi(y)\, dy\ , \quad \text{respectively} \right).$$

The function φ is unique in L_p and agrees with the corresponding fractional Weyl derivative of f of order α (cf. [1]).
The above definition appears to be independent of the choice of α and leads to all nice properties of an integral.
If g has bounded variation and some additional technical condition is fulfilled then (3) agrees with the Lebesgue–Stieltjes integral, in particular, for step functions f. Moreover, if H^λ denotes the space of Hölder continuous functions of order λ and $f \in H^\lambda$, $g \in H^\mu$ with $\lambda + \mu > 1$ then we can prove the convergence of the Riemann–Stieltjes sums to the expression in (3). Thus, we obtain the Riemann–Stieltjes integral for this large class of functions of unbounded variation.
The following two calculation rules are basic for solving the differential equations mentioned in the introduction.

If $f \in H^\lambda([a,b]),\ F \in C^1(\mathbb{R}\times[a,b]),\ F_1'(f(\cdot),\cdot) \in H^\mu([a,b]),\ \lambda+\mu>1$ we get the *chain rule* for the change of variables:

$$F(f(x),x) - F(f(a),a) = \int_a^x F_1'(f(y),y)\,df(y) + \int_a^x F_2'(f(y),y)\,dy \tag{5.4}$$

where F_1' and F_2' are the partial derivatives of F.
Now suppose that $f,\ \varphi \in H^\lambda$, $g \in H^\mu$ and $\lambda+\mu>1$. Then the *integral as function of the boundary* is Hölder continuous of the same order as the integrator, i.e.,

$$h(\cdot) := \int_a^{(\cdot)} f\,dg \in H^\mu \tag{5.5}$$

and a *density–type formula* holds:

$$\int_a^b \varphi\,dh = \int_a^b \varphi\cdot f\ dg. \tag{5.6}$$

(For more details and further properties see [3].)

5.3 Solution of differential equations

(The results of this section are obtained together with F. Klingenhöfer.)
We now consider the differential equation (1) in the sense of (2) on a finite interval $(-T,T)$ supposing $Z \in H^\lambda$ for some $\lambda > 1/2$, $Z(0)=0$, $a \in C(\mathbb{R}\times(-T,T))$ and is Lipschitz continuous in the first variable, $b \in C^1(\mathbb{R}\times(-T,T))$ and $b(x_0,0)\neq 0$. Under these conditions we prove in [2] existence and uniqueness of a local solution $x(t)$ with initial condition $x(0)=x_0$. Moreover, the following solution procedure reduces the problem to ordinary (partial) differential equations in the classical sense:
We construct two auxiliary differentiable functions. First we choose any local C^1–solution $\varphi(x,t)$ of the quasi–linear partial differential equation

$$\begin{aligned} \varphi_x'(x,t) &= b(\varphi(x,t),t) \\ \varphi(0,0) &= x_0. \end{aligned} \tag{5.7}$$

Secondly, for given φ as above we take the unique (maximal) solution $y(t)$ of the ordinary differential equation

$$\begin{aligned} \dot y(t) &= \frac{a(\varphi(y(t)+Z(t),t) - \varphi_t'(y(t)+Z(t),t)}{b(\varphi(y(t)+Z(t),t),t)} \\ y(0) &= 0 \end{aligned} \tag{5.8}$$

Then we get for the solution of (1)

$$x(t) = \varphi(y(t)+Z(t),t)$$

on the common interval of definition.

In the *autonomous case* $a(x,t) = a(x),\ b(x,t) = b(x)$ equations (7) and (8) reduce to

$$\varphi'(x) = b(\varphi(x))\,, \quad \varphi(0) = x_0$$

and

$$\dot{y}(t) = \frac{a(\varphi(y(t) + Z(t))}{b(\varphi(y(t) + Z(t))}\,, \quad y(0) = 0,$$

respectively, and the solution is given by

$$x(t) = \varphi(y(t) + Z(t)).$$

In particular, if $a = 0$ and we put here an ergodic stationary random process $Z(t)$ then the resulting process $x(t) = \varphi(Z(t))$ is also ergodic. This answers a question posed by M. Keane.
Finally, in the *linear case*

$$a(x,t) = r(t)\,x\ ,\ b(x,t) = \sigma(t)\,x$$

our procedure leads to the solution of (2) presented in the introduction.

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Chapter 6

Note on the Axiomatic Approach to Self-Similar Random Sets and Measures

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6.1 Introduction

In this brief note I would like to indicate how my axiomatic approach to random self-similar sets [2] generalises to measures, and how it complements the Palm measure approach of U. Zähle [15],[16],[17], thus answering a question which Martina Zähle raised on our meeting in November 1995. As it should be at the beginning of a cooperation, I shall focus on concepts and problems, plans and suggestions for future research, and very fresh results, if any.
At the end I try to point out how constructive and axiomatic approach to self-similarity complement each other, too. Special self-similar constructions in $\mathbb{R}^n$ were studied around 1900 by many people [8]. A general definition came only 1981 when Hutchinson studied compact sets A in a complete metric space X which satisfy an equation $A = f_1(A) \cup ... \cup f_m(A)$ with given contractive similarity maps f_i. A probability measure μ on X is self-similar with respect to given contractions f_i and nonnegative numbers p_i with $\sum p_i = 1$ if

$$\mu = p_1 f_1(\mu) + ... + p_m f_m(\mu) \qquad \text{where} \qquad f(\mu)(B) = \mu(f^{-1}(B)) \ . \tag{6.1}$$

The support of μ is then the self-similar set A. Random self-similar measures were introduced 1986 by Falconer, Graf, Mauldin and Williams [5], [7] as solutions of (1) where the p_i and f_i are random variables (i.e. they depend on ω taken from a probability space), and equality is *in distribution.* This kind of self-similarity does not apply to single figures but, on a higher level, to the whole construction process. For a rigorous definition, replace

μ in (1) by another letter, say M^*, to denote the distribution of the random variable μ. If the realizations $\mu(\omega)$ of μ live on $\mathbb{R}^n$ then M^* is a probability measure on the space $X = \mathcal{M}_1(\mathbb{R}^n)$ of probability measures on $\mathbb{R}^n$, and (1) with M^* instead of μ is again a ordinary linear equation in a Banach space.
Nevertheless, the use of these concepts for modelling reality remains restricted, due to the very specific choice of mappings. The desired symmetries f_i are put into the definition, and the resulting figures look somewhat artificial. (Also, for the random case, I always felt irritated by the fact that the choice of maps on the first level is more important for the shape of the figure than subsequent choices.) So far, almost all statements on and examples of self-similar sets or measures, deterministic or random, assume the rather peculiar *open set condition:* there should be an open U with $f_i(U) \subset U$ and $f_i(U) \cap f_j(U) = \emptyset$ for all i and $j \neq i$.
In stochastic processes, self-similarity sometimes comes up much more naturally. Limit theorems guarantee, for instance, that $B(r^2 t) = rB(t)$ holds for Brownian motion starting at zero. Equality is again in distribution, not for specific figures, but for the whole process. The measure which describes the process either lives on the space of continuous functions of t, and is self-affine [16], or, for trajectories of the motion in $\mathbb{R}^n$ with $n \geq 2$, on the space of all paths in $\mathbb{R}^n$, and is self-similar.
The important point here is that r can be *any* positive number. As concerns the centre of the similarity map (taken as zero here), however, we have to restrict ourselves to functions or paths passing through this centre. On the family of these paths the similarity map acts in a topologically transitive way, while in a family of paths outside the center there would be wandering sets, which is impossible for a probability distribution.

6.2 Axioms from similarities on hyperspaces

Now we consider *stationary* random sets φ and measures M on $\mathbb{R}^n$. Let $\mathcal{F}$ denote the space of closed subsets of $\mathbb{R}^n$ and $\mathcal{M}$ the space of locally finite measures on $\mathbb{R}^n$, with the usual Borel σ-algebras [15], [13], [6],[11]. Since a random variable is often identified with its distribution, we can define a random set as a Borel measure φ on $\mathcal{F}$, and a random measure as a Borel measure M on $\mathcal{M}$. Due to wandering sets in the action of similarities on $\mathcal{F}$ and $\mathcal{M}$, we cannot work with probability distributions [15], [2] but we require φ and M to be locally finite: $\varphi(\mathcal{F}_B) < \infty$ and $M(\mathcal{M}_B) < \infty$ for every bounded open set B, where $\mathcal{F}_B = \{F \in \mathcal{F} \mid F \cap B \neq \emptyset\}$ and $\mathcal{M}_B = \{\mu \in \mathcal{M} \mid \mu(B) > 0\}$. We assume φ to be stationary, that is, $\varphi(\mathcal{F}_B) = \varphi(\mathcal{F}_{B+x})$ for all x and B, and similarly for M. The spaces of all locally finite and stationary random sets φ and random measures M will be denoted by $\mathbb{F}$ and $\mathbb{M}$, respectively.
The similarity map $g(x) = rx$ acts on $\mathcal{F}$ by $g(F) = rF$ as usual, on the generating system of the Borel sets of $\mathcal{F}$ by $g(\mathcal{F}_B) = \mathcal{F}_{rB}$, and on $\mathbb{F}$ by

$$g(\varphi)(\mathcal{F}_B) = \varphi(\mathcal{F}_{r^{-1}B}) \; . \tag{6.2}$$

We would like to consider φ as self-similar if $g(\varphi) = \varphi$ for all similarity maps g. However, if we assume that $r < 1$ and B is a convex set containing the origin, then $B \subset r^{-1}B$ implies that $\mathcal{F}_B$ is strictly contained in $\mathcal{F}_{r^{-1}B}$, and (2) says that $g(\varphi)$ is strictly larger than φ. Thus we have to introduce a normalising factor $\eta(r)$ in order to compare $g(\varphi)$

with φ. It is easy to see that $\eta(rs) = \eta(r)\eta(s)$ and thus $\eta(r) = r^\alpha$ with $0 < \alpha < n$ [2]. So a random set φ is defined to be self-similar, with scaling exponent α, if

$$g(\varphi) = r^{-\alpha}\varphi \quad \text{for each similarity map} \quad g(x) = rx \ . \tag{6.3}$$

In [2] we used a slightly different approach. For each ball B, the random set φ makes $\mathcal{F}_B$ a probability space, by $\mathbf{P}_B(\mathcal{F}_C) = \varphi(\mathcal{F}_C)/\varphi(\mathcal{F}_B)$ for open subsets C of B. Self-similarity of φ was defined as the isomorphy of the probability spaces $(\mathcal{F}_B, \mathbf{P}_B)$ for all balls B, and the scaling exponent α was derived later from this definition. Moreover, φ can be reconstructed from the probability space $(\mathcal{F}_B, \mathbf{P}_B)$, where B denotes the unit ball, say.
For measures, the situation is still more intricate. $g(x) = rx$ acts on $\mathcal{M}$ by $g(\mu)(B) = \mu(r^{-1}B)$. For $r < 1$ and convex neighbourhoods B of zero this implies that $g(\mu)(B)$ is strictly greater $\mu(B)$, for all μ. Already here we have to introduce a normalising factor $\xi(r)$ which again is a power of r. We define

$$g_\beta(\mu)(B) = r^\beta \mu(r^{-1}B) \ . \tag{6.4}$$

For Borel sets $\mathcal{B} \subset \mathcal{M}$ we have $g_\beta(\mathcal{B}) = \{g_\beta(\mu) \mid \mu \in \mathcal{B}\}$, in particular $g_\beta(\mathcal{M}_B) = \mathcal{M}_{rB}$. The action of g_β on $\mathbb{M}$ is now defined as

$$g_\beta(M)(\mathcal{B}) = M(g_\beta^{-1}(\mathcal{B})) = M(\{r^{-\beta}\mu(r\,\cdot) \mid \mu \in \mathcal{B}\}) \ . \tag{6.5}$$

As above, a normalising factor r^α becomes necessary on the level of $\mathbb{M}$. *So let us call M self-similar with scaling exponents α and β if*

$$g_\beta(M) = r^{-\alpha}M \quad \text{for each similarity map} \quad g(x) = rx \ . \tag{6.6}$$

U. Zähle found that for many fractals μ, the exponent β is the Hausdorff dimension of μ and $\alpha = n - \beta$ is the codimension. Actually, he used a slightly different approach in [15]. The Palm measure M_0 of a stationary random measure M had been introduced 1967 by Mecke as, roughly speaking, "the probability describing realizations μ of M under the condition that 0 belongs to μ." U. Zähle saw that for the probability M_0, like for Brownian motion mentioned in sec. 1, no normalising factor r^α is necessary since g_β acts in a transitive way. He defined β-self-similarity of M_0 by the equation $g_\beta(M_0) = M_0$ and proved that this is equivalent to definition (6) with $\alpha = n - \beta$. Then he proved a number of theorems using Palm measure as a powerful technical tool.
For random sets there is no general concept of Palm measure, and even for examples where we can define an analogue φ_0, the equation $g(\varphi_0) = \varphi_0$ is not sufficient to define self-similarity. A scaling exponent cannot be derived in this case, it must be explicitly put into the definition [2]. One can of course argue that the theory of random measures embraces random sets as supports of the measures. For topological purposes, however, as for the count of "islands and lakes" [8] which results in a hyperbolic distribution [2], or the study of connectivity properties in connection with continuous percolation models, it seems convenient to have a direct approach to self-similar random sets.

6.3 Axioms for multifractals

We have seen that axioms for self-similarity can be formulated in three ways: for the infinite measure M on $\mathcal{M}$ which describes measures μ on all $\mathbb{R}^n$, for the probability

measure $\mathbf{P}_B$ which comes from conditioning M to a "circular window" B, and for the Palm measure M_0 of M. Let us formulate now the "window" approach for measures instead of sets, so that it becomes clear why the Palm measure approach is more special. If $B \subset \mathbb{R}^n$ is a bounded open set and M a random measure, we can define the probability space $(\mathcal{M}_B, \mathbf{P}_B)$ by conditioning M to "all realizations μ which can be seen in the window B", that is, $\mathbf{P}_B(\mathcal{B}) = M(\mathcal{B})/M(\mathcal{M}_B)$ for Borel subsets $\mathcal{B}$ of $\mathcal{M}_B$. A stationary M is self-similar if for all $r > 0$ the probability spaces $(\mathcal{M}_B, \mathbf{P}_B)$ and $(\mathcal{M}_{rB}, \mathbf{P}_{rB})$ are isomorphic. The correspondence between these spaces will be given by $g(\mu(\cdot)) = c\mu(r^{-1}\,\cdot)$. The constant c depends on r and is necessary since μ generally assigns larger values to larger sets.

This definition contains no scaling exponents. It is easy to prove, however, that there exists exponents $\alpha, \beta > 0$ such that $M(\mathcal{M}_{rB}) = r^\alpha M(\mathcal{M}_B)$ and $c(r) = r^{-\beta}$. This leads back to the definition (6). It turns out that $\beta \leq n - \alpha$, the equality assumed by U. Zähle is only a particular case. In typical examples, β is the carrying dimension of the realizations μ of M while $n - \alpha$ is the dimension of their support.

The Palm measure of M plays an important role also in the general case but it does not characterise self-similarity for $\beta < n - \alpha$ (cf. the last chapter in [1]). There are other concepts which may better describe essential features of M, as for instance the distribution of $\mu(B)$ under M for fixed B (conditioned to $\mu(B) > 0$). For the self-similar measures of sec. 1 this distribution could be lognormal, but it is not at all clear to me which distributions are admissible for different M and B. More generally, we could ask for the distribution of $\mu(B)$ conditioned to $\mu(C)$ for two fixed sets B and C. Such questions are very near in spirit to geometric statistics.

6.4 Tangential measure distributions

Self-similar constructions given by (1) fit into the axiomatic framework. The Palm measure can be obtained directly by "zooming in on a point x of μ." For an arbitrary measure ν on $\mathbb{R}^n$ and $r > 0$ let $\nu_r(C) = c(r)\nu(r(x + C))$ be the corresponding normalised measure. ν_r measures the unit ball as ν measures the Euclidean neighbourhood $U = U_r(x)$, and $c(r)$ is a normalising factor. When r runs down from 1 to t, say, we get a "zooming-in film" of μ at x [4]. As a probability distribution of "pictures" this can be written as

$$\mathbf{P}_t^x = \frac{1}{-\ln t} \int_t^1 \nu_r \, d \ln r$$

where integration is performed in $\mathcal{M}$ with respect to the Haar measure $d \ln r$ on the group of homotheties $g(y) = ry$. A limit point in $\mathcal{M}$ of the function $\mathbf{P}_t^x$ for $t \to 0$ and for $c(r) = r^{-\beta}$ is called a β-dimensional tangent measure distribution of μ at x.

The relevance of the axiomatic approach of U. Zähle is underlined by a very interesting recent theorem of P. Mörters and D. Preiss [11] which says that each β-dimensional tangent measure distribution of an arbitrary measure in $\mathbb{R}^n$ is a self-similar Palm measure $M_0 = g_\beta(M_0)$.

On the other hand, it was shown by Graf [6] and Arbeiter and Patzschke [1] that for deterministic and random self-similar measures μ fulfilling (1) and the (strong) open set condition, there is one and the same tangent measure distribution M_0 at μ-almost all

points x. For deterministic μ, M_0 is a more complicated but very symmetric object which comprises all essential features of μ, not only Hausdorff dimension, but also topological and geometric properties needed to build up harmonic analysis on μ. When μ is random with distribution M^* (see sec. 1) then M_0 is no more complicated than M^*. The stationary measure M derived from the Palm measure M_0 is the unbounded and stationary version of M^* – just the right way to look at μ.

6.5 Outward constructions without separation

Since [6], [1] prove the existence of M_0 by ergodic theorems, we indicate very briefly a definitive construction of M_0 for a deterministic or random self-similar μ fulfilling (1). While the usual construction of μ proceeds inward, defining smaller and smaller pieces, we shall go *outward,* building larger and larger "super-pieces" [3].
Look at a Sierpiński triangle S^0 ("gasket" [8]) which consists of a left part S^0_l, a right part S^0_r, and an upper part S^0_u. An outward construction has to specify whether S^0 is the left, right or upper part of the super-piece S^1 which contains S^0, which part S^1 occupies in the still larger S^2 etc. One possibility is $S^0 = S^1_r, S^1 = S^2_l$ and hence $S^0 = S^2_{lr}$. This way each left-infinite sequence $\mathbf{i} = ...i_2 i_1$ of symbols from $\{l, r, u\}$ defines an outward construction, and the choice of $\mathbf{i}$ makes the construction random even in the deterministic case. (In the random case, we have also to determine the distribution of inverse weights and mappings p_i^{-1} and f_i^{-1} corresponding to the distribution of the p_i and f_i.)
Combining each $\mathbf{i}$ with each right-infinite sequence $\mathbf{j} = j_1 j_2 ...$ which defines a point in S^0 we have a complete construction [3] which defines the tangential measure distribution M_0 of S^0 and the corresponding M. The important point in this procedure is that the *open set condition is not required* for defining outward constructions. What we need is that for a ball B and any $\epsilon > 0$ there is a level k such that

$$\mathbf{P}\{S^m \cap B = S^k \cap B \text{ for all } m \geq k\} \geq 1 - \epsilon \ . \tag{6.7}$$

Under this "transience" condition outward constructions are well-defined.
Finally, let us express the opinion that outward constructions are required for many other types of fractals, including plane self-avoiding Brownian motion, geometric versions of Aldous' self-similar Brownian tree, "Swiss cheese", measures coming from branching diffusions etc. This should bridge a gap between fractal geometry and stochastic processes. There is a lot of work to be done.

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Chapter 7

Holomorphic Scaling for Para-Hyperbolic Polynomials

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7.1 Holomorphic scaling

Self similarity and fractals are closely related terms. Some people claim that self similarity is 'characteristic' for fractals, cf. [13], other say that fractals have 'some form' of self similarity, cf. [5]. Julia sets — as a sort of fractals — are 'self similar' in the following sense: For each $z \in \mathcal{J}(f)$ where $\mathcal{J}(f)$ is the Julia set of some polynomial f and each open neighbourhood U of z there is some positive integer n such that $f^{\circ n}(U \cap \mathcal{J}(f)) = \mathcal{J}(f)$. That means that the Julia set is a holomorphic copy of each relatively open subset. Unfortunately, the degree of the iterates $f^{\circ n}$ — even when restricted to U — converges to infinity in general. In particular, the sequence of iterates does not belong to some compact family of holomorphic mappings and hence distortion theorems can not be applied. Thus it is quite natural to look for classes of polynomials where the iterates somehow lead to compact families.

A first step has been made by Carleson, Jones and Yoccoz in [3]. They introduced the class of semi-hyperbolic polynomials, that are polynomials which do not have parabolic cycles and whose Julia sets do not contain any recurrent critical point. For a moment let f be semi-hyperbolic. Using a lemma due to Mañe they proved the existence of some integer M such that for each $z \in \mathcal{J}(f)$ and for each $\varepsilon > 0$ there is some neighbourhood U of z and some $n \in \mathbb{N}$ satisfying

i.) $diam(U) < \varepsilon$;
ii.) $f^{\circ n}(U \cap \mathcal{J}(f)) = \mathcal{J}(f)$;
iii.) $\deg\left(f^{\circ n}|_U\right) \leq M$.

They called this phenomenon 'holomorphic scaling'. The purpose of the present note is to extend this result to a wider class namely 'para-hyperbolic' polynomials, that are polynomials which have no recurrent critical points in their Julia set. Note that the notion

of para-hyperbolicity is weaker than each other variation of the term 'hyperbolicity', cf. Section 7.3. At this place we should mention that the closing lemma as well as the shadowing lemma holds for para-hyperbolic polynomials, too, cf. [6] [9]. Para-hyperbolic polynomials admit holomorphic scaling.

Main Theorem
Let f be a para-hyperbolic polynomial. Then there exists an integer N such that for each $\varepsilon > 0$ there exists a finite covering $\{U_m\}_{m\in I}$ of $\mathcal{J}(f)$ and some integer M satisfying

$$\begin{aligned} diam(U_m) &< \varepsilon, \\ f^{\circ M}(U_m \cap \mathcal{J}(f)) &= \mathcal{J}(f) \text{ and} \\ \#\left(f^{\circ M}|_{U_m}\right)^{-1}(z) &\leq N \text{ for each } z \in \mathcal{J}(f). \end{aligned}$$

Of importantance in this theorem is the existence of an upper bound of the degree of the iterates $f^{\circ M}|_{U_m}$, while the sets in the covering have arbitararily small diameter.
This note is organised as follows. In the following section we recall some notations and basic facts. We then introduce and discuss the class of para-hyperbolic polynomials. The proof of the Main Theorem is based on a local version of Mañe's lemma, cf. [10], and the flower theorem for parabolic fixed points. Unfortunately, due to lack of space, we have to omit the complete proof, cf. [8] for the details.

7.2 Notations and basic facts

First, we set up some notations. For basic facts on iteration of rational functions and the definitions of Julia sets and Fatou sets we refer the reader to the monographs [1], [2], [12] and to the lecture notes [11].
For $S \subset\subset \mathbb{C}$, let $diam(S) := sup\{|z-w| \mid z,w \in S\}$ and $D_\varepsilon(S) := \cup_{z\in S} D_\varepsilon(z)$.
We now fix a polynomial $f : \mathbb{C} \to \mathbb{C}$. The set $O^+(\zeta) := \{f^{\circ n}(\zeta) \mid n > 0\}$ is called *orbit of ζ with respect to f*. The ω–limit set $\omega(\zeta)$ is defined to be the set of all accumulation points of the sequence $\{f^{\circ n}(\zeta)\}_{n\in\mathbb{N}}$. In general, neither $\overline{O(\zeta)} \setminus O(\zeta) = \omega(\zeta)$ nor $\overline{O(\zeta)} = \omega(\zeta)$ hold. Recall that ζ is *recurrent* if $\zeta \in \omega(\zeta)$ is satisfied. For a polynomial $f : \mathbb{C} \to \mathbb{C}$ let $Rec(f)$ and $Per(f)$, denote the set of recurrent resp. periodic points. It is a well–known fact that the Julia set $\mathcal{J}(f)$ contains a dense set of points ζ such that the orbit $O(\zeta)$ forms a dense subset of $\mathcal{J}(f)$: $\omega(\zeta) = \overline{O(\zeta)} = \mathcal{J}(f)$. Clearly, these points are recurrent, and we obtain $\overline{Rec(f) \cap \mathcal{J}(f)} = \mathcal{J}(f)$.
Crucial for the dynamics of meromorphic functions is the set $CV(f)$ of the so–called *critical values* of f. We write $\Omega := \overline{O^+(CV(f))}$. In particular, it is significant whether the critical points in the Julia set are recurrent or not. For example, an irrationally indifferent cycle requires at least one critical point in the Julia set to be recurrent, cf. [3] and compare Proposition 5.

7.3 Para-hyperbolic polynomials

The notion "para-hyperbolicity" is motivated by the term "semi-hyperbolicity", cf. [3] for further details, but combines (semi-)hyperbolic and parabolic features of holomorphic

functions. We proceed as in [3]. Let p be a polynomial and $z \in \mathbb{C}$. Now, let S^n be any branch of the inverse of $p^{\circ n}$ and let $B_n(z,\varepsilon)$ be a connected component of $S^n(D_\varepsilon(z))$ for some $\varepsilon > 0$. Then $p^{\circ n}|_{B_n(z,\varepsilon)} : B_n(z,\varepsilon) \to D_\varepsilon(z)$ is a finite ramified covering. We denote by $d_n(B_n(z,\varepsilon))$ its degree.

Definition 1 *A polynomial p is called* semi-hyperbolic *if there exists some $\varepsilon > 0$ and some $D \in \mathbb{N}$ such that $d_n(B_n(z,\varepsilon)) \leq D$ holds for all $z \in \mathcal{J}(p)$ and all choices of inverse branches.*

This yields the following lemma, compare [3].

Lemma 2 *Let p be semi-hyperbolic. Then there exist $\varepsilon > 0$, $C > 0$ and $c \in]0,1[$ such that $diam\,(B_n(z,\varepsilon)) \leq C \cdot c^n$ holds for every $z \in \mathcal{J}(p)$ and $n \in \mathbb{N}$.*

Definition 2 *a.) A polynomial $f : \mathbb{C} \to \mathbb{C}$ is called* para-hyperbolic at some point $z \in \mathbb{C}$ *if there exists some $\varepsilon_0 > 0$ such that for every $\delta > 0$ there is some $N \in \mathbb{N}$ with*

$$n \geq N \Longrightarrow diam\,(B_n(z,\varepsilon_0)) \leq \delta\,.$$

b.) A polynomial $f : \mathbb{C} \to \mathbb{C}$ is called para-hyperbolic *if $c \notin \omega(c)$ holds for each $c \in CV(f) \cap \mathcal{J}(f)$.*

Before we proceed we want to discuss this quite technical definition in more detail and thereby motivate it.

Proposition 3 *f is para-hyperbolic at each $\zeta \in \mathcal{J}(f) \setminus \overline{\Omega}$.*

Proof: Compare [1]. □

Proposition 4 *Let $\zeta \in \mathcal{J}(f) \setminus \overline{\Omega}$ and $n_0 := \sup\{n \in \mathbb{N} \mid f^{\circ n}(\zeta) \notin \omega(CV(f))\}$. Then f is para-hyperbolic at $f^{\circ n}(\zeta)$ for each $n = 0, \ldots, n_0 - 1$.*

In order to provide a better feeling for the dynamical consequences, we state the following proposition.

Proposition 5 *If f is para-hyperbolic then f has no Siegel disc.*

Proof: We assume f to have a Siegel disc S and f to be para-hyperbolic. Due to Theorem 7 there exists a point $z \in \partial S$ such that f is para-hyperbolic at z. Now, by definition, for $\varepsilon > 0$ sufficiently small,

$$\lim_{n\to\infty} diam\,(S^n(D_\varepsilon(z)) = 0\,.$$

But $f|_S$ is conjugate to an irrational rotation, hence for a suitable subsequence $\{n_k\}_{k\in\mathbb{N}}$ we obtain $S^{n_k}|_{\varepsilon(z)\cap S} \to id$ uniformly on compact subsets, which yields a contradiction. □
As an immediate Sullivan's classification theorem we derive the following corollary.

Corollary 6 *If a polynomial f is para-hyperbolic, then $\mathcal{F}(f)$ is the union of attracting and parabolic basins.*

Remarks

1. If f has some irrationally indifferent periodic point ζ, then f is clearly *not* para-hyperbolic at ζ and therefore f is *not* semi-hyperbolic in the sense of [3].

2. A para-hyperbolic polynomial might have an indifferent cycle. In fact, each quadratic polynomial which has a parabolic cycle is para-hyperbolic (but not semi-hyperbolic).

3. Corollary 6 implies that the mapping $f \mapsto \mathcal{J}(f)$ is continuous at f_0, provided f_0 is para-hyperbolic, cf. [7].

We note

Theorem 7 *Let f be a polynomial. Suppose that $c \notin \omega(c)$ for each critical point c of f, and let $\zeta \in \mathcal{J}(f)$ be no parabolic periodic point. Then f is para-hyperbolic at ζ.*

In particular, this theorem states that a para-hyperbolic polynomial f cannot have irrationally indifferent cycles at all. We refer the reader to [8] for the proof.
In iteration theory, several variations of the notion "hyperbolicity" have been introduced. In order to avoid confusion, we explain their relationship to para-hyperbolicity.

hyperbolic: that is, any critical point is absorbed by some attracting cycle.

This implies $|(f^{\circ N})'(z)| \geq C > 1$ for some $N \in \mathbb{N}$ and some constant C. This property is also called "expanding".

sub-hyperbolicity: that is, any critical value is either absorbed by some attracting cycle or it is pre-periodic.

Each $z \in \Omega \cap \mathcal{J}(f)$ has to be a pre-periodic point.

sub-expanding: that is, the restriction of f to the intersection of the Julia set with Ω is expanding with respect the spherical metric.

Due to Proposition 3, f is para-hyperbolic at $z \in \mathcal{J}(f) \setminus \Omega$. Now, the hyperbolicity of $f|_\Omega$ yields the para-hyperbolicity for each $z \in \Omega$, compare [1].

semi-hyperbolic: cf. Definition 1.

expansive: that is, there exists some $\eta > 0$ such that for all $z_1, z_2 \in \mathcal{J}(f)$ the following holds: If $|f^{\circ n}(z_1) - f^{\circ n}(z_2)| \leq \eta$ for all $n \in \mathbb{N}$, then $z_1 = z_2$.

This implies that each critical point of f is absorbed either by some attracting or by some parabolic cycle, cf. [4].

parabolic: that is, $\mathcal{J}(f) \cap CV(f) = \emptyset$ but $\Omega \cap \mathcal{J}(f) \neq \emptyset$.

In particular, f has no Siegel disc or Herman ring. Hence, each point in $\Omega \cap \mathcal{J}(f)$ is a parabolic periodic point.

critically finite — that is, $\mathcal{J}(f) \cap \Omega$ is finite.

This implies that each critical point either is absorbed either by some attracting or by some parabolic cycle or is is pre-periodic.

Note that each of the terms in the above list implies para-hyperbolicity. We summarise these results.

$$\begin{array}{ccccccccc} \text{hyp.} & \Leftrightarrow & \text{expanding} & & \Longrightarrow & & \text{expansive} & \Leftarrow & \text{parabolic} \\ \Downarrow & & \Downarrow & & & & \Downarrow & & \Downarrow \\ \text{sub-hyp.} & \Rightarrow & \text{sub-expanding} & \Rightarrow & \text{semi-hyp.} & \Rightarrow & \text{para-hyp.} & \Leftarrow & \text{crit. finite} \end{array}$$

Remarks

1. The reverse of each of the above implications does *not* hold. Counterexamples can be found in [1], [2], [3].

2. "Expansive" implies either "hyperbolic" or "parabolic", cf. [4].

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Chapter 8

Fractal Random Measures Generated by Time Continuous Branching and Diffusion

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8.1 Introduction

Fractal random measures are often defined in a constructive way. In the introduction I want to motivate the special kind of the branching diffusion model with the well known construction of self–similar random measures.
Throughout this lecture I use the following notation of the code space. Let N be a positive integer and $\Sigma_n = \{1, \dots, N\}^n$. Furthermore denote by $\Sigma^* = \bigcup_{n=0}^{\infty} \Sigma_n$ the family of all finite sequences with entries from $\{1, \dots, N\}$ (including the empty sequence $\emptyset$) and by $\Sigma = \{1, \dots, N\}^{\mathbb{N}}$ the code space over the indices $\{1, \dots, N\}$. To handle with these sequences, which will describe our particles, we denote the length (generation) of a sequence (particle) $\sigma = (\sigma_1, \dots, \sigma_n) \in \Sigma^*$ by $|\sigma| = n$. For $\sigma \in \Sigma^*$ and $\tau \in \Sigma^* \cup \Sigma$ we write $\sigma\tau$ for the juxtaposition of σ and τ. The truncation of a sequence $\sigma \in \Sigma^* \cup \Sigma$ to the length k will be denoted by $\sigma|k$.
Now I start with the construction of self–similar random measures.

Self–similar random measures

The primary object is a probability measure μ on the space

$$M = (0,1)^N \times (\mathrm{Sim}\mathbb{R}^d)^{\mathbb{N}},$$

where $\mathrm{Sim}\mathbb{R}^d$ denotes the space of all contractive similarities $S : \mathbb{R}^d \to \mathbb{R}^d$.

first step:

We choose a random variable $(p_1, \dots, p_N, S_1, \dots, S_N)$ w.r.t. μ, with $\mathbb{E} \sum_{i=1}^{N} p_i = 1$ and define a finite random measure Φ_1 on $\mathbb{R}^d$ by

$$\Phi_1 = \sum_{i=1}^{N} p_i \ \delta_{S_i(0)}.$$

The general construction model is given by the

n–th step:

For $\sigma \in \Sigma_{n-1}$ we choose a random variable $(p_{\sigma 1}, \dots, p_{\sigma N}, S_{\sigma 1}, \dots, S_{\sigma N})$ w.r.t. μ, independent of $\{p_\sigma, S_\sigma \ : \ \sigma \in \bigcup_{k=1}^{n-1} \Sigma_k\}$.
Now we define a random measure Φ_n on $\mathbb{R}^d$ by

$$\Phi_n = \sum_{\sigma \in \Sigma_n} p_{\sigma|1} \cdot \ldots \cdot p_\sigma \ \delta_{S_{\sigma|1} \circ \ldots \circ S_\sigma(0)}.$$

The sequence $(\Phi_n)_{n=1,2,\dots}$ leads to the so called self–similar random measure.

Theorem 8.1.1 (e.g. [5], [6], [1], [4])

1. *There exists a finite random measure Φ, such that the sequence $(\Phi_n)_{n=1,2,\dots}$ converges almost surely to Φ with respect to the weak topology.*

2. *Φ is the unique μ–self–similar random measure, that means*

$$\Phi \stackrel{d}{=} \sum_{i=1}^{N} p_i \ \Phi^i \circ S_i^{-1},$$

   where Φ^i, $i = 1, \dots, N$, are independent copies of Φ and independent of $(p_1, \dots, p_N, S_1, \dots, S_N)$.

3. *If the so called Open Set Condition is satisfied, the the Hausdorff dimension of Φ is given by*

$$\dim_H \Phi = \frac{\mathbb{E} \sum_{i=1}^{N} p_i \ln p_i}{\mathbb{E} \sum_{i=1}^{N} p_i \ln(\operatorname{Lip} S_i)}$$

   with probability one, where $\operatorname{Lip} S_i$ denotes the Lipschitz constant of S_i.

Now we asked whether it is possible to construct such self–similar measures by time discrete branching and jumping of particles. This is the next step I want to do.

Construction by time discrete branching and jumping

The construction model is analogously to the construction of the self–similar random measures but now the primary object is a sequence $(\kappa_n)_{1,2,\dots}$ of probability measures on the space

$$M = (0,1)^N \times (\mathbb{R}^d)^N,$$

equipped with the usual σ-algebra.

first step:

We choose a random variable

$$(p_1, \dots, p_N, \eta_1, \dots, \eta_N) \quad \text{w.r.t. } \kappa_1$$

and define a finite random measure Φ_1 on $\mathbb{R}^d$ by

$$\Phi_1 = \sum_{i=1}^{N} p_i \ \delta_{\eta_i}.$$

n-th step:

For $\sigma \in \Sigma_{n-1}$ we choose a random variable

$$(p_{\sigma 1}, \dots, p_{\sigma N}, \eta_{\sigma 1}, \dots, \eta_{\sigma N}) \quad \text{w.r.t. } \kappa_n$$

independent of $\{p_\sigma, \eta_\sigma \ : \ \sigma \in \bigcup_{k=1}^{n-1} \Sigma_n\}$.
Now we define the random measure Φ_n on $\mathbb{R}^d$ by

$$\Phi_n = \sum_{\sigma \in \Sigma_n} p_{\sigma|1} \dots p_\sigma \ \delta_{\eta_{\sigma|1} + \dots + \eta_\sigma}.$$

Interpretation

The particle σ splits into N new particles $\sigma 1, \dots, \sigma N$. The mass of the particle σi is the mass of the mother particle multiplied by the random factor $p_{\sigma i}$ and the position of the particle σi is the position of the mother particle translated by $\eta_{\sigma i}$.
Under some assumptions to the underlying probability measures $(\kappa_n)_{n=1,2,\dots}$ we get the same results as in Theorem 1. Especially a large class of self–similar random measures fits in the framework on this construction (for details see [2]). I want to demonstrate these results on the following simple examples.

Examples

Let ν be a probability measure on $\mathbb{R}^d$ with

1. $\int |x| \nu(dx) < \infty$
2. ν has a bounded density w.r.t. Lebesgue measure.

Furthermore for $c > 1$ we define probability measures ν_n on $\mathbb{R}^d$ by

$$\nu_n = \nu \times h_n^{-1},$$

where $h_n(x) = c^{-(n-1)} x \ , \ x \in \mathbb{R}^d, \ n = 1, 2, \dots$.
Now we take

$$\kappa_n = \bigtimes_{i=1}^{N} \delta_{1/N} \quad \times \quad \bigtimes_{i=1}^{N} \nu_n,$$

that means, that the random mass factors are $1/N$ and the random translations of the particle $\sigma \in \Sigma^*$ are independent and identically distributed according to $\nu_{|\sigma|+1}$. Then we obtain the following theorem.

Theorem 8.1.2 ([2])

1. *For* $\Phi_n = \sum_{\sigma\in\Sigma_n} \frac{1}{N^n}\delta_{\eta_{\sigma|1}+\ldots+\eta_\sigma}$ *there exits a finite random measure* Φ*, such that the sequence* $(\Phi_n)_{n=1,2,\ldots}$ *converges almost surely to* Φ *with respect to the weak topology.*

2. Φ *is (statistically) self–similar w.r.t.* $p_i = \frac{1}{N}$ *and* $S_i = \frac{1}{c}x + \eta_i,\ i = 1,\ldots,N$*, where the* η_i *are i.i.d. w.r.t.* ν*.*

3. *The Hausdorff dimension of the limit measure* Φ *is given by*

$$\dim_H \Phi = \min\{d, \frac{\ln N}{\ln c}\}$$

with probability one.

Remark The Open Set Condition may not to be satisfied in these examples.

The main topic of this lecture is the answer to the question, whether the time discrete branching and jumps can be replaced by time continuous branching and motion processes.

8.2 Construction by time continuous branching and diffusion

8.2.1 Model

The construction method is formally the same as in the previous cases. The primary object is now a probability measure μ on the measurable space $[M, \mathcal{M}]$, where

$$M = [0,\infty)^N \times (0,1)^N \times [(\mathbb{R}^d)^{[0,\infty)}]^N,$$

equipped with the usual σ–algebra $\mathcal{M}$. The particles of our model will be described again by finite sequences from Σ^*. For each of this particles $\sigma \in \Sigma^*$ we choose a random variable

$$(\lambda_{\sigma 1},\ldots,\lambda_{\sigma N}, p_{\sigma 1},\ldots,p_{\sigma N}, Y^{\sigma 1},\ldots,Y^{\sigma N}) \in M$$

according to the underlying distribution μ independent from $\{\lambda_\sigma, p_\sigma, Y^\sigma \ :\ \sigma \in \bigcup_{k=1}^{|\sigma|} \Sigma_k\}$. These random variables have the following interpretation.

Interpretation

$\lambda_{\sigma i}$	– is the duration of life span of the particle σi
$p_{\sigma i}$	– is the ratio of the mass of the particle σi and the mass of the mother particle σ
$(Y_t^{\sigma i})_{t\in[0,\infty)}$	describes the motion of the particle σi during its live time.

Throughout this lecture we assume the following.

Assumptions

1. $\lambda_i > 0$ for $i = 1,\ldots,N$, with probability one.

2. $\mathbb{E}\sum_{i=1}^{N} p_i = 1$

For defining a measure Φ_t at time $t \in [0,\infty)$ we need some more notations.

Notation

For $\sigma \in \Sigma^*$ denote

$\overline{\lambda}_\sigma = \lambda_{\sigma|1} + \ldots + \lambda_\sigma$ (the death time of the particle σ)
($\overline{\lambda}_\emptyset = \lambda_\emptyset = 0$)

$\overline{p}_\sigma = p_{\sigma|1} \cdot \ldots \cdot p_\sigma$ (the mass of the particle σ)
($\overline{p}_\emptyset = p_\emptyset = 1$)

$\overline{Y}_t^\sigma = Y_{\overline{\lambda}_{\sigma|1}}^{\sigma|1} - Y_{\overline{\lambda}_\emptyset}^{\sigma|1} + Y_{\overline{\lambda}_{\sigma|2}}^{\sigma|2} - Y_{\overline{\lambda}_{\sigma|1}}^{\sigma|2} + \ldots + Y_t^\sigma - Y_{\overline{\lambda}_{\sigma||\sigma|-1}}^{\sigma}$

(For $t \in [\overline{\lambda}_{\sigma||\sigma|-1}, \overline{\lambda}_\sigma]$ this is the position of the particle σ at time t.)

Furthermore we denote by

$$\Gamma_t = \{\sigma \in \Sigma^* \ : \ \overline{\lambda}_{\sigma||\sigma|-1} \le t < \overline{\lambda}_\sigma\}$$

the set of particles which are living at time t.

Now we are ready to define for each $t \in [0,\infty)$ a finite random measure Φ_t on $\mathbb{R}^d$ by

$$\Phi_t = \sum_{\sigma\in\Gamma_t} \overline{p}_\sigma \ \delta_{\overline{Y}_t^\sigma}.$$

Now we are interested in studying the long time behaviour of the measure valued process $(\Phi_t)_{t\in(0,\infty)}$.

8.2.2 Convergence

Some general convergence conditions are given in [3], in this lecture we restrict to the following case.

Special case

Let $c > 1$ and take

$$Y_t^\sigma = Z_{(1-c^{-t})}^\sigma \ , \quad t \in [0,\infty),$$

where $\{Z^\sigma \ : \ \sigma \in \Sigma^*\}$ is a family of independent symmetric γ–stable processes with stationary and independent increments and $1 \le \gamma \le 2$. (For $\gamma = 2$, Y_t^σ is a usual Brownian motion.)

For this special case we have the following result.

Theorem 8.2.1 ([3])
There exists a finite random measure Φ on $\mathbb{R}^d$, such that $(\Phi_t)_{t\in[0,\infty)}$ converges almost surely to Φ with respect to the weak topology.

Finally we want to give some results for some fractal parameters of the limit measure Φ.

8.2.3 Fractals properties of the limit measure Φ

Hausdorff Dimension

For a finite Borel measure φ on $\mathbb{R}^d$ we denote by $\underline{\dim}_H \varphi$ ($\overline{\dim}_H \varphi$) the lower (upper) Hausdorff dimension of φ. (For details see e.g. [3].)

Theorem 8.2.2 ([3])
If the assumptions of the "Special case" be satisfied, then we have

$$\underline{\dim}_H \Phi = \overline{\dim}_H \Phi = \min\{\alpha, d\}$$

with probability one, where

$$\alpha = \gamma \frac{\mathbb{E} \sum_{i=1}^N p_i \ln p_i}{\mathbb{E} \sum_{i=1}^N p_i \ln c^{-\lambda_i}}.$$

We can prove a further result for the correlation dimension.

Correlation dimension

For a finite Borel measure φ the correlation dimension $D_2(\varphi)$ is defined by

$$D_2(\varphi) = \lim_{r \downarrow 0} \frac{\ln \int \varphi(B(x,r))\varphi(dx)}{\ln r},$$

if the limit exists.

Theorem 8.2.3 *Let the assumptions of the "Special case" be satisfied. Then we have*

$$D_2(\Phi) = \min\{\alpha', d\},$$

with probability one, where α' is the unique solution of

$$\mathbb{E} \sum_{i=1}^N p_i^2 c^{-\frac{1}{\gamma}\lambda_i \alpha'} = 1.$$

We finish this lecture by considering the following easy example.

Example

Let λ_i, $i = 1, \ldots, N$ be independent and identically distributed with expectation $\frac{1}{\lambda}$ and take $p_i = c^{-\frac{1}{\gamma}\lambda_i \alpha}$, where α is the unique solution of $\mathbb{E} \sum\limits_{i=1}^N p_i^\alpha = 1$. Then we obtain

$$D_2(\Phi) = \underline{\dim}_H \Phi = \overline{\dim}_H \Phi = \min\{\frac{\gamma\lambda(N-1)}{\ln c}, d\},$$

with probability one.

Remark The last example is the typical case of a one point spectrum. It is an open question, whether it is possible to compute the whole multifractal spectrum of the limit measure Φ.

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Chapter 9

Multiple Fractal Aspects of Conformal Measures; a Survey.

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9.1 Introduction

Throughout this survey, let $G < Con(N)$ denote a non-elementary, geometrically finite Kleinian group. It is well known that the discontinuous action of such a group on hyperbolic space D^{N+1} gives rise to the perfect limit set $L(G) \subset S^N$ of G ((D^{N+1}, d) denotes the Poincaré model of hyperbolic space, and S^N its visible boundary).
The set $L(G)$ has rather interesting fractal properties, and the purpose of these notes will be to recall and clarify a few of these. In particular, we shall specify various fractal dimensions of the limit set, such as Hausdorff dimension ($\dim_H$), packing dimension ($\dim_P$) and box-counting dimension ($\dim_B$), as well as the generalised q-th Renyi dimension ($\mathcal{R}_\mu(q)$), the q-th logarithmic index ($\mathcal{L}_\mu(q)$) and the information dimension ($\mathcal{I}_\mu$), with respect to the associated Patterson measure μ and for $q \neq 0$. The derivation of these fractal quantities relies on a comprehensive understanding of the Patterson measure (specifically, the 'global measure formula' and the 'Khintchine-like law of iterated logarithm'), and as a corollary we obtain the result that this particular conformal measure is a regular measure, but in fact not a fractal measure. Furthermore, for $N = 1$ and 2 we relate the Patterson measure to the geometric measures Hausdorff measure and packing measure. Although the Patterson measure is not fractal, it still makes sense to consider certain weak singularity spectra of this measure. By using results from the metrical Diophantine analysis of Kleinian limit sets, we derive these spectra for the Patterson measure. For simplicity, we restrict the latter investigations to the special case in which G is a finitely generated 2nd kind Fuchsian group with parabolic elements.
We remark that analogous multiple fractal statements may be derived for geometrically finite Julia sets of rational functions. This follows since for such Julia sets there exist

conformal measures with essentially the same properties as in the Kleinian case (i.e. 'global measure formula' ([6], [7]) and 'Khintchine-like law of iterated logarithm ([21])).

9.2 The orbital measure

Throughout, let μ denote the Patterson measure associated to the geometrically finite Kleinian group G. We assume that the reader is familiar with the construction of this non-atomic probability measure ([14],[16],[17],[22],[24]). We only recall that this construction is based on the existence of the important quantity $\delta = \delta(G)$, the exponent of convergence of G, which is defined by

$$\delta := \inf\{ s \geq 0 : \sum_{g \in G} \exp(-sd(0, g0)) \text{ converges } \} \quad .$$

We now recall a crucial uniform estimate for the measure μ. In order to do so, we require some further notation. For $\xi \in L(G)$ and $t > 0$, let ξ_t denote the unique point on the ray between 0 and ξ of hyperbolic distance t from 0. Let $b(\xi_t)$ denote the intersection of S^N with the $(N+1)$-ball whose boundary is orthogonal to S^N and which intersects the ray between 0 and ξ orthogonally at ξ_t. Hence, $b(\xi_t)$ is a N-ball in S^N whose radius is comparable to e^{-t}. Also, $k(\xi_t)$ is defined to be equal to the rank $k(p)$ of the parabolic point p if ξ_t is contained in one of the horoballs in the G-orbit of the standard horoball at p; otherwise we let $k(\xi_t)$ be equal to δ. Furthermore, define $\Delta(\xi_t) := d(\xi_t, G(0))$, and let k_{min} and k_{max} denote the minimal and maximal occurring rank for the parabolic elements in G.

Theorem 1 ((GMF) Global measure formula ([22]))
For all $\xi \in L(G)$ and $t > 0$,

$$\mu(b(\xi_t)) \asymp e^{-t\delta}\, e^{-(\delta - k(\xi_t))\Delta(\xi_t)} \quad .$$

An immediate corollary is that the Patterson measure is a *Federer measure*, and also that the limit set of a geometrically finite Kleinian group is *uniformly perfect*. In order to derive more precise information concerning the fractal nature of the limit set, it is vital to give better approximations of the essential support of the Patterson measure.

Theorem 2 ((KLIL) Khintchine-like law of iterated logarithm ([22]))
For μ-almost all $\xi \in L(G)$,

$$\limsup_{t \to \infty} \frac{\Delta(\xi_t)}{\log t} = (2\delta - k_{max})^{-1}.$$

An immediate corollary is the 'ergodic law' (cf. [23]), which states that for μ-almost all $\xi \in L(G)$ we have that $\lim_{t \to \infty} \frac{\Delta(\xi_t)}{t} = 0$.

9.3 Fractal dimensions of the limit set

Theorem 3 *If $G < Con(N)$ denotes a geometrically finite Kleinian group, δ the exponent of convergence of G and μ the associated Patterson measure, then for $q \neq 0$,*

- $\dim_H(L(G)) = \dim_P(L(G)) = \dim_B(L(G)) = \delta$;
- $\mathcal{I}_\mu = \mathcal{R}_\mu(q) = \delta$ *and* $\mathcal{L}_\mu(q+1) = q\delta$.

Proof. The Hausdorff and packing dimension result follows immediately from a combination of (GMF) and (KLIL), using the Frostman-Billingsley lemma ([2],[9]). For the proof that the box-counting dimension equals the exponent of convergence, we refer to [20] (for an alternative proof, see [3]).
The q-th generalised Renyi dimension and logarithmic index, and the information dimension are obtained from (GMF) and (KLIL) as follows:

$$\begin{aligned}\mathcal{R}_\mu(q) &:= \lim_{t\to\infty} \frac{\log \int_{L(G)} \mu(b(\xi_t))^q d\mu(\xi)}{-qt} = \lim_{t\to\infty} \frac{\log \int_{L(G)} e^{-t\delta q} e^{-q\Delta(\xi_t)(\delta - k(\xi_t))} d\mu(\xi)}{-qt} \\ &= \delta + \lim_{t\to\infty} \frac{\log \int_{L(G)} e^{-q\Delta(\xi_t)(\delta-k(\xi_t))} d\mu(\xi)}{-qt} \quad = \quad \delta \quad ;\end{aligned}$$

$$\begin{aligned}\mathcal{I}_\mu(q) &:= \lim_{t\to\infty} \frac{\int_{L(G)} \log \mu(b(\xi_t)) d\nu(\xi)}{-t} = \delta - \lim_{t\to\infty} \frac{\log \int_{L(G)} e^{-\Delta(\xi_t)(\delta-k(\xi_t))} d\mu(\xi)}{t} \\ &= \quad \delta \quad .\end{aligned}$$

For the definition of the q-th logarithmic index $\mathcal{L}_\mu(q)$ we refer to [15]. Since μ is a Federer measure, we may apply [15] Theorem 2.24, from which we deduce that $\mathcal{L}_\mu(q+1) = q\delta$. □

The theorem shows in particular that the Patterson measure μ has a *flat Renyi dimension spectrum* equal to δ. In addition, as an immediate consequence of (GMF) and (KLIL), one has that μ is a *regular measure*[1].
Recall that by definition (cf.,[4],[13],[25]), a measure ν in $\mathbf{R}^N$ is called *fractal* if and only if $\mathcal{I}_\nu < \dim_H(supp\ \nu)$. Hence, we further obtain the following corollary.

Corollary 1 *For a non-elementary, geometrically finite Kleinian group $G < Con(N)$ the associated Patterson measure is not a fractal measure.*

9.4 Conformal measure versus geometric measures

For a geometric analysis of the Patterson measure it is crucial to study its relation to the more geometric Hausdorff measure $\mathcal{H}_s$ and packing measure $\mathcal{P}_s$. The following table gives these relations for the finitely generated Fuchsian case ($N = 1$) and the geometrically finite 'classical' Kleinian case ($N = 2$). Most of these relations were already stated in [24], the discussion of the case $(*)$ below is new. The table follows from an application of

[1] i.e., $\liminf \frac{\log \mu(B(\xi,r))}{\log r} = \limsup \frac{\log \mu(B(\xi,r))}{\log r}$ for μ-almost all $\xi \in$ *supp* μ (cf.,[4])

(GMF) and (KLIL), using the well known 'generalised mass distribution principle' (for a condensed statement of this principle we refer to [5] and/or [19]).

CUSPS vs. δ	$0 < \delta < 1$	$\delta = 1$	$1 < \delta < 2$	$\delta = 2$
no cusps	$\mu = \mathcal{H}_\delta = \mathcal{P}_\delta$	$\mu = \mathcal{H}_1 = \mathcal{P}_1$	$\mu = \mathcal{H}_\delta = \mathcal{P}_\delta$	$\mu = \mathcal{H}_2 = \mathcal{P}_2$
$k_{max} = 1$	$\mu = \mathcal{P}_\delta, \mathcal{H}_\delta = 0$	$\mu = \mathcal{H}_1 = \mathcal{P}_1$	$\mu = \mathcal{H}_\delta, \mathcal{P}_\delta = \infty$	$\nexists$
$k_{min} = 2$	$\nexists$	$\nexists$	$\mu = \mathcal{P}_\delta, \mathcal{H}_\delta = 0$	$\mu = \mathcal{H}_2 = \mathcal{P}_2$
$k_{min} < k_{max}$	$\nexists$	$\nexists$	! see $(*)$ below !	$\nexists$

$(*)$: For the case $(*)$ we have that $\mathcal{H}_\delta = 0$ and $\mathcal{P}_\delta = \infty$. Hence here the Patterson measure does not have an immediate geometric interpretation, and up to now it is still an open problem to give in this case a complete description of the Patterson measure in terms of geometric measures. It is expected that a solution to this problem may be found using techniques similar to those which led to the description of the conformal measure in terms of generalised Hausdorff measure ([12]). Nevertheless, our analysis leads to some fairly reasonable approximations of the 'right gauge function'. Namely, from (GMF) and (KLIL), and since $1 = k_{min} < k_{max} = 2$, we have that for μ-a.a. $\xi \in L(G)$ and $\forall \epsilon > 0$,

$$t^{(1-\epsilon)\frac{2-\delta}{2(\delta-1)}} \quad \ll_{inf.\,oft.\,(t)} \quad \frac{\mu(b(\xi_t))}{e^{-t\delta}} \quad \ll_{event.\,always\,(t)} \quad t^{(1+\epsilon)\frac{2-\delta}{2(\delta-1)}} \quad .$$

Using this estimate, we derive for the Hausdorff measure $\mathcal{H}_{\phi_\theta}$ with respect to the gauge function ϕ_θ, which is given by $\phi_\theta(r) = r^\delta (\log \frac{1}{r})^{(1+\theta)\frac{2-\delta}{2(\delta-1)}}$,

- $\mathcal{H}_{\phi_\epsilon}(L(G)) \gg 1$ and $\mathcal{H}_{\phi_{-\epsilon}}(L(G)) \ll 1$ for all $\epsilon > 0$.

(We remark that by similar arguments analogous approximations in terms of gauge functions may be given for the generalised packing measure of the limit set.)

9.5 Weak singularity spectra of conformal measures

The multifractal formalism usually requires the fractality of the underlying measure. As we have seen above, the system $(L(G), \mu)$ is strictly speaking not a multifractal system. Nevertheless, the Patterson measure μ may still be used to analyse certain interesting fractal subsets of the support of μ. These sets have been studied originally in terms of the metrical theory of Diophantine approximation for Kleinian limit sets, which has its historical origin in classical number theory ([1],[11]). Embedded into the theory of multifractals, these sets provide interesting examples where a 'naive approach' to multifractality seems to fail. This will be made more precise below. For simplicity we have restricted the discussion to finitely generated 2nd kind Fuchsian groups with parabolic elements. We remark that similar results may be derived for more general geometrically finite groups, and these results depend largely on how the Patterson measure relates to the geometric Hausdorff and packing measures (as discussed in the preceding section). (For a more complete discussion the results in [10] and [19] are helpful.)
We now first introduce the notion of weak singularity spectra of a measure.

Definition 1 *Let ν denote a Borel probability measure on $\mathbf{R}^N$.*
For $\sigma \geq 0$ write

$$INF^{\sigma}(\nu) := \{\xi \in supp(\nu) : \liminf_{r\to 0} \tfrac{\log \nu(B(\xi,r))}{\log r} \leq \sigma\} \quad ;$$

$$INF_{\sigma}(\nu) := \{\xi \in supp(\nu) : \liminf_{r\to 0} \tfrac{\log \nu(B(\xi,r))}{\log r} \geq \sigma\} \quad ;$$

$$SUP^{\sigma}(\nu) := \{\xi \in supp(\nu) : \limsup_{r\to 0} \tfrac{\log \nu(B(\xi,r))}{\log r} \leq \sigma\} \quad ;$$

$$SUP_{\sigma}(\nu) := \{\xi \in supp(\nu) : \limsup_{r\to 0} \tfrac{\log \nu(B(\xi,r))}{\log r} \geq \sigma\} \quad .$$

The collections of Hausdorff dimensions of these sets, for $\sigma \geq 0$, are referred to as the weak singularity spectra of ν.

For finitely generated 2nd kind Fuchsian group with parabolic elements, the weak singularity spectra of the associated Patterson measure are specified in the following theorem.

Theorem 4 *Let G be a finitely generated 2nd kind Fuchsian group with parabolic elements. Let δ denote the exponent of convergence of G. For the weak singularity spectra of the associated Patterson measure μ on the limit set $L(G)$ we have, for $\sigma \geq 0$,*

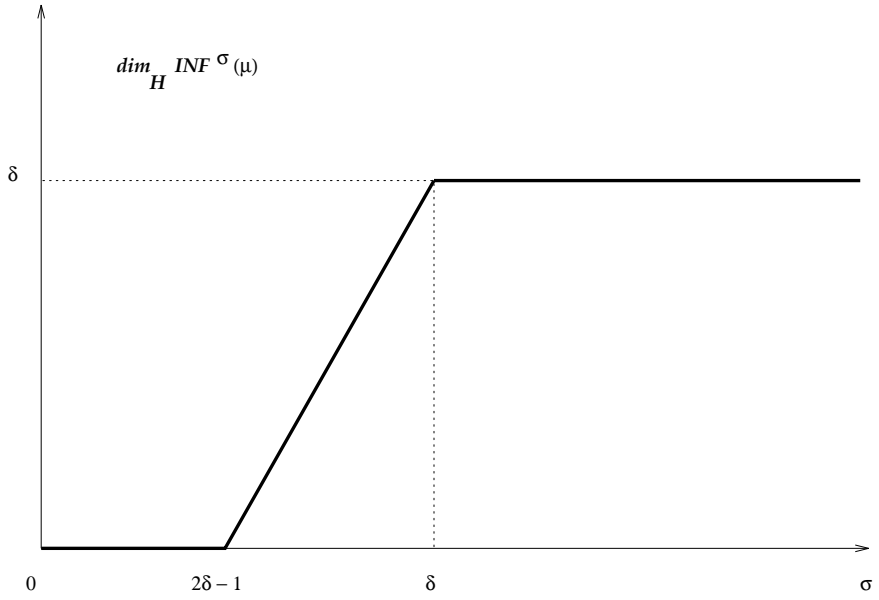
(1) $$\dim_H(INF^{\sigma}(\mu)) := \begin{cases} 0 & for \quad \sigma \leq 2\delta - 1 \\ \delta - \frac{\delta(\delta-\sigma)}{1-\delta} & for \quad 2\delta - 1 < \sigma \leq \delta \\ \delta & for \quad \sigma > \delta \end{cases} \quad ;$$

(2) $$\dim_H(INF_{\sigma}(\mu)) := \begin{cases} \delta & for \quad \sigma \leq \delta \\ 0 & for \quad \sigma > \delta \end{cases} \quad ;$$

(3) $$\dim_H(SUP^{\sigma}(\mu)) := \begin{cases} 0 & for \quad \sigma < \delta \\ \delta & for \quad \sigma \geq \delta \end{cases} \quad ;$$

(4) $$\dim_H(SUP_{\sigma}(\mu)) := \begin{cases} \delta & for \quad \sigma \leq \delta \\ 0 & for \quad \sigma > \delta \end{cases} \quad .$$

Proof. The estimates in (2), (3) and (4) are deduced immediately from (GMF) and (KLIL). The derivation of (1) is slightly more delicate, and we refer to [19] for the actual calculations. □



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