Weak Singularity Spectra of the Patterson Measure for Geometrically Finite Kleinian Groups with Parabolic Elements

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Abstract

In this note we determine the weak singularity spectra of the Patterson measure for geometrically finite Kleinian groups with parabolic elements. This 'weak multifractal analysis' is based on a further investigation of the Hausdorff dimension of the Jarník limit sets, which represent the natural generalisation of the sets of well-approximable irrational numbers to the theory of Kleinian groups.

1 Introduction and statement of results

In this paper we give a multifractal description of the Patterson measure μ supported on the limit set $L(G)$ of a geometrically finite Kleinian group G with parabolic elements. More precisely, we estimate the *weak singularity spectra* of μ, which means that for $\theta > 0$ we determine the Hausdorff dimensions of the following sets:-

$$\mathcal{I}^{\theta}(\mu) := \{\xi \in L(G) : \liminf_{r\to 0} \frac{\log \mu(B(\xi,r))}{\log r} \leq \theta\} \quad ,$$

$$\mathcal{I}_{\theta}(\mu) := \{\xi \in L(G) : \liminf_{r\to 0} \frac{\log \mu(B(\xi,r))}{\log r} \geq \theta\} \quad ,$$

$$\mathcal{S}^{\theta}(\mu) := \{\xi \in L(G) : \limsup_{r\to 0} \frac{\log \mu(B(\xi,r))}{\log r} \leq \theta\} \quad ,$$

$$\mathcal{S}_{\theta}(\mu) := \{\xi \in L(G) : \limsup_{r\to 0} \frac{\log \mu(B(\xi,r))}{\log r} \geq \theta\} \quad ,$$

(where $B(\xi, r)$ denotes the Euclidean ball of radius r centred at ξ).
This 'weak multifractal analysis' of the Patterson measure will be based on a further investigation of the Hausdorff dimension $\dim_H(\mathcal{J}_\sigma(G))$ of the associated *σ-Jarník limit sets* $\mathcal{J}_\sigma(G) \subset L(G)$, which represent the natural generalisation of the well-approximable real numbers to the theory Kleinian groups (for a definition of $\mathcal{J}_\sigma(G)$ we refer to the end of this introduction).
In [12] we derived a complete description of $\mathcal{J}_\sigma(G)$ in terms of the dimension with respect to μ. As a consequence, we were able to determine $\dim_H(\mathcal{J}_\sigma(G))$ for those cases in which $\dim_H(L(G))$ does not exceed the maximal rank of the parabolic

fixed points of G. The first aim of this paper will be to show how to modify the construction in [12] in order to deal with the remaining cases, i.e. based on the construction in [12] we compute $\dim_H(\mathcal{J}_\sigma(G))$ for *all* geometrically finite Kleinian groups with parabolic elements.
We then discuss how these estimates give access to the before mentioned multifractal aspects of the Patterson measure. The results obtained clearly suggest that the most natural way to view the derived Hausdorff dimension of $\mathcal{J}_\sigma(G)$ is exactly this interpretation in terms of the theory of multifractals.
In order to state this main results more explicit, we recall that $\dim_H(L(G))$ is proven to be equal to the *exponent of convergence* $\delta = \delta(G)$ of G, which is given by the exponent of convergence of the Dirichlet series

$$\sum_{g \in G} \exp(s\, d(0, g0));$$

where d denotes the hyperbolic distance in the Poincaré ball $\mathbb{D}^3$ (note that for simplicity we restrict the discussion to hyperbolic 3-space). In fact, a combination of the *global measure formula* (see section 2) and the *Khintchine law for geodesics* (cf. [15]) immediately shows that for μ-almost all $\xi \in L(G)$ we have that

$$\delta = \liminf_{r \to 0} \frac{\log \mu(B(\xi, r))}{\log r} = \limsup_{r \to 0} \frac{\log \mu(B(\xi, r))}{\log r}.$$

Also, recall that a parabolic element of a Kleinian group acting on $\mathbb{D}^3$ may be either of *rank 1* or of *rank 2*, depending on whether the stabiliser of the associated parabolic fixed point is isomorphic to a finite extension of $\mathbb{Z}$ or $\mathbb{Z}^2$. If k_{min} and k_{max} denote the minimal and maximal possible ranks among the parabolic elements occuring in G, then it is well known that $\delta > k_{max}/2$ [1]. Combining this fact with the global measure formula for μ then shows that the investigation of the weak singularity spectra of μ has to be split into the following five cases:-

(A): $k_{max} = 1$ and $\delta < 1$, **(B):** $k_{max} = 1$ and $\delta > 1$,
(C): $1 = k_{min} < k_{max} = 2$, **(D):** $k_{min} = k_{max} = 2$,
(E): $k_{max} = 1$ and $\delta = 1$.

Note that (A) includes as a special case all finitely generated 2nd kind Fuchsian groups with parabolic elements. Also, we shall see that from the multifractal point of view (B) and (C) provide the most interesting cases.
The following theorem is the main result in this paper.

Theorem 1
Let G be a geometrically finite Kleinian group with parabolic elements and with exponent of convergence δ. The weak singularity spectra of the associated Patterson measure μ are determined by the following.

(A): *If* $k_{max} = 1$ *and* $\delta < 1$, *then*

- $\dim_H(\mathcal{I}^\theta(\mu)) = \begin{cases} 0 & for \quad 0 < \theta \leq 2\delta - 1 \\ \frac{\delta}{1-\delta}\,\theta - \frac{\delta(2\delta-1)}{1-\delta} & for \quad 2\delta - 1 < \theta \leq \delta \\ \delta & for \quad \theta > \delta; \end{cases}$
- $\dim_H(\mathcal{S}_\theta(\mu)) = \begin{cases} \delta & for \quad 0 < \theta \leq \delta \\ 0 & for \quad \theta > \delta. \end{cases}$

(B): *If* $k_{max} = 1$ *and* $\delta > 1$, *then*

- $\dim_H(\mathcal{I}^\theta(\mu)) = \begin{cases} 0 & for \quad 0 < \theta < \delta \\ \delta & for \quad \theta \geq \delta; \end{cases}$
- $\dim_H(\mathcal{S}_\theta(\mu)) = \begin{cases} \delta & for \quad 0 < \theta \leq \delta \\ (1-\delta) - \frac{(1-\delta)(2\delta-1)}{\theta-1} & for \quad \delta \leq \theta \leq (2\delta-1) - \frac{\delta-1}{\delta} \\ \frac{\delta}{1-\delta}\,\theta - \frac{\delta(2\delta-1)}{1-\delta} & for \quad (2\delta-1) - \frac{\delta-1}{\delta} \leq \theta \leq 2\delta - 1 \\ 0 & for \quad 2\delta - 1 \leq \theta. \end{cases}$

(C): *If* $1 = k_{min} < k_{max} = 2$, *then*

- $\dim_H(\mathcal{I}^\theta(\mu)) = \begin{cases} 0 & for \quad 0 < \theta \leq 2\delta - 2 \\ \frac{\delta}{2-\delta}\theta - \frac{\delta(2\delta-2)}{2-\delta} & for \quad 2\delta - 2 < \theta \leq \delta \\ \delta & for \quad \theta > \delta; \end{cases}$
- $\dim_H(\mathcal{S}_\theta(\mu)) = \begin{cases} \delta & for \quad 0 < \theta \leq \delta \\ (1-\delta) - \frac{(1-\delta)(2\delta-1)}{\theta-1} & for \quad \delta \leq \theta \leq (2\delta-1) - \frac{\delta-1}{\delta} \\ \frac{\delta}{1-\delta}\,\theta - \frac{\delta(2\delta-1)}{1-\delta} & for \quad (2\delta-1) - \frac{\delta-1}{\delta} \leq \theta \leq 2\delta - 1 \\ 0 & for \quad 2\delta - 1 \leq \theta. \end{cases}$

(D): *If* $k_{min} = k_{max} = 2$, *then*

- $\dim_H(\mathcal{I}^\theta(\mu)) = \begin{cases} 0 & for \quad 0 < \theta \leq 2\delta - 2 \\ \frac{\delta}{2-\delta}\theta - \frac{\delta(2\delta-2)}{2-\delta} & for \quad 2\delta - 2 < \theta \leq \delta \\ \delta & for \quad \theta > \delta; \end{cases}$
- $\dim_H(\mathcal{S}_\theta(\mu)) = \begin{cases} \delta & for \quad 0 < \theta \leq \delta \\ 0 & for \quad \theta > \delta. \end{cases}$

Furthermore, in each of the cases (A) - (D) we have that

- $\dim_H(\mathcal{I}_\theta(\mu)) = \begin{cases} \delta & for \quad 0 < \theta \leq \delta \\ 0 & for \quad \theta > \delta; \end{cases}$
- $\dim_H(\mathcal{S}^\theta(\mu)) = \begin{cases} 0 & for \quad 0 < \theta < \delta \\ \delta & for \quad \theta \geq \delta. \end{cases}$

(E): *If* $k_{max} = 1$ *and* $\delta = 1$, *then the weak singularity spectra of* μ *are trivial. In this case we have for all* $\xi \in L(G)$ *that*

$$\lim_{r \to 0} \frac{\log \mu(B(\xi, r))}{\log r} = \delta.$$

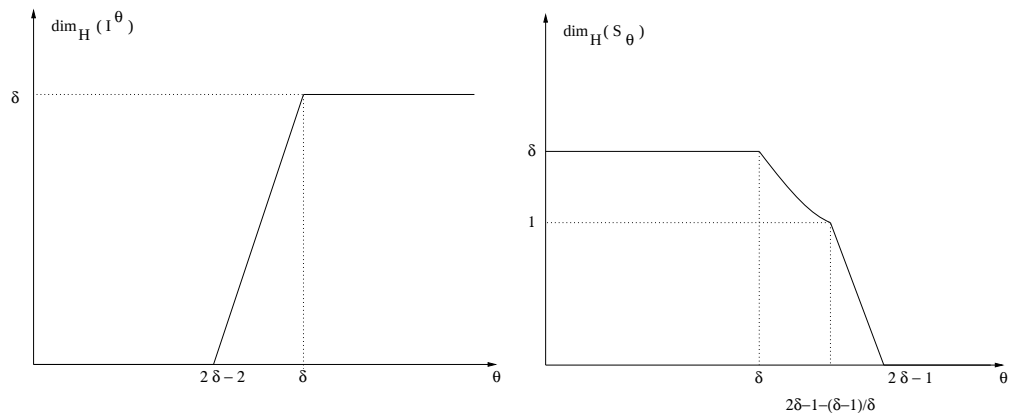

Figure 1: The interesting spectra in case (C)

Note that in (B) and (C) the lower lim sup -spectrum exhibits a *phase transition* at $\theta = \theta^*$, where $\theta^* := 2\delta - 1 - (\delta - 1)/\delta$. The significance of this value is that $\dim_H(\mathcal{S}_{\theta^*}(\mu)) = 1$.
As we have already mentioned, one of the main ingredients in the proof of this theorem will be the explicit calculation of the Hausdorff dimension of the σ-Jarník limit sets $\mathcal{J}_\sigma(G)$. In order to state this result more precisely, recall from [12] the actual definition of $\mathcal{J}_\sigma(G)$. For $t > 0$ let ξ_t denote the unique point on the ray between the origin $0 \in \mathbb{D}^3$ and $\xi \in L(G)$ whose hyperbolic distance from 0 is equal to t. Let Δ denote the *ray excursion function* which is given by $\Delta(\xi_t) := d(\xi_t, G(0))$. For $\sigma > 0$, the *σ-Jarník limit set* $\mathcal{J}_\sigma(G)$ is defined by (see section 2 for an alternative definition)

$$\mathcal{J}_\sigma(G) := \left\{ \xi \in L(G) : \limsup_{t \to \infty} \frac{\Delta(\xi_t)}{t} \geq \frac{\sigma}{1+\sigma} \right\}.$$

We obtain the following theorem.

Theorem 2
Let G be a geometrically finite Kleinian group with parabolic elements and with limit set of Hausdorff dimension δ. For $\sigma > 0$, the Hausdorff dimension of the σ-Jarník limit set $\mathcal{J}_\sigma(G)$ is determined by the following.

(1) *For $\delta \leq k_{max}$, we have that* $\dim_H(\mathcal{J}_\sigma(G)) = \frac{\delta}{1+\sigma}$.

(2) *For $\delta > k_{max}$, we have that* $\dim_H(\mathcal{J}_\sigma(G)) = \begin{cases} \frac{\delta}{1+\sigma} & \text{for} \quad \frac{\delta}{1+\sigma} \leq 1 \\ \frac{\delta+\sigma}{1+2\sigma} & \text{for} \quad \frac{\delta}{1+\sigma} \geq 1. \end{cases}$

Remarks:

- Our estimates of the weak singularity spectra of μ, combined with the fact that μ has a flat *Rényi dimension spectrum* equal to the exponent of convergence δ (cf. [14]), reveal that μ can not be analysed by means of the usual multifractal theory, whose goal it is to relate the *dimension spectrum* and the Rényi spectrum by means of a Legendre transform. Furthermore, at the moment a complete multifractal analysis for those cases in which the Patterson measure is equivalent to the 1-dimensional Lebesgue measure is not fully developed. Such an analysis relates closely to the multifractal analysis of the continued fraction map, and here we refer to [11] for some very interesting preliminary results.

- Recall that Jarník [6] (and later also Besicovitch [3]) showed that the Hausdorff dimension of the set of *well-approximable irrational numbers* is given by

$$\dim_H \left(\left\{ x \in \mathbb{R} : \left| x - \frac{p}{q} \right| < \left(q^{-2} \right)^{1+\sigma} \text{ for infinitely many reduced } \frac{p}{q} \right\} \right) = \frac{1}{1+\sigma}.$$

  Some first generalisations of this result to certain special cases of geometrically finite Kleinian groups with parabolic elements were derived in [7], [12]. Now, as explained above, in this paper we also deal with the remaining cases and hence we complete this generalisation to geometrically finite Kleinian groups with parabolic elements. We mention that for Kleinian group of this type the Hausdorff dimension of the set of well-approximable limit points has been computed also in [5], and that for finitely generated 2nd kind Fuchsian groups the results concerning the weak singularity spectra of the Patterson measure have already been stated in [14].

2 Preliminaries

In this section we recall a few well known results which are fundamental for what follows. Let G denote a geometrically finite Kleinian group with parabolic elements. For simplicity we assume that G acts on hyperbolic 3-space $\mathbb{D}^3$.

The Jarník set as a limsup-set: Let P be a complete set of inequivalent parabolic fixed points of G. For $p \in P$, let $\mathcal{T}_p \subset G$ denote a set of coset representatives of G_p chosen such that $|g(0)| \leq |h(0)|$, for each $g \in \mathcal{T}_p$ and for all $h \in G_{g(p)} := gG_pg^{-1}$. Also, for $p_o \in P$ such that the rank $k(p_o)$ of p_o is equal to k_{max}, let $\mathcal{T}_{max} := \mathcal{T}_{p_o}$, and define $\mathcal{T} := \bigcup_{p \in P} \mathcal{T}_p$. It is well known that to each $g \in \mathcal{T}_p$ we may associate a *standard horoball* $H_{g(p)}(r_g) \subset \mathbb{D}^3$ at $g(p)$ with Euclidean radius $r_g \asymp (1 - |g(0)|)$, such that the set $\{H_{g(p)}(r_g) : g \in \mathcal{T}_p, p \in P\}$ comprises a G-equivariant set of pairwise disjoint horoballs. If $\mathcal{H}_\xi(r)$ denotes the radial projection of a horoball $H_\xi(r)$ at ξ of radius r, projected from the origin in $\mathbb{D}^3$ onto the boundary S^2 of hyperbolic 3-space, then we can express $\mathcal{J}_\sigma(G)$ in terms of the set of standard horoballs as follows. Let $\mathcal{F}$ denote the set of functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$\lim_{x\to 0} \log\phi(x)/\log x = 0$. Using basic hyperbolic geometry, we see that $\xi \in \mathcal{J}_\sigma(G)$ if and only if $\xi \in \bigcup_{\phi\in\mathcal{F}} \limsup\{\mathcal{H}_{g(p)}(\phi(r_g) r_g^{1+\sigma}) : g \in \mathcal{T}_p, p \in P\}$, i.e. we have that

$$\mathcal{J}_\sigma(G) = \bigcup_{\phi\in\mathcal{F}} \bigcap_{n\in\mathbf{N}} \bigcup_{p\in P} \bigcup_{\substack{g\in\mathcal{T}_p \\ r_g\le\frac{1}{n}}} \mathcal{H}_{g(p)}(\phi(r_g) r_g^{1+\sigma}) \quad .$$

The Patterson measure: For the construction and basic properties of the *Patterson measure* μ we refer to [8],[9],[10] and [16]. Recall that μ is a non atomic probability measure supported on the limit set $L(G)$, for which we derived in [15] (see also [17]) a formula concerning its scaling properties. This formula describes in a uniform way the decay of μ around arbitrary limit points. In order to be able to restate this formula, we require the following notation. For $\xi \in L(G)$ and positive t define $k(\xi_t)$ to be equal to $k(p)$ if $\xi_t \in H_{g(p)}(r_g)$ for some $p \in P$ and $g \in \mathcal{T}_p$, and let $k(\xi_t)$ be equal to δ otherwise.

The global measure formula:

For $\xi \in L(G)$ and positive t , we have that

$$\mu(B(\xi, e^{-t})) \asymp e^{-t\delta}\, e^{-\Delta(\xi_t)\,(\delta - k(\xi_t))}.$$

3 Hausdorff dimensions of Jarník limit sets

In this section we give the proof of Theorem 2 , i.e. for arbitrary geometrically finite Kleinian groups with parabolic elements we calculate the Hausdorff dimension of the Jarník limit set $\mathcal{J}_\sigma(G)$. As already mentioned in the introduction, part (1) of Theorem 2 has already been dealt with in [12], and we refer to that paper for the proof in this cases. Hence, we now concentrate on the proof of part (2) of Theorem 2 , i.e. we assume that $\delta > k_{max}$, or, equivalently, that $k_{max} = 1$ and $\delta > 1$. Our arguments are based on certain parts of the construction in [12]. In particular, we shall recall two lemmata which were derived in [12], and restate a construction of a probability measure supported on a Cantor-like subset contained in $\mathcal{J}_\sigma(G)$.
For simplicity, we adapt the following notation from [12]. For $n \in \mathbb{N}, \tau > 0$ and $g \in \mathcal{T}_{max}$ we define

$$A_n(\tau) := \{h \in \mathcal{T}_{max} : \tau^{n+1} \le r_h < \tau^n\} ,$$

$$Q_n(g,\sigma,\tau) := \{h \in A_n(\tau) : \mathcal{H}_{h(p)}(r_h) \subset \mathcal{H}_{g(p)}(r_g^{1+\sigma})\} \quad .$$

The following result was obtained in [12] (Proposition 1).

Lemma 1
There exist $\rho, k_o, k_1, k_2 > 0$ and an increasing function $\iota : \mathbb{N} \to \mathbb{R}^+$ such that if $g \in A_n(\rho)$ for some $n > k_o$, then we have for $m > \iota(n)$ that

$$k_1\, \rho^{\delta(n-m)+\sigma n(2\delta-k_{max})} \le \mathrm{card}(Q_m(g,\sigma,\rho)) \le k_2\, \rho^{\delta(n-m)+\sigma n(2\delta-k_{max})} \quad .$$

For the rest of this section let us fix the number ρ derived in this lemma.
Following the construction in [12], let $\{n_k\}_{k\in\mathbf{N}}$ denote a sequence of positive integers such that $n_0 > \max\{k_o; 2\sigma^{-1}\}$, $n_k > \iota(n_{k-1})$ for all k, and such that $\lim_{m\to\infty}\left(\frac{1}{n_m}\sum_{j=0}^{m-1} n_j\right) = 0$. For $k \in \mathbb{N}$, let

$$N_k := \min_{g\in A_{n_{k-1}}(\rho)} \operatorname{card} Q_{n_k}(g,\sigma,\rho) \quad .$$

Further, simply by deleting certain elements if necessary, we reduce for an arbitrary $h \in A_{n_{k-1}}(\rho)$, the number of elements in $Q_{n_k}(h,\sigma,\rho)$ to N_k. The resulting set is denoted by $\tilde{Q}_{n_k}(h)$. Hence by definition, for all $k \in \mathbb{N}$ and $h \in A_{n_{k-1}}(\rho)$ we have that $N_k = \operatorname{card}\tilde{Q}_{n_k}(h)$. For a fixed $g_o \in A_{n_o}(\rho)$, we define:-

- $I_o^\sigma := \{\mathcal{H}_{g_o(p_o)}(r_{g_o}^{1+\sigma})\}$;
- if I_{k-1}^σ is defined for k in $\mathbb{N}$, then let
  $I_k^\sigma := \{\mathcal{H}_{h(p_o)}(r_h^{1+\sigma}) : h \in \tilde{Q}_{n_k}(g) \quad \text{for some} \quad g \in A_{n_{k-1}}(\rho)$
  $\text{such that} \quad \mathcal{H}_{g(p_o)}(r_g^{1+\sigma}) \in I_{k-1}^\sigma\}$.

By construction, each element in I_{k-1}^σ contains exactly N_k elements of I_k^σ. This allows us to define the Cantor-like set $I_\sigma := \bigcap_{k\geq 0}\bigcup_{I\in I_k^\sigma} I$. Next, in order to construct a probability measure on I_σ, renormalise the Patterson measure on every I_k^σ, i.e. for each $k \in \mathbb{N}$ define a probability measure $\nu_\sigma^{(k)}$ on I_k^σ such that for Borel sets $E \subset S^2$ we have that

$$\nu_\sigma^{(k)}(E) = \sum_{I\in I_k^\sigma} (N_1 \cdots N_k)^{-1}(E \cap I)/\mu(I).$$

Using *Helly's Theorem*, we obtain a probability measure ν_σ on I_σ as the weak limit of the sequence of measures $\{\nu_\sigma^{(k)}\}_{k\in\mathbf{N}}$. Note that $\nu_\sigma^{(k)}(I) = \nu_\sigma(I)$, for each $k \in \mathbb{N}$ and $I \in I_k^\sigma$.
Now, recall the following lemma obtained in [12] (Lemma 6).

Lemma 2
There exists $k_3 > 0$ such that if $\xi \in I_\sigma$ and r has the property $\rho^{n_k} \leq r < \rho^{n_{k-1}}$ for some $k \in \mathbb{N}$, then $B(\xi,r)$ intersects at most one element in I_{k-1}^σ and

$$\operatorname{card}\{I \in I_k^\sigma : I \cap B(\xi,r) \neq \emptyset\} \leq k_3\,\rho^{-\delta n_k}\,\mu(B(\xi,r)) \quad .$$

The following lemma marks the difference between the construction here and the construction in [12]. The reader is asked to compare this result with Lemma 3 in [12].

Lemma 3
There exists $k_4 > 0$ such that for each $\epsilon > 0$ there exists $r_o(\epsilon) > 0$ with the property that for all $\xi \in I_\sigma$ and for all $0 < r < r_o(\epsilon)$ such that $\rho^{n_k} \leq r < \rho^{n_{k-1}}$ for some $k \in \mathbb{N}$, we have that

$$\nu_\sigma(B(\xi,r)) \leq k_4\,\mu(B(\xi,r))\,\rho^{-n_{k-1}(\sigma(2\delta-k_{max})+\epsilon)} \quad .$$

Proof.
Let ξ and r be given as stated in the lemma. By construction of the measure ν_σ and using Lemma 2, we have that

$$\begin{aligned}\nu_\sigma(B(\xi,r)) &\leq \prod_{j=0}^{k} N_j^{-1}\ \mathrm{card}\{I \in I_k^\sigma : I \cap B(\xi,r) \neq \emptyset\} \\ &\leq k_3 \rho^{-n_k\delta}\mu(B(\xi,r)) \prod_{j=0}^{k} N_j^{-1}\ .\end{aligned}$$

Using Lemma 1, it follows that

$$\begin{aligned}\nu_\sigma(B(\xi,r)) &\leq k_3 k_1^{-1}\rho^{-n_k\delta}\mu(B(\xi,r))k_1^{-(k-1)}\rho^{\delta(n_k-n_{k-1})}\rho^{-\sigma n_{k-1}(2\delta-k_{max})} \\ &\qquad \cdot\rho^{\delta(n_{k-1}-n_o)}\rho^{-\sigma(2\delta-k_{max})\sum_{j=0}^{k-2} n_j} \\ &= k_3 k_1^{-1}\mu(B(\xi,r)) \\ &\qquad \cdot\rho^{n_{k-1}(-\sigma(2\delta-k_{max})-n_{k-1}^{-1}(n_o+\sigma(2\delta-k_{max})\sum_{j=0}^{k-2} n_j+(k-1)(\log k_1)(\log\rho)^{-1}))}\ .\end{aligned}$$

By construction of the sequence (n_k), we have for each $\epsilon > 0$ and for sufficiently large k that

$$n_o + \sigma(2\delta - k_{max})\sum_{j=0}^{k-2}(n_j + (k-1)(\log k_1)(\log\rho)^{-1}) < \epsilon\ n_{k-1}.$$

Using this fact in the above estimate, the statement of the lemma follows. □

Proposition 1
For each $\epsilon > 0$ there exists $r_1(\epsilon) > 0$ such that for each $\xi \in I_\sigma$ and $0 < r < r_1(\epsilon)$ we have that

$$\nu_\sigma(B(\xi,r)) \ll \begin{cases} r^{\delta/(1+\sigma)-\epsilon} & \text{for} \quad \sigma \geq \delta - 1 \\ r^{(\delta+\sigma)/(1+2\sigma)-\epsilon} & \text{for} \quad \sigma \leq \delta - 1. \end{cases}$$

Proof.
Let $\xi \in I_\sigma$ be given. Without loss of generality assume that $\rho^{n_k} \leq r < \rho^{n_{k-1}}$ and that $\xi \in \mathcal{H}_{g(p)}(r_g^{1+\sigma})$ for some $g \in \mathcal{T}_p$ with $r_g \asymp \rho^{n_{k-1}}$. We then have the following.

(1) (*'Before the visit to $H_{g(p)}(r_g^{1+\sigma})$ '*):
If $(\rho^{n_{k-1}})^{1+\sigma} \leq r < \rho^{n_{k-1}}$, then $\nu_\sigma(B(\xi,r)) \ll r^{\delta/(1+\sigma)-\epsilon/(1+\sigma)}$.

Proof: Since by construction $\nu_\sigma(\mathcal{H}_{g(p)}(r_g)) \asymp \nu_\sigma(\mathcal{H}_{g(p)}(r_g^{1+\sigma}))$, it follows that

$$\begin{aligned}\nu_\sigma(B(\xi,r)) &\ll \nu_\sigma(\mathcal{H}_{g(p)}(r_g^{1+\sigma})) \\ &\ll \mu(\mathcal{H}_{g(p)}(r_g^{1+\sigma}))\ \rho^{-n_{k-1}(\sigma(2\delta-1)+\epsilon)} \\ &\asymp r_g^{\delta}\ r_g^{\sigma(2\delta-1)}\ \rho^{-n_{k-1}(\sigma(2\delta-1)+\epsilon)} \\ &\asymp r_g^{\delta}\ r_g^{-\epsilon} \\ &\asymp \left(r_g^{1+\sigma}\right)^{\delta/(1+\sigma)}\ r_g^{-\epsilon} \\ &\ll r^{\delta/(1+\sigma)}\ r^{-\epsilon/(1+\sigma)}.\end{aligned}$$

♣

(2) (*'After having met* $H_{g(p)}(r_g^{1+\sigma})$ *, while still visiting* $H_{g(p)}(r_g)$ *'*):
If $(\rho^{n_{k-1}})^{1+2\sigma} \leq r < (\rho^{n_{k-1}})^{1+\sigma}$, then

$$\nu_\sigma(B(\xi,r)) \ll \begin{cases} r^{\delta/(1+\sigma)}\, r^{-\epsilon/(1+\sigma)} & \text{for} \quad \sigma \geq \delta - 1 \\ r^{(\delta+\sigma)/(1+2\sigma)}\, r^{-\epsilon/(1+\sigma)} & \text{for} \quad \sigma \leq \delta - 1. \end{cases}$$

Proof: Let $r = \rho^{n_{k-1}(1+\sigma+\tau)}$ for some $0 < \tau \leq \sigma$. Then in this case the fluctuation of the Patterson measure is maximal for $\Delta(\xi_t) = -(\sigma - \tau) n_{k-1} \log \rho$ (note that $r \asymp e^{-t}$, where $t = -n_{k-1}(1+\sigma+\tau)\log\rho$). This gives that

$$\begin{aligned} \nu_\sigma(B(\xi,r)) &\ll \mu(B(\xi,r))\, \rho^{-n_{k-1}(\sigma(2\delta-1)+\epsilon)} \\ &\ll r^\delta\, \rho^{n_{k-1}(\sigma-\tau)(\delta-1)}\, \rho^{-n_{k-1}(\sigma(2\delta-1)+\epsilon)} \\ &\ll \rho^{n_{k-1}(1+\sigma+\tau)\delta}\, \rho^{n_{k-1}(\sigma-\tau)(\delta-1)}\, \rho^{-n_{k-1}(\sigma(2\delta-1)+\epsilon)} \\ &= (\rho^{n_{k-1}})^{\delta+\tau}\, \rho^{-n_{k-1}\epsilon} \\ &= \left(\rho^{n_{k-1}(1+\sigma+\tau)}\right)^{(\delta+\tau)/(1+\sigma+\tau)} \left(\rho^{n_{k-1}(1+\sigma+\tau)}\right)^{-\epsilon/(1+\epsilon+\tau)} \\ &\ll \begin{cases} r^{\delta/(1+\sigma)}\, r^{-\epsilon/(1+\sigma)} & \text{for} \quad \sigma \geq \delta - 1 \\ r^{(\delta+\sigma)/(1+2\sigma)}\, r^{-\epsilon/(1+\sigma)} & \text{for} \quad \sigma \leq \delta - 1. \end{cases} \end{aligned}$$

♣

(3) (*'After the visit to* $H_{g(p)}(r_g)$ *'*):
If $\rho^{n_k} \leq r < (\rho^{n_{k-1}})^{1+2\sigma}$, then $\nu_\sigma(B(\xi,r)) \ll r^{(\delta+\sigma)/(1+2\sigma)-\epsilon/(1+2\sigma)}$.

Proof: Since $\delta > 1$, the global measure formula implies that $\mu(B(\xi,r)) \ll r^\delta$. Using this and Lemma 3, it follows that

$$\begin{aligned} \nu_\sigma(B(\xi,r)) &\ll \mu(B(\xi,r))\, \rho^{-n_{k-1}(\sigma(2\delta-1)+\epsilon)} \\ &\ll r^\delta\, r^{-(\sigma(2\delta-1)+\epsilon)/(1+2\sigma)} \\ &\ll r^{(\delta+\sigma-\epsilon)/(1+2\sigma)} \\ &\ll r^{(\delta+\sigma)/(1+2\sigma)}\, r^{-\epsilon/(1+2\sigma)}. \end{aligned}$$

♣

The statement of the proposition now follows by summing up the above three considerations. □

Proof of Theorem 2:
Recall that in the above we obtained that for each $\sigma > 0$ there exists a set $I_\sigma \subset \mathcal{J}_\sigma(G)$ which supports a probability measure ν_σ with the property that for each $\epsilon > 0$ and for each $\xi \in I_\sigma$ there exists $r_1(\epsilon)$ such that for all $0 < r < r_1(\epsilon)$ we have that

$$\nu_\sigma(B(\xi,r)) \ll \begin{cases} r^{\delta/(1+\sigma)-\epsilon'} & \text{for} \quad \sigma \geq \delta - 1 \\ r^{(\delta+\sigma)/(1+2\sigma)-\epsilon'} & \text{for} \quad \sigma \leq \delta - 1. \end{cases}$$

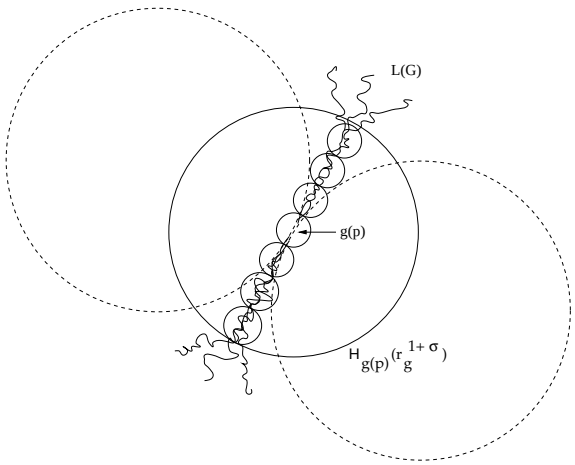

Figure 2: The 'alternative cover'

Hence, by the *mass distribution principle* (see e.g. [4]), we have for each $\sigma > 0$ that

$$\dim_H(\mathcal{J}_\sigma(G)) \geq \begin{cases} \frac{\delta}{1+\sigma} & \text{for} \quad \sigma \geq \delta - 1 \\ \frac{\delta+\sigma}{1+2\sigma} & \text{for} \quad \sigma \leq \delta - 1. \end{cases}$$

In order to obtain the upper bounds for the Hausdorff dimension of $\mathcal{J}_\sigma(G)$, note first that for this it is sufficient to give upper bounds for the Hausdorff dimension of the set

$$\mathcal{W}_\sigma(G) := \bigcap_{n\in\mathbf{N}} \bigcup_{p\in P} \bigcup_{\substack{g\in\mathcal{T}_p \\ r_g \leq \frac{1}{n}}} \mathcal{H}_{g(p)}(r_g^{1+\sigma}) \quad .$$

Now, $\{\mathcal{H}_{g(p)}(r_g^{1+\sigma}) : p \in P, g \in \mathcal{T}_p\}$ provides a 'natural cover' of $\mathcal{W}_\sigma(G)$. Using this and using the fact that $\sum_{p\in P}\sum_{g\in\mathcal{T}_p} \left(r_g^{1+\sigma}\right)^{\delta/(1+\sigma)+\epsilon}$ converges for any $\epsilon > 0$, it follows that

$$\dim_H(\mathcal{W}_\sigma(G)) \leq \frac{\delta}{1+\sigma}.$$

On the other hand note that the intersection of $L(G)$ with the shadow of any σ-reduced horoball $\mathcal{H}_{g(p)}(r_g^{1+\sigma})$ is contained in some 'tie-shaped region' (see Figure 2). This observation together with the fact that the largest horoball which one can possibly meet after having travelled through $H_{g(p)}(r_g^{1+\sigma})$ is of the size $r_g^{1+2\sigma}$, gives that we may cover $L(G) \cap \mathcal{H}_{g(p)}(r_g^{1+\sigma})$ by $r_g^{-\sigma}$ Euclidean balls of the size $r_g^{1+2\sigma}$ (these balls are not necessarily shadows of horoballs!). Clearly, this gives us an

alternative way of covering $\mathcal{W}_\sigma(G)$ (see Figure 2), and for this cover we obtain that

$$\sum_{p\in P}\sum_{g\in\mathcal{T}_p} r_g^{-\sigma}\; r_g^{(1+2\sigma)\,s} \begin{cases} \text{converges} & \text{for} \quad s > (\delta+\sigma)/(1+2\sigma) \\ \text{diverges} & \text{for} \quad s \le (\delta+\sigma)/(1+2\sigma). \end{cases}$$

In particular, this implies that

$$\dim_H(\mathcal{W}_\sigma(G)) \le \frac{\delta+\sigma}{1+2\sigma}.$$

Combining the two upper bounds for the Hausdorff dimension, an elementary calculation now gives that

$$\dim_H(\mathcal{W}_\sigma(G)) \le \begin{cases} \frac{\delta}{1+\sigma} & \text{for} \quad \sigma \ge \delta-1 \\ \frac{\delta+\sigma}{1+2\sigma} & \text{for} \quad \sigma \le \delta-1, \end{cases}$$

which proves the theorem.

□

4 Weak singularity spectra of the Patterson measure

In this section we give the estimates which lead to the weak singularity spectra of the Patterson measure μ. The results are a consequence of a combination of Theorem 2 and the global measure formula for μ.

Proof of Theorem 1:
Since in each of the cases (A) up to (D) the arguments are similar, we restrict our discussion to the cases (B) and (C), which are hardest to derive and where the outcome is most interesting.
We consider case (B) first, i.e. we assume that G has exclusively rank 1 parabolic fixed points and that $\delta > 1$. In this case the global measure formula for μ gives the existence of a universal constant $c > 1$, such that for each $\xi \in L(G)$ and every $t > 0$ we have that

$$\delta + (\delta-1)\frac{\Delta(\xi_t)}{t} - \frac{\log c}{t} \le \frac{\log \mu(B(\xi, e^{-t}))}{\log e^{-t}} \le \delta + (\delta-1)\frac{\Delta(\xi_t)}{t} + \frac{\log c}{t}.$$

From this we immediately see that

$$\xi \in \mathcal{J}_\sigma(G) \quad \text{if and only if} \quad \limsup_{t\to\infty} \frac{\log \mu(B(\xi, e^{-t}))}{\log e^{-t}} \ge \delta + (\delta-1)\frac{\sigma}{1+\sigma}.$$

Hence, if we let $\theta = \delta + (\delta-1)\frac{\sigma}{1+\sigma}$, then with an elementary calculation we see that Theorem 2 implies that

$$\dim_H(\mathcal{S}_\theta(\mu)) = \begin{cases} \delta & \text{for} \quad 0 < \theta \le \delta \\ (1-\delta) - \frac{(1-\delta)(2\delta-1)}{\theta-1} & \text{for} \quad \delta \le \theta \le (2\delta-1) - \frac{\delta-1}{\delta} \\ \frac{\delta}{1-\delta}\,\theta - \frac{\delta(2\delta-1)}{1-\delta} & \text{for} \quad (2\delta-1) - \frac{\delta-1}{\delta} \le \theta \le 2\delta-1 \\ 0 & \text{for} \quad 2\delta-1 \le \theta. \end{cases}$$

In order to determine the remaining spectra in case (B), note that the global measure formula gives in particular that for all $\xi \in L(G)$ and $t > 0$ we have that $\mu(B(\xi, e^{-t})) \ll e^{-\delta t}$. Also, note that for $\delta < \theta \leq 2\delta - 1$ the inequality $\mu(B(\xi, e^{-t})) \ll e^{-\theta t}$ holds *t-eventually* (i.e. uniformly for large values of t) only if $\xi \in G(P)$, and for $\theta > 2\delta - 1$ we even have that this inequality never holds. Using these observations, we derive that

$$\dim_H(\mathcal{I}_\theta(\mu)) = \begin{cases} \delta & \text{for} \quad 0 \leq \theta \leq \delta \\ 0 & \text{for} \quad \theta > \delta. \end{cases}$$

In a similar way we see that for $\theta \geq \delta$ the inequality $\mu(B(\xi, e^{-t})) \gg e^{-\theta t}$ is satisfied for each radial limit point $\xi \in L_r(G) := L(G) \setminus G(P)$ for some sequence $\{t_n\}$ tending to infinity (where $\{t_n\}$ depends on ξ). For $\theta < \delta$ no such sequence $\{t_n\}$ exists for any $\xi \in L(G)$. Hence, we have that

$$\dim_H(\mathcal{I}^\theta(\mu)) = \begin{cases} 0 & \text{for} \quad 0 \leq \theta < \delta \\ \delta & \text{for} \quad \theta \geq \delta. \end{cases}$$

Finally, we see that for $\theta \geq \delta$ the inequality $\mu(B(\xi, e^{-t})) \gg e^{-\theta t}$ is t-eventually satisfied for any ξ in the *uniformly-radial limit set*[1] $L_{ur}(G) := \{\xi \in L(G) : \Delta(\xi_t) \ll 1 \text{ for all } t > 0\}$. For $\theta < \delta$ this inequality is never t-eventually satisfied for any $\xi \in L(G)$. Using the fact that $\dim_H(L_{ur}(G)) = \delta$, which we obtained in [13] (for an alternative proof see [2]), it follows that

$$\dim_H(\mathcal{S}^\theta(\mu)) = \begin{cases} 0 & \text{for} \quad 0 < \theta < \delta \\ \delta & \text{for} \quad \theta \geq \delta. \end{cases}$$

This gives the weak singularity spectra of μ in the case (B).

Next, we consider the case (C), i.e. we assume that $1 = k_{min} < k_{max} = 2$. Note that, since in general $\delta > k_{max}/2$, we have that $\delta > 1$. In this case the global measure formula for μ gives the existence of a universal constant $c > 1$, such that for each $\xi \in L(G)$ and every $t > 0$ we have that

$$c^{-1}\, e^{-t(\delta + (\delta-1)\Delta(\xi_t)/t)} \leq \mu(B(\xi, e^{-t})) \leq c\, e^{-t(\delta - (2-\delta)\Delta(\xi_t)/t)}.$$

Obviously, the scaling of μ in rank 1 cusps differs from the scaling of μ in rank 2 cusps. Because of this, we now decompose $\mathcal{J}_\sigma(G)$ as follows. For $l = 1, 2$, let Δ_l denote the ray excursion function detecting only excursions into rank l cusps, i.e. for $\xi \in L(G)$ and $t > 0$ we let $\Delta_l(\xi_t) = d(\xi_t, G(0))$ if ξ_t is contained in a standard horoball associated with some rank l parabolic fixed point, otherwise we let Δ_l be equal to some fixed constant. Then the *(σ, l)-Jarník limit set* $\mathcal{J}_{\sigma,l}(G)$ is defined by

$$\mathcal{J}_{\sigma,l}(G) := \left\{ \xi \in L(G) : \limsup_{t \to \infty} \frac{\Delta_l(\xi_t)}{t} \geq \frac{\sigma}{1+\sigma} \right\}.$$

Clearly, we have that $\mathcal{J}_\sigma(G) = \mathcal{J}_{\sigma,1}(G) \cup \mathcal{J}_{\sigma,2}(G)$, which is of course not a disjoint union. The Hausdorff dimensions of $\mathcal{J}_{\sigma,1}(G)$ and $\mathcal{J}_{\sigma,2}(G)$ are derived following essentially the same string of arguments given in the previous section and [12]. Here it is important to note that as an immediate consequence of the *decoupling*

[1] which is also referred to as *the set of badly approximable limit points* (cf. [13])

lemma in [15] we have that for $l = 1, 2$ the exponent of convergence of the Dirichlet series $\sum_{p \in P, rk(p)=l} \sum_{g \in \mathcal{T}_p} r_g^s$ remains equal to δ (this fact gives the upper bounds of $\dim_H(\mathcal{J}_{\sigma,l}(G))$). Hence, for computing $\dim_H(\mathcal{J}_{\sigma,2}(G))$ we may proceed as we did in the case '$k_{min} = 2$', and we obtain the result that

$$\dim_H(\mathcal{J}_{\sigma,2}(G)) = \frac{\delta}{1+\sigma}.$$

For computing $\dim_H(\mathcal{J}_{\sigma,1}(G))$ we proceed as we did in the case '$k_{max} = 1$ and $\delta > 1$', which gives that

$$\dim_H(\mathcal{J}_{\sigma,1}(G)) = \begin{cases} \frac{\delta}{1+\sigma} & \text{for} \quad \sigma \geq \delta - 1 \\ \frac{\delta+\sigma}{1+2\sigma} & \text{for} \quad \sigma \leq \delta - 1. \end{cases}$$

Using these observations, we can now determine the weak singularity spectra as follows. Namely, exactly as in case (B) we have that

$$\xi \in \mathcal{J}_{\sigma,1}(G) \quad \text{if and only if} \quad \limsup_{t \to \infty} \frac{\log \mu(B(\xi, e^{-t}))}{\log e^{-t}} \geq \delta + (\delta - 1)\frac{\sigma}{1+\sigma},$$

and hence,

$$\dim_H(\mathcal{S}_\theta(\mu)) = \begin{cases} \delta & \text{for} \quad 0 < \theta \leq \delta \\ (1-\delta) - \frac{(1-\delta)(2\delta-1)}{\theta-1} & \text{for} \quad \delta \leq \theta \leq (2\delta-1) - \frac{\delta-1}{\delta} \\ \frac{\delta}{1-\delta}\,\theta - \frac{\delta(2\delta-1)}{1-\delta} & \text{for} \quad (2\delta-1) - \frac{\delta-1}{\delta} \leq \theta \leq 2\delta - 1 \\ 0 & \text{for} \quad 2\delta - 1 \leq \theta. \end{cases}$$

Also, using once again the global measure, we have that

$$\xi \in \mathcal{J}_{\sigma,2}(G) \quad \text{if and only if} \quad \liminf_{t \to \infty} \frac{\log \mu(B(\xi, e^{-t}))}{\log e^{-t}} \leq \delta - (2-\delta)\frac{\sigma}{1+\sigma},$$

which gives, with $\theta = \delta - (2-\delta)\frac{\sigma}{1+\sigma}$, that

$$\dim_H(\mathcal{I}^\theta(\mu)) = \begin{cases} 0 & \text{for} \quad 0 < \theta \leq 2\delta - 2 \\ \frac{\delta}{2-\delta}\theta - \frac{\delta(2\delta-2)}{2-\delta} & \text{for} \quad 2\delta - 2 < \theta \leq \delta \\ \delta & \text{for} \quad \theta > \delta. \end{cases}$$

For the remaining parts of the spectra, note that for $0 < \theta \leq \delta$ we have that $\mu(B(\xi, e^{-t})) \ll e^{-\theta t}$ is t-eventually satisfied for at least all $\xi \in L_{ur}(G)$. For $\delta < \theta \leq 2\delta - 1$ this inequality holds t-eventually only for rank 1 parabolic fixed points, and for $\theta > 2\delta - 1$ we have that this inequality never holds t-eventually. Hence, it follows that

$$\dim_H(\mathcal{I}_\theta(\mu)) = \begin{cases} \delta & \text{for} \quad 0 \leq \theta \leq \delta \\ 0 & \text{for} \quad \theta > \delta. \end{cases}$$

The computation of $\dim_H(\mathcal{S}^\theta(\mu))$ is essentially the same as in case (B). Here, the only difference is that for $2\delta - 2 \leq \theta < \delta$ the inequality $\mu(B(\xi, e^{-t})) \gg e^{-\theta t}$ is satisfied t-eventually by all rank 2 parabolic fixed points (which are of course the only elements in $L(G)$ with this behaviour). It follows that

$$\dim_H(\mathcal{S}^\theta(\mu)) = \begin{cases} 0 & \text{for} \quad 0 < \theta < \delta \\ \delta & \text{for} \quad \theta \geq \delta. \end{cases}$$

□

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