Singular bidouble covers and the construction of interesting algebraic surfaces.

Fabrizio Catanese

Friedrich Hirzebruch zum 70. Geburtstag gewidmet

Abstract. A bidouble cover is a finite flat Galois morphism with Galois group $(\mathbf{Z}/\mathbf{2})^2-$. The structure theorem for smooth Galois $(\mathbf{Z}/\mathbf{2})^2-$ covers was given in [**Cat2**] [pag. 491-493] where bidouble covers of $\mathbf{P}^1 \times \mathbf{P}^1$ were introduced in order to find interesting properties of the moduli spaces of surfaces of general type. In this paper we develop general formulae for the case of resolutions of singular bidouble covers. P. Burniat used singular bidouble covers in order to fill out sectors of surface geography. In this paper instead, the main application is for the construction of surfaces with birational canonical map (so called simple canonical surfaces) and high K^2, for instance we construct such surfaces with $p_g = 4, 11 \leq K^2 \leq 28$, against a prediction of F. Enriques that 24 should be the maximum allowed.

Moreover, we find, among several new examples of surfaces, some surfaces with $p_g = q = 1$, $K^2 = 4, 5$, and also some infinite series of surfaces whose canonical map is composed of a pencil of curves of genus 2 or 3, with non costant moduli.

1. Introduction

In this paper we present work in progress, motivated by the following problem raised by Federigo Enriques ([**En**], chapter VIII, page 284)

Problem 1. Try to construct simple canonical algebraic surfaces (i. e., surfaces having a birational canonical map ϕ_1), with geometric genus $p_g = 4$ and K^2 as high as possible.

One knows that if the canonical map ϕ_1 is birational, then necessarily $p_g \geq 4$ and the Castelnuovo inequality $K^2 \geq 3p_g - 7$ holds. More generally, one can ask for which values of the invariants $\chi = 1 - q + p_g$ and K^2 does there exist a simple canonical surface. Recently, while writing the survey article [**Cat5**], I realized that the question is quite open for small values of p_g, and I noticed that one can ask a stronger question

---

1991 *Mathematics Subject Classification.* 14J29, 14J10, 14J25, 14J17.

The research of the author was performed in the realm of the Project AGE HCM, Contract ERBCHRXT 940557.

Problem 2. For which values of the invariants χ and K^2 do there exist algebraic surfaces such that a general projection of the canonical map ϕ_1 to a 3-dimensional projective space is such that its image Σ_1 has ordinary singularities.

If then S is a simple canonical surface and p_g attains its minimal value $= 4$, then $K^2 \geq 5$ and simple canonical surfaces were constructed for $K^2 = 6, 7$ by Enriques (the case $K^2 = 5$ being the trivial case of surfaces of degree 5 either smooth or with at worst Du Val singularities). The existence problem for $6 \leq K^2 \leq 8$ was solved by Franchetta ([**Fran**]), whereas other constructions for $K^2 = 7$ were also done by Maxwell and Kodaira ([**Max**],[**Kod**]), and finally Ciliberto gave an affirmative answer to problem 2 in the interval $6 \leq K^2 \leq 10$ ([**Cil1**]) showing moreover that in this range surfaces with ordinary singularities give rise to an irreducible unirational open set of the moduli space. Recently, I was informed by Ciliberto that P. Burniat had also obtained results in this direction, thus via a bibliographical search I found out that Burniat ([**Bur1**]) constructed simple canonical surfaces with $p_g = 4$ and $11 \leq K^2 \leq 16$.

Enriques indeed had conjectured that $K^2 = 24$ should be the maximum possible value for a simple canonical surface with $p_g = 4$, but we show in this paper that his prediction, based on the conjecture that the expected number of moduli should be strictly positive, does not hold true.

Theorem 1. There do exist algebraic surfaces with geometric genus $p_g = 4$, and with a birational canonical map ϕ_1, for the following values of K^2: $11 \leq K^2 \leq 28$.

The above highest value 28 for K^2 is very likely to be substantially improved with the same methods of the present paper. As the reader may easily guess from the title, the above surfaces arise as minimal desingularizations of singular bidouble covers, in fact of Q $= \mathbf{P^1} \times \mathbf{P^1}$. Here, a bidouble cover is a finite flat Galois morphism with Galois group $(\mathbf{Z/2})^2$. The structure theorem for smooth Galois $(\mathbf{Z/2})^2$ covers was given in [**Cat2**] [pag. 491-493], including a calculation of their invariants, and a study of their deformation theory. There, smooth bidouble covers of $\mathbf{P}^1 \times \mathbf{P}^1$ were introduced in order to find interesting properties of the moduli spaces of surfaces of general type. Later, Hahn and Miranda ([**H-M**]) extended the structure theorem to the general case, and we present here this result in the nice formulation found by F. Schreyer. The general set up we are mostly interested in here is as follows: we have a bidouble cover $f : Y \rightarrow X$ where X is a smooth surface, Y is singular and we let $\pi : S \rightarrow Y$ be a minimal resolution of singularities. The main problem is to calculate the invariants of S in terms of the 3 branch curves D_1, D_2, D_3 and of the singularities of $D = D_1 \cup D_2 \cup D_3$ (observe that Y is smooth just when the 3 branch curves are smooth and have normal crossings). Thus, one obtains formulae where there is a global contribution (yielding the value of the invariants when Y is smooth) and a local contribution, depending on the local structure of the singularities of D. We mostly analyse the case where D has ordinary singularities, and thus we can attach to any point P of X a triple $[\mu_1(P), \mu_2(P), \mu_3(P)]$, where $\mu_i(P)$ is the multiplicity of D_i at P.

Once we obtain such formulae we convince ourselves that the most significant singularities are the following ones:

Singularity type	drop of K^2	drop of χ
(1, 1, 1)	$K^2 \downarrow 1$	$\chi =$ same,
(3, 1, 0)	$K^2 \downarrow 1$	$\chi \downarrow 1$,
(2, 2, 0)	$K^2 \downarrow 4$	$\chi \downarrow 1$.

The usefulness of imposing such singularities to the branch curve was best illustrated by Burniat's beautiful construction of surfaces with $p_g = 0$ and $K^2 = 6 - m$ $(m = 0, ...4)$ obtained by choosing a bidouble cover of the projective plane branched on three cubic curves having three $(3, 1, 0)$ points and m $(1, 1, 1)$ points (then the branch curve must be a union of lines).

Choosing smooth bidouble covers of the quadric Q $= \mathbf{P^1} \times \mathbf{P^1}$ has the advantage that we have 6 discrete parameters (giving the 3 bidegrees of the branch curves), upon which the two basic surface invariants χ and K^2 depend, but as one tries to fill in sectors of the surface geography chessboard a lot of gaps are encountered: however, imposing $(3, 1, 0)$ and $(1, 1, 1)$ points, gives chess moves which allow to fill these gaps, and hopefully to solve in an easier and better way the geography problem attacked by Persson in [**Per**] (who considered surfaces with a pencil of curves of genus 2, thus with non birational canonical and bicanonical map). This will be the object of a forthcoming article, with an eye also to the construction of simple canonical surfaces. In this respect, imposing $(1, 1, 1)$ points has the great advantage that if the most singular surface yields a simple canonical surface, then also the less singular ones achieve this goal ; whereas imposing $(3, 1, 0)$ points has the great advantage of improving the slope K^2/χ.

These examples are particularly interesting in order to analyse the equations of canonical surfaces in $\mathbf{P^r}$ for $r = 3, 4, 5$ (cf. [**C-S**] [**Cat5**]). Another application is towards the construction of surfaces with low invariants, classically done with the technique of singular double covers. For instance, surfaces with $p_g = q = 1$ have been classified by the author and also by E. Horikawa in the case $K^2 = 2$ (cf. [**Cat1**], [**Hor**]), and by the author and Ciliberto in the case $K^2 = 3$ ([**Ca-Ci1**], [**Ca-Ci2**]).

But for higher values of K^2, only a sporadic example was known, due to Xiao ([**Xiao1**]), having $K^2 = 4$ and Albanese map a pencil of curves of genus 2. Here we find new examples having $K^2 = 3, 4, 5$ and Albanese map also equal to a pencil of curves of genus 2. It was surprising that, just by recalling the theory of smooth bidouble covers and giving several numerical examples, we stomped on several noteworthy examples of families of algebraic surfaces.

We want to mention for instance example 2, containing some infinite series of surfaces for which the canonical map is composed of a pencil of curves of genus 2 and 3. The principal novelty here is that, unlike the previously known examples, due to Pompilj, Beauville, Xiao and Zucconi (cf. e.g. [**Bea**] , [**Xiao2**], [**M-Y**]) our examples provide fibrations with non constant moduli and with $q = 0$.

One interesting generalization of the theory would be to consider the case of singular Abelian covers (the smooth case having been treated by Comessatti [**Com**] in 1930, and later more thoroughly by Pardini [**Par1**]). This generalization should be viewed not only as a nice algebraic calculation, but pretty much in the spirit of Hirzebruch's beautiful constructions of algebraic surfaces as Kummer coverings ([**Hir**], [**B-H-H**]). Finally, another interesting direction of study which one may pursue is the investigation of the surgeries arising when the branch curves acquire more singularities, or become non reduced.

2. The basic model of a bidouble cover and the structure theorem

The simplest example of a bidouble cover is the mapping $f : \mathbf{P}^2 \to \mathbf{P}^2$ which is given by squaring the coordinates, i. e. $f(y_1, y_2, y_3) = (x_1^2, x_2^2, x_3^2)$.

The most convenient way is to think of it as a projection of the embedding of $\mathbf{P}^2$ to the Veronese surface V. The cone over the Veronese surface consists in the algebraic set of rank 1 (3×3) matrices, thus V is defined by the determinantal equation

$$rk \begin{pmatrix} x_1 & w_3 & w_2 \\ w_3 & x_2 & w_1 \\ w_2 & w_1 & x_3 \end{pmatrix} = 1. \qquad (*)$$

The isomorphism of $\mathbf{P}^2$ with V is given by setting

$$(2.1) \qquad \begin{aligned} x_i &= y_i^2 \\ w_k &= y_i \cdot y_j \end{aligned}$$

($\{i, j, k\}$ being a permutation of $\{1, 2, 3\}$).

The mapping f is then the projection $f : V \to \mathbf{P}^2$ given by the diagonal entries of the symmetric matrix.

$f : \mathbf{P}^2 \to \mathbf{P}^2$ is branched on the three coordinate lines $D_j = div(x_j)$, whereas the ramification divisor R on the source is given by the three coordinate lines $R_j = div(y_j)$, which represent the respective Fix point sets of the three non trivial elements σ_j in the Galois group.

The above toy model contains all the basic features of the general case, as illustrated by the following structure theorem which is proven in [**Cat2**] [pag. 491-493] and [**H-M**][pag. 27-29] (except for the last statements, not explicitly stated in these references, and for which we give hints of proof).

Especially, the toy model illustrates the profound difference between the birational and the biregular point of view: birationally, in the toy model we take the two square roots of the rational functions $x_1 x_2/x^2$, $x_1 x_3/x^2$ (x being a general linear form). But, biregularly, f is not a fibre product of two double coverings, only the normalization of such a fibre product given by equations

$u^2 = x_1 x_2, v^2 = x_1 x_3$.

Theorem 2. Let $f : Y \to X$ be a finite flat $(\mathbf{Z}/\mathbf{2})^2$- Galois cover with X an algebraic variety in char $(k) \neq 2$. Then there exist 3 Cartier divisors $D_j = div(x_j)$ on X and 3 line bundles L_i, with fibre coordinate w_i, such that Y is embedded in the total space of the vector bundle $L_1 \oplus L_2 \oplus L_3$, direct sum of the line bundles L_i. Moreover Y is defined there by the determinantal equation $(*)$. Conversely, assume that the above data satisfy the following linear equivalences on X,

- $L_i + D_i \equiv L_j + L_k$.
  for each permutation (i, j, k) of $(1, 2, 3)$:

then they determine a bidouble cover defined by the above determinantal equations $(*)$. One has a decomposition according to the characters of $(\mathbf{Z}/\mathbf{2})^2$:

- $f_* \mathcal{O}_Y \cong \mathcal{O}_X \oplus \bigoplus_i \mathcal{O}_X(-L_i)$.

Assume in addition that X is a smooth variety, then:

- Y is smooth if and only if the divisors D_j are smooth, they do not have a common intersection and have pairwise transversal intersections.

- Y is Cohen - Macaulay (this follows from the determinantal equations) and for its dualizing sheaf ω_Y (which, if Y is normal, equals the sheaf of Zariski differentials) we have (cf. [**H**], ex. III. 6. 10 and III. 7. 2)
  $f_*\omega_Y \cong \mathcal{H}om(f_*\mathcal{O}_Y, \omega_X) \cong \omega_X \oplus \bigoplus_i \omega_X(L_i)$.
- Y is normal if and only if the divisors D_j are reduced and have no common components (since by the Cohen - Macaulay property, normality is equivalent to non singularity in codimension 1).

Remark 1. The determinantal equation $(*)$ obviously implies some linear equivalences on X, namely (using the additive notation for Cartier divisors) that

- $2L_i \equiv D_j + D_k$
- $L_i + D_i \equiv L_j + L_k$.

But, (cf. the statement of the structure theorem) the first linear equivalences follows from the second ones: this shows that one can choose the L_i 's with the sole restriction that the linear system $|L_j + L_k - L_i|$ is non empty, and then pick up a divisor D_i in the system $|L_j + L_k - L_i|$.

Whereas the first equations show that the linear equivalence classes of the branch divisors cannot be chosen freely, being subject to $(mod 2)$ congruences.

Definition 1. A bidouble cover is said to be of *simple type* if there is a permutation of the indices (1, 2, 3) such that the divisor D_k is trivial (equivalently, $L_k \equiv L_i + L_j$).

Remark 2. Simple bidouble covers are useful for many applications, but especially in dimension higher than 2, where otherwise the condition that the three branch divisors have an empty intersection may be hard to fulfill.

From now on, we shall constantly assume that X is smooth.

Remark 3. Assume that Y is not normal, i. e., that either the divisors D_j are not reduced, or that they have a common component. Then the recipe for finding the normalization of Y is quite easy:

- if $D_i = D'_i + 2A$, replace D_i by D'_i and L_j by $L_j - A$ for $j \neq i$
- if there is a divisor A with $D_i \geq A$ for all i, replace each D_i by $D_i - A$ and each L_j by $L_j - A$
- if $D_i = D'_i + A$, $D_j = D'_j + A$, but $D_k - A$ is not effective, replace D_i by D'_i, D_j by D'_j, and D_k by $D_k + A$, accordingly replace L_k by $L_k - A$.

We then obtain a bidouble cover which is nonsingular in codimension 1, whence normal (cf. theorem 2).

Remark 4. The structure of bidouble covers is easier to understand locally, where all line bundles are trivial, and therefore one can take three square roots $y_1^2 = x_1, y_2^2 = x_2, y_3^2 = x_3$, which define a Galois $(\mathbf{Z/2})^3$-cover Z.

Y is the quotient of Z by the involution multiplying all the three functions y_i by -1.

In fact, the functions $w_i = y_j y_k$ are just needed to obtain a set of generators as $\mathcal{O}_X$-module of the ring $\mathcal{O}_Y$ of functions on Z invariant under the above involution. These observations show that if X is Gorenstein (e. g. smooth, as in our case), then also Z is Gorenstein: therefore Y is then 1/2-Gorenstein.

In particular, if we set $N = 2K_X + D = 2K_X + \sum_{j=1}^3 L_j$, $2K_Y$ is the Cartier divisor $f^*(N)$.

Observe moreover that in this context one verifies immediately the determinantal equations

$w_i^2 = x_j x_k$, $w_i w_j = w_k x_k$

which can be interpreted as giving the ring structure to the direct sum

$f_* \mathcal{O}_Y = \mathcal{O}_X \oplus \bigoplus_i \mathcal{O}_X(-L_i)$.

Assume now that our bidouble cover is smooth, or just Gorenstein and locally factorial. Then the above functions y_1, y_2, y_3 globalize to sections of line bundles on Y, since the Weil divisors

$R_i = div(y_i)$

are by hypothesis Cartier divisors.

We should point out the importance of these functions, since they are precisely the eigenfunctions for the action of the Galois group, indeed they satisfy

- $\sigma_i y_i = -y_i$
- $\sigma_k y_i = y_i$ for $k \neq i$.

Using those, we can write more precisely:

$f_* \omega_Y = y_1 y_2 y_3 \omega_X \oplus \bigoplus_i y_i \omega_X(L_i)$.

This means that, if $u_1, ..u_n$ are local coordinates on X, then the sections of ω_Y can be written, locally on X, uniquely as

$(y_1 y_2 y_3 f_0(u_1, ..u_n) + \sum_{j=1}^3 y_j f_j(u_1, ..u_n))\eta$,

$\eta = (du_1 \wedge .. \wedge du_n)(y_1 y_2 y_3)^{-1}$ being a local generator of ω_Y.

The above way of writing sections of the canonical sheaf is the most appropriate to analyse the behaviour of the canonical map, and in particular its base points.

In fact, one can properly understand it by analogy with the absolute case of direct image, that is, global sections: if $F = div(y)$ is the fixed part of a linear system $|D| = F + |M|$ on a variety Y, then $H^0(\mathcal{O}_Y(D) = yH^0(\mathcal{O}_Y(M)$.

An equivalent formula, which is used to prove the former one, and does not require the assumption that the ramification divisors R_i be Cartier, is:

$f_0(u_1, ..u_n)(du_1 \wedge .. \wedge du_n) + \sum_{j=1}^3 f_j(u_1, ..u_n))(du_1 \wedge .. \wedge du_n)(w_j)^{-1}$).

On the other hand, since $\eta^2 = (du_1 \wedge .. \wedge du_n)^2 (y_1 y_2 y_3)^{-2} = (du_1 \wedge .. \wedge du_n)^2 (x_1 x_2 x_3)^{-1} = (du_1 \wedge .. \wedge du_n)^2 (w_1 w_2 w_3)^{-1}$, setting as above

- $N = 2K_X + D = 2K_X + \sum_{j=1}^3 L_j$

  we obtain also the important formulae

- $H^0(\mathcal{O}_Y((2m+1)K_Y) =$
  $= y_1 y_2 y_3 H^0(\mathcal{O}_X(K_X + mN) \oplus \bigoplus_i y_i H^0(\mathcal{O}_X(K_X + mN + L_i)$,

  and
- $H^0(\mathcal{O}_Y((2m)K_Y) = H^0(\mathcal{O}_X(mN) \oplus \bigoplus_i w_i H^0(\mathcal{O}_X(mN - L_i)$.

  Please note also that, since $f^*(N) = 2K_Y$, we have
- $K_Y^2 = N^2$,

  whereas
- $p_g(Y) = p_g(X) + \sum_{i=1}^3 h^0(\mathcal{O}_X(K_X + L_i)$.

EXAMPLE 1. Let $X = Q = \mathbf{P^1} \times \mathbf{P^1}$: then we have three pairs of integers, the bidegrees (n_i, m_i) of D_i, or equivalently

the bidegrees (a_i, b_i) of L_i.

Notice that the n_i 's are either all even or all odd, and likewise the m_i's, and non negative, and $(a_i, b_i) = 1/2[(n_j, m_j) + (n_k, m_k)]$.

Let Y be a smooth bidouble cover of type $(n_1, m_1), (n_2, m_2), (n_3, m_3))$: then

$$\textit{setting } n = \sum_{j=1}^{3} n_j, m = \sum_{j=1}^{3} m_j, \tag{2.2}$$

$$\chi(Y) = \frac{1}{4}((n-4)(m-4) + \sum_{j=1}^{3} n_j m_j) \tag{2.3}$$

$$p_g(Y) = \sum_{j=1}^{3} max(0, (a_j - 1))max(0, (b_j - 1)) \tag{2.4}$$

$$K_Y^2 = 2(n-4)(m-4).$$

We are now interested to see which surfaces of general type with low p_g can we obtain. We see immediately that no surface with $p_g = 0$ can be thus obtained, whereas for $p_g = 1$ we get the following list

L_i = (1, 2) (2, 1) (2, 2) $K^2 = 2, p_g = 1, \chi = 2$ D_i = (3, 1) (1, 3) (1, 1)
L_i = (1, 2) (3, 1) (2, 2) $K^2 = 4, p_g = 1, \chi = 2$ D_i = (4, 1) (0, 3) (2, 1),
L_i = (1, 3) (3, 1) (2, 2) $K^2 = 8, p_g = 1, \chi = 2$ D_i = (4, 0) (0, 4) (2, 2).

where the first surfaces have been considered in [**C-D**].

For $p_g = 2$ there is already some overlapping for the values of K^2 and χ.

L_i = (1, 2) (a, 1) (3, 2) $K^2 = 2a, p_g = 2, \chi = 3$ D_i = (a+2, 1) (4-a, 3) (a-2, 1)
where $2 \leq a \leq 4$, so $K^2 = 4, 6, 8$
L_i = (0, 2) (3, 1) (3, 2) $K^2 = 4, p_g = 2, \chi = 2$ D_i = (6, 1) (0, 3) (0, 1),
L_i = (0, 3) (3, 1) (3, 2) $K^2 = 8, p_g = 2, \chi = 1$ D_i = (6, 0) (0, 4) (0, 2),
L_i = (1, 3) (a, 1) (3, 2) $K^2 = 4a, p_g = 2, \chi = 3$ D_i = (a+2, 0) (4-a, 4) (a-2, 2)
where $2 \leq a \leq 4$, so $K^2 = 8, 12, 16$
L_i = (1, t) (2, 2) (2, 2) $K^2 = 2t, p_g = 2, \chi = 3$ D_i = (3, 4-t) (1, t) (1, t)
where $1 \leq t \leq 4$, so $K^2 = 2, 4, 6, 8$

Remark 5. Observe that in the last case the canonical pencil has no fixed part, but it has $2t$ base points. Moreover let us point out that, in the fourth case, where the three branch divisors are divisible by 2, then (cf. prop. 2. 7 of [**Cat2**]) the fundamental group of Y is equal to $(\mathbf{Z/2})$, while in the other cases with $\chi = 3$ the surface can be shown to be simply connected.

Example 2. We give here series of surfaces of general type with canonical map composed of a pencil of genus $a = 2$ or $a = 3$. To this purpose, choose

L_i = (m+1, 2) (1, a) (m+k, 1) D_i = (k, a-1) (2m+k, 3-a) (2-k, 1+a) whence in particular $2 \leq a \leq 3, 0 \leq k \leq 2$, $K^2 = 2(a-1)(2m+k-2)$, $p_g = m$, $\chi = m+1$.

Notice that the fibres of the canonical pencil $\phi_1 : Y \to \mathbf{P^1}$ have non constant moduli.

Next, we proceed to search for canonical surfaces: in order to do this, we need to make a preliminary observation

Lemma 3. Let $f : Y \to X$ be a smooth bidouble cover, where $H^0(\omega_X) = 0$. Assume moreover that the canonical map ϕ_1 of Y is birational: then for each $i = 1, 2, 3$, $H^0(\omega_X(L_i)) \neq 0$.

Proof. Let y be a point of Y which does not map to the branch divisor D. If, say, $H^0(\omega_X(L_1)) = 0$ we take the element σ_1 in the Galois group, such that it takes the same value (-1) on the two characters χ_2, χ_3.

Then the image point $\phi_1(\sigma_1(y))$ is the same as $\phi_1(y)$ (since in projective space $v = -v$), contradicting the birationality of ϕ_1. □

Example 3. The first possibility for the canonical map ϕ_1 of Y to be birational is then the case where $p_g = 4$, and where moreover

L_i = (3, 2) (2, 2) (2, 2), $K^2 = 12, p_g = 4, \chi = 5$, D_i = (1, 2) (3, 2) (3, 2).

We shall verify that for a general choice of the branch divisors, ϕ_1 is indeed birational. It is also interesting to look at the other cases where $p_g = 4$ and the canonical map has a 2-dimensional image

L_i = (3, 2) (3, 2) (n, 1) $K^2 = 2(n+2), \chi = 5$ D_i = (n, 1) (n, 1) (6-n, 3)
where $0 \leq n \leq 6$
L_i = (3, 2) (3, 2) (1, m) $K^2 = 6m, \chi = 5$ D_i = (1, m) (1, m) (5, 4-m),
where $1 \leq m \leq 4$
L_i = (3, 2) (2, 3) (1, a) $K^2 = 4(1+a), \chi = 5$ D_i = (0, 1+a) (2, a-1) (4, 5-a),
where $1 \leq a \leq 5$
L_i = (4, 2) (2, 2) (n, 1) $K^2 = 2(n+2), \chi = 5$ D_i = (n-2, 1) (n+2, 1) (6-n, 3)
where $2 \leq n \leq 6$
L_i = (3, 3) (1, n) (m, 1) $K^2 = 2mn, \chi = 5$ D_i = (m-2, n-2) (m+2, 4-n) (4-m, n+2)
where $2 \leq n, m \leq 4$.

In the first two cases the image Σ_1 of the canonical map is a smooth quadric, ϕ_1 has $2n$, respectively $2m$ base points, and therefore degree 2, respectively $2m$.

In the third case ϕ_1 has $2 + 2a$ base points, and degree 2 onto a ruled surface Σ_1 of degree $1 + a$ whose equation can be easily written, as follows.

If one takes coordinates $(u_0, u_1)(t_0, t_1)$ on $X = Q = \mathbf{P^1} \times \mathbf{P^1}$ and takes into account that the canonical map is given by setting

$(z_0, z_1, z_2, z_3) = (y_1 u_0, y_1 u_1, y_2 t_0, y_2 t_1)$,

one obtains the following equations for Σ_1:

$x_1(z_2, z_3) = x_2(z_0, z_1, z_2, z_3)$.

In fact, $x_1(z_2, z_3) = (y_2^{a+1}) x_1(t_0, t_1) = (y_2^{a+1}) y_1^2 =$
$x_2(u_0, u_1, t_0, t_1)(y_2^{a-1}) y_1^2 = x_2(z_0, z_1, z_2, z_3)$.

In the fourth case, the canonical image Σ_1 is a quadric cone, ϕ_1 has $2n$ base points, and degree 2.

Finally, in the fifth case, the canonical system has a fixed part, equal to R_1, and the canonical map coincides with the bidouble cover f followed by the standard embedding of Q as a quadric.

We prove now the assertion about birationality of ϕ_1 made in the previous example. In doing so, we shall establish a quite general method and criterion, valid for all bidouble covers with $p_g = 4$.

Proposition 4. Let Y be a general bidouble cover of $X = Q = \mathbf{P^1} \times \mathbf{P^1}$, of type (1, 2) (3, 2) (3, 2): then the canonical map ϕ_1 of Y is birational. The canonical image Σ_1 does not have ordinary singularities.

Proof. Observe that the canonical map is given by setting

$(z_0, z_1, z_2, z_3) = (y_1 u_0, y_1 u_1, y_2, y_3)$. Therefore there are no base points and it suffices, to show the second assertion, to show that the map $\phi : R_1 \to \mathbf{P^1} = \{z_0 =$

$z_1 = 0\})$ has degree 8. To show this, we change coordinates on $Q = \mathbf{P^1} \times \mathbf{P^1}$ so that $D_1 = \{u_1 t_0^2 = u_0 t_1^2\}$, and we compose ϕ with the map which squares the coordinates on $\mathbf{P^1}$.

It follows that, since we have a degree 2 cover $R_1 \rightarrow D_1$, the degree of ϕ equals the degree of $\psi : D_1 \rightarrow \mathbf{P^1}$, where in affine coordinates $\psi(t^2, t) = (x_2(t^2, t), x_3(t^2, t))$. It is then clear that, varying x_2, x_3, we obtain all possible polynomials of degree 8.

To verify the birationality of the canonical map, let us consider two general points y, y' of Y. There are two cases to consider:

Case I: $f(y) = f(y')$.

Then there is an involution σ in the Galois group such that $y' = \sigma(y)$; moreover, since there are three non trivial eigenspaces in $H^0(\omega_Y)$, we can choose two corresponding characters χ, χ' such that $\chi(\sigma) = 1, \chi'(\sigma) = -1$. Letting s, s' be non zero sections in the corresponding eigenspaces, we obtain $(s(y'), s'(y')) = (s(y), -s'(y))$: if y, y' are general, then s, s' do not vanish at y and $\phi_1(y') \neq \phi_1(y)$. Thus we are reduced to consider

Case II: $f(y) = x \neq x' = f(y')$.

The case where x, x' do not belong to the same curve of the pencil $C_u = \{u_1/u_0 = u\}$ is immediate, therefore it suffices to show the birationality of the restriction of ϕ_1 to C_u. We have the following diagram:

$$\begin{tikzcd}
(C_u) \arrow[r, "\phi"] \arrow[d] & \mathbf{P^2} \arrow[d, "\beta"] \\
(L_u) \arrow[r, "\psi"] & \mathbf{P^2}
\end{tikzcd}$$

where ϕ is given by $(y_1 u_0, y_2, y_3)$, β is the standard bidouble cover obtained by squaring the coordinates, L_u is the curve image of C_u on X, and finally, then, ψ is given by $(x_1 u_0^2, x_2, x_3)$.

The content of Case I is that the map β, which has degree 4, remains also of degree 4 when restricted to the image of C_u, since indeed the Galois orbit of a general point has cardinality 4. The conclusion is that $deg\phi = deg\psi$, whence it is equivalent to show that the map ψ is birational onto its image.

Now, in this particular case, ψ is given, on $L_u = \mathbf{P^1}$, by three polynomials of degree 2, so the only possibility is that the image of L_u should be a line for each u. (In this case the image of $Q = \mathbf{P^1} \times \mathbf{P^1}$ under the map ψ_1 given by $(x_1 u_0^2, x_1 u_1^2, x_2, x_3)$ would be a sextic surface ruled by lines and with a 4-ple line).

We can directly show that this case does not occur for general choice of the branch loci, since on a general curve L_u we can fix the three divisors $D_i \cap L_u$ as we want, therefore the three sections (x_1, x_2, x_3) do not map L_u to a line. ☐

EXAMPLE 4. We restrict now to smooth covers Y of Q which have $p_g = 5, 6$ and fulfill the necessary condition for the birationality of ϕ_1 given in lemma 3.

$L_i = (4,\ 2)\ (2,\ 2)\ (2,\ 2) \quad K^2 = 16, p_g = 5, \chi = 6 \quad D_i = (0,\ 2)\ (4,\ 2)\ (4,\ 2)$
$L_i = (3,\ 2)\ (3,\ 2)\ (2,\ 2) \quad K^2 = 16, p_g = 5, \chi = 6 \quad D_i = (2,\ 2)\ (2,\ 2)\ (4,\ 2),$
$L_i = (2,\ 3)\ (3,\ 2)\ (2,\ 2) \quad K^2 = 18, p_g = 5, \chi = 6 \quad D_i = (3,\ 1)\ (1,\ 3)\ (3,\ 3).$

$L_i = (3,\ 3)\ (2,\ 2)\ (2,\ 2)$ $K^2 = 18, p_g = 6, \chi = 7$ $D_i = (1,\ 1)\ (3,\ 3)\ (3,\ 3)$
$L_i = (3,\ 2)\ (3,\ 2)\ (3,\ 2)$ $K^2 = 20, p_g = 6, \chi = 7$ $D_i = (3,\ 2)\ (3,\ 2)\ (3,\ 2)$,
$L_i = (4,\ 2)\ (3,\ 2)\ (2,\ 2)$ $K^2 = 20, p_g = 6, \chi = 7$ $D_i = (1,\ 2)\ (3,\ 2)\ (5,\ 2)$,
$L_i = (3,\ 2)\ (3,\ 2)\ (2,\ 3)$ $K^2 = 24, p_g = 6, \chi = 7$ $D_i = (2,\ 3)\ (2,\ 3)\ (4,\ 1)$
$L_i = (4,\ 2)\ (2,\ 3)\ (2,\ 2)$ $K^2 = 24, p_g = 6, \chi = 7$ $D_i = (0,\ 3)\ (4,\ 1)\ (4,\ 3)$.

The two surfaces with branch divisors of respective types ((1, 2) (3, 2) (5, 2)) and ((3, 2), (3, 2), (3, 2)) are homeomorphic but they belong to different irreducible components of the moduli space (cf. [**Cat2**] [p. 500] where it is shown that for the first surfaces the local dimension of the moduli space is at least 39, whereas for the second ones the local moduli space is smooth of dimension equal to 38).

The last calculations about local moduli spaces are based on the concept of *natural deformations of bidouble covers* which consists in modifying the determinantal equations to require the following matrix to have rank equal to 1:

$$\begin{pmatrix} x_1 + \gamma_1 w_1 & w_3 & w_2 \\ w_3 & x_2 + \gamma_2 w_2 & w_1 \\ w_2 & w_1 & x_3 + \gamma_3 w_3 \end{pmatrix} = 1 \qquad (*)$$

where γ_i is a section of $H^0(\mathcal{O}_X(D_i - L_i))$, for $i = 1, 2, 3$.

Example 5. Example (a, b, c) consists of two smooth simple covers Y, Y' of respective types ((2a, 2b), (2c, 2b)), and ((2a + 2, 2b), (2c - 2, 2b)).

By the previous formulae, these two surfaces have the same invariants $\chi(S) = 2(a+c-2)(b-1) + 4b(a+c)$, $K_S^2 = 16(a+c-2)(b-1)$.

If one moreover assumes that $a \geq 2c+1$, $a \geq b+2$, $c \geq b+2$, and that a, b, c are even and ≥ 3, one gets different irreducible components of the moduli space (this again follows directly from the results of [**Cat2**]).

3. Resolutions and Adjunction ideals

Throughout this section we assume that $\pi : S \longrightarrow Y$ is the minimal resolution of a normal bidouble cover $f : Y \longrightarrow X$.

Moreover, here we assume throughout that S, Y, X are surfaces, in order to simplify the treatment.

We recall first of all that, in view of the characterization of ω_Y as the sheaf of $2-$ forms regular on the smooth locus, one has immediately:

$\pi_* \omega_S \subset \omega_Y$

(in the Gorenstein case, ω_Y is invertible, therefore we can write

$\pi_* \omega_S = \mathcal{A}\ \omega_Y$, and $\mathcal{A}$ is called the adjunction ideal).

Recalling moreover that

$f_* \omega_Y = y_1 y_2 y_3 \omega_X \oplus \bigoplus_i y_i \omega_X(L_i)$,

we can write

$f_* \pi_* \omega_S = y_1 y_2 y_3 \mathcal{A}_0 \omega_X \oplus \bigoplus_i \mathcal{A}_i y_i \omega_X(L_i)$.

In order to obtain the explicit determination of the "adjunction" ideals $\mathcal{A}_i$, observe that, if u, v are local coordinates on X, then the sections of ω_S can be written, locally on X, uniquely as

$f_0(u,v)(du \wedge dv) + \sum_{j=1}^{3} f_j(u,v))(du \wedge dv)(w_j)^{-1}$).

Therefore the ideals $\mathcal{A}_j$ are precisely the ideals of the functions $f_j(u,v)$ for which we do not get on S a $2-$form with poles on the exceptional divisors.

It is therefore straightforward to observe that

$\mathcal{A}_0 = \mathcal{O}_X$.

In general, though, it happens that $\mathcal{A}_j$ is a proper ideal, and since Y is normal, it is an ideal of finite colength. What we shall now do will be to determine the ideals $\mathcal{A}_j$ for the singular points of D, and in particular calculate the colength

$l(\mathcal{O}_X/\mathcal{A}_i)$.

The interpretation of this calculation will be that the imposition of certain singularities on the branch curve D will produce a "drop" of the value of χ equal to $r = \sum_i l(\mathcal{O}_X/\mathcal{A}_i)$. Moreover, calculating the dimension of the vector spaces $H^0(\mathcal{A}_i\ \omega_X(L_i))$ one can compute p_g, whence also the irregularity q of S.

To calculate the contribution to the drop of K^2, since we are not sure whether the surface S is minimal, it is better to calculate the bigenus $P_2(S) = H^0(\omega_S^2)$: in fact, we have

$K^2 + \chi = \chi(\omega_S^2)$

and for minimal surfaces of general type, indeed holds

$K^2 + \chi = P_2$.

To this purpose, we need to calculate similar formulae (i. e., ideals of "biadjunction") for $f_*\pi_*\omega_S^2$.

We remind the reader that the local generator of (ω_Y^2) (in the sense of sheaves associated to Weil divisors) was found to be $\eta^2 = (du \wedge dv)^2(y_1y_2y_3)^{-2} = (du \wedge dv)^2(x_1x_2x_3)^{-1}$.

Whence we obtained

$H^0(\omega_Y^2) = H^0(\mathcal{O}_X(2K_X + D)) \oplus \bigoplus_i w_i H^0(\mathcal{O}_X(2K_X + D - L_i))$.

Therefore we obtain

$f_*\pi_*\omega_S^2 = \mathcal{B}_0(2K_X + D) \oplus \bigoplus_i w_i\mathcal{B}_i(2K_X + D - L_i)$,

and a rather immediate description of the ideals $\mathcal{B}_j$, as follows:

$f_0 \in \mathcal{B}_0$ iff $div(f_0) \geq div(x_1x_2x_3) - 2div(du \wedge dv)$

$f_i \in \mathcal{B}_i$ iff $div(f_i) \geq div(x_1x_2x_3) - 2div(du \wedge dv) - div(w_i)$, i. e.,

$f_i \in \mathcal{B}_i$ iff $div(f_i) \geq 2div(y_i) + div(y_jy_k) - 2div(du \wedge dv)$.

Also in this case the conclusion is that the sum of the colengths of the ideals $\mathcal{B}_j$ contributes to the drop of $\chi(\omega_S^2)$, whence, subtracting the drop of χ, we get the drop for K^2 of the minimal model (in case S is of general type); whereas, the biadjunction ideals do not necessarily impose independent conditions when S is not minimal.

We shall analyse now the case where D has ordinary singularities, therefore we can attach to any point P of X a triple $[\mu_1(P), \mu_2(P), \mu_3(P)]$, where $\mu_i(P)$ is the multiplicity of the curve D_i in P.

It will be convenient to write $\mu_i = \mu_i(P) = 2m_i + \nu_i$, where $0 \leq \nu_i \leq 1$.

We have four possible cases:

$$\begin{aligned} [\nu_1, \nu_2.\nu_3] &= (0,0,0) \\ [\nu_1, \nu_2.\nu_3] &= (1,1,1) \\ [\nu_1, \nu_2.\nu_3] &= (0,1,1) \\ [\nu_1, \nu_2.\nu_3] &= (1,0,0). \end{aligned}$$

On the blow up X' of X at P the full transforms of the D_i 's would not give a normal cover, therefore we apply the recipe indicated in remark 3, by which the multiplicities of E in the four cases can be respectively reduced to be (0, 0, 0), (0, 0, 0), (1, 0, 0), (1, 0, 0).

In the first two cases (since locally the blow up and the reduction process have brought us to the normal crossings condition) the resolution of singularities is obtained by taking the corresponding cover of X', which introduces a single smooth curve C, obtained as the preimage of the exceptional curve E.

The curve C has self intersection $= -4$, and its genus g satisfies

$2g-2 = -8 + 4\sum_{j=1}^{3} m_j$ in the first case,

respectively $2g-2 = -8 + 4\sum_{j=1}^{3} m_j + 6$ in the second case.

Thus $g = 2\sum_{j=1}^{3} m_j - 3$, resp. $g = 2\sum_{j=1}^{3} m_j$.

We calculate:

$(du \wedge dv)$	has order of zero on C $= 1$
x_i	has order of zero on C $= \mu_i$
$(du \wedge dv)/w_i$	has order on C $= 1 - (\mu_j + \mu_k)/2$
$f_i \in \mathcal{A}_i$	iff ord $f_i \geq (\mu_j + \mu_k - 2)/2$
$f_i \in \mathcal{B}_i,\ i \geq 1$	iff ord $f_i \geq \mu_i - 2 + (\mu_j + \mu_k)/2$
$f_0 \in \mathcal{B}_0$	iff ord $f_0 \geq \mu_i + \mu_j + \mu_k - 2$

Therefore in all these cases the adjoint and biadjoint ideals are just powers of the maximal ideal $\mathcal{M}_P$ of the point P. Since the colength of the ideal $\mathcal{M}_P^r$ equals $r(r+1)/2$, which fortunately vanishes for $r = 0, -1$, it follows that χ drops by the following amount:

$\sum_{i=1}^{3}(1/8)(\mu_j + \mu_k - 2)(\mu_j + \mu_k) = (1/8)[\sum_{i=1}^{3}(\mu_j + \mu_k)^2 - 2(\mu_j + \mu_k)] = (1/4)\sum_{i=1}^{3}(\mu_i)^2 - (1/2)\sum_{i=1}^{3}\mu_i + (1/4)\sigma_2(\mu) =$
$= ((1/2)\sum_{i=1}^{3}\mu_i)^2 - (1/2)\sum_{i=1}^{3}\mu_i - (1/4)\sigma_2(\mu)$

where σ_2 is the second elementary symmetric function.

For the calculation of how much K^2 drops, recall the formula

$r(r+1)/2 - s(s+1)/2 = (1/2)(r-s)(r+s+1)$.

Whence (apply the formula to $r = l(\mathcal{O}_X/\mathcal{B}_i), l = l(\mathcal{O}_X/\mathcal{A}_i)$) we obtain that K^2 drops by:

$(1/2)((\sum_{i=1}^{3}\mu_i) - 2)((\sum_{i=1}^{3}\mu_i) - 1) + (1/2)(\sum_{i=1}^{3}(\mu_i - 1)((\sum_{i=1}^{3}\mu_i) - 2)) =$
$= ((\sum_{i=1}^{3}\mu_i) - 2)^2$.

There remains now to treat the third and fourth cases. Here, after we perform a blow up of the point P, and the reduction process, it happens that the exceptional curve E is part of the branch locus, and that the branch locus does not consist of three smooth curves having normal crossings.

Indeed, by our choice, the first branching divisor D'_1 consists of the union of E and of the strict transform of D_1, while, for $i = 2, 3$, D'_i equals the strict transform of D_i.

Therefore the bidouble cover Z of X' corresponding to the branch divisors D'_i is normal with only $2\mu_1$ nodes as singularities, corresponding to the inverse images of the intersection points of E and of the strict transform of D_1.

But we can observe that Z is Gorenstein, and ω_Z is the direct image of ω_S under the minimal resolution of singularities of Z. Therefore everything works similarly to the first two cases: the major difference being that, since C is part of the ramification locus, then

$(du \wedge dv)$	has order on C $= 3$
x_i	has order of zero on C $= 2\mu_i$
$(du \wedge dv)/w_i$	has order on C $= 3 - \mu_j - \mu_k$
$f_i \in \mathcal{A}_i$	iff ord $f_i \geq (\mu_j + \mu_k - 3)/2$
$f_i \in \mathcal{B}_i,\ i \geq 1$	iff ord $f_i \geq \mu_i - 3 + (\mu_j + \mu_k)/2$
$f_0 \in \mathcal{B}_0$	iff ord $f_0 \geq \mu_i + \mu_j + \mu_k - 3$

Moreover, the numbers $(\mu_j + \mu_k)/2$ can just be half integers.

This remark gives rise to a simplification of the situation, in the sense that the third and fourth cases give rise (with the exception of the third case with $\mu_1 = 0$) to singularities which are not worth their price!

PROPOSITION 5. Given an ordinary singularity of the branch curve D of type $(2m_1+1, 2m_2, 2m_3)$, the adjoint and biadjoint ideals are the same as for a singularity of type $(2m_1, 2m_2, 2m_3)$. In particular, χ and K^2 drop exactly by the same amounts as for a singularity of type $(2m_1, 2m_2, 2m_3)$. Similarly an ordinary singularity of the branch curve D of type $(2m_1, 2m_2+1, 2m_3+1)$ has the same adjoint and biadjoint ideals as a singularity of type $(2m_1 - 1, 2m_2 + 1, 2m_3 + 1)$. In the latter case, when $m_1 = 0$, this means that the drops of χ and K^2 are given by the same formula as in the first two cases, but with multiplicities $(-1, \mu_2, \mu_3)$, i. e.,

χ drops by $(1/4)[(\mu_2 + \mu_3 - 1)^2 - (\mu_2 + \mu_3) + 2 - \mu_2\mu_3]$,
while K^2 drops by $[(\mu_2 + \mu_3 - 3)^2$.

PROOF. Note that in both cases we replace a triple (μ_1, μ_2, μ_3) where the μ_i's do not have the same parity by a new triple where the μ_i's have the same parity, but the sum $(\mu_1 + \mu_2 + \mu_3)$ goes down by 1.

Therefore, the ideal $\mathcal{B}_0$ does not change. We also notice that $r = (\mu_1 + \mu_2 + \mu_3) - 3$ is always at least -1, since we can disregard the case $(1, 0, 0)$ (it yields no singular point) ; so the calculation that $r(r+1)/2$ conditions are imposed is correct (for $r = -2$ we would obtain a positive number instead of 0).

The same reasoning yields that the ideals $\mathcal{A}_i$, for $i = 2, 3$, and the ideal $\mathcal{B}_1$ do not change. In the case of $\mathcal{A}_1$, we may observe that $(\mu_2 + \mu_3)$ is an even integer, whence an integer d is at least $(\mu_2 + \mu_3 - 3)/2$ iff it is at least $(\mu_2 + \mu_3 - 2)/2$. A completely similar argument works for $\mathcal{B}_i$ when $i = 2, 3$.

□

We can summarize our results as follows:

μ_1, μ_2, μ_3 of the same parity	
K^2 drops by	$((\sum_{i=1}^{3} \mu_i) - 2)^2$
χ drops by	$((1/2)\sum_{i=1}^{3} \mu_i)^2 - (1/2)\sum_{i=1}^{3} \mu_i - (1/4)\sigma_2(\mu)$
μ_1, μ_2, μ_3 of different parity	
K^2 drops by	$((\sum_{i=1}^{3} \mu_i) - 3)^2$
χ drops by	$(1/4)[(\sum_{i=1}^{3} \mu_i)^2 - 3(\sum_{i=1}^{3} \mu_i) - \sigma_2(\mu) - \mu_1 + 3]$.

For the effective construction of algebraic surfaces, it is convenient to have at hand a numerical table of the numerical changes of the invariants χ and K^2 for singularities of small multiplicities ; clearly, in view of proposition 5, we will not consider the triples of three strictly positive multiplicities having different parities, and also those (like $(0, 1, 2)$) which do not affect any of the two invariants. We will moreover restrict ourselves to the cases where the drop of K^2 is at most 25 (recall that this drop is a perfect square): this means, in terms of the sum of the multiplicities, $(\sum_{i=1}^{3} \mu_i) \leq 7$, resp. ≤ 8 in the case where $\mu_1 = 0, \mu_2, \mu_3$ are odd.

Singularity type	drop of K^2	drop of χ
(1, 1, 1)	$K^2 \downarrow 1$	$\chi =$ same,
(0, 1, 3)	$K^2 \downarrow 1$	$\chi \downarrow 1$,
(2, 2, 0)	$K^2 \downarrow 4$	$\chi \downarrow 1$,
(0, 0, 4)	$K^2 \downarrow 4$	$\chi \downarrow 2$,
(3, 1, 1)	$K^2 \downarrow 9$	$\chi \downarrow 2$,
(3, 3, 0)	$K^2 \downarrow 9$	$\chi \downarrow 3$,
(0, 1, 5)	$K^2 \downarrow 9$	$\chi \downarrow 4$,
(2, 2, 2)	$K^2 \downarrow 16$	$\chi \downarrow 3$,
(4, 2, 0)	$K^2 \downarrow 16$	$\chi \downarrow 4$,
(0, 0, 6)	$K^2 \downarrow 16$	$\chi \downarrow 6$,
(3, 3, 1)	$K^2 \downarrow 25$	$\chi \downarrow 5$,
(0, 3, 5)	$K^2 \downarrow 25$	$\chi \downarrow 7$,
(0, 1, 7)	$K^2 \downarrow 25$	$\chi \downarrow 9$,

REMARK 6. Of course, one can also search for non ordinary singularities. For instance, if we have that the multiplicities of the D_i's are $(1,1,1)$ and they have a common tangent which gives an ordinary singularity of type $(1,1,1)$ on the blow up, then $K^2 \downarrow 2$, $\chi =$ same.

If instead the multiplicities are $(2,1,1)$ and D_2, D_3 have a common tangent while D has at least two distinct tangents, then $K^2 \downarrow 1$, $\chi =$ same.

Finally, if the multiplicities are $(2,1,1)$, all the tangents are equal, then $K^2 \downarrow 8$ $\chi \downarrow 2$.

This means that the first singularity is a superposition of two ordinary singularities of type $(1,1,1)$, the second is a useless degeneration of this singularity, the third could be convenient.

We end this section with a useful

LEMMA 6. Assume that we have a singular bidouble cover $f : Y \to X$ where $p_g(X) = 0$. Let P be a point of type $(1,0,3)$ for D: then, on the desingularization S of Y, ϕ_1 has no base points on the elliptic curve $\mathcal{E}$ which is the inverse image of P in S, if the following conditions are satisfied

- there is a section f_1 of $\mathcal{A}_1\omega_X(L_1)$ not vanishing at P, and
- there is a section f_2 of $\mathcal{A}_2\omega_X(L_2)$ vanishing simply at P but with tangent direction different from the one of the divisor $D_1 = div(x_1)$.

If instead Q is a point of type $(1,1,1)$ for D, and A is the rational curve lying over Q in the resolution S of Y, ϕ_1 has no base points on A if there are at least two sections f_i, f_j of $\mathcal{A}_i\omega_X(L_i)$, resp. $\mathcal{A}_j\omega_X(L_j)$ not vanishing at Q.

PROOF. Note that $\mathcal{E}^2 = -1$ and $K_S\mathcal{E} = 1$; observe moreover that we are looking on S at the linear system generated by the divisors of the 2−forms $f_i(du \wedge dv)/w_i$.

In the case of the point P of type $(1,0,3)$, $f_1(du \wedge dv)/w_1$, if f_1 does not vanish at P, yields a divisor which does not contain $\mathcal{E}$, but contains the divisor R_1, thus this divisor meets $\mathcal{E}$ in the two points corresponding to the inverse image of the tangent direction of $D_1 = div(x_1)$. Whereas $f_2(du \wedge dv)/w_2$ if f_2 vanishes simply at P, yields also a divisor which does not contain $\mathcal{E}$, but contains the divisor R_2, and the inverse image F_2 of the proper transform of $div(f_2)$. We are done since $R_1 \cap R_2 \cap \mathcal{E} = \emptyset$, and moreover $R_1 \cap F_2 \cap \mathcal{E} = \emptyset$ under our assumption on the tangent of $div(f_2)$.

In the case of the point Q of type $(1,1,1)$, we use the fact that $R_i \cap R_j \cap A = \emptyset$. □

4. Simple canonical surfaces

In this section we shall give some existence results for simple canonical surfaces with $p_g = 4$ and $K^2 \geq 11$. Our results are not yet extensive, but we present the basic methods in order to obtain a large segment of values for high K^2. There is the following strategy principle which is best illustrated in the next theorem. Since the $(1,1,1)$ points do not give adjoint conditions, assume that we get a connected family of singular bidouble covers yielding equisingular deformations of all the singularities of the branch curves, with the exception of a certain number m of $(1,1,1)$ points each deforming to a triple of double points: if the canonical map is birational for the resolution of a singular cover Y^o with m $(1,1,1)$ points, then it is still birational for all the surfaces which occur as resolutions of some sufficiently small deformation of Y^o.

Theorem 7. There do exist algebraic surfaces with geometric genus $p_g = 4$, $K^2 = 11, 12...17$, $q = 0$, and with birational canonical map ϕ_1.

Proof. Let, once again, $X = Q = \mathbf{P^1} \times \mathbf{P^1}$: and take three divisors D_1, D_2, D_3 of respective bidegrees $(1,3)(3,1)(3,3)$. Then the respective bidegrees of L_1, L_2, L_3 are $(3,2)(2,3)(2,2)$, thus for a smooth such bidouble cover one would have $K^2 = 18$, $p_g = 5$ and $q = 0$. We consider the case where D_3 has a triple point P, $P \in D_1$, (whence P gives then a $(1,0,3)$ point of D) and moreover we have $0 \leq m \leq 6$ $(1,1,1)$ points of D. If we take coordinates $(u_0, u_1)(t_0, t_1)$ on $X = Q = \mathbf{P^1} \times \mathbf{P^1}$, such that P is the point $u_1 = t_1 = 0$, we have that the canonical map of the minimal resolution S of Y is given by $(y_1 u_0, y_1 u_1, y_2 t_1, y_3)$, since the adjoint ideal $\mathcal{A}_2$ is the maximal ideal of the point P. We may also rewrite $(y_1 u_0, y_1 u_1, y_2 t_1, y_3) = (w_3 u_0, w_3 u_1, x_2 t_1, w_1)$ and thus notice that ϕ_1 is induced by a family of rational maps defined on the vector bundle where Y is embedded. Clearly these rational maps have as base points on Y exactly the point P' corresponding to the $(1,0,3)$ point P $(y_1 = t_1 = y_3 = 0)$ and the $(1,1,1)$ points where $y_1 = y_2 = y_3 = 0$.

We observe preliminarly that ϕ_1 has no base points on S. This follows in fact from lemma 6, which ensures that there are no base points on the exceptional curves (for the other points, it is evident).

Let us then consider the following family of branch divisors, given, in affine coordinates $t = t_1/t_0, u = u_1/u_0$ as follows:

$D_3 = div[u^3(1-t^2) + u^2(b_1 t + b_2 t^2 + b_3 t^3) + u(c_2 t^2 + c_3 t^3) - (b_3 + c_3)t^3]$,

so that D_3 has an ordinary triple point at P with tangent distinct from $t = 0$, then

$D_2 = div(t_1 u_0 (u_1 - u_0)(u_1 + u_0)$

so that D_2 intersects D_3 at P and at 9 points

$(u = \infty, t = \infty)$, $(u = \infty, t = 1)$, $(u = \infty, t = -1)$, $(u = 1, t = \infty)$, $(u = 1, t = \alpha)$, $(u = 1, t = \beta)$,

$(u = -1, t = \gamma)$, $(u = -1, t = \delta)$, $(u = -1, t = \eta)$, such that (for general choice of the parameters, as it is easy to verify) the seven numbers

$1, -1, \alpha, \beta, \gamma, \delta, \eta$ are all distinct.

We consider now the divisors D_2, D_3 as fixed, and we consider a family of divisors D_1 given as follows

$D_1 = div[(t_0 - \lambda t_1)(\mu t_0 - t_1) f(u,t)]$

where $f(u,t)$ is a general equation of bidegree $(1,1)$ (corresponding to the choice of a plane in 3-space).

Clearly, for $\lambda = 0, \mu = \alpha$ there is an appropriate choice f^o of f such that we get 6 points of type $(1,1,1)$ (i. e., we take a plane passing through three of the remaining six points). Moreover, in a neighbourhood of this point in the parameter space for (λ, μ, f) we obtain bidouble covers with any number $0 \leq m \leq 6$ of $(1,1,1)$ points. There remains therefore, since birationality is an open property, to show that the canonical map for the minimal resolution S^o of Y^o is birational.

This last fact is indeed obvious since we have seen that there are no base points and $K^2_{S^o} = 11$, a prime number.

□

Remark 7. The reader might have noticed that indeed the previous construction could easily be pushed to yield also canonical surfaces with $8 \leq K^2 \leq 10$. We did not treat this case since $8, 9, 10$ are not prime numbers, thus the proof would have been more complicated (but it can be done along the lines of proposition 4 and Theorem 9). A natural question, however, would be whether these surfaces belong to the previously known families of canonical surfaces in those degrees, for instance to the closure of the ones considered by Ciliberto in [**Cil1**].

The next theorem goes entirely along the lines of the foregoing one, by a specialization argument and by the fact that 17 is again a prime number.

Theorem 8. There do exist algebraic surfaces with geometric genus $p_g = 4, q = 0$ and $17 \leq K^2 \leq 22$, having a birational canonical map ϕ_1.

Proof. We consider as base surface $X = Q = \mathbf{P^1} \times \mathbf{P^1}$ and bidouble covers Y with the D_j's of respective bidegrees $(2,3),(2,3),(4,1)$. Thus the L_i's have bidegrees $(3,2),(3,2),(2,3)$ and if Y were smooth, it would have $K^2_Y = 24$ and $p_g = 6$, by the formulae we have previously given.

We shall impose two points of respective types $(3,1,0),(1,3,0)$ (thus they both lie in $D_1 \cap D_2$, and moreover $0 \leq m \leq 5$ points of type $(1,1,1)$ on D.

We first fix the choice of D_1, D_2.

We set D_1 to be the sum of the line $\{t_1 = 0\}$ and of two irreducible plane sections π_1, π_2 of the quadric Q. π_1 is chosen to pass through the points $(0,0),(\infty,\infty)$, π_2 is chosen to pass through the points $(0,0),(u_2,\infty)$.

Similarly, we let D_2 be the sum of the line $\{t_0 = 0\}$ and of two irreducible plane sections π_3, π_4 of Q, where π_3 passes through the points $(0,0),(\infty,\infty)$, π_4 passes through $(\infty,\infty),(u_1,0)$.

It is easy to verify that, for the 6 points given by $(u_1,0),(u_2,\infty)$ and by the four intersection points $\pi_i \cap \pi_j$, the values of the first, resp. second, coordinate are all distinct.

It suffices therefore to take a divisor D_3 of type $(4,1)$ consisting of 5 lines passing through any required number $0 \leq m \leq 5$ of these points.

For $m = 5$ we get a resolution S which has $K^2_S = 17$, and for which, by lemma 6 again, the canonical map ϕ_1 is base point free. Therefore ϕ_1 is then birational, and the theorem follows by the usual specialization argument.

□

We continue by producing an example of a canonical surface in $\mathbf{P^3}$ with very high degree, $K^2 = 28$, for which we give a direct proof of birationality of the canonical map.

THEOREM 9. There do exist algebraic surfaces with geometric genus $p_g = 4$ and $K^2 = 28, q = 0$, having a birational canonical map ϕ_1.

PROOF. We consider as base surface $X = \mathbf{P^1} \times \mathbf{P^1}$ and bidouble covers Y with the D_j's of respective bidegrees $(4,2),(2,4),(2,2)$. Thus the L_i's have bidegrees $(2,3),(3,2),(3,3)$ and if Y were smooth, it would have $K_Y^2 = 32$ and $p_g = 8$. We shall impose four points of type $(0,1,3)$, one on $D_1 \cap D_3$, one on $D_2 \cap D_3$, two on $D_1 \cap D_2$: thus we get surfaces with $K^2 = 28$ and $p_g = 4$.

Choose coordinates $(x_0, x_1), (y_0, y_1)$ on $X = \mathbf{P^1} \times \mathbf{P^1}$, and let z_1, z_2, z_3 be then the equations of the ramification divisors on Y (so, $div(z_i^2) = D_i$).

We let $D_1 = div(y_0 x_1 (x_1 - x_0)[x_1 y_1 (x_1 - x_0) + y_0 x_0 b_1(x)])$, where $b_1(x)$ is a linear function in (x_0, x_1).

Similarly, we let $D_2 = div(x_0 y_1 (y_1 - y_0)[x_1 y_1 (y_1 - y_0) + x_0 y_0 n_1(y)]$, where $n_1(y)$ is a linear function in (y_0, y_1), finally we let $D_3 = div(G(x,y))$ where the polynomial G of bidegree $(2,2)$ vanishes on the point $P_{1,\infty}$ which is a triple point for D_1, and on the point $P_{\infty,1}$ which is a triple point for D_2. In particular, the equation of G can be written as follows:

$G(x,y) = x_0 y_0 \alpha(x,y) + (x_1 - x_0)(y_1 - y_0)\gamma(x,y)$, where α, γ are homogeneous of bidegree $(2,2)$.

The reader will notice that $P_{\infty,0}$ and $P_{0,\infty}$ are points of respective multiplicities $(1,3,0),(3,1,0)$.

Then the canonical map of the resolution S of Y is given by:

$(z_1(y_1 - y_0), z_2(x_1 - x_0), z_3 x_0 y_0, z_3 x_1 y_1)$.

The proof proceeds now as for proposition 4: we consider the pencil L_λ on $X = \mathbf{P^1} \times \mathbf{P^1}$ given by $L_\lambda = div(\lambda_0 x_0 y_0 + \lambda_1 x_1 y_1)$.

Let moreover C_λ be the inverse image of L_λ on S. The pencil L_λ is a pencil of conics on the quadric, with base points $P_{\infty,0}$ and $P_{0,\infty}$. The divisors cut by the branch loci on the general L_λ are, respectively

$3P_{0,\infty} + P_{\infty,0} + B'$, $P_{0,\infty} + 3P_{\infty,0} + N'$, G', where B', N' have degree 2, but G' has degree 4.

The conclusion is that C_λ has genus $= 5$, indeed $C_\lambda^2 = 0$, $C_\lambda K_S = 8$.

Since the canonical system has no fixed part, there are a priori only three possibilities:

i) the restriction ϕ of ϕ_1 to a general C_λ is birational, whence ϕ_1 is birational, or

ii) ϕ has degree 2, whence deg $\phi_1 = 2$

iii) ϕ is a degree 4 covering of a conic Q_λ, whence deg $\phi_1 = 4$.

Case ii) leads to a contradiction as follows.

There is then an involution σ of which the canonical map is composed, thus σ is central in $Aut(S)$.

Since ϕ_1 separates the four points of a general orbit of $(\mathbf{Z/2})^2$, we get an action of $(\mathbf{Z/2})^3$ on S, whence an action of $(\mathbf{Z/2})$ on $X = \mathbf{P^1} \times \mathbf{P^1}$. Such involution τ on X must leave the three branch divisors invariant, since they are the images of the fixed point sets of the non trivial involutions in $(\mathbf{Z/2})^2$. Moreover, τ is biregular on the blow up of X in the four essential singularities of the branch divisor (since it comes from an involution which is biregular on S). Let us denote by Z such a blow up: it is a del Pezzo surface, and the anticanonical system, which has projective dimension equal to 4, is free from base points and contracts exactly two divisors, Δ_x and Δ_y, Δ_x being the proper transform of the line through $P_{\infty,1}$ and $P_{\infty,0}$, while Δ_y is the proper transform of the line through $P_{1,\infty}$ and $P_{0,\infty}$.

We want to derive a contradiction by showing that τ is already biregular on X.

To this purpose we look then at the lines on Z (for the anticanonical mapping), i. e., the irreducible curves C with $CK_Z = -1 = C^2$ (since these are preserved under any automorphism). Among the lines are the exceptional divisors E_i of the blow up; the other lines (and also the curves contracted by $|-K_Z|$) have class $a\Delta_x + b\Delta_y + \sum_{i=1}^4 (2 - r_i)E_i$, and since they are contained in a two dimensional system of hyperplanes, it follows that $a, b \leq 1$, and clearly $0 \leq r_i$.

The diophantine equations

$\sum_{i=1}^4 r_i^2 = 1 + 2ab \leq 3$

$\sum_{i=1}^4 r_i = 2(a + b) - 1 \leq 3$

show that $r_i \leq 1$ and that the only other lines on Z, beyond the four we already considered, are the proper transforms of the four lines on X passing exactly through one of the blown up points. In fact, the solutions with $a = b = 1$ and three $r_i = 1$ should be discarded, since they correspond to planes on the quadric X passing through three of the blown up points, whence they give reducible curves.

The conclusion is now that τ preserves the four lines which meet Δ_x, Δ_y, that is, the four exceptional curves E_i of the blow up. After composing with an automorphism of X, we get a τ' with $\tau'(\Delta_x) = \Delta_x$, $\tau'(\Delta_y) = \Delta_y$ and $\tau'(E_i) = E_i$.

But then it follows that τ is an automorphism of X.

Since the branch divisors are left invariant, it follows easily that τ does not exchange the two factors, leaves the four essential singularities fixed, whence it is the identity, a contradiction.

Let us now consider case iii), where ϕ is a degree 4 covering of a conic Q_λ.

Then Q_λ is irreducible, and its equation is an eigenvector for the Galois group, therefore it must have the form

$\sum_{i=1}^3 a_i(\lambda) t_i^2$

where ϕ is given by (t_1, t_2, t_3) with the t_i's eigenvectors for the Galois group.

In fact, if the equation were not invariant, it would have the form $t_i t_j$, against the irreducibility of Q_λ.

We obtain therefore a rational map

$\lambda \to (a_1(\lambda), a_2(\lambda), a_3)(\lambda)$ such that

$a_1(\lambda)G_1(y_1 - y_0)^2 + a_2(\lambda)G_2(x_1 - x_0)^2 + a_3(\lambda)G_3 = 0$,

where $D_i = div(G_i)$, and G_3 equals our previous G.

Since $(\lambda_0, \lambda_1) = (x_1 y_1, x_0 y_0)$,

we may get rid of denominators in the rational functions and substitute. We obtain an equation of the form

$A_1(x_1 y_1, x_0 y_0)G_1(y_1 - y_0)^2 + A_2(x_1 y_1, x_0 y_0)G_2(x_1 - x_0)^2$
$+ A_3(x_1 y_1, x_0 y_0)G_3 = 0$,

where $deg A_1 = deg A_2 = m$, $deg A_3 = m+2$ and m is minimal with this property.

We observe now that x_0 divides G_2, therefore we obtain:

$A_1 G_1 (y_1 - y_0)^2 = -A_3 G_3 \ (mod(x_0))$.

Now, we can vary $G = G_3$ in a linear system so that the intersection with $div(x_0)$ moves: we get a contradiction unless x_0 divides both A_1, A_3.

By the special form of these polynomials, it follows that indeed x_0y_0 divides both A_1, A_3.

But then y_0 divides A_2, and by the same principle x_0y_0 divides A_2. The conclusion is that x_0y_0 divides A_1, A_2, A_3 against the minimality of m, and we are done. ☐

We have given above a geometric proof, but one can indeed mimic the idea of proof of Theorem 6 (replacing the prime number 11 by 23) and obtain also the following result:

THEOREM 10. There do exist algebraic surfaces with geometric genus $p_g = 4$, $23 \leq K^2 \leq 28$, having a birational canonical map ϕ_1.

PROOF. We consider everything, $X = \mathbf{P^1} \times \mathbf{P^1}$, the D_j's of bidegrees

$(4,2), (2,4), (2,2)$, the L_i's of bidegrees $(2,3), (3,2), (3,3)$, as in the previous theorem, except that we specialize the branch curves so as to obtain, beyond the given four points of type $(0,1,3)$, also r points of type $(1,1,1)$.

We choose coordinates $(x_0, x_1), (y_0, y_1)$ on $X = \mathbf{P^1} \times \mathbf{P^1}$ exactly as before, and calculate the further intersection points of

$D_1 = div(y_0x_1(x_1 - x_0)[x_1y_1(x_1 - x_0) + y_0x_0b_1(x)]$ with

$D_2 = div(x_0y_1(y_1 - y_0)[x_1y_1(y_1 - y_0) + x_0y_0n_1(y)]$.

Note that $n_1(y)$, $b_1(x)$ are linear functions: therefore our points are, beyond

$P_{\infty,\infty}$, $P_{1,1}$, $P_{0,1}$, $P_{1,0}$, $P_{0,0}$, the points $P_{0,\infty}$, $P_{o,n}$, P_{1,ν_1}, P_{1,ν_2}, $P_{\infty,0}$, $P_{b,0}$, $P_{\mu_1,1}$, $P_{\mu_2,1}$,

and the two points which are solutions of the system

$n_1(y)(x_1 - x_0) = b_1(x)(y_1 - y_0)$,

$[x_1y_1(x_1 - x_0) + y_0x_0b_1(x)] = [x_1y_1(y_1 - y_0) + x_0y_0n_1(y)] = 0$.

Our notation above is aimed to be almost self explanatory: the point b is the root of $b_1(x)$, n is the root of $n_1(y)$, ν_1, ν_2 are the roots of $y_1^2 - y_1y_0 + y_0n_1(y)$, while μ_1, μ_2 are the roots of $x_1(x_1 - x_0) + x_0b_1(x)$.

We choose $D_3 = div(G(x,y))$, where

$G(x,y) = x_0y_0\alpha(x,y) + (x_1 - x_0)(y_1 - y_0)\gamma(x,y)$, and in order to let it pass through

$0 \leq r \leq 5$ of these points, we impose one after the other the following specialization conditions:

I) $P_{\infty,\infty} \in D_3$:

this amounts to $\gamma(x,y) = x_0\gamma_2(y) + y_0\gamma_1(x)$

II) $P_{0,1} \in D_3$:

this amounts to $\alpha(x,y) = x_1\alpha_2(y) + (y_1 - y_0)\alpha_1(x)$

III) $P_{1,0} \in D_3$:

this amounts to $\alpha(x,y) = a_1x_1y_1 + a_2(y_1 - y_0)(x_1 - x_0)$, since $P_{0,1}$ and $P_{1,0}$ form the complete intersection $x_1y_1 = (y_1 - y_0)(x_1 - x_0) = 0$.

IV) $P_{b,0} \in D_3$:

this amounts to b being a root of (set $y_0 = 1, y_1 = 0$) the polynomial

$(a_2 + \gamma_2(1,0))x_0 + \gamma_1(x)$. This condition is easily solved by setting

$b_1(x) = (a_2 + \gamma_2(1,0))x_0 + \gamma_1(x)$.

V) $P_{0,n} \in D_3$:

this amounts to n being a root of (set $x_0 = 1, x_1 = 0$) the polynomial

$(\alpha_1(1,0) - \gamma_1(1,0))y_0 - \gamma_2(y)$. This condition is easily solved by setting

$n_1(y) = (\alpha_1(1,0) - \gamma_1(1,0))y_0 - \gamma_2(y)$.

At this point it is easy to verify that once we have imposed further $r = 5$ points of type $(1, 1, 1)$, namely $(P_{\infty,\infty}, P_{0,1}, P_{1,0}, P_{b,0}, P_{0,n})$, for general choice of the remaining parameters we get a divisor D with ordinary singularities.

It follows as in theorem 6 that we get a surface S^o with $|K_{S^o}|$ free from base points, and with $K^2 = 23$.

Therefore our surface S^o has a birational canonical morphism, and the same holds for all the surfaces S with $K^2 = 24, ...28$ which are the minimal resolutions of singular bidouble covers in a suitable neighbourhood of Y^o.

□

5. Burniat's surfaces and Burniat type surfaces

This short section is dedicated to some constructions of surfaces of general type with $p_g = 0$, or, when the singular points do not impose independent conditions, to surfaces with $p_g = q = 1$ with $K^2 = 3, 4, 5$ and Albanese map equal to a pencil of genus 2 curves. We call all these Burniat type surfaces, since in the book by Barth Peters and van de Ven [**B-P-V**] the name of Burniat surfaces refers precisely to certain surfaces S of general type with $p_g = 0$ and $K^2 = 6 - m, 0 \leq m \leq 4$ constructed by Paul Burniat as resolutions of singular bidouble covers of the plane (cf. [**Bur2**]). A more modern exposition was later given by Peters in [**Pet**], but unfortunately only based on the theory of double coverings. Let us see then how the Burniat surfaces occur rather canonically in the framework of bidouble covers.

Example 6. Consider $X = \mathbf{P}^2$ and consider a bidouble cover Y of type $(3, 3, 3)$ (in this cases the degrees of the L_i's and D_j's are all equal to 3). If Y is smooth, then it has $K_Y^2 = 9$ and $p_g = 3$, by our standard formulae.

We would like to make p_g drop by 3, but K^2 by at most 8. A quick inspection of the table shows that this is only possible if we impose exactly three $(0, 1, 3)$ points and then we can try to further impose m $(1, 1, 1)$ points, where m should be at most 5. But if a cubic curve C has a triple point, then it is a union of three lines. Therefore must be $D_1 = A_1 + A_2 + A_3$, where the three lines go through a point a, and similarly $D_2 = B_1 + B_2 + B_3$ consists of three lines through b, $D_3 = C_1 + C_2 + C_3$ consists of three lines through c.

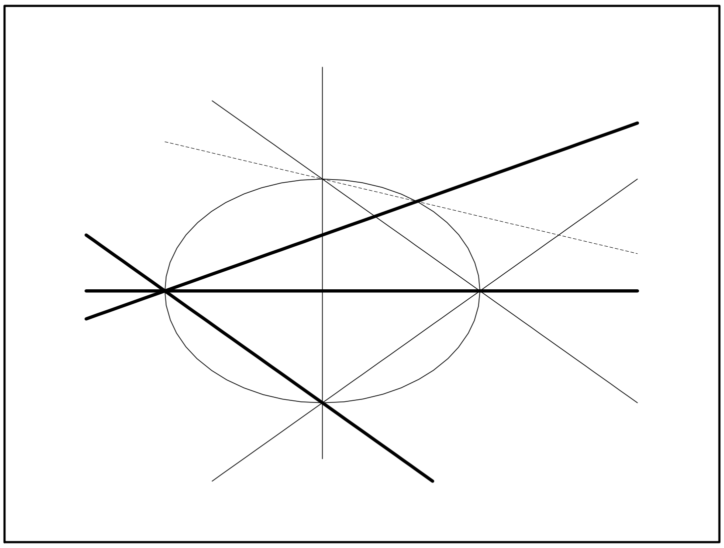
To have the desired three $(0, 1, 3)$ points we must moreover assume that A_1 joins a and b, and similarly B_1 joins b and c, C_1 joins c and a. For special positions of the configuration one can obtain that the two lines B_2, B_3 go through any number m of the four points of intersection $A_i \cap C_j, i, j = 2, 3$.

Therefore Burniat obtains in this way surfaces S with $p_g = 0$ and $K^2 = 6 - m, 0 \leq m \leq 4$.

Example 7. Consider $X = \mathbf{P}^2$ and consider a bidouble cover Y with the L_i's of degrees $(3, 2, 4)$ and the D_j's of degrees $(3, 5, 1)$. If Y is smooth, then it has $K_Y^2 = 9$ and $p_g = 4$, by our standard formulae.

We make p_g and K^2 drop by 4 by imposing four $(0, 1, 3)$ points. This is only possible if the quintic curve $D_2 = B$ has three triple points (which are then non collinear, since B is reduced), and $D_1 = A$ consists of three lines $A = A_1 + A_2 + A_3$ through a point P. But then B splits into a triangle $B_1 + B_2 + B_3$ and a conic B' through the vertices of the triangle. We let the point P belong to B', the lines A_2, A_3 go through two of the vertices of the triangle, and D_3 go through the third vertex. According to whether D_3 does or does not go through the intersection point $B' \cap A_1$ we get a surface S with S with $K^2 = 4$, respectively $K^2 = 5$. We get $p_g = 0$

(thus also $q = 0$) since the canonical system is given by the lines through the points of type $(0,1,3)$ lying in $B \cap A$ and the constants vanishing on the $(0,3,1)$ point $B \cap D_3$.



$K^2 = 4, \quad p_g = 0.$

The following examples yield surfaces with $p_g = q = 1$, $K^2 = 3,4,5$ and genus $g = 2$ of the fibres of the Albanese pencil, thus answering Problem 5. 10 raised in [**Ca-Ci1**]. In fact, surfaces with $p_g = q = 1$, were classified in [**Cat1**], [**Ca-Ci1**], [**Ca-Ci2**] for the values $K^2 \leq 3$, and only one example, due to Xiao ([**Xiao1**]), was known for higher K^2: namely, $K^2 = 4$ and genus $g = 2$ of the Albanese fibres (notice that under the last assumption holds: $K^2 \leq 6$).

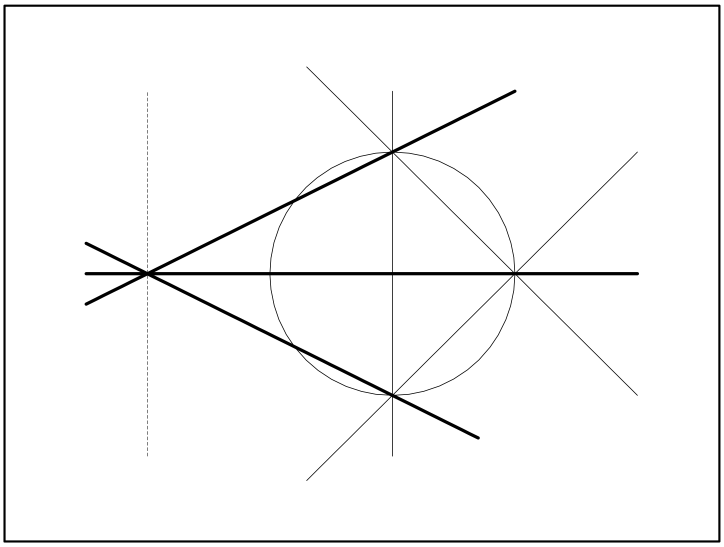
EXAMPLE 8. Here again we consider $X = \mathbf{P^2}$ and a bidouble cover Y of the same degrees as in the previous example, namely $(3,2,4)$ for the L_i's, and $(3,5,1)$ for the D_j's.

Again we impose four points of type $(0,1,3)$, B splits into a triangle $B_1+B_2+B_3$ and a conic B' through the vertices of the triangle, $D_1 = A$ consists of three lines $A = A_1 + A_2 + A_3$ through a point P.

We have two cases:

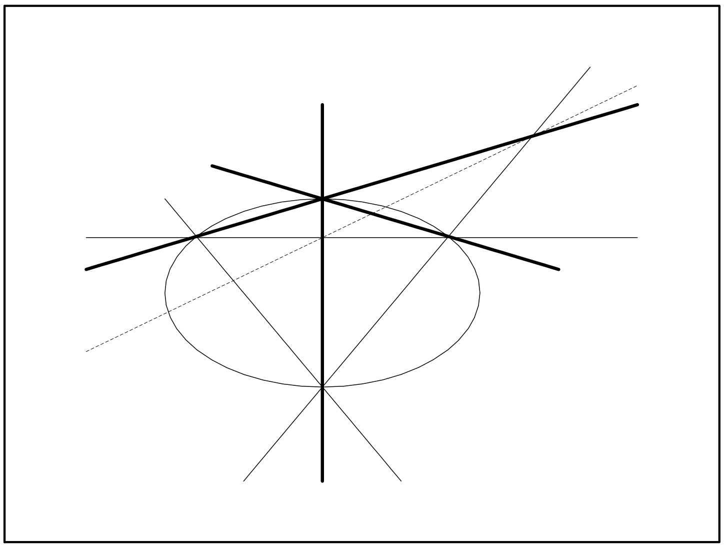
Case I): P does not belong to B' and the three lines A_1, A_2, A_3 pass through the three respective vertices of the triangle; finally, D_3 is a general line through P. The difference from the previous example is that there is no essential singularity in $B \cap D_3$, thus χ drops to 1, but also $p_g = 1$.

Here, $K^2 = 5$. Moreover, consider the pencil of lines through P: we can decompose the Galois covering as first taking a double cover branched on four lines through P (thus we obtain a ruled surface of irregularity 1), and then taking a double cover branched on the $5-$ section given by B and on the section coming from the blow up of the point P. Therefore, the above pencil lifts to the Albanese pencil, which is a pencil of genus 2 curves.



Case I: $K^2 = 5, \quad p_g = q = 1.$

Case II): P belongs to B', and the three lines A_1, A_2, A_3 pass through the three respective vertices of the triangle; the line D_3 can be chosen to go through zero, one or two of the points of transversal intersection of B with A. Accordingly, we get a surface S with $K^2 = 5, 4, or 3$, and again $p_g = q = 1$. The pencil of conics through the four points of type $(0,1,3)$ induces as before the Albanese pencil, and the Albanese fibres have genus 2, since they are double covers of those conics branched on 6 points (2 come from the intersection with D_3, the other 4 from the four sections coming from the blow ups of the four points).



Case II: $K^2 = 3, \quad p_g = q = 1.$

Example 9. Here we consider instead as base surface $X = \mathbf{P^1} \times \mathbf{P^1}$ and consider a bidouble cover Y with the L_i's and the D_j's of bidegrees $(2,2)$. If Y is smooth, then it has $K_Y^2 = 8$ and $p_g = 3$, by our standard formulae.

We make p_g and K^2 drop by 3 by imposing three points of type $(0,1,3)$. We are moreover able to impose up to three points of type $(1,1,1)$.

The way to do so is to let any divisor D_i be a union of a horizontal line H_i, a vertical line V_i, and a curve C_i of type $(1,1)$ passing through $P_i = V_i \cap H_i$; notice that C_i is a plane section of the quadric surface Q in $\mathbf{P^3}$, hence we can let these curves pass through any three non collinear points (that is, lying neither on a vertical line V nor on a horizontal line H).

Case I): let C_i go through P_i, P_{i+1} (here $1,2,3$ are viewed as rest classes modulo 3), and possibly through a fourth point P. Accordingly, we get three points of type $(0,1,3)$ and zero or one point of type $(1,1,1)$.

Thus $K^2 = 5$ or 4, $p_g = 0$.

Case II): C_1 goes through P_1, P_2, P_3, C_2 goes through P_1, P_2, C_3 goes through P_3 and the two points $V_1 \cap H_2$, $V_2 \cap H_1$. Here, we get three points of type $(0,1,3)$ and two points of type $(1,1,1)$.

Thus $K^2 = 3$, $p_g = 0$.

There remains to calculate the fundamental groups for the above surfaces (these are known for the classical Burniat surfaces, cf. [**B-P-V**] or [**Pet**]). In particular, is the last example essentially different from the one given by Burniat?

Acknowledgements. I would like to thank C. Ciliberto for pointing out Burniat's work to my attention, and M. Lehn for helping with the pictures of the Burniat type surfaces.

References

[Ashi] T. Ashikaga, *A remark on the geography of surfaces with birational canonical morphisms*, **Math. Ann. 290** (1991), 63-76.

[B-P-V] W. Barth, C. Peters and A. Van de Ven, *Compact Complex Surfaces*, **Springer-Verlag**, (1984).

[B-H-H] G. Barthel, F. Hirzebruch, T. Höfer, *Geradenkonfigurationen und algebraische Flächen* **Vieweg Aspekte der Mathematik, Bd. 4.** (1987).

[Bea] A. Beauville, "L'application canonique pour les surfaces de type general", **Inv. Math. 55** (1979), 121-140.

[Bur1] P. Burniat, *Modeles de surfaces canoniques normales de S_3 et de genre line'aire* $11 \leq p^{(1)} \leq 17$, in ' 2-eme Colloque Geom. algebrique, Liege du 9 au 12 juin 1952', **Centre Belge Rech. Math** (1952), 185-210.

[Bur2] P. Burniat, *Sur les surfaces de genre* $P_{12} > 1$ **Ann. Mat. Pura Appl., IV. Ser. 71** (1966), 1-24.

[Cat1] F. Catanese, *On a class of surfaces of general type.* in 'C. I. M. E. 1977: Algebraic surfaces', **Liguori, Napoli (1981)**, 267-284.

[Cat2] F. Catanese, *On the Moduli Spaces of Surfaces of General Type*, **J. Diff. Geom 19** (1984) 483–515.

[Cat3] F. Catanese, *Commutative algebra methods and equations of regular surfaces*, in 'Algebraic Geometry - Bucharest' 1982, **Springer L. N. M., 1056** (1984), 68-111.

[Cat4] F. Catanese, *Automorphisms of Rational Double Points and Moduli Spaces of Surfaces of General Type*, **Comp. Math. 61** (1987), 81-102.

[Cat5] F. Catanese, *Homological Algebra and Algebraic Surfaces*, **A. M. S. Proc. Symp. Pure Math. 62, 1** (1997), 3-56.

[Ca-Ci1] F. Catanese-C. Ciliberto, *Surfaces with* $p_g = q = 1$. , in Proc. of the Meeting "Problems on surfaces and their classification", **Symposia Mathematica 32, INdAM, Academic Press**, 49-79 (1991), Catanese, Ciliberto, Cornalba editors.

[Ca-Ci2] F. Catanese-C. Ciliberto, *Symmetric products of elliptic curves and surfaces with* $p_g = q = 1$. **Jour. Alg. Geom. 2** (1993), 389- 411.

[C-D] F. Catanese-O. Debarre, *Surfaces with* $K^2 = 2, p_g = 1, q = 0$. **Crelle's Jour. 395** (1989), 1-55.

[C-S] F. Catanese-F. O. Schreyer, *Canonical rings of irregular algebraic surfaces and canonical projections*, to appear

[Cil1] C. Ciliberto, *Canonical surfaces with* $p_g = p_a = 4$ *and* $K^2 = 5, \ldots, 10$, **Duke Math. J. 48** (1981), 121-157.

[Cil2] C. Ciliberto, *Canonical surfaces with* $p_g = p_a = 5$ *and* $K^2 = 10$, **Ann. Sc. Norm Sup. s. IV, v. IX, 2** (1982), 287-336.

[Com] A. Comessatti *Sui rivestimenti abeliani delle superficie algebriche* **Rend. Semin. Mat. Univ. Padova 1** (1930), ?.

[En] F. Enriques, *Le Superficie Algebriche*, **Zanichelli, Bologna** (1949)

[H-M] D. Hahn, R. Miranda *Quadruple covers of algebraic varieties*, **J. Alg. Geom 8, 1** (1999), 1-30.

[Fran] A. Franchetta, *Su alcuni esempi di superficie canoniche*, **Rend. Sem. Mat. Roma 3** (1939), 23-28.

[H] Hartshorne, R., *Algebraic Geometry*, **G. T. M. 52, Springer-Verlag** (1977).

[Hir] F. Hirzebruch, *Arrangements of lines and algebraic surfaces* in 'Arithmetic and geometry, Papers. dedic. to I. R. Shafarevich', Vol. II: Geometry, **Prog. Math. 36, Birkhauser** (1983), 113-140.

[Hor] E. Horikawa, *Algebraic surfaces of general type with small* c_1^2, *V* **J. Fac. Sci. Univ. Tokyo, Sect. A. Math. 283** (1981), 745-755.

[Kod] K. Kodaira, *On characteristic systems of families of surfaces with ordinary singularities in a projective space*, **Amer. J. Math. 87** (1965) , 227-256.

[Max] E.A. Maxwell, *Regular canonical surfaces of genus three and four* ,**Proc. Camb. Phil. Soc. 23** (1937), 306-310.

[M-Y] M. Miyanishi-J. Yang, *Surfaces of general type whose canonical map is composed of a pencil of genus 3 with small invariants*, **Kyoto J. Math.** , to appear.

[Miya] Y. Miyaoka, *On the Chern numbers of surfaces of general type*, **Inv. Math. 42** (1977), 225-237.

[Par1] R. Pardini, *Abelian covers of algebraic varieties*, **J. Reine Angew. Math. 417** (1991), 191-213.

[Par2] R. Pardini, *Canonical images of surfaces*, **J. reine angew. Math. 417** (1991), 215-219.

[Per] U. Persson, *On Chern invariants of surfaces of general type*, **Comp. Math. 43** (1981), 3-58.

[Pet] C. Peters, *On certain examples of surfaces with* $p_g = 0$ *due to Burniat*, **Nagoya Math. Jour. 66** (1977), 109-119.

[Xiao1] G. Xiao *Surfaces fibrees en courbes de genre deux*, **Springer L. N. M. 1137** (1985).

[Xiao2] G. Xiao *Irregularity of surfaces with a linear pencil*, **Duke Math. Jour. 55, 3** (1987), 597-602.

MATHEMATISCHES INSTITUT DER GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN, BUNSENSTR. 3/5, D-37073 GÖTTINGEN, GERMANY

E-mail address: `catanese@cfgauss.uni-math.gwdg.de`