Growing a Baker domain from attracting islands I:

The dynamics of the limit function

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Abstract

We consider the dynamics of the entire transcendental function $f(z) = z - 1 + (1 - 2z)e^z$. This function features the origin as a superattracting fixed point together with a Baker domain $\mathcal{B}$ (on which the family of iterates converges to the constant function ∞). The function f can be approximated by the polynomials $p_d(z) = z - 1 + (1 - 2z)(1 + z/d)^d$ and we are interested in the convergence of the associated Julia and Fatou sets. To this end, we pursue a detailed study of the dynamics of f. In particular, we establish that f has no periodic components other than the Baker domain $\mathcal{B}$ and the immediate basin of attraction $A_f^*(0)$ of the superattracting fixed point $z_0 = 0$, and no wandering domain. We also prove a theorem on the structure of the Baker domain $\mathcal{B}$. Finally, we study the location of the critical points, the critical values, and the fixed points.

Keywords: Baker domain, exponential function, Fatou set, iteration, Julia set, transcendental function

MSC2000: 30D05, 37F10, 37C25, 30D20

1 Introduction

An entire function $g : \mathbb{C} \to \mathbb{C}$ defines a dynamical system by iteration. The main objects of study are the Fatou set $\mathcal{F}(g)$, defined as the set of all points in $\mathbb{C}$ on which the iterates g^n of g, where n runs through the non–negative integers $\mathbb{N}$, form a normal family, and the Julia set $\mathcal{J}(g) = \mathbb{C} \setminus \mathcal{F}(g)$. If g is a polynomial, then by Sullivan's celebrated non–wandering theorem, cf. [23], every component of its Fatou set is eventually periodic. A cycle of Fatou components can either be an attracting basin of an attracting cycle, an attracting basin of a parabolic cycle, or a cycle of Siegel disks. In particular, ∞ is a superattracting fixed point. On the other hand, if g is transcendental, then ∞ is never in the Fatou set and there may be wandering domains and/or Baker domains in $\mathcal{F}(g)$. A wandering domain is a domain that never becomes periodic under iteration. A Baker domain is a domain on which the sequence of iterates converges uniformly on compact subsets to the constant limit function ∞. Examples of functions with Baker domains go back to Fatou, cf. [13, Example 1], but wandering domains have been found later in [1, 2]. We refer the reader to the papers [1, 4, 3, 12] for more details on Baker and wandering domains, and the iteration of transcendental functions.

In this paper we focus on the entire transcendental function $f(z) = z-1+(1-2z)e^z$. It features a Baker domain while the origin is superattracting, which constitutes a level of complication not studied earlier. In order to learn more about this function we also consider the polynomials $p_d(z) = z - 1 + (1 - 2z)(1 + z/d)^d$ that converge to $f(z)$ uniformly on compact subsets of the complex plane $\mathbb{C}$. The question is whether or not one can learn something about the Julia set of the limit function f by looking at the sequence of Julia sets of the approximating functions p_d. Convergence processes of this type have been intensively studied in the past, e. g. cf. [11, 10, 16, 17, 18, 20]. In particular, it is well known that these polynomials must have an attracting fixed point whose basin converges to the basin of the superattracting fixed point of G. However, there is no general theory to predict how the Baker domain is approximated. To better understand this convergence process, we have performed a number of computer experiments utilizing the SGI's at the National Center for Supercomputing Applications, University of Illinois at Urbana-Champaign. The results have lead to a detailed study of the Julia set $\mathcal{J}(f)$ of the limit function f as well as the Julia sets $\mathcal{J}(p_d)$ of the polynomials p_d. In the present paper we establish their findings on the dynamics of the limit function f (see Figure 1) and some statements about the polynomials p_d. In the forthcoming paper [15] the convergence process will be discussed.

Computer animations illustrating this limiting process can be found on the web at
http://www.math.uiuc.edu/~kriete/anim.baker/ (US site) or
http://www.uni-math.gwdg.de/kriete/anim.baker/ (German site).

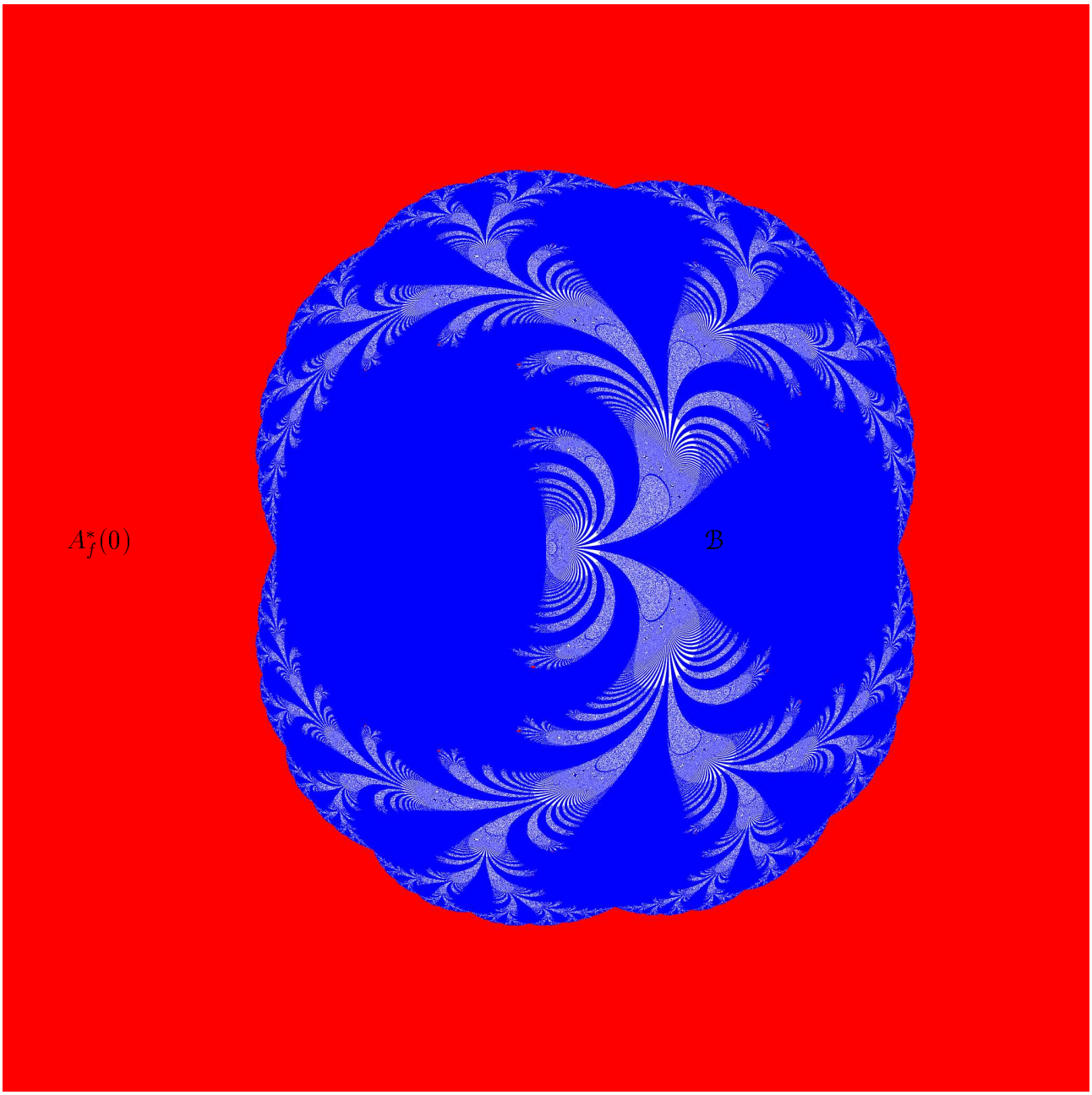

Figure 1: The Julia set $\mathcal{J}(f)$ of f in white in a chart near ∞. The Baker domain $\mathcal{B}$ and its preimages are in blue. The basin of attraction of the super attracting fixed point $z_0 = 0$ is in red. The large exterior red region is the immediate basin of attraction $A_f^*(0)$. The other components of the basin of attraction $A_f(0)$ are the very tiny red spots.

For the convenience of the reader, the main results of this paper are given now. Please see Section 2 for the notions and notations.

Theorem 1 (On the Fatou components)

(i) *Each component of the Fatou set $\mathcal{F}(f)$ is simply connected,*

(ii) *f has no wandering domains, and*

(iii) *$\mathcal{F}(f)$ has exactly two periodic components:*

– *the Baker domain $\mathcal{B}$ where the iterates of f tend to $-\infty$,*

– *the immediate basin $A_f^*(0)$ of the super attracting fixed point $z_0 = 0$.*

In other words, f has only two periodic domains: the Baker domain $\mathcal{B}$ and the basin of attraction $A_f^*(0)$. Note that the latter is a simply connected bounded subset of the complex plane. The proof of Theorem 1 is given in Section 3. It follows that the study of the structure of the Fatou set reduces to the study of the Baker domain $\mathcal{B}$, which is the topic of Section 4. It turns out, the $\mathcal{B}$ contains a large portion of the left half plane, see Proposition 5 below. In addition, $\mathcal{B}$ stretches to $+\infty$ along countably many narrow channels; compare Figure 1. Furthermore, for the location of these channels relative to the critical and repelling points we have the following.

Theorem 2 (Structure Theorem) *There exists a family $\{\Gamma_k\}_{k\in\mathbb{Z}\setminus\{0\}}$ of Jordan arcs with the following properties:*

(i) *All Γ_k are pairwise disjoint,*

(ii) *Each Γ_k is forward invariant,*

(iii) *Each Γ_k contains exactly one critical point,*

(iv) *Each component of $\mathbb{C}\setminus\Gamma$, where $\Gamma = \cup_{k\in\mathbb{Z}\setminus\{0\}}\Gamma_k$,*
contains exactly one repelling fixed point z_k, and $z_k \in \partial\mathcal{B}$,

(v) *$\Gamma_k \subset \mathcal{B}$ for every $k \in \mathbb{Z} \setminus \{0\}$.*

We remark that the curves Γ_k are approximately horizontal: Their imaginary parts are bounded while the real part runs through the all the real line, that is to say, from $-\infty$ to $+\infty$; see Proposition 17 in Section 4.

It is well known that the location and the orbits of the singular values of f play a crucial role. Note that f has no asymptotic value (see Lemma 10 in Section 4), thus we need to worry about the critical points only. Their location is studied in Section 5. In fact, all the critical points and values of f which are distinct from the origin in fact lie in the Baker domain $\mathcal{B}$.

Theorem 3 (Location of critical values) *If $f'(z) = 0 \neq z$ then*

$$\begin{array}{ll} (i) & z - 2 \in D \text{ and} \\ (ii) & f(z) \in D. \end{array}$$

Theorem 4 (Location of critical points) *If $z \neq 0 = f'(z)$ then $z \in \mathcal{B}$.*

The main ingredient for the proof the above two results in Section 5 is finding a suitable – actually in some sense optimal – subdomain $D \subset \mathcal{B}$ which is invariant with respect to f and on which the dynamics of f are basically a translation to the left.

Proposition 5 *If*

$$D := D_{2,0.9} = \{\, x + iy \,:\, x < -2 \text{ and } |(1-2z)e^z| = |f(z) - (z-1)| < 0.9 \,\}$$

then $f(\overline{D}) \subset D$, and $D \subset \mathcal{B}$ where $\mathcal{B}$ is a Baker domain for f. In addition, $dist_{HD}(\partial D, \partial \mathcal{B}) > \sigma$ for some constant $\sigma > 0$.

In Section 6, we prove some useful results on the approximation polynomials p_d which will be needed in the forthcoming second part [15] of this paper. Finally, in Section 7 we add a number of technical lemmas used in Section 3 and Section 4.

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2 Notions and notations

On the complex plane $\mathbb{C}$ with the Euclidean metric, the Hausdorff distance $\mathrm{dist}_{\mathrm{HD}}$ between two closed sets $A, B \subset \mathbb{C}$ is defined in the usual way as

$$\mathrm{dist}_{\mathrm{HD}}(A, B) := \inf\{\varepsilon > 0 | A \subset V_\varepsilon(B) \text{ and } B \subset V_\varepsilon(A)\}\,,$$

where $V_\varepsilon(X) \subset \mathbb{C}$ is the ε-neighbourhood (with respect to the Euclidean metric) of the closed set $X \subset \mathbb{C}$. Throughout this paper, for some subset $S \subset \mathbb{C}$ let $\overline{M}$ denote the closure of S with respect to the Euclidean metric on $\mathbb{C}$. Consequently, the notions 'boundary', 'closed' and 'open' are taken with respect to the Euclidean topology, too.

For $r > 0$, $\zeta \in \mathbb{C}$, we write $B(\zeta, r) = \{t \in \mathbb{C} : |t - \zeta| < r\}$, $\overline{B}(\zeta, r) = \{t \in \mathbb{C} : |t - \zeta| \leq r\}$, and $s(\zeta, r) = \{t \in \mathbb{C} : |t - \zeta| = r\}$.

Let g be some entire function, i. e. either a polynomial or an entire transcendental function. As mentioned in the introduction, $g : \mathbb{C} \to \mathbb{C}$ defines a dynamical system. The Fatou set $\mathcal{F}(g)$ of g is the maximal open subset of the complex plane such that for every compact subset K the family of iterates $\{g^n|_K\}_{n\in\mathbb{N}}$ forms a normal family. Note that some authors call the Fatou set the 'set of normality' and denote it by $N(f)$. The complement $\mathbb{C} \setminus \mathcal{F}(g)$ is called the Julia set of g. It will be denoted by $\mathcal{J}(g)$. Recall that $\mathcal{J}(g)$ is always non-empty, closed, and perfect. It is unbounded and therefore not a compact subset of the complex plane. However, after adding the point ∞, it becomes a compact subset of the Riemann sphere $\widehat{\mathbb{C}}$.

Both, $\mathcal{F}(g)$ and $\mathcal{J}(g)$ are completely invariant with respect to g. Furthermore, $\mathcal{J}(g)$ can also be described as the closure of the set of all repelling periodic points of g. We refer the reader to [5, 6, 8, 9, 22] as general references to iteration theory. All important in the study of complex dynamics are the critical points of the function in question, say, g, and the finite asymptotic values. Recall that a point $c \in \mathbb{C}$ is called *critical point* if and only if $g'(c) = 0$. The value $g(c)$ is then called *critical value*. Some point $v \in \widehat{\mathbb{C}}$ is called asymptotic value if and only if there exists some path $\gamma : [0, \infty[\to \mathbb{C}$ such that $\lim_{t\to\infty} |\gamma(t)| = \infty$ and $\lim_{t\to\infty} f(\gamma(t)) = v$ holds. Let denote $SV(g)$ the set of all critical and finite asymptotic values of g and $P(g) = \cup_{n\in\mathbb{N}} g^n(SV(f))$, the postsingular set.

Remark. Since f is a real function, i. e. it maps the real line onto itself, $\overline{f(\overline{z})} = f(z)$ holds for all $z \in \mathbb{C}$. Thus f is symmetric with respect to the real line and so are $\mathcal{J}(f)$, $\mathcal{F}(f)$, $\mathcal{B}$, $A_f(0)$, and so on. Therefore, in all the proofs, we may and will restrict to the upper halfplane without stating this explicitly.

3 The dynamics of the limit function

We want to study the dynamics of the entire transcendental function

$$f(z) = z - 1 + (1 - 2z)e^z.$$

Since f is an entire function, all Fatou components – except for possible existing multiply connected wandering domains – are simply connected. Talking about wandering domains, it is worth to note that Sullivan's non–wandering theorem holds for f.

In this section we prove Theorem 1. Before we do this we consider some of its consequences and prove some helpful lemmas.

In fact, both periodic components are invariant with respect to f. Note that f maps the real line into itself, thus Fatou components intersecting the real line have to be symmetric to the real line. Later, we will need

$$f'(z) = 1 + (1 - 2z)e^z - 2e^z = 1 - (1 + 2z)e^z$$

and

$$f''(z) = -(1 + 2z)e^z - 2e^z = -(3 + 2z)e^z.$$

This yields

$$f(0) = f'(0) = 0 > f''(0) = -3.$$

Thus $z_0 = 0$ is a superattracting fixed point. Let $A_f^*(0)$ denote its immediate basin of attraction. We shall later see, cf. Section 5, that $z_0 = 0$ is the only critical point of f contained in $A_f^*(0)$. It will turn out that all other critical points lie in the Baker domain. Therefore, in view of Theorem 1, we obtain

Lemma 6 *$f|_{A_f^*(0)} : A_f^*(0) \to A_f^*(0)$ is biholomorphically conjugated to $q : \mathbb{D} \to \mathbb{D}$, $z \longmapsto z^2$.*

In particular, on $\partial A_f^*(0)$ there is exactly one repelling fixed point r_0 of f. For symmetry reason, it is located on the real axis. Since $f''(0) < 0$, it has to be on the negative real axis. Mathematica

gives the approximation

$$r_0 \approx -1.25634.$$

Corollary 7 *The origin z_0 is a superattracting fixed point of f. There is a unique repelling fixed point $r_0 \approx -1.25634$ on the boundary of the immediate basin of attraction $A_f^*(0)$.*

Next, we establish

Lemma 8 *The domain $D_0 = \{y^2 \leq (\frac{1}{4}e^{-x} - \frac{1}{2})^2 - x^2,\ x \leq -3\}$ is forward invariant with respect to f.*

Remark. In Section 5 we will optimize the method to find this domain D_0. Actually, the role of D_0 will be taken by some larger domain $D \subset \mathcal{B}$ which is invariant with respect to f, too.

Proof of Lemma 8.
In order to show that D_0 is non–empty and connected, we note

Claim 0. $x \leq -3 \Longrightarrow (\frac{1}{4}e^{-x} - \frac{1}{2})^2 - x^2 > 0.$

Consequently, the following two clains yield the lemma:

Claim 1 $z_0 \in \overline{D}_0 \Longrightarrow |f(z_0) - (z_0 - 1)| \leq \frac{1}{2}$.

and

Claim 2 *For each $z_0 \in \partial D_0$, the tangent of ∂D_0 has slope of modulus larger than one.*

Proof of Claim 1.
We have to show $|(1-2z)e^z| \leq \frac{1}{2}$. Let $z = x + iy$ and $r = |z|$. Then

$$\begin{aligned}
& |(1-2z)e^z| = |1-2z||e^z| && \leq (1+2r)e^x \leq \tfrac{1}{2} \\
\Longleftarrow \quad & 1 + 2\sqrt{x^2+y^2} && \leq \tfrac{1}{2}e^{-x} \\
\Longleftrightarrow \quad & \sqrt{x^2+y^2} && \leq \tfrac{1}{4}e^{-x} - \tfrac{1}{2} \\
\Longleftrightarrow \quad & y^2 && \leq \left(\tfrac{1}{4}e^{-x} - \tfrac{1}{2}\right)^2 - x^2.
\end{aligned}$$

□

Proof of Claim 2.
For symmetry reason it suffices to show that $x \le -3$ and $g(x) = \sqrt{(\frac{1}{4}e^{-x} - \frac{1}{2})^2 - x^2}$ implies $g'(x) \le -1$. An elementary calculation gives:

$$\begin{aligned}
g'(x) = \ & \tfrac{1}{2g(x)}\left[2\left(\tfrac{1}{2}e^{-x} - \tfrac{1}{2}\right) \cdot \tfrac{-1}{4} \cdot e^{-x} - 2x\right] && \le -1 \\
\iff \quad & 2\left(\tfrac{1}{4}e^{-x} - \tfrac{1}{2}\right) \cdot \tfrac{-1}{4} \cdot e^{x} - 2x && \le -2\sqrt{\left(\tfrac{1}{4}e^{-x} - \tfrac{1}{2}\right)^2 - x^2} \\
\iff \quad & \tfrac{1}{2}e^{-x}\left(\tfrac{1}{2}e^{-x} - \tfrac{1}{2}\right) + 2x && \ge 2\sqrt{\left(\tfrac{1}{4}e^{-x} - \tfrac{1}{2}\right)^2 - x^2} \\
\impliedby \quad & \tfrac{1}{2}e^{-x}\left(\tfrac{1}{4}e^{-x} - \tfrac{1}{2}\right) + 2x && \ge 2\left(\tfrac{1}{4}e^{-x} - \tfrac{1}{2}\right) \\
\impliedby \quad & \left(\tfrac{1}{2}e^{-x} - 2\right)\left(\tfrac{1}{4}e^{-x} - \tfrac{1}{2}\right) && \ge -2x \\
\impliedby \quad & 8\left(\tfrac{1}{4}e^{-x} - \tfrac{1}{2}\right) && \ge -2x \\
\iff \quad & e^{-x} - 2 && \ge -x
\end{aligned}$$

The last inequality is true for $x = -3$, it then reads $e^3 \ge 20 \ge -(-3) + 2)$. By looking at the derivatives $(e^{-x} - 2)' = -e^{-x} \le -20$ and $(-x)' = -1$ we conclude that in fact $e^{-x} - 2 \ge -x$ holds for all $x \le -3$. □

We add the

Proof of Claim 0.
We have to show (recall $x \le -3$)

$$\frac{1}{4}e^{-x} - \frac{1}{2} > -x \tag{1}$$

or, equivalently,

$$\frac{1}{4}e^{-x} > \frac{1}{2} - x\,. \tag{2}$$

Again we check for $x = -3$ first:

$$e^3 \ge 20 \Longrightarrow \frac{1}{4}e^3 > 5 > \frac{1}{2} + 3.$$

Note that $(\frac{1}{2} - x)' = -1$ and

$$\left(\frac{1}{4}e^{-x}\right)' = -\frac{1}{4}e^{-x} \le -\frac{1}{4}e^{-3} \le -5.$$

This completes the proof of Claim 0 (and of Lemma 8). ■

As a consequence of Claim 1 we obtain

$$\begin{aligned} z = x < -3 \quad &\Longrightarrow \quad f^n(x) \;\; > x - \tfrac{n}{2} \\ &\Longrightarrow \quad \lim_{n\to\infty} f^n(x) \;\; = -\infty\,. \end{aligned}$$

Thus, Lemma 8 yields

Proposition 9 *On D_0, $\lim\limits_{n\to\infty} f^n(z) = -\infty$ holds. In particular, f has a Baker domain $\mathcal{B}$ satisfying $D_0 \subset \mathcal{B}$.*

Later, in Section 5, we will establish that all critical points distinct from the superattracting fixed point $z_0 = 0$ are in $\mathcal{B}$, see Theorem 4. In addition, we note

Lemma 10 *f has no finite asymptotic value.*

Remark. So, at this point we know all about the singular values and their orbits. One, $z_0 = 0$, is a fixed points, and all the other tend to $-\infty$ under iteration of f.

Proof of Lemma 10.
We assume that f has some finite asymptotic value, say, $a \in \mathbb{C}$. In other words, there exists some curve $\gamma : [0, \infty[\to \mathbb{C}$ such that $\lim_{t\to\infty} |f(\gamma(t))| = \infty$ and

$$a = \lim_{t\to\infty} f(\gamma(t)) = \lim_{t\to\infty} \left(\gamma(t) - 1 + (1 - 2\gamma(t))e^{\gamma(t)}\right)$$

or, equivalently,

$$0 = \lim_{t\to\infty} \frac{f(\gamma(t))}{\gamma(t)} = \lim_{t\to\infty} \left(1 - \frac{1}{\gamma(t)} + \left(\frac{1}{\gamma(t)} - 2\right) e^{\gamma(t)}\right)$$

Since $\lim_{t\to\infty} \gamma(t) = \infty$, we obtain

$$1 = 2 \lim_{t\to\infty} e^{\gamma(t)}$$

and this in turn would imply $1/2$ to be an asymptotic value of the exponential function, a contradiction. ■

Altogether we obtain

Lemma 11 *Let $SV(f)$ denote the set of all finite singular values of f and*

$$P(f) := \cup_{v \in SV(f)} \{f^n(v) : n \in \mathbb{N}\}.$$

Then $P(f)$ is discrete in $\mathbb{C}$ and a subset of $\mathcal{B} \cup A_f^*(0)$.

Actually, Theorem 3 and Propostion 5 yield

Proposition 12 *There exists some $\sigma > 0$ such that*

$$V_\sigma(P(f)) \subset \mathcal{B} \cup A_f^*(0)\,.$$

In particular, f cannot have further (different from $z_0 = 0$) attracting periodic points since such a point would have to be an element of or a cluster point of $P(f)$. By the same argument, f cannot have a parabolic periodic point. f cannot have a Cremer point since this would require the Julia set to contain a (recurrent) singular value, cf. [19, 21]. Neither can f have a Siegel disc since its boundary would have to consist of cluster points of $P(f)$, too. The only remaining possibilities, namely that f has another Baker domain or some wandering domain, will be ruled out later in this section.

Next we discuss the dynamics of f on the real line. Note that $f''(x) < 0$ on $]-\frac{3}{2}, 0[$, and

$$-1.6075 \approx f\left(-\frac{3}{2}\right) < -\frac{3}{2}\,.$$

Thus, the interval $\left[-\frac{3}{2}, 0\right[$ does contain exactly one fixed point which has to be repelling. So, r_0 and z_0 are the only fixed points in $\left[-\frac{3}{2}, 0\right]$.

Since $f''(x) > 0$ on $]-\infty, -\frac{3}{2}[$ and $\lim_{x\to\infty} \frac{f(x)}{x-1} = 1$, we conclude that there cannot be fixed points on the interval $]-\infty, -\frac{3}{2}[$. Furthermore, we conclude

$$x - 1 < f(x) < x \quad \text{for} \quad x < r_0$$

and

$$x < f(x) < 0 \quad \text{for} \quad r_0 < x < 0\,.$$

Consequently,

$$]-\infty, r_0[\subset \mathcal{B} \quad \text{and} \quad]r_0, 0] \subset A_f^*(0)\,.$$

Finally, note that $f''(x) > 0$ for $x > 0$ and $f(0) = f'(0) = 0$. Thus, $f(x) < 0$ for $x > 0$. Obviously, $\lim_{x\to\infty} f(x) = -\infty$. Altogether we obtain

$$f([0, \infty[) =]-\infty, 0].$$

Consequently,

$$\mathbb{R} \subset \overline{\mathcal{B} \cup f^{-1}(\mathcal{B}) \cup A_f^*(0)}\,. \tag{3}$$

We are now prepared for the

Proof of Theorem 1:
Recall that if f would have a multiply connected wandering domain W, then by [3, Theorem 3.1] each component of $\mathcal{F}(f)$ would have to be bounded, a contradiction to inclusion (3). This proves Theorem 1, (i). We have already ruled out that f has Herman rings, Siegel discs, parabolic basins, and attracting basins other than $A_f(0)$. Actually, one can show that f cannot have wandering domains at all, and that $\mathcal{B}$ is the only Baker domain of f. Consequently, the next Proposition completes the proof of the Theorem. ■

Proposition 13 *f has no wandering domains and $\mathcal{B}$ is the only Baker domain of f.*

The main ingredients in the proof are the following two lemmas.

Lemma 14
$\left\{z \in \mathbb{C} : \mathrm{Im}(z) = \pm(2k - \frac{1}{2})\pi\right\} \subset \mathcal{B}$ *holds for all but finitely many positive integer k.*

This lemma is a consequence of Lemma 26 and Proposition 28. □

Lemma 15 *Let $U \subset \mathbb{C}$ be an open subset of the plane and invariant with respect to g. We assume $P(f) \subset U$ and*

$$dist(P(f), \partial U) := \inf\{|z - w| : z \in P(f), w \in \partial U \cap \mathbb{C}\} > \sigma$$

for some positive constant σ. Let $\zeta_0 \in \mathcal{F}(f)$ such that $f^n(\zeta_0) \notin U$ for every $n \in \mathbb{N}$. If

$$dist(f^n(\zeta_0), \partial U) := \inf_{z \in \partial U} |z - \zeta_0| < C$$

for some constant $C > 0$, then ζ_0 is not contained in a Baker domain or in a wandering domain of f.

Proof of Proposition 13:
We choose $U = \mathcal{B} \cup A_f^*(0)$. If f would have a Baker domain different from $\mathcal{B}$ or a wandering domain, then we could choose z_0 to be in this Baker or wandering domain, respectively. Then by Lemma 14 we find the constant C, and by Proposition 12 a constant σ, as required in the hypothesis of Lemma 15, and from this lemma we derive that z_0 can not be an element of this Baker or wandering domain, respectively. ■

Proof of Lemma 15: (This proof is a modification of the proof of Lemma 3 in [7].)
Throughout this proof, let $\mathrm{dist}(z, S)$ denote the distance of a finite point $z \in \mathbb{C}$ to the set $S \subset \mathbb{C}$ with respect to the euclidean distance. We write $\zeta_n := f^n(\zeta_0)$ and $\delta_n := \mathrm{dist}(\zeta_j, \partial U)$. By the hypothesis of the lemma we find some constant $\sigma > 0$ such that

$$B(\zeta_n, \sigma + \delta_n) \cap P(f) = \emptyset \,.$$

Hence there exist the branches ϕ_n of $(f^n)^{-1}$ satisfying $\phi_n(\zeta_n) = \zeta_0$. In addition, there exists points ξ_n satisfying

(i) $\xi_n \in \mathcal{J}(f)$,
(ii) $|\xi_n - \zeta_n| = \delta_n$, and
(iii) $\phi_n(\xi_n) \in \mathcal{J}(f)$.

Note that the branches ϕ_n are holomorphic (actually schlicht) on $B(\zeta_n, \sigma + \delta_n)$, and that $\phi_n(B(\zeta_n, \sigma + \delta_n)) \cap P(f) = \emptyset$. Then the family of the schlicht mappings

$$\psi_n : \mathbb{D} \to \mathbb{C}; z \mapsto \phi_n\left((\sigma + \delta_n)(z + \zeta_n)\right)$$

is normal. Since $\psi_n(0) = \zeta_0$ for every $n \in \mathbb{N}$, the constant function ∞ cannot occur as a limit function. Note that $\widehat{\xi}_n := \phi_n^{-1}(\phi_n(\xi_n)) \in \mathbb{D}$ is well defined. By construction,

$$0 < |\zeta_n - \xi_n| = \delta_n < \delta_n + \sigma < C + \sigma \,.$$

and

$$0 < |\widehat{\xi}_n| < \frac{\delta_n}{\sigma + \delta_n} < \frac{C}{\sigma + C} < 1\,.$$

For every limit function ψ of the normal family $\{\psi_n\}_{n\in\mathbb{N}}$ there is some $\widehat{\xi}$ with $|\widehat{\xi}| = r \in]0,1[$ and $\psi(\widehat{\xi}) \in \mathcal{J}(f)$. This in turn implies that no limit function can be constant. In particular, $0 < r_1 < |\psi'(0)| < R_1 < \infty$ for some constants r_1 and R_1. For simplicity, we assume $\psi = \lim_{n\to\infty} \psi_n$. Then, for all but finitely many $n \in \mathbb{N}$, the iterates f^n are univalent on some fixed disc $D := B(\widehat{\xi}, 1-\widehat{r})$ for some arbitrary but fixed $\widehat{r} \in]0, r[$. Furthermore, we have $f^n(D) \subset B(\zeta_n, \sigma + \delta_n)$ for all but finitely many $n \in \mathbb{N}$. In particular, we have $f^n(D) \cap P(f) = \emptyset$ for all but finitely many $n \in \mathbb{N}$, so that $\psi(\widehat{\xi}) \in D \subset \mathcal{F}(f)$, a contradiction. ■

We have already shown, that $]-\infty,\ r_0[=: I$ has an unbounded preimage I_0 in the positive real axis. Actually, *all* preimages (different from I itself) stretch to $+\infty$.

Lemma 16 *Let $\widetilde{I}$ a component of $f^{-1}(I)\backslash I$. Then* $\sup_{z\in\widetilde{I}} \mathrm{Re}(z) = \infty$.

In fact, a more general statement holds:

Proposition 17 *Let $\Gamma : [0,\infty[\to D_0$ a curve invariant with respect to f, $\gamma : [0,\infty[\to \mathbb{C}$ the parameterization of some unbounded component of $f^{-1}(\Gamma) \setminus \Gamma$. Then*

$$\begin{array}{lrcl}
(i) & \lim_{t\to\infty} \mathrm{Re}(\Gamma(t)) & = & -\infty \\
(ii) & \sup_{t\in[0,\infty[} |\mathrm{Im}(\Gamma(t))| & < & \infty \\
(iii) & \lim_{t\to\infty} \mathrm{Re}(\gamma(t)) & = & \infty \\
(iv) & \sup_{t\in[0,\infty[} |\mathrm{Im}(\gamma(t))| & < & \infty
\end{array}$$

Proof of Proposition 17:
Statement (i) is clear. (ii) is a consequence of Lemma 27. Statement (iv) follows from Lemma 25, the injectivity of $f|_{D_0}$, and the Lemma (29). Finally, we turn our attention to the third statement. By construction, $\lim_{t\to\infty} |\gamma(t)| = \infty$. The injectivity of f_{D_0} gives $\gamma \subset \mathbb{C} \setminus D_0$. Now, (iv) yields that $|\mathrm{Im}(\gamma(t))|$ is bounded. Altogether, we obtain that $\mathrm{Re}(\gamma(t))$ is bounded from below. So, $\lim_{t\to\infty} \mathrm{Re}(\gamma(t)) = \infty$ is the only remaining possibility. ■

Finally, we note

Lemma 18 *Each critical point c of f is a simple critical point, that is to say,*

$$f'(c) = 0 \Longrightarrow f''(c) \neq 0.$$

Proof of Lemma 18.
Recall $f'(z) = 1 - (1+2z)e^z$ and $f''(z) = -(3+2z)e^z$. If $f'(c) = c$, then $\frac{1}{1+2c} = e^c$. Thus

$$f''(c) = \frac{-(3+2c)}{1+2c}e^c.$$

So, if $f''(c) = 0$, then $c = \frac{-3}{2}$. However, $f'(\frac{-3}{2}) = 1 + 2e^{-3/2} > 1 > 0$. ■

In Section 5 we will refine the above methods for a more detailed study of the location and the orbits of the critical points.

4 Structure of the Baker domain

In this section, we deal with the structure of the Baker domain and prove Theorem 2. The main ingredients are the results of the next section, in particular, Proposition 5, Theorem 3, and Theorem 4. The main result is, that $\mathcal{B}$ does not only contain the large subdomain D, but also countably many channels stretching out to $+\infty$.

Note that property (v) in Theorem 2 implies that each fixed point of f is contained in some component of $\mathbb{C}\backslash\Gamma$. By Proposition 17, these curves are approximately horizontal lines: For each parameterization $\Gamma_k : \mathbb{R} \to \mathbb{C}$ we obtain

$$\sup \{ \, |\mathrm{Im}(\Gamma_k(t))| \, : \, t \in \mathbb{R} \, \} < \infty \tag{4}$$

and

$$\lim_{t \to \pm\infty} \mathrm{Re}(\Gamma_k(t)) = \pm\infty \, . \tag{5}$$

Proof of Theorem 2:
Preparations. First, we label the critical points c_k such that $\mathrm{Im}(c_k) < \mathrm{Im}(c_{k+1})$ and such that $\mathrm{Im}(c_k) > 0$ (resp. < 0) provided $k > 0$ (resp. $k < 0$). Let $v_k = f(c_k)$ and $w_k = f(v_k)$.

Recall that $\mathrm{Im}(c_k) \approx 2\pi(k - \frac{1}{2})$, $v_k \approx c_k - 2$ and $w_k \approx v_k - 1$. Consequently, the line segments

γ_k connecting v_k and w_k are in D (and, of course, in $\mathcal{B}$). We define $\Gamma_k^+ = \cup_{n\in\mathbb{N}} f^n(\gamma_k) \subset D$. Then, Γ_k^+ is a forward invariant Jordan arc. By construction, all γ_k are pairwise disjoint. Since $f|_D$ operates injectively, all Γ_k^+ are pairwise disjoint. Furthermore, for each $k \in \mathbb{Z}^*$, there exist a parametrization $g_k :]-\infty, 0] \to \Gamma_k^+$ satisfying

$$\sup_{t\in]-\infty,0[} |\mathrm{Im}(g_k(t))| < \infty$$

and

$$\lim_{t\to-\infty} \mathrm{Re}(g_k(t)) = -\infty.$$

This is because $f(z) \approx z - 1$ on D (and therefore on each Γ_k^+), see Lemma 27.

Construction of Γ_k. Note that each critical value v_k is contained in exactly one Γ_k^+, and that each critical point c_k is simple. Therefore, for each $k \in \mathbb{Z}^*$, there exists a component Γ_k of Γ_k^+ containing c_k. Furthermore we have

(i) Γ_k are pairwise disjoint

(ii) Each Γ_k is mapped $2:1$ onto Γ_k^+.

For symmetry reasons, we restrict to $k > 0$ and prove that each "strip" S_k bounded by Γ_k and Γ_{k+1} contains exactly one fixed point (which has to be repelling, compare Lemma 20). An additional argument will give, that the strip S_0 bounded by $\mathbb{R}$ and Γ_1 does not contain a fixed point. Since the real line carries exactly one repelling fixed point, r_0, this yields the strip bounded by Γ_{-1} and Γ_1 to contain no further repelling fixed points.

Let $k > 0$. We fix the strip S_k. Recall that ∂S_k is forward invariant and that S_k does not contain any critical point. Consequently, each branch of f^{-1} is well-defined on S_k.

Let g be the unique branch of f^{-1} which maps S_k into itself. $\zeta_0 \in S_k$ and $\zeta_1 = g(\zeta_0)$. We connect ζ_0 and ζ_1 by some curve $\widetilde{G}_k \subset S_k$ and define $G_k = \cup_{m\in\mathbb{N}} g^m(\widetilde{G}_k)$. Note that $\widetilde{G}_k$ does not contain a critical point or critical value of f. By Fatou's snail lemma, cf. [14, Section 54f],

$$\lim_{m\to\infty} g^m(\widetilde{G}_k)$$

exists. Furthermore, it is a singleton, say, ξ_k. More precisely, it has to be (again by Fatou's argument) a weakly repelling fixed point, that is to say, either parabolic fixed point with multiplier equal to one or a repelling fixed point.

Recall that f does not have parabolic fixed points. Thus, ξ_k is a repelling fixed point. By construction, $\xi_k \in \overline{S}_k$. However, $\partial S_k \subset B$, so we conclude $\xi_k \in \mathcal{B}$ and $\xi_k \in S_k$. Note that $\widetilde{S}_k = S_k \backslash f^{-1}(\Gamma_k^+ \cup \Gamma_{k+1}^+)$ is a simply connected and that $S_k \backslash \widetilde{S}_k$ is a finite union of disjoint Jordan arcs. Again, $\partial \widetilde{S}_k$ does not contain any critical point, critical value, or fixed point. Consequently, $\widetilde{S}_k \subset S_k$ and $f|_{\widetilde{S}_k}$ is a *conformal* map from $\widetilde{S}_k$ onto $\mathbb{C} \backslash (\Gamma_k^+ \cup \Gamma_{k+1}^+)$, so, there is exactly one branch, namely g, of f^{-1} which takes S_k into itself.

Recall that g is well-defined on the simply connected domain S_k, that $g(S_k) \subset S_k$, and that $g(C_k) = \xi_k$. Applying Schwarz Lemma yields that ξ_k is the only fixed point of g (and therefore of f) in S_k. Once again, by Schwarz Lemma, it has to be attracting with respect to g, i. e. it is repelling for f_0.

We now turn the attention to the strip S_0 bounded by $\mathbb{R}$ and Γ_1. As above, there exists exactly one branch of f^{-1} which maps S_0 into itself. We denote this branch again by g. Note that ∂S_0 contains r_0, the repelling fixed point of f. So, for each point $z \in S_0$ near r_0, we have $\lim_{n\to\infty} g^n(z) = r_0$. Assume that S_0 contains a fixed point $\tilde{z}$ of f. This would have to be repelling, so, equivalently, it's attracting for g. Applying Schwarz Lemma gives

$$\partial S_0 \ni r_0 \neq \lim_{n\to\infty} g^n(z) = \widetilde{z} \in S_0 \,,$$

a contradiction. This completes the proof of the structure theorem. ■

5 Critical points of f lie in the Baker domain

This section is the crucial one. The results established here are needed to prove the main results of the previous sections. In this section, we search for a 'large' subdomain $D \subset \mathcal{B}$ invariant with respect to f, and discuss the location of the critical points. The first subsection contains some preparations. Then we construct the domain D from Proposition 5. In the next two sections we prove Teorems 3 and 4, that is, we show that all critical values and critical points lie in D and $f^{-1}(D) \cap \mathcal{B}$, respectively.

5.1 A priori estimates

Recall

$$f(z) = z - 1 + (1 - 2z)e^z \tag{6}$$

and

$$f'(z) = 1 - (1 + 2z)e^z \,. \tag{7}$$

If $f(z) = z$ then $e^z = 1/(1-2z)$ and $f'(z) = 2 + \frac{1}{z-\frac{1}{2}}$. If $f'(z) = 0$ then $e^z = 1/(1+2z)$ and $f(z) = z - 2 + \frac{1}{z+\frac{1}{2}}$.

Numerical calculations with Mathematica show that the first few critical points are at:

$$\begin{aligned} -2.23 &\pm 4.33i \\ -3.10 &\pm 10.76i \\ -3.55 &\pm 17.10i \\ -3.86 &\pm 23.42i \end{aligned}$$

Appearently the imaginary part of the critical points is approximately $\pm(2k-\frac{1}{2})\pi$, where k runs through the positive integers. The first few fixed points distinct from r_0 and z_0 are similarly found to be:

$$\begin{aligned} -1.25634 & \\ -2.79 &\pm 7.44i \\ -3.36 &\pm 13.87i \\ -3.72 &\pm 20.22i \end{aligned}$$

Appearently the imaginary part of the fixed points is approximately $\pm(2k+\frac{1}{2})\pi$ where k runs through the positive integers. Let us define a domain D by

$$D_{M,\rho} = \{\, z = x + iy \,:\, x < -M \quad \text{and} \quad |f(z) - (z-1)| = |(1-2z)e^z| < \rho \,\}$$

where the parameters $M > 0$ and $\rho \in]0,1[$ will be chosen later. We try to determine M and ρ so as to have $f(\overline{D_{M,\rho}}) \subset D_{M,\rho}$. This will then imply that $\overline{D_{M,\rho}} \subset \mathcal{F}(f)$, the Fatou set of f, so that $\overline{D_{M,\rho}} \subset \mathcal{B}$ for some component $\mathcal{B}$ of $\mathcal{F}(f)$. Then it is clearly seen that $\mathcal{B}$ must be a Baker domain since $f(z) = z - 1 + 0(1)$ as $z \to \infty$, e. g. in a Stolz angle contained in the left half plane, cf. Section 3.

5.2 Finding a 'large' subdomain of the Baker domain

The condition $|(1-2z)e^z| < \rho$ is equivalent to (since $x < 0$)

$$1 + 4|x| + 4r^2 = 1 - 4x + 4r^2 < \rho^2 e^{2|x|}$$

where $r^2 = x^2 + y^2$, $z = x + iy$. So $\partial D_{M,\rho}$ consists of pieces on which $x = -M$ or

$$1 + 4|x| + 4(x^2 + y^2) = \rho^2 e^{2|x|}\,. \tag{8}$$

At a point of intersection of such pieces we have

$$1 + 4M + 4(M^2 + y^2) = \rho^2 e^{2M}$$

or

$$y^2 = -M^2 + \frac{1}{4}\left(\rho^2 e^{2M} - 1 - 4M\right)\,. \tag{9}$$

Considering the case $y > 0$ and letting (8) determine y as a function of $|x|$, we get from (8) that

$$4 + 8|x| + 8y\frac{dy}{d|x|} = 2\rho^2 e^{2|x|}$$

so that

$$\frac{dy}{d|x|} = \frac{2\rho^2 e^{2|x|} - 4 - 8|x|}{8y}.$$

Hence if $|x| = M$ and (9) holds, we have

$$\frac{dy}{d|x|} = \frac{2\rho^2 e^{2M} - 4 - 8M}{8\sqrt{\frac{1}{4}(\rho^2 e^{2M} - 1 - 4M) - M^2}}\,.$$

Assuming also that

$$\frac{1}{4}(\rho^2 e^{2M} - 1 - 4M) - M^2 > 0\,. \tag{10}$$

If $\tan\beta = \frac{dy}{d|x|}$, we want to have $\alpha \le \beta < \frac{\pi}{2}$ to ensure that $f(z) \in D_{M,\rho}$, where (as in the above

picture),

$$\sin\alpha = \rho, \quad \alpha = \sin^{-1}\rho.$$

Thus

$$\tan\alpha = \frac{\rho}{\sqrt{1-\rho^2}}$$

and we want to have

$$\frac{\rho}{\sqrt{1-\rho^2}} \leq \frac{dy}{d|x|} = \frac{2\rho^2 e^{2M} - 4 - 8M}{8\sqrt{\frac{1}{4}(\rho^2 e^{2M} - 1 - 4M) - M^2}}, \tag{11}$$

that is,

$$0 < 64\rho^2 \left(\frac{1}{4}(\rho^2 e^{2M} - 1 - 4M) - M^2\right) \leq (1-\rho^2)(2\rho^2 e^{2M} - 4 - 8M)^2 . \tag{12}$$

A numerical calculation with Mathematica shows that if

$$\rho = 0.9 \quad \text{and} \quad M = 1.5$$

then (12) holds and the right hand side of each of (11) and (9) is strictly positive.

At points where $|x| > M$ and $y > 0$ we have

$$\frac{dy}{d|x|} = \frac{2\rho^2 e^{2|x|} - 4 - 8|x|}{8\sqrt{\frac{1}{4}(\rho^2 e^{2|x|} - 1 - 4(|x|) - |x|^2}} = \psi(|x|), \text{ say.}$$

Note that to show that here the right hand side is at least $\frac{\rho}{\sqrt{1-\rho^2}}$ for $|x| = M_1 > M$, say, is equivalent to proving (11) with M replaced by M_1.

Now a calculation shows that

$$\left(\frac{8}{\psi(|x|)}\frac{8}{d|x|}\psi(|x|)\right) \cdot (2\rho^2 e^{2|x|} - 4 - 8|x|)\left(\frac{1}{4}(\rho^2 e^{2|x|} - 1 - 4|x|) - |x|^2\right)$$

$$= 4\rho^2 e^{2|x|}(\rho^2 e^{2|x|} - 2 - 8|x|^2).$$

Write $\psi_1(|x|) = \rho^2 e^{2|x|} - 2 - 8|x|^2$. Then $\frac{1}{2}\psi_1'(|x|) = \rho^2 e^{2|x|} - 8|x|$ and $\frac{1}{4}\psi_1''(|x|) = \rho^2 e^{2|x|} - 4 > 0$, for $\rho = 0.9$, when $|x| > \frac{1}{2}\log\frac{4}{\rho^2} \approx 0.7985$, hence for $|x| \geq 0.8$. So $\psi_1'(|x|)$ is increasing for

$|x| \geq 0.8$. We have $\frac{1}{2}\psi_1'(1.5) \approx 4.27 > 0$ so that $\psi_1'(|x|) > 0$ for $|x| \geq 1.5$. Hence $\psi_1(|x|)$ is increasing in $|x|$ for $|x| \geq 1.5$. We have $\psi_1(2) \approx 10.22 > 0$ so that $\psi_1(|x|) > 0$ for $|x| \geq 2$.

When $|x| \geq 2$, we have

$$2\rho^2 e^{2|x|} - 4 - 8|x| > \rho^2 e^{2|x|} - 2 - 8|x|^2 = \psi_1(|x|) > 0,$$

and

$$4\left(\frac{1}{4}(\rho^2 e^{2|x|} - 1 - 4|x|) - |x|^2\right) = \rho^2 e^{2|x|} - 1 - 4|x| - 4|x|^2 > \psi_1(|x|) > 0,$$

so that $\psi(|x|)$ is well defined and $\psi(|x|) > 0$, and furthermore, $\psi'(|x|) > 0$ so that $\psi(|x|)$ is increasing.

Take $\rho = 0.9$ and $M = 2$. Again a numerical calculation shows that then (12) holds, while we have already checked above that the right hand side of each of (9) and (11) is strictly positive. We have proved Proposition 5.

5.3 On critical values in the Baker domain

Next, we show that z_0 is the only critical point of f in $\{\mathrm{Re}(z) \geq -2\}$. To this end we study three cases.

Suppose that $f'(z) = 0$, that is, $(1+2z)e^z = 1$. If $\mathrm{Re}(z) = x \geq 0$ and $z \neq 0$ then $|z-(-\frac{1}{2})| > \frac{1}{2}$ so that $|(1+2z)e^z| = e^x 2|z - \frac{-1}{2}| > 1$ and so $f'(z) \neq 0$. Thus, if $f'(z) = 0 \neq z$, then $\mathrm{Re}(z) = x < 0$.

Case 1: $x < 0$ and $|z - \frac{-1}{2}| \leq \frac{1}{2}$
We obtain $e^x < 1$ and $|1 + 2z| \leq 1$, so $|e^z(1 + 2z)| < 1$ and therefore $f'(z) \neq 0$.

Case 2: $-2 \leq x < 0$ and $|z - (-\frac{1}{2})| > e^2/2$
Now we obtain $|e^z(1 + 2z)| > 1$, so $f'(z) \neq 0$.

Case3: $-2 \leq x < 0$ and $|z + \frac{1}{2}| \leq e^2/2 < 4$
Then $|y| < 4$. If also $(1 + 2z)e^z = 1$ then, considering arguments, we see that there is $k \in \mathbb{Z}$ with

$$k2\pi = y + \arg\ z + \arg(1 + \frac{1}{2z}) \quad (13)$$
$$= y + \arg(1 + 2z). \quad (14)$$

We have

$$-3 \leq \mathrm{Re}(1 + 2z) < 1, \quad -8 < \mathrm{Im}(1 + 2z) < 8$$

By symmetry, we may assume that $y \geq 0$. Then we may take $0 \leq \arg(1 + 2z) \leq \pi$, and since $0 \leq y < 4$, we can only have $k = 0$ or $k = 1$. For a moment we assume $k = 0$. Then $y = 0$. Since $\arg(1 + 2z) = 0$ we obtain $-\frac{1}{2} \leq x < 0$. But

$$\frac{d}{dx}((1 + 2x)e^x) = (3 + 2x)e^x \geq \frac{2}{\sqrt{e}} > 0, \text{ on } \left[-\frac{1}{2}, 0\right[,$$

so that for $-\frac{1}{2} \leq x < 0$, we have $(1 + 2x)e^x < (1 + 2x)e^x|_{x=0} = 1$. Thus $f'(z) \neq 0$ for $z = x \in [-\frac{1}{2}, 0[$. In other words, $k = 0$ is not possible. Thus we have $k = 1$ and $y = 2\pi - \arg(1 + 2z)$. Consequently $\pi < y < 4$ and $\pi > \arg(1 + 2z) = 2\pi - y > 2\pi - 4 > \frac{2\pi}{3}$. Hence $-2 \leq x < -\frac{1}{2}$ and

$$1 > |\cos(\arg(1 + 2z))| = \frac{|1 + 2x|}{|1 + 2z|} > \left|\cos\frac{2\pi}{3}\right| = \frac{1}{2}$$

which yields

$$\begin{array}{rrcl} & (1 + 2x)^2 & > & \frac{1}{4}|1 + 2z|^2 \\ \iff & 4(1 + 4x + 4x^2) & > & 1 + 4(x^2 + y^2) + 4x \\ \iff & 3(1 + 4x^2 - 4|x|) & > & 4y^2 > 4\pi^2 \\ \iff & \left(\frac{4}{3}\pi^2 - 1\right)/4 & < & |x|(|x| - 1) \leq 2 \\ \iff & 4\pi^2 & < & 27, \end{array}$$

which is false. Thus $f'(z) \neq 0$ whenever $-2 \leq x < 0$.

If $z = x < -\frac{1}{2}$ then $(1 + 2x)e^x < 0$, hence $f'(z) \neq 0$. We have proved the following result.

Lemma 19 *If $f'(z) = 0 \neq z$ then $x = \mathrm{Re}(z) < -2$ and $y \neq 0$.*

Note that if $f'(z) = 0 \neq z$ then $z \notin D$ since $f(z) - (z - 1) = \frac{1-2z}{1+2z} \approx 1$ while $\rho = 0.9 < 1$.

However, we now prove the stronger statement in Theorem 3 for the critical values.

Proof of Theorem 3.
If $f'(z) = 0 \neq z$ then $x < -2$, so $x - 2 < -4$ by Lemma 19, and $\operatorname{Re} f(z) = x - 2 + \frac{x+\frac{1}{2}}{|z+\frac{1}{2}|^2}$ since $f(z) = z - 2 + \frac{1}{z+\frac{1}{2}}$ (see begin of this section) so that

$$2 + Re\ f(z) = x + \frac{x + \frac{1}{2}}{x^2 + x + \frac{1}{4}} < -2 < 0.$$

So it only needs to be shown that $\zeta = z - 2$ and $\zeta = f(z)$ satisfy $|(1 - 2\zeta)e^{\zeta}| < \rho$, that is, if $\zeta = \xi + i\eta$,

$$\begin{aligned} & \rho^2 > e^{2\xi}(1 + 4(\xi^2 + \eta^2) - 4\xi) \\ \iff & 1 - 4\xi + 4\xi^2 + 4\eta^2 < \rho^2 e^{-2\xi} = \rho^2 e^{2|\xi|}. \end{aligned}$$

Part (i): $\zeta = z - 2$

Noting that $1 + 2z = e^{-z}$, so that $1 + 4x^2 + 4y^2 + 4x = e^{-2x} = e^{2|x|} = 1 + 4x^2 + 4y^2 - 4|x|$, and $\eta = y$, $\xi = x - 2$, $|\xi| = -\xi = -x + 2 = |x| + 2$, we get

$$\begin{aligned} \rho^2 e^{2|\xi|} &= \rho^2 e^4 e^{2|x|} = \rho^2 e^4 (1 + 4x^2 + 4y^2 - 4|x|) \\ &= \rho^2 e^4 (1 + 4(\xi + 2)^2 + 4\eta^2 - 4(|\xi| - 2)) \\ &= \rho^2 e^4 (1 + 4\xi^2 + 16\xi + 16 + 4\eta^2 - 4|\xi| + 8) \\ &= \rho^2 e^4 (1 + 4(\xi^2 + \eta^2) + 12\xi + 24). \end{aligned}$$

Hence

$$\rho^2 e^{2|\xi|} > 1 - 4\xi + 4(\xi^2 + \eta^2) \tag{15}$$

if, and only if,

$$\begin{aligned} & 0 < 4(\rho^2 e^4 - 1)(\xi^2 + \eta^2) + (\rho^2 e^4 - 1) - 4|\xi|(3\rho^2 e^4 + 1) + 24\rho^2 e^4 \\ \Longleftarrow\ & 0 < 4(\rho^2 e^4 - 11)\xi^2 - 4|\xi|(3\rho^2 e^4 + 1) \end{aligned}$$

which is certainly true (since $\rho^2 e^4 > 1$) if

$$|\xi| \geq \frac{3\rho^2 e^4 + 1}{\rho^2 e^4 - 1} = \frac{3 + \frac{1}{\rho^2 e^4}}{1 - \frac{1}{\rho^2 e^4}}. \tag{16}$$

Now $|\xi| > 4$ while

$$\frac{3 + 1/(\rho^2 e^4)}{1 - 1/(\rho^2 e^4)} > 3.1$$

so that (16) holds. We deduce that $f'(z) = 0 \neq z$ implies that $z - 2 \in D$.

Part (ii): $\zeta = f(z)$

Suppose that $f'(z) = 0 \neq z$ and $\zeta = \xi + i\eta = f(z) = z - 2 + \frac{1}{z+\frac{1}{2}}$. Recall that $f'(z) = 0$ implies $e^z = \frac{1}{1-2z}$ which in turn yields $e^{2|\xi|} = 1 + 4|z|^2 - 4|x|$. Now

$$\xi = x - 2 + \frac{x + \frac{1}{2}}{|z + \frac{1}{2}|^2} < x - 2 < -4$$

so $|\xi| = -\xi > |x| + 2$. Thus

$$\rho^2 e^{2|\xi|} > \rho^2 e^4 e^{2|x|} = \rho^2 e^4 (1 + 4|z|^2 - 4|x|)\,. \tag{17}$$

Write $\frac{x+\frac{1}{2}}{|z+\frac{1}{2}|} = -\epsilon$. Then $0 < \epsilon < \frac{2}{3}$ since $x < -2$. Write $|y| \left|\mathrm{Im}\left(\frac{1}{z+\frac{1}{2}}\right)\right| = \frac{|y|^2}{|z+\frac{1}{2}|^2} = \theta < \frac{|y|^2}{y^2+\frac{9}{4}} \in [0, 1]$. We have

$$|x| = -x = -2 - \xi - \epsilon = |\xi| - 2 - \epsilon$$

and $\eta = y - \frac{\theta}{y}\,\mathrm{sgn}\, y$. We may assume that $y > 0$, then $0 < \theta/y^2 < 1$ and

$$0 < \eta = y\left(1 - \left|\frac{\theta}{y^2}\right|\right),$$

so

$$y = \frac{\eta}{1 - \frac{\theta}{y^2}} > \eta.$$

Hence

$$\begin{aligned} 1+4|z|^2-4|x| &= 1-4|\xi|+8+4\epsilon+4\left(|\xi|^2+(2+\epsilon)^2-2(2+\epsilon)|\xi|+\frac{\eta^2}{(1-\frac{\theta}{y^2})^2}\right) \\ &> 9-4|\xi|+4|\zeta|^2+16-\frac{64}{3}|\xi|. \end{aligned}$$

Recalling (17) we obtain

$$\rho^2 e^{2|\xi|} > \rho^2 e^4\left(25+4|\zeta|^2-4|\xi|-\frac{64}{3}|\xi|\right)$$

which is, compare (15),

$$\geq 1+4|\zeta|^2-4\xi = 1+4|\zeta|^2+4|\xi|$$

if, and only if,

$$0 < (25\rho^2e^4-1)+(\rho^2 3^4-1)4|\zeta|^2-|\xi|\left(4+\rho^2e^4\frac{76}{3}\right). \tag{18}$$

Since

$$\frac{25\rho^2e^4-1}{4(\rho^2e^4-1)} > 6.388 \quad \text{and} \quad \frac{4+\rho^2e^4\frac{76}{3}}{4(\rho^2e^4-1)} < 6.503,$$

(18) is certainly true if

$$0 < 6.388+|\xi|(|\xi|-6.503). \tag{19}$$

This is true if $|\xi| \geq 6.503$.

If $4 < |\xi| < 6.503$, then the right hand side of (19) is

$$> 6.388+4(4-6.503) > -3.624.$$

Hence (18) holds if $\eta^2 > 3.624$, or $\eta > 1.904$.

Finally, we deal with the case $2 < |x| = |\xi| - 2 - \epsilon < 4.503$, and $\eta \le 1.904$. Here we obtain

$$0 < y = \eta - \mathrm{Im}\frac{1}{z+\frac{1}{2}} = \eta + \frac{y}{|z+\frac{1}{2}|^2} = \eta + \frac{y}{(x+\frac{1}{2})+y^2} < \eta + \frac{1}{3} < 2.238.$$

(because $\frac{y}{(x+\frac{1}{2})^2+y^2} < \frac{y}{y^2+\frac{9}{4}} \le \frac{y}{y^2+\frac{9}{4}}\Big|_{y=3/2} = \frac{1}{3}$.) Assuming that there is $z = x + iy \neq 0$ with $f'(z) = 0$, with $0 < y < 2.2371$ and $2 < -x < 4.503$, return to (13), we get a contradiction since $0 < y + \arg(1+2z) < 2.2371 + \pi < 2\pi$. This proves Theorem 3. ■

5.4 On critical points in the Baker domain

The previous theorem states that the critical values (except the origin) are located in the Baker domain $\mathcal{B}$. Now we prove that the corresponding critical points are located in the Baker domain, too, which is the statement of Theorem 4.

Proof of Theorem 4.
Pick z with $0 \neq z = f'(z)$, and set $L = \{z - t : 0 \le t \le 2\}$. We shall show that $f(L) \subset D$. Since $z - 2 \in L \cap D \subset L \cap \mathcal{B}$ we then have $L \subset \mathcal{B}$, so $z \in L \subset \mathcal{B}$.

We may assume, by symmetry, that $y > 0$. We have

$$f(z-t) = z - t - 1 + \frac{1-2z+2t}{1+2z}e^{-t}$$

since $e^z = \frac{1}{1+2z}$. Further,

$$f(z-t) = z - t - 1 - e^{-t} + \frac{2e^{-t}(1+t)}{1+2z}\,.$$

Thus

$$Re\ f(z-t) = x - t - e^{-t} - 1 + \frac{2e^{-t}(1+t)(1+2x)}{1+4x+4|z|^2} < x - 2 < -4 \tag{20}$$

(since $1 + 2x < 0$ and $\frac{d}{dt}(t + e^{-t} - 1) = 1 - e^{-t} > 0$, $(t + e^{-t} - 1)|_{t=o} = 0$, so $t + e^{-t} - 1 > 0$ for $0 < t < 2$) and

$$\Im f(z-t) = y - \epsilon_1 \quad \text{where} \quad 0 < \epsilon_1 = \frac{4e^{-t}(1+t)y}{1+4x+4|z|^2} \le \frac{4y}{|1+2z|^2} < y,$$

so $\mathrm{Im}(z) > \mathrm{Im} f(z-t) > 0$ (note: $e^{-t}(1+t)$ is decreasing for $t \geq 0$). Since $\mathrm{Re}\, f(z-t) < -4$, in order to show that $f(z-t) \in D$, it suffices to prove that with $\zeta = \xi + i\eta = f(z-t)$, we have

$$\rho^2 e^{2|\xi|} > 1 + 4\xi^2 + 4\eta^2 + 4|\xi| = 1 + 4|\zeta|^2 - 4\xi = |1 - 2\zeta|^2 .$$

Now $\eta^2 < y^2$, and $\rho^2 e^{2|\xi|} > \rho^2 e^4 e^{2|x|} = \rho^2 e^4 (1 + 4x^2 + 4y^2 - 4|x|) > \rho^2 e^4 (1 + 4\eta^2 + 4x^2 - 4(|\xi| - 2))$. Further,

$$\begin{aligned}
|x| = -x &= -\xi - t - e^{-t} - 1 + \frac{2e^{-t}(1+t)(1+2x)}{1 + 4x + 4|z|^2} \\
&= |\xi| - (t + e^{-t} + 1) - \frac{2e^{-t}(1+t)(2|x| - 1)}{1 - 4|x| + 4|z|^2} \\
&\geq |\xi| - (3 + e^{-2}) - 2\frac{2|x| - 1}{1 - 4|x| + 4|z|^2} \\
&\geq |\xi| - (3 + e^{-2}) - \frac{2}{2|x| - 1} \\
&\geq |\xi| - (3 + e^{-2}) - \frac{2}{3} \\
&> |\xi| - 3.81 \\
&> 0
\end{aligned}$$

since $|\xi| > 4$ by (20). Thus

$$\begin{aligned}
\rho^2 e^{2|\xi|} &> \rho^2 e^4 (9 + 4\eta^2 - 4|\xi| + 4x^2) \\
&\geq \rho^2 e^4 (9 + 4\eta^2 - 4|\xi| + 4|\xi|^2 + 4 \cdot 3.81^2 - 7.62|\xi|)
\end{aligned}$$

which is

$$\geq 1 + 4\xi^2 + 4\eta^2 + 4|\xi|$$

provided that

$$0 < (9\rho^2 e^4 - 1) + 4 \cdot (3.81)^2 \rho^2 e^4 + 4\eta^2(\rho^2 e^4 - 1) + 4\xi^2(\rho^2 e^4 - 1) - |\xi|(11.62\rho^2 e^4 + 4)$$

which is certainly true if

$$|\xi| \geq \frac{11.62\rho^2 e^4 + 4}{4(\rho^2 e^4 - 1)} \approx 2.99534,$$

and this holds since $-\xi < -4$. This proves Theorem 4. ■

Finally, we turn our attention to the fixed points of f and show that the origin is the only non–repelling fixed point of f.

Lemma 20 *If* $f(z) = z \neq 0$ *then* $|f'(z)| = |2 + \frac{1}{z-\frac{1}{2}}| > 1$.

Proof of Lemma 20. If $f(z) = z \neq 0$ and $|z - \frac{1}{2}| > 1$ then $|f'(z)| > 2 - 1 = 1$. When $|z - \frac{1}{2}| = 1$, we have $|(1-2z)e^z| = 2e^x \geq 2/\sqrt{e} > 1$ so that the functions $(1-2z)e^z$ and $-1+(1-2z)e^z$ have the same number of zeros (one zero) in $B(\frac{1}{2}, 1)$, by Rouche's theorem, and for $-1 + (1-2z)e^z$ this must be the zero at $z = 0$. Hence there are no further fixed points to be considered. This completes the proof of the lemma. ■

6 On the approximating polynomials p_d

Define

$$p_d(z) = z - 1 + (1 - 2z)\left(1 + \frac{z}{d}\right)^d$$

for a large $d \in \mathbb{Z}_+ = \{1, 2, 3, \dots\}$. We note

$$p_d'(z) = 1 - 2\left(1 + \frac{z}{d}\right)^d + (1 - 2z)\left(1 + \frac{z}{d}\right)^{d-1} \tag{21}$$

and

$$p_d''(z) = -4\left(1 + \frac{z}{d}\right)^{d-1} + \frac{d-1}{d}(1 - 2z)\left(1 + \frac{z}{d}\right)^{d-2}. \tag{22}$$

If $p_d(z) = z$, then $(1 + \frac{z}{d})^d = \frac{1}{1-2z}$ (and conversely) and hence

$$p_d'(z) = 1 - \frac{2}{1-2z} + \frac{1-2z}{1+\frac{z}{d}}\frac{1}{1-2z} = 1 + \frac{1}{z-\frac{1}{2}} + \frac{1}{1+\frac{z}{d}}.$$

We also write

$$w = 1 + \frac{z}{d}, \quad \text{so} \quad z = d(w-1) = -d + dw.$$

So if $p_d(z) = z$ then

$$p_d'(z) = 1 + \frac{1}{w} + \frac{1}{-(d+\frac{1}{2}) + dw} = 1 + \frac{1}{w} - \frac{1}{2w^d}$$

and

$$w^d = \frac{1}{1-2z} = \frac{1}{2d+1-2dw} = \frac{1}{2d}\,\frac{1}{1-w+\frac{1}{2d}}.$$

We have

$$\begin{aligned} \deg\ p_d = \deg(p_d - z) &= d+1, \\ \deg\ p_d' &= d. \end{aligned}$$

Also $p_d(0) = p_d'(0) = 0 \neq -4 + \frac{d-1}{d} = p_d''(0)$.

Thus outside the origin (and ∞), p_d has d fixed points and $d-1$ critical points. Here note that

$$(p_d - z)' = p_d' - 1 = \frac{(1+\frac{1}{d})z + \frac{1}{2}}{(z-\frac{1}{2})(1+\frac{z}{d})} \neq 0$$

when $p_d(z) = z$. Namely, note that

$$p_d\left(\frac{1}{2}\right) = -\frac{1}{2} \neq \frac{1}{2}, \quad p_d(-d) = -d - 1 \neq -d,$$

$$p_d\left(-\frac{d}{2(d+1)}\right) = \frac{-d}{2(d+1)} - 1 + \frac{1}{2^d}\left(\frac{2d+1}{d+1}\right)^{d+1} \neq \frac{-d}{2(d+1)},$$

the point $\frac{-d}{2(d+1)}$ being the zero of $(1+\frac{1}{d})z + \frac{1}{2}$. Also when $p_d(z) = z \neq 0$, we have

$$p_d'(z) = \frac{z + (\frac{d}{2}+2)}{d(z-\frac{1}{2})(1+\frac{z}{d})} z \neq 0$$

since

$$(p_d(z) - z)\left(\frac{-d}{2} - 2\right) = -1 + (5+d)\left(\frac{1}{2} - \frac{2}{d}\right)^d \neq 0.$$

So the fixed points of p_d outside 0 (and ∞) are neither superattracting nor parabolic, in particular, they are simple as fixed points, so there are d distinct fixed points outside 0 (and ∞).

For d large, say, $d \geq 2$, we have $p_d'(z) = 0$ if, and only if,

$$\begin{aligned} 0 &= 1 - 2w^d + w^{d-1}(2d + 1 - 2dw) \\ &= 1 + (2d + 1 - 2(d+1)w)w^{d-1} \\ &= (1-w)(1 + w + w^2 + \cdots + w^{d-2} + 2(d+1)w^{d-1}) \\ &= (1-w)q(w)\,, \end{aligned}$$

say, where $w = 1$ corresponds to $z = 0$.

For $|w| = 1$, we have

$$|1 + w + \cdots + w^{d-2}| \leq d - 1 < 2(d+1) = |2(d+1)w^{d-1}|$$

so that, by Rouche's theorem, $q(w)$ and $2(d+1)w^{d-1}$ have the same number, $d-1$, of zeros in the open unit disk. Hence the critical points of p_d all lie in $\{0\} \cup B(-d, d)$. For $|w| = r < 1$ we have

$$|1 + w + \cdots + w^{d-2}| \leq 1 + r + \cdots + r^{d-2} = \frac{1 - r^{d-1}}{1 - r} < 2(d+1)r^{d-1} = |2(d+1)w^{d-1}|$$

provided that

$$1 - r^{d-1} < 2(d+1)(1-r)r^{d-1} \Longleftrightarrow 1 < r^{d-1}(1 + 2(d+1)(1-r))$$

so that again, q has all of its $d-1$ zeros in $B(0, r)$, and p_d has its critical points in

$$\{0\} \cup B(-d, rd).$$

We may take, e. g. , (as a calculation shows)

$$1 - r = \frac{1}{d-1},\ r = 1 - \frac{1}{d-1} = \frac{d-2}{d-1}.$$

Also we may take $r = 1/2^{\frac{1}{d-1}}$ since $\frac{1}{2(d+1)} + \frac{1}{2^{\frac{1}{d-1}}} < 1$ as one can check. In fact, $\frac{1}{2(d-1)} + \frac{1}{2^{\frac{1}{d-1}}} \leq 1$ when $d \geq 2$. Note that equality holds only in the case $d = 2$.

On the other hand, if $0 < \rho < 1$ and $\rho^{d-1}(2d+1+2(d+1)\rho) < 1$, then it follows from Rouche's theorem that $1 + (2d + 1 - 2(d+1)w)w^{d-1}$ has no zeros in $\overline{B}(0, \rho)$, and so $p_d'(z) \neq 0$ for all $z \in B(-d, \rho d)$. Here we may take, e. g. ,

$$\rho = \frac{1}{(4d+3)^{\frac{1}{d-1}}} \to 1 \text{ as } d \to \infty.$$

Note, that for large d, say, $d \geq 3$, r and ρ depend monotonically increasing on d. In addition,

we have $r = \frac{d-2}{d-1} > \frac{1}{(4d+3)^{\frac{1}{d-1}}} = \rho$. Consequently, all critical points of p_d disctinct from 0 lie in the annulus $B(-d, rd) \setminus \overline{B}(-d, \rho d)$. r and ρ can be easily computed. E. g. if $d = 64$, then $\rho = 0.915774$ and $r = 0.984127$. We have (almost) proved the following

Proposition 21 *The finite critical points of p_d are at 0, and $d-1$ other points in*

$$\left\{\zeta : \rho < \left|\frac{\zeta + d}{d}\right| < r\right\}$$

where

$$0 < \rho = \frac{1}{(4d+3)^{\frac{1}{d-1}}} < \frac{d-2}{d-1} = \rho < 1\,.$$

We have $\rho \to 1$ and $r \to 1$ as $d \to \infty$. All critical points are simple zeros of p_d'.

Proof of Proposition 21.
Only the last statement requires a proof any more. From the representation (22) we see that p_d'' has a zero of order $d-2$ at $z = -d$, and a simple zero at $z = -(3d+1)/(2d+2)$, and no other zeros. But $p_d'(-d) = 1 \neq 0$ and

$$\begin{aligned} p_d'\left(-\frac{3d+1}{2d+2}\right) &= 1 - \left(1 - \frac{3d+1}{2d(d+1)}\right)^{d-1}\left(1 - \frac{3d+1}{d}\right) \\ &= 1 + \left(1 - \frac{3d+1}{2d(d+1)}\right)^{d-1}(2d-1) \\ &\neq 0\,. \end{aligned}$$

This completes the proof of Proposition 21. ∎

Since $r < 1$ in Proposition 21, it follows that if $p_d'(z) = 0 \neq z$ then $\mathrm{Re}(z) < 0$.

One may note that

$$rd = d - \frac{d}{d-1} = d - 1 - \frac{1}{d-1} \simeq d - 1 + O(\frac{1}{d})$$

and

$$-\log\rho = \frac{\log 4 + \log d + \log(1 + \frac{3}{4d})}{d-1} \simeq \frac{\log d + O(1)}{d}$$

so that $\rho \simeq 1 - \frac{\log d + O(1)}{d}$ and $d(r - \rho) = \log d + O(1)$. We note

$$1\rho \simeq O\left(\frac{\log d}{d}\right) \tag{23}$$

.

Next we consider the *fixed points* of p_d other than 0 and ∞. At those,

$$1 = w^d(2d + 1 - 2dw).$$

If $0 < \rho < 1$ and $\rho^d(2d + 1 + 2d\rho) < 1$, then by Rouche's theorem, 1 and $1 - w^d(2d + 1 - 2dw)$ have the same number of zeros (actually no zeros) in $\overline{B}(0, \rho)$. We may take

$$\rho = \frac{1}{(4d+1)^{1/d}} \to 1 \quad \text{as} \quad d \to \infty.$$

The solution $w = 1$ corresponds to $z = 0$. We have

$$1 - w^d(2d + 1 - 2dw) = (w - 1)(2dw^d - (1 + w + w^2 + \cdots + w^{d-1})).$$

If $0 < r < 1$ and

$$1 + r + r^2 + \cdots + r^{d-1} = \frac{1 - r^d}{1 - r} < 2dr^d \Leftrightarrow 1 < r^d(1 + 2d(1 - r))$$

then by Rouche's theorem, $2dw^d$ and $2dw^d - (1 + \cdots + w^{d-1})$ have the same number, d, of zeros in $B(0, r)$ so that all the zeros of p'_d, other than $z = 0$, correspond to these points w in$B(0, r)$. We may take, e. g. , $r = \frac{d-2}{d-1}$, or

$$r = \frac{1}{2^{1/d}} \to 1 \quad \text{as} \quad d \to \infty,$$

since $\frac{1}{2d} + \frac{1}{2^{1/d}} < 1$ for $d > 1$ as one can verify (since $\frac{d}{dx}x^{\frac{1}{d}}|_{x=1} = \frac{1}{d}$ and $x^{1/d}$ is concave). We have proved the result.

Proposition 22 *If* $p_d(z) = z \neq 0$ *then*

$$z \in \left\{\zeta : \rho < \left|\frac{\zeta + d}{d}\right| < r\right\}$$

where $0 < \rho < r < 1$ *and* $\rho = \frac{1}{(4d+1)^{1/d}}$, $r = \frac{1}{2^{1/d}}$.

We now consider

$$p'_d(z) = 1 + \frac{1}{w} - \frac{1}{2w^d} = 1 + \frac{1}{w} + \frac{1}{dw - (d + \frac{1}{2})}$$

when $p_d(z) = z \neq 0$. We have seen that then $\rho < |w| < r$. The expressions for ρ and r show

that

$$\frac{1}{4d+1} < |w|^d < \frac{1}{2}$$

but in fact, $1/w^d = 2d + 1 - 2dw$ so that $1 + 2d(1-r) \leq |w|^{-d} \leq 1 + 2d(1+r)$.

Next we want to show tht all fixed points distinct from z_0 are repelling. To this end, we estimate $|p_d'(z)|$. Recall that for fixed points $p_d'(z) = 1 + \frac{1}{w} + \frac{1}{dw-(d+\frac{1}{2})}$. We introduce $A = dw - (d + \frac{1}{2})$. Consequently

$$\begin{aligned}
|p_d'(z)|^2 &= p_d'(z) \cdot \overline{p_d'(z)} \\
&= 1 + \frac{1}{w} + \frac{1}{A} + \frac{1}{\overline{w}} + \frac{1}{w\overline{w}} + \frac{1}{\overline{w}A} + \frac{1}{\overline{A}} + \frac{1}{w\overline{A}} + \frac{1}{A\overline{A}} \\
&= 1 + \frac{2\mathrm{Re}(w)+1}{w\overline{w}} + \frac{A+\overline{A}}{A\overline{A}} + \frac{1}{A\overline{A}} + \frac{w\overline{A}+\overline{w}A}{w\overline{w}A\overline{A}} \\
&= 1 + \frac{2\mathrm{Re}(w)+1}{w\overline{w}} + \frac{1+2d\mathrm{Re}(w)-2(d+\frac{1}{2})}{A\overline{A}} + \frac{2dw\overline{w}-2\mathrm{Re}(w)(d+\frac{1}{2})}{w\overline{w}A\overline{A}} \\
&= 1 + \frac{2\mathrm{Re}(w)+1}{w\overline{w}} + \frac{2d\mathrm{Re}(w)}{A\overline{A}} - \frac{2\mathrm{Re}(W)(d+\frac{1}{2})}{w\overline{w}A\overline{A}}
\end{aligned}$$

and

$$|p_d'(z)|^2 - 1 = \frac{2\mathrm{Re}(w)+1}{w\overline{w}} + \frac{2d\mathrm{Re}(w)}{A\overline{A}} - \frac{2\mathrm{Re}(W)(d+\frac{1}{2})}{w\overline{w}A\overline{A}}\,.$$

We have to show that the latter expression is strictly positive. Note that

$$\begin{aligned}
A\overline{A} &= \left(dw - (d+\frac{1}{2})\right) \cdot \left(d\overline{w} - (d+\frac{1}{2})\right) \\
&= d^2 w\overline{w} - dw\left(d+\frac{1}{2}\right) - d\overline{w}\left(d+\frac{1}{2}\right) + \left(d+\frac{1}{2}\right)^2 \\
&= d^2 w\overline{w} - 2\mathrm{Re}(w)d\left(d+\frac{1}{2}\right) + \left(d+\frac{1}{2}\right)^2 .
\end{aligned}$$

We obtain

$$
\begin{aligned}
& d^{-2} w\overline{w} A\overline{A}\,(|p_d'(z)|^2 - 1) \\
&= d^{-2}\left((2\mathrm{Re}(w)+1)A\overline{A} + 2d\mathrm{Re}(w)|w\overline{W} - 2\mathrm{Re}(w)\left(d+\frac{1}{2}\right)\right) \\
&= (2\mathrm{Re}(w)+1)\left(|w|^2 - 2\mathrm{Re}(w)\left(1+\frac{1}{2d}\right) + \left(1+\frac{1}{2d}\right)^2\right) + \\
&\qquad \frac{2}{d}\mathrm{Re}(w)\left(w\overline{w} - \left(1+\frac{1}{2d}\right)\right) \\
&= 2\mathrm{Re}(w)|w|^2 + |w|^2 - 4(\mathrm{Re}(w))^2\left(1+\frac{1}{2d}\right) - 2\mathrm{Re}(w)\left(1+\frac{1}{2d}\right) + \\
&\qquad 2\mathrm{Re}(w)\left(1+\frac{1}{2d}\right)^2 + \left(1+\frac{1}{2d}\right)^2 + \frac{2}{d}\mathrm{Re}(w)|w|^2 - \frac{2}{d}\mathrm{Re}(w)\left(1+\frac{1}{2d}\right) \\
&= 2\mathrm{Re}(w)|w|^2 + |w|^2 - 4(\mathrm{Re}(w))^2 + 1 + \frac{2}{d}\mathrm{Re}(w)|w|^2 - \frac{1}{d}\mathrm{Re}(w)\left(1+\frac{1}{2d}\right) - \\
&\qquad \frac{2}{d}(\mathrm{Re}(w))^2 + \frac{1}{d} + \frac{1}{4d^2} \\
&= \Big(-4(\mathrm{Re}(w))^2 + 2\mathrm{Re}(w) + 2\Big) + (|w|^2-1)2\mathrm{Re}(w) + (|w|^2-1) + O\left(\frac{1}{d}\right) \\
&= -2\big(2(\mathrm{Re}(w))^2 - \mathrm{Re}(w) - 1\big) + O\left(\frac{\log d}{d}\right)
\end{aligned}
$$

because $0 < \rho < |w| < r < 1$ and the estimate (23).

Now

$$(\mathrm{Re}(w))^2 - \frac{1}{2}\mathrm{Re}(w) - \frac{1}{2} = 0 \Longleftrightarrow \mathrm{Re}(w) = 1 \quad \text{or} \quad \mathrm{Re}(w) = \frac{-1}{2}$$

and $u^2 - \frac{1}{2}u - \frac{1}{2} < 0 \Leftrightarrow -\frac{1}{2} < u < 1$. We conclude that the following holds.

Proposition 23 *If $p_d(z) = z \neq 0$ then $|p_d'(z)| > 1$ if, and only if,*

$$\mathrm{Re}(w) > -\frac{1}{2} + \psi(w) \quad \textit{where} \quad \psi(w) = O\left(\frac{\log d}{d}\right),$$

that is,

$$\mathrm{Re}(z) > -\frac{3d}{2} + \psi_1(z) \quad \textit{where} \quad \psi_1(z) = O\left(\log d\right).$$

Furthermore, $\rho < |w| < r$ and so $\rho d < |z+d| < rd$.

Finally, we give a first statement on the size of the basins of the attraction of the attracting fixed points.

Lemma 24 *Let z_0 be an attracting fixed point of p_d (so $z_0 \neq 0$). Let $A(z_0)$ be the component of the Fatou set $\mathcal{F}(p_d)$ containing z_0. Then $B(z_0, r) \subset A(z_0)$, where*

$$r = \frac{1 - |p_d'(z_0)|}{4d+1}.$$

If $\varepsilon \in (0,1)$, we have $1 - |p_d'(z_0)| > \varepsilon$ provided that $d \geq d_0(\varepsilon)$ and $\mathrm{Re}(z_0) < (-\frac{3}{2} - \delta)d$, where $\delta = \delta(\varepsilon) > 0$. Here $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Proof of Lemma 24: Write $w = 1 + \frac{z}{d}$ and $p_d(z) = p_d(w)$. Now

$$p_d''(z) = -4w^{d-1} + \frac{d-1}{d} w^{d-2}(2d+1-2dw) = (2d+1)\left(1 - \frac{1}{d}\right) w^{d-2} - \left(4 + 2d\left(1 - \frac{1}{d}\right)\right) w^{d-1}.$$

Hence for $2 \leq k \leq d+1$, we have

$$\begin{aligned} p_d^{(k)}(z) &= \frac{1}{d^{k-2}} \Bigg[-(2d+1) w^{d-k+1} (d-1)(d-2)\dots(d-k+2) \\ &\qquad + (2d+1)\frac{d-1}{d^{k+1}} w^{d-k} (d-2)\dots(d-k+1) \Bigg] \\ &= \frac{w^{d-k}}{d^{k-1}} (k-2)! \binom{d-1}{k-2} \Big[(2d+1)(d-k+1) - 2d(d+1)w \Big] \end{aligned}$$

so that

$$\frac{p_d^{(k)}(z)}{k!} = \binom{d-1}{k-2} \frac{w^{d-k}}{k(k-1)d^{k-1}} \Big[2d+1)(d-k+1) - 2d(d+1)w \Big], \ 2 \leq k \leq d+1,$$

while

$$p_d'(z) = 1 - 2w^d + w^{d-1}(2d+1-2dw).$$

When $p_d(z_0) = z_0 \neq 0$ and $w_0 = 1 + \frac{z_0}{d}$, we have

$$p_d'(z_0) = 1 + \frac{1}{w_0} - \frac{1}{2w_0^d}.$$

We may write

$$p_d(z_0 + \zeta) = z_0 + p_d'(z_0)\zeta + \sum_{k=2}^{d+1} \binom{d-1}{k-2} \frac{w_0^{d-k}\zeta^k}{k(k-1)d^{k-1}} \left[(2d+1)(d-k+1) - 2d(d+1)w_0 \right]$$

which implies that (with $j = k-2,\ 0 \le j \le d-1$)

$$\left|p_d(z_0+\zeta) - z_0 - p_d'(z_0)\zeta\right| \le \sum_{j=0}^{d-1} \binom{d-1}{j} \frac{|\zeta|^2}{2d|w_0|} \frac{|w_0|^{d-1-j}|\zeta|^j}{d^j} \left[\underbrace{(2d+1)(d-1)+2d(d+1)}_{d(4d+1)}\right]$$
$$= \frac{4d+1}{2}\frac{|\zeta|^2}{|w_0|}\left(\frac{|\zeta|}{d}+|w_0|\right)^{d-1}.$$

We have $|w_0| < r = 2^{-1/d} = e^{-(\log 2)/d} < 1 - \frac{\log 2}{2}d^{-1}$ when $d \ge d_0$ for some absolute constant d_0. Choosing $d \ge d_0$ and $|\zeta| < \frac{1}{2}\log 2$, we get $\frac{|\zeta|}{d} + |w_0| < 1$. Also $|w_0| > \rho > \frac{1}{2}$ when $d \ge d_0$. Hence

$$|p_d(z_0+\zeta) - z_0 - p_d'(z_0)\zeta| < (4d+1)|\zeta|^2.$$

Then $(4d+1)|\zeta|^2 < (1-|p_d'(z_0)|)\,|\zeta|$ and the above inequality gives

$$|p_d(z_0+\zeta) - z_0| < |p_d'(z_0)\zeta| + (4d+1)|\zeta|^2.$$

provided

$$|\zeta| < \frac{1-|p_d'(z_0)|}{4d+1}.$$

From this it is easy to see that the iterates of p_d tends to z_0 locally uniformly in $B\left(z_0, \frac{1-|p_d'(z_0)|}{4d+1}\right)$, so this disk is a subset of $A(z_0)$.

Recall

$$1 - |p_d'(z_0)| > \frac{1-|p_d'(z_0)|^2}{2} =$$
$$\frac{1}{2|w_0|^2}\frac{1}{|w_0 - 1 + \frac{1}{2d}|^2}\left[4\left((\mathrm{Re}(w_0))^2 - \frac{1}{2}(\mathrm{Re}(w_0)) - \frac{1}{2}\right) + O\left(\frac{\log d}{d}\right)\right].$$

Now if $\mathrm{Re}(w) < -\frac{1}{2} - \delta$ for a suitable $\delta = \delta(\varepsilon) > 0$, and if $d \ge d_0(\varepsilon)$, we can achieve $1 - |p_d'(z_0)| > \varepsilon > 0$. If $\varepsilon \to 0$, we can let $\delta \to 0$ provided that $d \ge d_0(\varepsilon)$ is large enough. This proves Lemma 24. ■

7 Some technical lemmas

Lemma 25 *Let* $z = x + \left(\left(2k - \frac{1}{2}\right)\pi - \frac{1}{\ln(k)}\right)i$ *with* $x \ge 0$ *Then* $\mathrm{Im}(f(z)) \ge \mathrm{Im}(z)$ *for all but*

finitely many $k \in \mathbb{N}$.

Proof of Lemma 25:
Under the hypothesis of the lemma, z is in the upper right quarter of the complex plane, thus

$$0 \leq \arg(z) \leq \frac{1}{2}\pi$$

and

$$0 \leq \arg(2z-1) \leq \frac{1}{2}\pi + O\left(\frac{1}{k}\right)$$

$\mathrm{Im}(z) = \left(\left(2k-\frac{1}{2}\right)\pi - \frac{1}{\ln(k)} \equiv \left(\frac{3}{2} - \frac{1}{\ln(k)}\right) \bmod 2\pi\right.$ yields $\arg(e^{-z} = \left(\frac{1}{2} - \frac{1}{\ln(k)}\right)$. Altogether we obtain

$$0 \leq \frac{1}{2} - \frac{1}{\ln(k)} \leq \arg\left((1-2z)e^z\right) \leq \pi - \frac{1}{\ln(k)} + O\left(\frac{1}{k}\right) \leq \pi$$

for all but finitely many $k \in \mathbb{N}$. Therefore, $\mathrm{Im}(f(z)) - \mathrm{Im}(z) \geq 0$. ■

Lemma 26 *Let* $z = x + \left(2k-\frac{1}{2}\right)\pi i$ *and* $x \geq 0$. *Then* $f(z) \in D_0$ *for all but finitely many* $k \in \mathbb{N}$.

Proof of Lemma 26:
We have $e^z = -ie^x$ and

$$(1-2z)e^z = -(4k-1)\pi e^x + i(2x-1)e^x\,.$$

This yields

$$f(z) = [x - 1 - (4k-1)\pi e^x] + \left[\left(2k-\frac{1}{2}\right)\pi + (2x-1)e^x\right] i$$

which in turn implies

$$0 \leq \mathrm{Im}(f(z)) \leq 2k\pi + 2xe^x\,.$$

In addition, we obtain

$$-4k\pi e^x \leq \mathrm{Re}(f(z)) \leq -(4k\pi - 2)e^x \leq -3k\pi e^x\,.$$

In order to show $f(z) \in D_0$ we check

$$\begin{array}{rrcl} & (2k\pi + 2xe^x)^2 & \leq & \left(\frac{1}{4}\exp\left(\left(2k\pi - \frac{1}{2}\right)e^x - x\right) - \frac{1}{2}\right)^2 - (4k\pi e^x)^2 \\ \Longleftarrow & (2k\pi + 2xe^x)^2 + (4k\pi e^x)^2 & \leq & \frac{1}{16}\exp\left((4k\pi - 3)e^x\right) \\ \Longleftarrow & (2k\pi + 2x)^2 + (4k\pi)^2 & \leq & \frac{1}{16}\exp\left(9ke^x\right) \\ \Longleftarrow & O((k+x)^2) & \leq & \exp(ke^x) \end{array}$$

and the latter is true for all $x \geq 0$ and all but finitely many $k \in \mathbb{N}$. ■

Lemma 27 *There exists some constant $C > 0$ such that for each $z \in D_0$ and every $n \in \mathbb{N}$ we have $|\mathrm{Im}(f^n(z) - z)| \leq C$.*

Proof of Lemma 27:
If $z \in D_0$, then $|(1-2z)e^z| \leq \frac{1}{2}$. Therefore we immediately obtain

$$|\mathrm{Im}(f^n(z) - z)| \leq \frac{n}{2}$$

and

$$|\mathrm{Re}(z)| + \frac{3n}{2} \leq |\mathrm{Re}(f^n(z))| \leq |\mathrm{Re}(z)| + \frac{n}{2}\,.$$

Altogether this yields

$$|f^n(z)| \leq n\sqrt{\frac{9}{4} + \frac{1}{4}} + O(1) \leq 2n$$

which in turn implies, recall $\mathrm{Re}(f^n(z)) \leq 0$,

$$\begin{aligned} |f^{n+1}(z) - f^n(z)| &\leq |1 - 2f^n(z)| \exp\left(-|\mathrm{Re}(f^n(z))|\right) \\ &\leq (4n+1)\exp\left(-|\mathrm{Re}(z)| - \frac{n}{2}\right) \\ &\leq (4n+1)e^{\frac{-n}{2}} \end{aligned}$$

and

$$|\mathrm{Im}(f^{n+1}(z)| \leq |\mathrm{Im}(z)| + C$$

where

$$C = \sum_{n=1}^{\infty}(4n+1)e^{\frac{-n}{2}} < \infty\,.$$

■

Proposition 28 *For $k \in \mathbb{N}$ we define R_k to be the rectangle*

$$\left\{ z = x + iy \,:\, -\left(\ln(k) + \frac{7}{2}\right) \le x \le 0 \text{ and } \left(2k - \frac{1}{2}\right)\pi - \frac{1}{\ln(k)} \le \mathrm{Im}(z) \le \left(2k - \frac{1}{2}\right)\pi \right\}.$$

Then $f(R_k) \subset D_0$, and $D_0 \cup R_k \subset \mathcal{B}$ holds for all but finitely many $k \in \mathbb{N}$.

Proof of Proposition 28:
Due to the maximum principle, for $f(R_k) \subset D_0$ it suffices to show

$$f(z) \in D_0 \tag{24}$$

for every $z \in \partial R_k$. Note that R_k is a rectangle. We shall prove (24) for each of the four sides of R_k separately. As we will see, the left side of the rectangle is contained in $\mathcal{B}$. Therefore, $f(R_k) \subset D_0 \subset \mathcal{B}$ implies $R_k \subset \mathcal{B}$. Thus we need to show (24), only.

Step 1 – left side Let $z = -\left(\ln(k) + \frac{7}{2}\right) + iy \in R_k$. In particular, $|\mathrm{Im}(z)| < \left(2k - \frac{1}{2}\right)\pi$. We have $|x| \ge \ln(k) + \ln(8\pi) + \eta$ for some $\eta > 0$. Then

$$\begin{array}{rrcl}
 & y^2 & \le & \left(\frac{1}{4}e^{|x|} - \frac{1}{2}\right)^2 - x^2 \\
\Longleftarrow & \left(\left(2k - \frac{1}{2}\right)\pi\right)^2 & \le & \left(2\pi k e^{\eta} - \frac{1}{2}\right)^2 - x^2 \\
\Longleftrightarrow & \left(1 - \frac{1}{4k}\right)^2 & \le & \left(e^{\eta} - \frac{1}{2\pi k}\right)^2 - \left(\frac{x}{2\pi k}\right)^2 \\
\Longleftrightarrow & 1 - \frac{1}{2k} + \frac{1}{16k^2} & \le & e^{2\eta} - \frac{e^{\eta}}{2\pi k} + \frac{1}{16\pi^2 k^2} - \left(\frac{x}{2\pi k}\right)^2 \\
\Longleftrightarrow & 0 & \le & (e^{2\eta} - 1) + \frac{1}{k}\left(\frac{1}{2} - \frac{e^{\eta}}{2\pi}\right) + \frac{1}{16k^2}\left(\frac{1}{\pi} - 1\right) - \left(\frac{x}{2\pi k}\right)^2 \\
 & & = & (e^{2\eta} - 1) + O\left(\frac{1}{k}\right).
\end{array}$$

The latter inequality is true for all but finitely many $k \in \mathbb{N}$ since $\eta > 0$. Consequently, the left side of R_k is contained in D_0, which – by the invariance of D_0 – yields that it is mapped by f into D_0.

Step 2 – right side In this case we have $z = iy$ where $\left(2k - \frac{1}{2}\right)\pi - \frac{1}{\ln(k)} \le y \le \left(2k - \frac{1}{2}\right)\pi$. For simplicity, we choose some number $\eta \in \left]0, \frac{1}{4}\right]$ and note $\frac{1}{\ln(k)} < \eta$ for all but finitely many $k \in \mathbb{N}$.

We have

$$\begin{aligned}
(1 - 2z)e^z &= (1 - 2yi)(\cos(y) + i\sin(y)) \\
&= (\cos(y) + 2y\sin(y)) + i(\sin(y) - 2y\cos(y))
\end{aligned}$$

This yields

$$|\mathrm{Re}\,((1-2z)e^z)| \geq (2k-1)\pi(1-\eta)$$

and

$$-1 \leq \mathrm{Im}((1-2z)e^z) \leq 2y\eta \leq \frac{y}{2}$$

which in turn implies, recall $-1 \leq \sin(y) \leq -(1-y)$ and $-y \leq \cos(y) \leq 0$:

$$\begin{aligned} |\mathrm{Im}(f(z))| &\leq (1+\eta)y \\ |\mathrm{Re}(f(z))| &\geq 2y(1-\eta) \end{aligned}$$

where $2y = (2k-1)\pi + O(1)$. Finally we prove that this inequalities imply $f(z) \in D_0$. Therefore it suffices to show

$$((1+\eta)y)^2 + ((1-\eta)y)^2 \leq \left(\frac{1}{4}e^{(1-\eta)} - \frac{1}{2}\right)^2 . \tag{25}$$

Note that the left hand side is of magnitude $O(y^2)$ and the right hand side of magnitude $O(e^{2(1-\eta)y})$. Therefore (25) holds for y sufficiently large, that is to say, for all but finitely many $k \in \mathbb{N}$.

Step 3 – upper side In this case we have $z = x + i\left(2k - \frac{1}{2}\right)\pi$, which yields $e^z = -ie^x$. Note $-\ln(k) + \frac{7}{2} \leq x \leq 0$. We obtain

$$\begin{aligned} (1-2z)e^x &= -i\left[(1-2x) - i\left(2k - \frac{1}{2}\right)\pi\right] e^x \\ &= -\left(2k - \frac{1}{2}\right)\pi e^x + i(2x-1)e^x \end{aligned}$$

and therefore

$$\begin{aligned} &|\mathrm{Im}(f(z))| \leq \mathrm{Im}(z) \\ &\mathrm{Re}(f(z))x - 1 - \left(2k - \frac{1}{2}\right)\pi e^x . \end{aligned}$$

So, in order to prove $f(z) \in D_0$ it suffices to show for $-\ln(k) + \frac{7}{2} \leq x \leq 0$

$$x - 1 - \left(2k - \frac{1}{2}\right)\pi e^x \leq -\left(\ln(k) + \frac{7}{2}\right)$$

or, equivalently,

$$\underbrace{x - \left(2k - \frac{1}{2}\right)\pi e^x}_{g(x)} \leq 1 - \left(\ln(k) + \frac{7}{2}\right) .$$

We search for the maximum of the left hand side and compute

$$g'(x) = 1 - \left(2k - \frac{1}{2}\right)\pi e^x .$$

So, $g'(x_0) = 0$ if and only if $e^{-x_0} = \left(2k - \frac{1}{2}\right)\pi$, that is to say, $-x_0 = \ln\left(\left(2k - \frac{1}{2}\right)\pi\right)$. Then

$$g(x) \leq g(x_0) = -\ln\left(\left(2k - \frac{1}{2}\right)\pi\right) - 1 .$$

We have to show $g(x_0) \leq -\left(\ln(k) + \frac{7}{2}\right)$ and this is equivalent to

$$\frac{7}{2} \leq 2 + \ln(2\pi) \approx 3.837 .$$

This is certainly true.

Step 4 – lower side We have $z = x + iy = x + i\left(\left(2k - \frac{1}{2}\right)\pi - \frac{1}{\ln(k)}\right)$. This yields

$$e^z = e^x\left(\cos(y) + i\sin(y)\right) \tag{26}$$

and

$$1 - 2z = 1 - 2x - \left((4k-1)\pi - \frac{2}{\ln(k)}\right) i \tag{27}$$

which gives

$$(1-2z)e^z = e^x\left((1-2x)\cos(y) + \left((4k-1)\pi - \frac{2}{\ln(k)}\right)\sin(y)\right) \tag{28}$$

$$ie^x\left((1-2x)\sin(y) - \left((4k-1)\pi - \frac{2}{\ln(k)}\right)\cos(y)\right) . \tag{29}$$

Recall

$$\frac{-1}{\ln(k)} \leq \cos(y) \leq 0 \text{and} -1 \leq \sin(y) \leq -1 + \frac{1}{\ln(k)} . \tag{30}$$

Therefore we obtain

$$\begin{aligned}\mathrm{Im}((1-2z)e^z) &\leq \left(-(2\ln(k)+O(1)) + \left((4k-1)\pi - \frac{2}{\ln(k)}\right)\frac{1}{\ln(k)}\right)\\ \Longrightarrow |\mathrm{Im}((1-2z)e^z)| &\leq 4e^x\frac{k}{\ln(k)}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Re}((1-2z)e^z) &= -e^x\left((1+2|x|)|\cos(y)| + \left((4k-1)\pi - \frac{2}{\ln(k)}\right)|\sin(y)|\right)\\ \Longrightarrow |\mathrm{Re}((1-2z)e^z)| &\geq e^x\left(2|x|\,|\cos(y)| + (4k\pi - \pi - 1)|\sin(y)|\right)\\ &\geq e^x(3k\pi + O(1))\,.\end{aligned}$$

Consequently, we derive

$$\begin{aligned}|\mathrm{Im}(f(z))| &\leq 2k\pi + 4e^x\frac{k}{\ln(k)}\\ &\leq 4k\left(\pi + \frac{e^x}{\ln(k)}\right)\\ &\leq 20k\pi\end{aligned}$$

and

$$\begin{aligned}|\mathrm{Re}(f(z))| &\geq |x| + 3k\pi e^x\\ &\geq 3\pi\frac{k}{\ln(k)}\,.\end{aligned}$$

In order to show $f(z) \in D_0$ it suffices to show

$$-3\pi\frac{k}{\ln(k)} + i20k\pi \in D_0$$

or, equivalently,

$$(20k\pi)^2 + \left(3\pi\frac{k}{\ln(k)}\right)^2 \leq \left(\frac{1}{4}\exp\left(3\pi\frac{k}{\ln(k)}\right) - \frac{1}{2}\right)^2\,.$$

This is true for all but finitely many $k \in \mathbb{N}$ because $\lim_{k\to\infty} k^2/\exp\left(\frac{k}{\ln(k)}\right) = 0$.

This completes the proof of Proposition 28 ■

As a consequence of equation (30) and inequality (29) we note

Lemma 29 *If* $z = x + ((2k-\frac{1}{2})\pi - \frac{1}{\ln(K)})I$, $0 \geq x \geq -(\ln(k)+\frac{7}{2})$, *then* $|\mathrm{Im}(f(z))| > |\mathrm{Im}(z)|$.

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