arXiv:math.AG/0005096 10 May 2000

Focal Loci of Algebraic Varieties I

Fabrizio Catanese - Cecilia Trifogli

april 2000

Abstract

The focal locus Σ_X of an affine variety X is roughly speaking the (projective) closure of the set of points O for which there is a smooth point $x \in X$ and a circle with centre O passing through x which osculates X in x. Algebraic geometry interprets the focal locus as the branching locus of the endpoint map ϵ between the Euclidean normal bundle N_X and the projective ambient space (ϵ sends the normal vector $O - x$ to its endpoint O), and in this paper we address two general problems :

1) Characterize the "degenerate" case where the focal locus is not a hypersurface

2) Calculate, in the case where Σ_X is a hypersurface, its degree (with multiplicity)

1 Introduction

The goal of the present paper is to introduce a general theory of focal loci of algebraic varieties in Euclidean space.

The theory of focal loci was classically considered only for plane curves and surfaces in 3-space (cf. [Coolidge] , [Salmon-Fiedler]), and Hilbert himself lectured in the Winter Semester 1893-94 at the University of Göttingen on the focal loci of curves and surfaces of degree two in 3-space.

Recently the theory was considered in ([Fantechi], [Trifogli]) for the respective cases of plane curves and hypersurfaces.

We would like to first briefly present the relevant concepts.

Usually the focal locus of a submanifold X (cf. [Milnor], 6, pp. 32-38, or also [D-F-N], vol. II 11, sections 2-3) is defined in Euclidean differential geometry as either the locus of centres of principal curvatures, or, more geometrically, as the locus where the infinitely near normal spaces intersect each other. Equivalently, the focal locus can also be defined as the complement of the set of points p such that the square of the distance function from p induces a local Morse function on X, or also as the union of the singular points of the parallel varieties to X.

To make the definition algebraic, one picks up the second geometrical definition, where the notion of length is not needed, just the notion of orthogonality is sufficient.

To explain this in more detail, let us consider (complex) affine space as the complement of a hyperplane (the "hyperplane at infinity") in projective space. In the hyperplane at infinity $\mathbf{P}_\infty$, we give a non degenerate quadric Q_∞.

These data allow , for each projective linear subspace L , to define the orthogonal $L^{\perp_x}$ to L in a point x as the join of x with the "orthogonal direction" to L (this is the subspace of $\mathbf{P}_\infty$ given by the polar of $L \bigcap \mathbf{P}_\infty$ with respect to Q_∞).

Given now an irreducible algebraic variety $X_d^n \subset \mathbf{P^m}$, of dimension n and degree d and not contained in the hyperplane at infinity , for each smooth point $x \in X - \mathbf{P}_\infty$ we define the normal space $N_x(X)$ as the orthogonal in x of the projective tangent space to X at x. The condition that x is a point in affine space ensures that $N_x(X)$ has the correct dimension $m - n$.

The normal variety N_X is then defined as the irreducible algebraic set in $\mathbf{P}^m \times \mathbf{P}^m$, closure of the set N_X^{good} consisting of the pairs (x, y) where x is a smooth point of X, $x \in X - \mathbf{P}_\infty$ and $y \in N_x(X)$.

Clearly, N_X is a projective variety of dimension m and the second projection induces a map π whose image is the closure of the union of the normal spaces to the smooth points of $X - \mathbf{P}_\infty$. Observe moreover that N_X^{good} is a projective bundle over $X - \mathbf{P}_\infty - Sing(X)$, in particular N_X^{good} is smooth of dimension m : therefore we can consider the ramification locus Y_X^{good} of $\pi : N_X^{good} \to \mathbf{P}^m$, and we define the ordinary ramification locus as the closure Y_X of Y_X^{good}.

Defining the good focal locus as $\Sigma_X^{good} = \pi(Y_X^{good})$, and the focal locus Σ_X as the closure of Σ_X^{good} (thus Σ_X is contained in the branch locus of $\pi : N_X \to \mathbf{P}^m$), we have a priori at least four cases:

- 1) $\pi : N_X \to \mathbf{P}^m$ is not dominant : in this case we say that the variety X is isotropically focally degenerate (for short : isotropically degenerate), and observe that the focal locus Σ_X of X is then simply the image of π (whence, Σ_X is an irreducible variety in this case!).

- 2) $\pi : N_X \to \mathbf{P}^m$ is dominant , but the focal locus Σ_X (respectively, the branch locus of $\pi : N_X \to \mathbf{P}^m$) has dimension at most $m-2$: in this case we say that X is strongly focally degenerate (respectively, completely strongly focally degenerate).

- 3) $\pi : N_X \to \mathbf{P}^m$ is dominant , whence surjective , and the focal locus Σ_X "is not a hypersurface", in the sense that not every component Z of the ordinary ramification divisor Y_X (closure of Y_X^{good}) maps to a hypersurface. In this case we shall say that X is weakly focally degenerate. We shall moreover say that we have the vertical case if Z does not dominate X.

- 4) When none of the above occurs, in particular $\pi : N_X \to \mathbf{P}^m$ is surjective , and the focal locus Σ_X is a hypersurface, we shall say that X is focally non degenerate. In this case, defining the focal hypersurface as a divisor, consisting as the image of the ramification divisor Y_X with multiplicities (if $Y_X = \Sigma_{i=1,..k} n_i Y_i$, and $d_i := degree(Y_i \to \pi(Y_i)$, then, setting $\Sigma_i := \pi(Y_i)$, we get $\Sigma_X := \Sigma_{i=1,..k} d_i n_i \Sigma_i$), the main problem is to describe Σ_X.

The first main result of this paper consists in calculating the degree (with multiplicity) of the focal hypersurface under a certain hypothesis upon X, which we call of being "orthogonally general", and which ensures that X is focally non strongly degenerate if it is not a linear subspace. This concept is important because, if X is smooth and not a linear subspace, then for a general projectivity g the translate $g(X)$ of X by g satisfies this condition whence it is not focally strongly degenerate and we have a divisor Σ_X. The hypothesis that X be "orthogonally general" is indeed very easy to verify since it simply amounts to three requirements: the smoothness of X, plus the two general position properties that X be transversal to $\mathbf{P}_\infty$, respectively to Q_∞.

More precisely, we have the following Theorem

Theorem 1 *Let $X \subset \mathbf{P}^m$ be a variety of dimension $n \geq 1$ which is orthogonally general. Then $dim\, \Sigma_X < m-1 \Leftrightarrow X$ is a linear space. If X is a linear space, Σ_X is a linear space of dimension equal to $codim\, X - 1$.*

One can ask in the above theorem whether one can replace the condition $dim\, \Sigma_X < m-1$ (i.e., that X be strongly focally degenerate) by the weaker condition that X be focally degenerate.

As a corollary of the full description given in Theorem 3 of the focally degenerate varieties, it turns out that if X is an orthogonally general and focally degenerate variety, then either X or X_∞ should be a developable variety rather explicitly described, but we have not yet had the time to look at the existence question for such very special varieties.

It is rather clear (e.g., from the case of plane curves) that the condition of being orthogonally general is a sufficient but not necessary condition in order that X be non focally strongly degenerate. When X is non orthogonally general, but focally non degenerate, what happens is that the degree of the focal divisor can drop (in this case, for plane curves we have Plücker type formulae, cf. [Fantechi]).

Naturally, what we have said insofar opens a series of problems. To some of them we give an answer in the present paper, to some others we hope to return in a sequel to this paper :

- 1) Try to completely classify the focally isotropically degenerate varieties. In section 7 we give a structure Theorem (Theorem 4) stating that the isotropically focally degenerate hypersurfaces are exactly the isotropically developable hypersurfaces. We observe thus that there are plenty of intriguing examples already in the case of surfaces in 3-space: these are obtained as the tangential developable surface of any space curve whose tangent direction is always an isotropic vector. We give moreover a description in section 8, Theorem 5, of the general case, in terms of the inverse focal construction applied to the focal variety Σ and to an algebraic function r on Σ. We get thus an implicit classification of these varieties as developable varieties, but for this we need to start with a variety Σ whose normal spaces are totally isotropic, and the function r must also satisfy a suitable condition.

- 2) Try to classify the weakly and the strongly focally degenerate varieties. In section 6 we give a complete classification for the weakly focally

degenerate varieties, showing in Theorem 3 how they can be divided into some"primitive" classes (cases 1), 2), 6), 7)) and some "derived" classes (cases 3),4),5)), related for instance by some tangential conditions to some primitive focally degenerate varieties. The primitive focally degenerate varieties can be described starting from fibrations in spheres or in affine spaces "around" the degenerate component Σ of the focal locus.

The question of classifying the strongly focally degenerate varieties seems harder.

- 3) Determine whether for a general projective deformation of X the focal hypersurface is reduced of degree equal to the virtual degree, and moreover answer more specific questions such as :

- 3a) can we also obtain that for a general deformation the focal hypersurface has generic Lagrangian singularities ?

- 3b) can we obtain the above good properties for the focal hypersurface Σ_{gX} of a general translate gX of X by a general projectivity g ?

Concerning the first problem, the situation seems to us rather hard (although quite interesting) as soon as the dimension of the ambient space grows: for instance, whereas a focally isotropically degenerate plane curve C is necessarily a line through a cyclic point (these are the two points of Q_∞ , satisfying the equations $z = x^2 + y^2 = 0$), in the case of a surface in 3-space we obtain the tangential developable of a space curve C which is "isotropic" in the following sense: C is just a curve such that any of its tangent lines L has the property that L intersects $\mathbf{P}_\infty$ in a point of Q_∞. Therefore, if we write the point of the curve C as a vector function $x(t)$ of a parameter t, we just have to solve the differential equation

$Q_\infty(dx/dt) = 0$.

Thus such a curve C yields a curve Γ in Q_∞ parametrizing the projective tangent lines to C, and the question reduces to: for which Γ can one find an algebraic integral ? (however, since the ring of polynomials in t is stable by d/dt , the above observation easily allows us to construct a lot of focally isotropically degenerate surfaces, which are tangential surfaces of rational space curves, cf. Example 10).

In higher dimension, as we already remarked, Theorem 5 partially reduces the quest to the search of varieties with totally isotropic normal spaces.

Turning to the other problems, the situation is clear for the plane curves (cf. [Fantechi]) : the only focally degenerate plane curves, which are not lines, are the circles (conics through the two cyclic points), and moreover, for an irreducible plane curve C the map of C to the focal curve Σ_C is non birational exactly for a well classified class of curves (by the way, Fantechi shows that this class is non empty, contrary to a statement made in [Coolidge]).

As we said, we characterize (cf. Theorem 3) the weakly focally degenerate varieties, distinguishing six essentially different cases :

- two vertical cases, where the exceptional component R of Y_X does not dominate X, but is instead the restriction of the normal bundle N_X to a divisor $X"$. In both cases, $X"$ is focally degenerate, and the focal degeneracy of X is determined by the first order neighbourhood of X along $X"$ (see Theorem 3 for more details).

- the case where X consists of a family of $(m-1-a)$-dimensional spheres parametrized by the a-dimensional degenerate component Σ of the focal locus: this family is moving according to a simple differential equation which can be explicitly solved, and it turns out that we get a family of spheres each obtained as the intersection of the big sphere with centre $O \in \Sigma$ with an affine subspace orthogonal to the tangent space to Σ in O.

- The case where X is a "transversal" divisor in a focally isotropically degenerate variety.

- The asymptotic case, i.e., the case where Σ lies at infinity, and then X is a developable variety whose intersection X_∞ with the hyperplane at infinity "is" the dual variety of Σ in $\mathbf{P}_\infty$. In this case there is another simple process, called the "asymptotic inverse focal construction", describing X in terms of the data of Σ and of an algebraic function $r(s)$ on Σ.

- The isotropically asymptotic case, where Σ lies at infinity, and a component Δ of X_∞ is projectively isotropically degenerate. This case is characterized by the property that $\Delta \subset X_\infty$ be obtained via the isotropic

projective inverse focal construction, starting from $\Sigma, r(s)$ satisfying suitable conditions.

The characterization given in Theorem 3 (where also the case of the focally isotropically degenerate varieties is considered) is expressed in terms of the "inverse focal construction", which, starting from a variety Σ of dimension a, and an algebraic function $r(s)$ on Σ, considers the union X' of the family of spheres each obtained as the intersection of the big sphere with centre $O \in \Sigma$ and radius equal to the square root of $r(s)$ with an affine subspace orthogonal to the tangent space to Σ in O, and whose position is determined by the differential of the function $r(s)$.

It turns out that for the focally isotropically degenerate varieties the above spheres degenerate to affine spaces,and X equals X', whereas in the case where these spheres have the right dimension $m-1-a$ X' is focally degenerate.

For hypersurfaces in higher dimensions the second author ([Trifogli]) showed that the focal hypersurface of a general hypersurface is reduced (indeed that this holds for a general diagonal hypersurface, i.e., for a translate of the Fermat hypersurface by a projectivity in the diagonal torus).

Concerning problem 3a), this is a global problem which is however related to a local problem which has been extensively studied: the theory of Lagrangian singularities. In fact the Normal variety N_X is a Lagrangian variety for the symplectic form on the product $\mathbf{A^m} \times \mathbf{A^m}$ which is associated to Q_∞, namely ${}^t x Q_\infty y - {}^t y Q_\infty x$, and the second projection is also Lagrangian (cf. [Arnold et al.]).

Partial results concerning problem 3a) have been obtained by the second author for surfaces in 3- space([Trif2]).

2 Notation

$V' :=$ a fixed vector space of dimension m

$V :=$ the vector space $V = V' \bigoplus \mathbf{C}$

$\mathbf{P}(V) = \mathbf{P}^m :=$ the projective space whose points correspond to the 1-dimensional vector subspaces of V

$\mathbf{P}(V') = \mathbf{P}_\infty \subset \mathbf{P}(V)$ ($\cong \mathbf{P^{m-1}}$) the complement of the affine space $\mathbf{P}(V) - \mathbf{P}_\infty \cong V'$.

$X_d^n \subset \mathbf{P^m}$ a quasi-projective algebraic variety of dimension n and degree d which does not lie at infinity , i.e.,

$X_d^n \not\subset \mathbf{P}_\infty$

$Q_\infty :=$ a non degenerate quadratic form on V', yielding an isomorphism $Q : V' \stackrel{\cong}{\to} (V')^\vee$.

By slight abuse of notation, the corresponding quadric

$Q_\infty \subset \mathbf{P}_\infty$.

$W :=$ a vector subspace of V',

$Ann(W) :=$ the vector subspace of V' which is the orthogonal space of W with respect to the quadratic form Q_∞

$L' := \mathbf{P}(W)$ a linear subspace at infinity

$L'^\perp := \mathbf{P}(Ann(W))$, the polar subspace of L' , also called the orthogonal direction to L'

$L \subset \mathbf{P}(V) :=$ a projective linear subspace , $L \bigcap \mathbf{P}_\infty$ the direction of L

$L^{\perp_x} :=$ the orthogonal to L in x, defined as the smallest linear subspace containing x and the orthogonal direction of L (i.e., the polar of $L \bigcap \mathbf{P}_\infty$) .

3 "Normal Bundle" in Euclidean Setting

In this section, we shall consider a smooth quasi- projective variety $X_d^n \subset \mathbf{P^m}$ and we shall define its projective normal variety $N_X \subset \mathbf{P}^m \times \mathbf{P}^m$, and its Euclidean Normal sheaf $\mathcal{N}_X$.

Under some assumptions that we are going to specify, the first projection of the normal variety N_X to X yields a projective bundle over X, which is the projectivization of the Euclidean Normal sheaf :

$N_X = \mathbf{P}(\mathcal{N}_X) \subset \mathbf{P}(V \bigotimes \mathcal{O}_X) \subset \mathbf{P}(V \bigotimes \mathcal{O}_{\mathbf{P}^m}) = \mathbf{P}^m \times \mathbf{P}^m$.

Start from the Euler sequence

(1) $0 \to \mathcal{O}_{\mathbf{P}}(-1) \to V \bigotimes \mathcal{O}_{\mathbf{P}} \to T_{\mathbf{P}}(-1) \to 0$:

setting $\mathcal{L} = \mathcal{O}_X(-1)$, the restriction to X of ¿the Euler sequence and the inclusion of the tangent bundle of X in the restricted tangent bundle of $\mathbf{P}^m$ define the bundle $\widehat{T}_X(-1)$ whose projectivization is the projective tangent bundle to X.

We get thus two exact sequences, the second included into the first:

$$(2)\quad \begin{array}{l} 0 \to \mathcal{L} \to V \bigotimes \mathcal{O}_X \to T_{\mathbf{P}}(-1) \bigotimes \mathcal{O}_X \to 0 \\ 0 \to \mathcal{L} \to \widehat{T}_X(-1) \to T_X(-1) \to 0 \end{array}$$

Assumption $0 =$ *smoothness*: X is smooth, whence T_X and $\widehat{T}_X$ are subbundles.

Recalling that $V = V' \bigoplus \mathbf{C}$,we state the further

Assumption 1 (= transversality of the intersection $X \bigcap \mathbf{P}_\infty$ with the hyperplane at infinity) : $\overline{T}_x := V' \cap \hat{T}_x$ is a hyperplane in $\hat{T}_x \ \forall x \in X$.

This means that we have two more exact sequences

$$(3)\quad \begin{array}{l} 0 \to V' \bigotimes \mathcal{O}_X \to V \bigotimes \mathcal{O}_X \to \mathcal{O}_X \to 0 \\ 0 \to \overline{T}_X(-1) \to \widehat{T}_X(-1) \to \mathcal{O}_X \to 0 \end{array}$$

At this stage we can define the bundle of normal directions $\mathcal{N}'_X$ as a twist of the annihilator of $\overline{T}_X$.

We define it through the exact sequence

(4) $0 \to \mathcal{N}'_X(-1) \to V' \bigotimes \mathcal{O}_X \cong (V')^\vee \bigotimes \mathcal{O}_X \to (\overline{T}_X(-1))^\vee \to 0$

In order to obtain a projective normal bundle from the bundle of normal directions we need a last

Assumption 2 (= transversality of X with Q_∞) : The natural map $\mathcal{L} \bigoplus \mathcal{N}'_X(-1) \to V \bigotimes \mathcal{O}_X$ is a bundle embedding, thus its image $\mathcal{N}_X(-1)$ is a subbundle of $V \bigotimes \mathcal{O}_X$, isomorphic to $\mathcal{L} \bigoplus \mathcal{N}'_X(-1)$

We notice thus that if assumption 2) holds, then $\mathcal{N}_X \cong \mathcal{O}_X \bigoplus \mathcal{N}'_X$

Definition 1 *X is said to be orthogonally general if it satisfies Assumptions $0 - 2$ above.*

Remark 1 *For every algebraic variety $X \subset \mathbf{P^m}$ which is not contained in the hyperplane at infinity there is a maximal nonempty Zariski open set U of X which is orthogonally general (U obviously contains the open set $X - \mathbf{P}_\infty - Sing(X)$).*

Remark 2 *The situation can be slightly generalized as follows : let Z be a singular projective variety, let Z' be its normalization, and let X be the open set of $Z' - Sing(Z')$ where the natural morphism to $\mathbf{P}^m$ has maximal rank. In this case, restrictions of bundles have to be understood as pull backs. If instead one wants to generalize to the case where X is the resolution of Z, many things change substantially because one does not get bundle maps any longer.*

Thus we can give the following definition

Definition 2 *Let X be an algebraic variety, not contained in the hyperplane at infinity, U a Zariski open set of X which is orthogonally general, and N_U the projective normal bundle of U. Then the projective normal variety N_X of X is defined as the Zariski closure of N_U.*

We can easily verify that the above definition is indeed independent of the choice of U.

Assume now that X is orthogonally general: in particular, X is smooth and we have a vector bundle (locally free sheaf) $\mathcal{N}_X$ on X, which is called the EUCLIDEAN NORMAL BUNDLE of X.

Remark 3 *The Euclidean Normal Bundle differs from the usual Normal Bundle (of a smooth subvariety $X \subset \mathbf{P^m}$) defined in algebraic geometry(cf. [Hartshorne]): the reader may in fact notice that their respective ranks differ first of all by 1. However, as we shall shortly see in the forthcoming example, they are somehow related to each other.*

We can therefore compute now the total Chern class of $\mathcal{N}_X$:

$c(\mathcal{N}_X) = c(\mathcal{N}'_X)$ and $c(\mathcal{N}_X(-1)) = c(\mathcal{L})c(\overline{T}_X(-1)^\vee)^{-1}$ by (4).

But (3) yields $c(\overline{T}_X(-1)) = c(\widehat{T}_X(-1))$ which by (2) equals $c(\mathcal{L})c(T_X(-1))$. Thus

$$c(\mathcal{N}_X(-1)) = c(\mathcal{L})c(-\mathcal{L})^{-1}c(\Omega^1_X(1))^{-1}$$

Let us verify this formula for a hypersurface of degree d. Then we have

$$0 \to \mathcal{O}_X(1-d) \to \Omega^1_{\mathbf{P}}(1) \bigotimes \mathcal{O}_X \to \Omega^1_X(1) \to 0$$

and $c(\Omega^1_P(1)) = c(\mathcal{O}(1))^{-1}$.

So, for a hypersurface, the rank 2 bundle $\mathcal{N}_X$ has

$$c(\mathcal{N}_X(-1)) = c(\mathcal{L})c(-\mathcal{L})^{-1}c(-\mathcal{L})c(\mathcal{O}_X(-(d-1))) = (1-H)(1-(d-1)H)$$

(indeed, the previous formulae show $\mathcal{N}_X \cong \mathcal{O}_X \bigoplus \mathcal{O}_X(-(d-2))$).
In general we have an exact sequence

$$0 \to \mathcal{N}^*_X(1) \to \Omega^1_{\mathbf{P}}(1) \bigotimes \mathcal{O}_X \to \Omega^1_X(1) \to 0,$$

where $\mathcal{N}^*_X$ is the usual conormal bundle of X.
Hence, $c(\mathcal{N}_X(-1)) = c(\mathcal{L})c(-\mathcal{L})^{-1}c(\mathcal{N}^*_X(1))c(-\mathcal{L})$, and we obtain the

$\underline{FINAL\ FORMULA}$: $c(\mathcal{N}_X) = c(\mathcal{N}^*_X(2))$.

Corollary 1 *If X is a general complete intersection of degrees $d_1, \ldots d_{m-n}$, then $c(\mathcal{N}_X) = \prod_i (1-(d_i-2)H)$, where H is the hyperplane divisor.*

We recall once more the definition of the Focal Locus Σ_X of X.

Definition 3 *Continue to assume that X is orthogonally general, let $N_X \subset \mathbf{P}^m \times \mathbf{P}^m$ be the projectivization of the Euclidean Normal Bundle, and let $\pi = p_2 : N_X \to \mathbf{P}^m$ be the second projection. Denote then by Y_X the ramification locus of π (recall: N_X is smooth and dim $N_X = m$). Clearly, if X is projective, $Y_X \neq \emptyset$, since $rk\, Pic(N_X) \geq 2$, and therefore π cannot be an isomorphism. We define in general the focal locus as $\Sigma_X := \pi(Y_X)$.*

Definition 4 *Let now Z be any projective variety, possibly singular. Let X be a maximal orthogonally general open set of the normalization Z' of Z (cf. remark 2): then the focal locus Σ_Z is defined as the closure of Σ_X.*

Remark 4 *In order to verify whether the definition would be the same when one would replace X by any orthogonally general open set of Z,i.e., independent of the chosen open set X, we may observe:*

- *For X orthogonally general, the projective normal bundle N_X has a canonical section, provided by the diagonal of X, and corresponding to the tautological sheaf $\mathcal{L} \subset \mathcal{N}_X(-1)$.*

- *In a neighbourhood of the canonical section, the morphism π is of maximal rank if and only if $\mathcal{N}'_X(-1)$ and $(\overline{T}_X(-1))$ yield a direct sum, i.e., $(\overline{T}_X(-1))$ contains no isotropic vectors.*
  We shall say that a point $x \in X$ is totally non isotropic if the above situation occurs.

It follows that, in the open set of totally non isotropic points, the ramification divisor cannot contain the fibre of the projection to X. Therefore, in this locus, the ramification divisor is the closure of its restriction to the inverse image of an open set in X.

Instead, when there is a divisor D of isotropic points of X, the inverse image of D may yield a component of the ramification divisor, as happens in the following example.

Consider the plane curve C given, in a standard system of Euclidean coordinates, by the parametrization $(t, it + t^3)$.

Then the normal vector is proportional to the vector $(i + 3t^2, -1)$ and the endpoint map π associates to (t, λ) the point

$x = t + \lambda(i + 3t^2)$

$y = it + t^3 - \lambda$,

and the Jacobian determinant equals

$J = -(1 + 6t\lambda) - (i + 3t^2)^2 = -3t(2\lambda + 2it + 3t^3)$.

Thus the focal locus consists of the evolute $\mathcal{E}$ (image of the curve $\lambda = -it - 3/2t^3$) and of the isotropic line $\{(x, y)|ix - y = 0\}$.

$\mathcal{E}$ is here the parametrical curve $(2t - 9/2it^3 - 9/2t^5, 2it + 5/2t^3)$.

The previous remark and example justify the following

Definition 5 *Let now Z be any projective variety, possibly singular. We say that an open set X of Z is excellent if X is orthogonally general and X is contained in the set of totally non isotropic points . If there exists an*

excellent open set X, then the strict focal locus Σ_Z^s is defined as the closure of Σ_X.

We define instead the large focal locus Σ_Z^L as the branch locus of the second projection π from $N_{Z'} \subset Z' \times \mathbf{P}^m$ to $\mathbf{P}^m$, where as before Z' is the normalization of Z.

Obviously one has inclusions $\Sigma_Z^s \subset \Sigma_Z \subset \Sigma_Z^L$.

Example 1 *In the case of a plane curve C, the strict focal locus is precisely the evolute of the curve C, as in [Fantechi]. Whereas, even if all the points are totally non isotropic, the large focal locus can be larger, as we shall now see in the case where the curve has as a singularity a higher order cusp.*

Let our curve C be locally given by (t^2, t^5), with respect to some standard Euclidean coordinates; then the normal vector is , for $t \neq 0$, proportional to $(-5t^4, 2t)$, i.e., to $(-5t^3, 2)$, and thus the large focal locus is provided by the image of the jacobian determinant of the map

$x = t^2 - 5t^3\lambda$,

$y = t^5 + 2\lambda$.

The equation of the Jacobian determinant equals therefore

$t(4 - 30t\lambda - 25t^6) = 0$, whence the large focal

locus consists of the evolute plus the line obtained for $t = 0$, namely the $y-$ axis.

Remark 5 *Assume now that Z is any projective variety and assume that there is a non empty excellent open set $X \subset Z$. If Σ_Z^L has dimension $\leq m-2$, then π is a birational morphism, since then $\prod_1(\mathbf{P}^m - \Sigma_Z^L) = \{1\}$. Thus if Z is not isotropically focally degenerate and $\dim \Sigma_Z^L < m-1 \Rightarrow N_X$ is rational $\Rightarrow Z$ is unirational, and indeed stably rational.*

Example 2 *If X is a smooth hypersurface of degree d and Σ_X^L has dimension $\leq m-2$, then $d \leq m$.*

Example 3 *If X is a smooth complete intersection of multidegree $(d_1, \ldots, d_{m-n})$, and Σ_X^L has dimension $\leq m-2$, then $\sum d_i \leq m$.*

Remark 6 *Let $X' \subset \mathbf{P}^m$ be a smooth variety not necessarily satisfying the non degeneracy conditions, i.e., Assumptions 1 and 2. Then $\exists g \in \mathbf{PGL}(m+1)$ such that $X = gX'$ satisfies the non degeneracy conditions.*

Proof
The non degeneracy conditions are equivalent to $(1')$ X is transversal to $\mathbf{P}_\infty$ and $(2')$ $X_\infty := X \bigcap \mathbf{P}_\infty$ is transversal to Q_∞. By Bertini's theorem, we can find a hyperplane H and a smooth quadric $Q \subset H$ such that X' is transversal to H and $X' \bigcap H$ is transversal to Q. Then choose $h, k \in \mathbf{PGL}(m+1)$ such that $hH = \mathbf{P}_\infty$ and $k\mathbf{P}_\infty = \mathbf{P}_\infty$, $khQ = Q_\infty$ and set $g = kh$. $\square$

Let us continue now to assume that X is orthogonally general. Moreover, we shall from now on assume that X is indeed projective. Then we can calculate $deg\, \Sigma_X\, deg\, \pi_{|Y_X}$ (notice that π is a morphism) by working in the Chow (or cohomology) ring of N_X.

Observe that, by the Leray-Hirsch theorem, the cohomology algebra of the projective normal bundle is generated by $H^*(X)$ and the relative hyperplane divisor H_2, and holds

$$H^*(N_X) \cong H^*(X)[H_2]/(\sum c_i(\mathcal{N}_X(-1))H_2^{m-n+1-i})$$

We denote by Π the first projection $\Pi : N_X \to X$, and for commodity we also set $p := \pi$.

Let $H_1 = \Pi^*(hyperplane)$, and observe that, since $\mathcal{N}_X(-1)$ is a subbundle of $V \bigotimes \mathcal{O}_X$, we have $H_2 = p^*(hyperplane)$.

Moreover, setting $N = N_X$, we have also the ramification formula

$$Y = K_N - p^*(K_{\mathbf{P}^m}) = K_N + (m+1)H_2.$$

In order to determine the canonical divisor K_N of $N = N_X$, we write as usual

$$K_N = K_{N|X} + \Pi^*(K_X),$$

where $K_{N|X}$ can be calculated through the Euler exact sequence for the relative tangent bundle $T_{N|X}$ of N

$$0 \to \mathcal{O}_N(-H_2) \to \Pi^*(\mathcal{N}_X(-1)) \to T_{N|X}(-H_2) \to 0,$$

whence

$K_{N|X} = -c_1(T_{N|X}) = -[c_1(T_{N|X}(-H_2)+(m-n)H_2)] = -(m-n+1)H_2 - \Pi^*(c_1(\mathcal{N}_X(-1))) = -(m-n+1)H_2+H_1-\Pi^*(c_1(\mathcal{N}_X^*(1))) = -(m-n+1)H_2+H_1-c_1(\mathcal{N}_X^*)-(m-n)H_1 = -(m-n+1)H_2+H_1+(m+1)H_1+K_X-(m-n)H_1$

In the end we obtain:

$K_N = 2\Pi^* K_X - (m-n+1)H_2 + (n+2)H_1$
thus we get the

$\underline{CLASS-FORMULA}$: $Y_X = 2\Pi^* K_X + nH_2 + (n+2)H_1$,

and the

$\underline{DEGREE-FORMULA}$:

$(deg\,\Sigma)(deg\,p_{|Y}) = H_2^{m-1}(2\Pi^* K_X + nH_2 + (n+2)H_1)$.

In the sequel (section 5) we shall see how the above cited Leray-Hirsch Theorem allows to make the degree formula more explicit.

4 Non degeneracy of Focal Loci

Throughout this section we assume that X is projective and orthogonally general, i.e., the non degeneracy conditions $0-2$ above are satisfied, in particular we have that $N = N_X$ is a bundle . Our aim is then to determine for which X it is possible that Σ_X is degenerate, that is, has dimension strictly less than $m-1$. It is easy to see that, if X is a linear space, then Σ_X is degenerate and is a linear space of dimension equal to $codim\, X - 1$. In what follows, we shall prove that if X is orthogonally general also the converse holds, i.e., if Σ_X is strongly degenerate, then X is a linear space.

Notation 1 *Let* $\mathbf{C^m} = \mathbf{P^m} - \mathbf{P}_\infty$, $N_\infty = p^{-1}(\mathbf{P}_\infty)$, $N_a = p^{-1}(\mathbf{C^m})$.

We have

Lemma 1 *dim* $p^{-1}(y) = 0\ \forall y \in \mathbf{C^m}$

Proof
After identifying $p^{-1}(y)$ with the set $\Gamma = \{x \in X : y \in N_x\}$, it is easy to see that Γ has empty intersection with the hyperplane $\mathbf{P}_\infty$. Indeed, if $x \in X_\infty$, then $N_x \subset \mathbf{P}_\infty$ $\square$

Corollary 2 *If* Σ *is a component of the focal locus, image of a component* Y *of the divisor* Y_X*, and dim* $\Sigma < m-1$*, then*
(i) $Y \subset N_\infty$ *(since* $\forall y \in \Sigma$ *dim* $p^{-1}(y) > 0$*)*
(ii) $\Sigma \subset\neq \mathbf{P}_\infty$ *(hence* Σ *is degenerate).*
(iii) if for every component Σ *of the focal locus holds dim* $\Sigma < m-1$*, then* $p : N_a \to \mathbf{C^m}$ *is an isomorphism.*

Remark 7 *The divisor* N_∞ *splits as* $N_{|X_\infty} \bigcup N'$*, where* $N' = \mathbf{P}(\mathcal{N}'_X)$*.*

Let us first consider the case where X is a curve (for this case we shall give a different proof in the sequel, showing that then either C is a line, or C is a circle, what contradicts the hypothesis that C be orthogonally general).

CASE: $X = C$ curve.

Let C be an irreducible (and orthogonally general) curve of degree d. Then $N_{|C_\infty}$ consists of d distinct copies of $\mathbf{P}_\infty$, $p : N_{|C_\infty} \to \mathbf{P}_\infty$ is a finite map, and by the transversality of C to $\mathbf{P}_\infty$, the divisor Y_C does not contain any component of $N_{|C_\infty}$.

Therefore we get

Corollary 3 *If* C *is an irreducible (and orthogonally general) curve and* $dim\, \Sigma_C < m-1$*, then* $Y_C = N'$ *(set-theoretically)*

Proof
Indeed, $Y_C \subset N_\infty$, but no component of $N_{|C_\infty}$ is contained in Y_C. Thus $Y_C \subset N'$. We can conclude since $dim\, Y_C = dim\, N'$ and N' is irreducible (being a projective bundle on the curve C). $\square$

Proposition 1 *Assume again that C is an irreducible (and orthogonally general) curve . Then* (1) *$dim\,\Sigma < m-1 \Leftrightarrow$* (2) *$C$ is a line.*
(3) *In this case Σ is a linear space of dimension $m-2 = codim\,C - 1$.*

Proof
(1) $\Leftarrow$ (2) being clear, let's prove the other implication (1) $\Rightarrow$ (2):
Let N'_p be the fibre of N' over $p \in C$, which is a hyperplane in $\mathbf{P}_\infty$. Now $\Sigma_C = p(N')$ is irreducible, has dimension $< m-1$ and contains N'_p, which has dimension equal to $m-2$. Therefore, $\Sigma_C = N'_p$ and $N'_p = N'_q\ \forall p, q \in C$. This implies $T_pC \bigcap \mathbf{P}_\infty = T_qC \bigcap \mathbf{P}_\infty\ \forall p, q \in C$.

This clearly implies that C is a line, since then for each point $p \in C$ the projective tangent line T_pC is the join of p and of a fixed point p_∞ (whence one can find then $m-1$ independent linear forms vanishing on C).

□

CASE: $dim\,X = n \geq 2$
Since X_∞ is smooth, by Bertini's theorem X_∞ is irreducible. Therefore also $N|_{X_\infty}$ and N' are irreducible.
We have

Lemma 2 *If $n \geq 2$ and $dim\,\Sigma_X < m-1$, then*
(i) $Y_X = N'$ set-theoretically.
(ii) $[Y_X] = [nN']$ in $Pic(N)$.
(iii) $2(K_X + (n+1)H) = 0$ in $Pic(X)$, where $H = H_1$ is the hyperplane divisor on X.

Proof
Since p is surjective, we have one and only one of the following two cases: (a) $p(N|_{X_\infty}) \subset\neq \mathbf{P}_\infty$; (b) $p(N') \subset\neq \mathbf{P}_\infty$. But (a) cannot hold. Indeed, since $[H_1] = [N|_{X_\infty}]$ in $Pic(N)$, (a) implies $[Y] = [\alpha H_1]$ for some $\alpha > 0$. But then from the class formula $(*)$ $Y_X = 2\Pi^* K_X + nH_2 + (n+2)H_1$, it follows that $2\Pi^* K_X + nH_2 + (n+2-\alpha)H_1 = 0$, contradicting the Leray-Hirsch Theorem.

Therefore, (b) holds and hence $Y = N'$ set-theoretically. (ii) and (iii) follow immediately from the class formula $(*)$, because $H_2 = H_1 + [N']$ in $Pic(N)$. □

¿From point (iii) it follows that

Corollary 4 *If $\dim \Sigma_X < m-1$, then X is a linear space.*

Proof
Let $C = X \bigcap H_1 \bigcap \ldots \bigcap H_{n-1}$ be a smooth curve. By successive applications of the adjunction formula (iii) yields $2(K_C + 2H) = 0$. Extracting degrees, we get $2(2g(C) - 2 + 2\,deg(C)) = 0$, which is equivalent to $g(C) = 0$ and $deg(C) = 1$. □

We can conclude

Theorem 1 *Let $X \subset \mathbf{P}^m$ be a projective variety of dimension $n \geq 1$ which is orthogonally general. Then $\dim \Sigma_X < m-1 \Leftrightarrow X$ is a linear space. In this case, Σ_X is a linear space of dimension equal to $\operatorname{codim} X - 1$.*

5 The Degree of the Focal Locus of a Surface

Let $X^2 = S \subset \mathbf{P}^m$ be a surface and assume that S satisfies the non-degeneracy conditions. Setting $n = 2$ in the Degree-Formula given in Section 1, we get (recall $H = H_1$)

$(F1)$ $deg\,\Sigma_S\, deg\, p_{|Y_S} = 2H_2^{m-1}(K_S + H_2 + 2H)$

Our first aim in this section is to express the right-hand side of $(F1)$ in terms of the Chern classes $c_1(S)$, $c_2(S)$ and of the hyperplane divisor H of S.

By the Leray-Hirsch theorem $H_2^{m-1} = -c_1(\mathcal{N}_S(-1))H_2^{m-2} - c_2(\mathcal{N}_S(-1))H_2^{m-3}$.
Using this relation, the right-hand side of $(F1)$ becomes

$(*)$ $2H_2^{m-2}(c_1(\mathcal{N}_S(-1))^2 - c_2(\mathcal{N}_S(-1)) - K_S c_1(\mathcal{N}_S(-1)) - 2H c_1(\mathcal{N}_S(-1)))$

Recall that $c(\mathcal{N}_S(-1)) = c(\mathcal{L})c(\mathcal{N}_S^*(1))$, where $\mathcal{L} = \mathcal{O}_S(-1)$ and $\mathcal{N}_S^*$ is the conormal bundle of S. Thus,

$$
\begin{array}{ll}
 & c_1(\mathcal{N}_S(-1)) = c_1(\mathcal{N}_S^*(1)) - H = c_1(\mathcal{N}_S^*) + (m-3)H \\
(1) & c_2(\mathcal{N}_S(-1)) = c_2(\mathcal{N}_S^*(1)) - H c_1(\mathcal{N}_S^*(1)) = \\
 & c_2(\mathcal{N}_S^*) + (m-4)H c_1(\mathcal{N}_S^*) + \tfrac{1}{2}(m-2)(m-5)H^2
\end{array}
$$

Using the normal-bundle sequence we get

$$
\begin{array}{ll}
 & c_1(\mathcal{N}_S^*) = c_1(S) - (m+1)H \\
(2) & c_2(\mathcal{N}_S^*) = -c_2(S) + \frac{1}{2}m(m+1)H^2 + c_1(S)c_1(\mathcal{N}_S^*) = \\
 & -c_2(S) + \frac{1}{2}m(m+1)H^2 + c_1(S)^2 - (m+1)Hc_1(S)
\end{array}
$$

and substituting in (1), we get

$$
\begin{array}{ll}
(3) & c_1(\mathcal{N}_S(-1)) = -4H + c_1(S) \\
 & c_2(\mathcal{N}_S(-1)) = 9H^2 - 5Hc_1(S) + c_1^2(S) - c_2(S)
\end{array}
$$

Hence $(*)$ becomes

$$(**)\ 2H_2^{m-2}(15H^2 + c_1^2(S) + c_2(S) - 9Hc_1(S))$$

We recall that $H_2 = [N'] + H$, so that $(**)$ can be rewritten as

$$(***)\ 2[N']^{m-2}(15H^2 + c_1^2(S) + c_2(S) - 9Hc_1(S))$$

Finally, since $[N']_{|F}^{m-2} = 1$, where F is a fibre of ¿$\pi : N \to S$, we conclude

$(DF)\ deg\,\Sigma_S\, deg\, p_{|Y_S} = 2(15H^2 + c_1^2(S) + c_2(S) - 9Hc_1(S)) = 2(15d + c_1^2(S) + c_2(S) - 9Hc_1(S))$,

where $d = deg(S)$.

By Noether's formula, we can also write

$(DF')\ deg\,\Sigma_S\, deg\, p_{|Y_S} = 2(15d + 12\chi(\mathcal{O}_S) - 9Hc_1(S))$.

We can express also our formula in terms of the sectional genus π of our surface S (recall that $2\pi - 2 = H^2 - Hc_1(S)$) as

$(DF'')\ deg\,\Sigma_S\, deg\, p_{|Y_S} = 2(18(\pi - 1) + 6d + 12\chi(\mathcal{O}_S))$

Example 4 *For $m = 3$, we have $c_1(S) = (4-d)H$ and $c_2(S) = 6H^2 - d(4-d)H^2$. Thus*

$deg\,\Sigma_S\, deg\, p_{|Y_S} = 2d(d-1)(2d-1)$

Example 5 *For $m = 4$, we have the formula $c_2(S) = c_1(S)^2 - 5Hc_1(S) + 10d - d^2$ [$Hartshorne, p.434$], or , equivalently,*
$d^2 - 5d + 2(6\chi(\mathcal{O}_S) - c_1(S)^2) = 10(\pi - 1)$
which gives

$deg\,\Sigma_S\, deg\, p_{|Y_S} = 2/5(9d^2 - 15d + 168\chi(\mathcal{O}_S) - 18c_1(S)^2)$

6 Weakly focally degenerate varieties

In this section we shall first consider the case of a hypersurface X of dimension n , and we shall characterize the case where X is weakly focally degenerate. The characterization of the hypersurfaces X which are isotropically focally degenerate will be given in the next section.

Later on in this section we shall deal with focally degenerate varieties of any codimension.

We shall essentially use very classical tools such as the implicit function theorem, dimension counts and the standard method of obtaining new equations by differentiating old ones .

Let $F(x_1, ...x_{n+1}) = 0$ be the affine polynomial equation of a hypersurface X. We shall in this section be mostly interested about a birational description of X, whenceforth we might, by abuse of notation, not distinguish between a projective variety and its affine part (or any nonempty Zariski open set of it).

In this case the gradient ∇F of F gives a trivialization of the Normal Bundle N_X at the smooth points of X, and the second projection $\pi : N_X \to \mathbf{P^{n+1}}$ coincides with the endpoint map

$\epsilon(x, \lambda) = x + \lambda \nabla F(x)$,

where $x = (x_1, ...x_{n+1})$ is a point of X, and λ is a scalar coordinate $= \lambda_1/\lambda_0$, (λ_0, λ_1) being homogeneous coordinates on $\mathbf{P^1}$.

As a warm up, let us investigate when does it occur that the endpoint map is not finite. That is, let us assume that Γ is a curve in N_X which is

mapped to a point O by the endpoint map ϵ, and that this point does not lie at infinity.

Choosing a parameter t for Γ, we have functions $x(t), \lambda(t)$ such that

1) $F(x(t)) \equiv 0$

1')$x(t) + \lambda(t)\nabla F(x(t)) \equiv O$.

If $x(t)$ is a smooth point of X, then the gradient $\nabla F(x(t))$ does not vanish, whence $x(t)$ is not constant: thus at a general point of Γ we may assume that the derivative $\dot{x}(t) := dx(t)/dt$ is non vanishing.

Let us use the scalar product $<,>$ associated to the quadratic form Q_∞, and let us choose affine coordinates such that $<,>$ is the standard scalar product (i.e., the matrix of Q_∞ is the identity matrix); since

2) $x(t) - O \equiv -\lambda(t)\nabla F(x(t))$, and $< \nabla F(x(t)), \dot{x}(t) > \equiv 0$ we infer that

3) $< x(t) - O, x(t) - O > \equiv constant.$

Therefore, the basis curve $\gamma \subset X = \{x | F(x) = 0\}$ is a curve contained in a sphere with centre the point O (note that the sphere may also have radius zero !).

Conversely, if we have such a spherical curve γ meeting X and with the property that the two vectors $x(t) - O, \nabla F(x(t))$ are proportional, then we find $\lambda(t)$ so that 1'), 1) hold, whence we find Γ which is mapped to the point O by the endpoint map (and moreover it follows from 1) that γ is contained in X). Finally, since Γ is mapped to a point, it is obviously contained in the ramification divisor Y of the endpoint map.

We have therefore the following

Proposition 2 *Given a smooth affine hypersurface X , the positive dimensional irreducible components of the fibres $\pi^{-1}(O)$ of the map to the affine part of the Focal Locus correspond exactly to the subvarieties Φ contained in a sphere S with centre O, and such that X is everywhere tangent to S along Φ.*

Proof

Let Ψ be a component of the fibre $\pi^{-1}(O)$. Then consider that Ψ is the union of the curves Γ contained in it : each of these projects to $\gamma \subset X$ contained in a sphere S_c with centre O and radius c. But the image of Ψ, call it Φ, is irreducible, whence all the radii are equal, and we get the desired sphere S. Conversely, the tangency condition provides a rational function λ on Φ whose graph is the required variety Ψ. $\square$

It is rather clear that the previous proposition allows easily to construct examples where the map $\pi : Y \to \Sigma_X$ is not finite.

Remark 8 *If instead the point O is at infinity, let's identify it with one vector in V', then we get the equation*

$O \equiv \lambda(t)\nabla F(x(t))$, *whence*

$< O, x(t) > \equiv constant$.

So, in this case, the positive dimensional irreducible components of the fibres $\pi^{-1}(O)$ correspond exactly to the subvarieties Φ contained in a hyperplane H with normal direction O, and such that X is everywhere tangent to H along Φ.

We can push the previous calculations to describe the weakly focally degenerate hypersurfaces.

Let us thus assume that $X = \{x|F(x) = 0\}$ is weakly focally degenerate. This simply means that there is a component Σ of the focal locus which has dimension

$dim\Sigma = a < n$.

Arguing as before, we notice that Σ will simply be any maximal irreducible variety such that its inverse image in N_X has a dominating component Z of dimension n. We can analogously treat the case where this dimension is bigger than n, i.e., when $Z = N_X$, or equivalently X is isotropically focally degenerate : in this case we may also allow $dim\Sigma = n$.

We have thus an irreducible component Z of the ramification divisor, with $\pi(Z) = \Sigma$.

To start with, let us assume that $\Sigma \not\subset \mathbf{P}_\infty$.

Therefore, at the general point of Z we can choose local coordinates

$s = (s_1, ..., s_a)$ and $t = (t_1, ..t_{\nu-a})$ ($\nu = n$ or $n + 1$)

such that the fibres of π are locally given by setting $s = constant$, in other words we have functions

$x(s,t), \lambda(s,t)$ parametrizing the points of Z ,

and a function $O(s)$ parametrizing the image $\pi(Z) = \Sigma$ of the end-point map. This means that the following equations hold :

1") $F(x(s,t)) \equiv 0$

2") $x(s,t) - O(s) \equiv -\lambda(s,t)\nabla F(x(s,t))$,

differentiating 1") with respect to both sets of variables s, t, we infer that

$< \nabla F(x(s,t)), (dx(s,t)/dt_i) > \equiv 0$ as well as

$< \nabla F(x(s,t)), (dx(s,t)/ds_j) > \equiv 0$.

We argue as we did before :

since $x(s,t) - O(s)$ is proportional to $\nabla F(x(s,t))$, we obtain that $x(s,t) - O(s)$ is orthogonal to all the partial derivatives of $x(s,t)$.

Since however $(dx(s,t)/dt_i) = (d(x(s,t) - O(s))/dt_i)$, it follows that there is a function $r(s)$ such that

3") $< x(s,t) - O(s), x(s,t) - O(s) > \equiv r(s)$.

What we have done insofar is to write down the family of spheres containing the projections X_s to X of the fibres over $O(s) \in \Sigma$.

On the other hand, we can use the other partial derivatives $(dx(s,t)/ds_j)$ in order to obtain a complete description of X_s.

In fact, let us calculate the partial derivatives $(\partial r(s)/\partial s_j)$

They are $= 2 < x(s,t) - O(s), (\partial(x(s,t) - O(s)/\partial s_j) >$

$= -2 < x(s,t) - O(s), (\partial O(s)/\partial s_j) >$.

We have therefore established

4") $(\partial r(s)/\partial s_j) = -2 < x(s,t) - O(s), (\partial O(s)/\partial s_j) >$,

whose geometric meaning is the following: if $O(s)$ is a smooth point of Σ, whence all the partial derivatives $(dO(s)/ds_j)$ are linearly independent, then X_s is contained in the intersection of the sphere given by 3") with the codimension a affine subspace given by 4").

If this intersection has the expected dimension $n - a$, then it has the same dimension as X_s and if it is moreover irreducible it will coincide with X_s.

Lemma 3 *Consider an affine subspace $L = \{x| < x - O, v_j >= c_j$ for $j = 1, ..a\}$ of codimension a and assume that L is contained in the sphere $S(O, r^{1/2}) = \{x| < x - O, x - O >= r\}$. Then*

(*) the direction W of L is an isotropic subspace for $<,>$, and there exists $x_0 \in L$ such that $x_0 - O$ is orthogonal to W (equivalently, W is isotropic and $L \subset O + W^{\perp}$).

Observe moreover that the orthogonal $W^{\perp}$ is the vector space U generated by the vectors v_j.

Also the converse holds, in the sense that if (*) is verified, then there exists a constant R such that L is contained in the sphere $S(O, R^{1/2})$.

Proof
Let $x_0 \in L$ and write $L = x_0 + W$. Since $< x - O, x - O > \equiv r$ for $x \in L$, we get

$< x_0 - O, x_0 - O > + 2 < w, x_0 - O > + < w, w > \equiv r$ for each vector $w \in W$.

Thus the quadratic polynomial $< w, w >$ is identically zero on W , what amounts to say that the subspace W is isotropic; the vanishing of the linear form yields the desired orthogonality of $x_0 - O$ to W.

Conversely , $< x - O, x - O > \equiv < x_0 - O, x_0 - O >$ and L is contained in the sphere

$\{x | < x - O, x - O > = R\}$ once we set $R = < x_0 - O, x_0 - O >$. □

Lemma 4 *Consider an affine subspace $L = \{x | < x - O, v_j > = c_j\}$ as in the previous lemma 3, and assume that the affine quadric $L \cap S(O, r^{1/2})$ is reducible. Then either*

(i) $dim(W/W \cap W^{\perp}) = 1$ and there exists $x_0 \in L$ such that $x_0 - O$ is orthogonal to W and $< x_0 - O, x_0 - O > \neq r$ or

(ii) $dim(W/W \cap W^{\perp}) = 2$ and there exists $x_0 \in L$ such that $x_0 - O \in W^{\perp}$ and $< x_0 - O, x_0 - O > = r$

Proof
As before , for each choice of $x_0 \in L$ we can write $L = x_0 + W$. Since the equation of our affine quadric is

$< x_0 - O, x_0 - O > + 2 < w, x_0 - O > + < w, w > = r$ for each vector $w \in W$, and we impose the condition that the quadric be the union of two affine hyperplanes, it follows that the quadratic form $< w, w >$ on W has rank either 1 or 2.

In the latter case, since the rank of the complete quadric equals the rank of the quadratic form, acting with a translation on W, we can kill the terms of lower degree.

In the former case, if the linear part of the equation would not belong to the image under Q_{∞} of $W/W \cap W^{\perp}$, the rank would be at least 3. Whence, acting with a translation on W, we may kill the linear part and then the constant must be non zero.

□

We have therefore found that the projection of Z is contained in the locus X' given by

3"") $\{x|\exists s, < x - O(s), x - O(s) > \equiv r(s)$

4"") $(dr(s)/ds_j) = -2 < x - O(s), (\partial O(s)/\partial s_j) > \}$.

If moreover Z surjects onto X and X' is irreducible, then X' equals X unless we are in the exceptional case where (cf. Lemma 3) for each point $O(s)$ the (vector) tangent space V_s to Σ at $O(s)$ satisfies the condition that V_s contains its orthogonal $W_s := V_s^{\perp}$, and moreover then $(x(s,t) - O(s))$, for each t belongs to the subspace $V_s := W_s^{\perp}$.

The locus X', as written, is the projection of the locus

$Z' \subset \mathbf{P}^m \times \Sigma$ defined as

3"') $\{(x,s)| < x - O(s), x - O(s) > \equiv r(s)$

4"') $(\partial r(s)/ds_j) = -2 < x - O(s), (\partial O(s)/\partial s_j) > \}$.

If we calculate the tangent space to Z' at the point (x,s) we obtain that it is contained in the hyperplane:

5"') $\{(\xi,\sigma)|2 < x - O(s), \xi > -2 < x - O(s), (\partial O(s)/\partial\sigma) > -(\partial r(s)/\partial\sigma) = 0\} = \{(\xi,\sigma)|2 < x - O(s), \xi >= 0\}$

(since (x,s) is a point of Z').

By Sard's lemma, X' has dimension at most n: whence, if we assume that the component Z dominates X, and thus $X \subset X'$, we conclude that $X = X'$ (in the exceptional case, or if X' is irreducible) or at least that X is a component of X'.

We are now in the position to explain the main constructions which are underlying the characterization of the focally degenerate varieties.

Definition 6 *THE INVERSE CONSTRUCTION TO FOCAL DEGENERACY.*

Start from the following data :

i) Let Σ be an irreducible affine variety of dimension a , and let $\Sigma^\star$ be an irreducible subvariety of the product $\Sigma \times \mathbf{C}$ which is the graph of an algebraic function r on Σ.

Proceed constructing an algebraic set X' as follows:

ii) The subvariety $\Sigma^\star$ defines a family of spheres

$Z^\star \subset \mathbf{C}^m \times \Sigma \times \mathbf{C}$ defined as

3"') $\{(x,O,r)|(O,r) \in \Sigma^\star, < x - O, x - O >= r\}$.

iii) Consider in $\mathbf{C}^m \times \mathbf{C}$ the tangent space to $\Sigma^\star$ at a point (O,r), and its orthogonal with respect to the quadratic form $Q_\infty \bigoplus 1$: under the embedding

of $\mathbf{C}^m$ *in* $\mathbf{C}^m \times \mathbf{C}$ *sending* x *to* $(x-O, 1/2)$*, its pull back is precisely an affine space given by an equation as 4"'). We can in this way define a bundle (if* Σ *is smooth) of affine spaces*

$A^\star \subset \mathbf{C}^m \times \Sigma^\star$,

$A^\star = \{(x, O, r) | (O, r) \in \Sigma^\star, (x - O, 1/2) \in T\Sigma^\star_{(O,r)}{}^\perp\}$.

iv) define Z' *as the intersection* $Z^\star \cap A^\star$ *(a divisor in* $A^\star$*)*

v) define X' *as the projection of* Z' *on the first factor* $\mathbf{C}^m$*;*

vi) observe that, by the argument we gave above, $dim X' \leq m - 1$.

vii) assume finally that Σ, r *are* **admissible***, which amounts to the requirement that* Z' *dominate* Σ.

Remark 9 *The condition that* Σ, r *be* **admissible** *is obviously satisfied unless* Z' *is a union of fibres of the projection* $A^\star \to \Sigma$*. This means, unless the quadratic function* $< x - O(s), x - O(s) >$ *is constant on the affine spaces* $A^\star_s$*. Therefore, the pair* Σ, r *is* **admissible** *unless we are in the situation of Lemma 3 , whence* $< x - O(s), x - O(s) > \equiv R(s)$ *on* $A^\star_s$*, but* $R(s) \not\equiv r(s)$.

There remains however to see what happens in the case where Σ lies at infinity .

In this case, we derive (cf. remark 8) the following equations, where $O(s)$ is a V'-valued function leading to a parametrization of Σ :

6) $< x(s,t), O(s) > \equiv r(s)$

7) $(\partial r(s)/\partial s_j) \equiv < x(s,t), (\partial O(s)/\partial s_j) >, for j = 1, ...a.$

In this case, if $O(s)$ is a smooth point of Σ, then the $a + 1$ vectors $O(s), \partial O(s)/\partial s_j$ are linearly independent and 6) and 7) imply that X_s is contained in the affine space

8) $X'_s = \{x | < x, O(s) > \equiv r(s),$
$(\partial r(s)/\partial s_j) \equiv < x, (\partial O(s)/\partial s_j) >, for j = 1, ...a\}$.

Since Σ lies at infinity , X is not isotropically focally degenerate, whence Z has dimension $m - 1$: it follows that X_s, X'_s have the same dimension $m - 1 - a$, whence they coincide.

Moreover, Z must dominate X, else a whole fibre of $N_X \to X$ is contained in Z, and therefore its projection cannot lie at infinity (remember that X is here supposed to be affine).

Therefore, it follows that X equals X', the closure of the union of the X'_s.

We are therefore led to the following

Definition 7 *THE ASYMPTOTIC INVERSE CONSTRUCTION TO WEAK FOCAL DEGENERACY.*

Start from the following data :

i) Let Σ be an irreducible variety of dimension a , contained in $\mathbf{P}_\infty$, and let $\Sigma^\star$ be an irreducible subvariety of the product $\Sigma \times \mathbf{C}$ which is the graph of an algebraic section r of $\mathcal{O}_\Sigma(1)$.

Then consider the algebraic set X' which is the closure of the union of the family of affine spaces X'_s defined by 8).

Remark 10 *The attentive reader will find a slight abuse of notation above, which can be explained as follows : in the case where Σ does not lie at infinity, since we have a privileged affine chart $= \mathbf{P^m} - \mathbf{P}_\infty$, we consider $r(s)$ just as an algebraic function on Σ. If however $\Sigma \subset \mathbf{P}_\infty$, then there is no favourite standard affine chart and we make clear that r is not really a function, but a section of $\mathcal{O}_\Sigma(1)$ (possibly multivalued and with poles!).*

Remark 11 *Consider a variety $X = X'$ obtained from the asymptotic inverse focal construction.*

Then its part $X_\infty = X \cap \mathbf{P}_\infty$ consists of the points

$\{x \in \mathbf{P}_\infty | < x, O(s) > \equiv r(s),$
$(\partial r(s)/\partial s_j) =< x, (\partial O(s)/\partial s_j) >, for j = 1, ...a\}$.

If we therefore identify $\mathbf{P}_\infty$ with its dual space via the quadratic form Q_∞, it follows that X_∞ is the dual variety of Σ !

Observe moreover, that if X is a linear space, then Σ_X equals $X^\perp_\infty$.

In this case the section $r(s)$ is just induced by a linear form on Σ_X (i.e., a vector in $(V')^\vee$).

We observe now that we have insofar proved the following

Theorem 2 *Let X be a focally degenerate hypersurface in $\mathbf{C}^{n+1}$ and let Σ be a component of the strict focal locus (i.e., we are in the non vertical case and the corresponding component Z of Y_X projects onto X). Then*

- *either Σ is contained in $\mathbf{P}_\infty$ and X is obtained from Σ, r via the asymptotic inverse focal construction associated to an algebraic section r of $\mathcal{O}_\Sigma(1)$*

- *or, Σ is not contained in $\mathbf{P}_\infty$ and there is an algebraic function $r(s)$ on Σ such that, applying the inverse construction to focal degeneracy, we get a hypersurface X' such that X is a component of X'.*

Conversely, start from any admissible pair (Σ, r), and assume that an irreducible hypersurface is a component of the algebraic set X' obtained from the inverse construction or from the asymptotic inverse construction : then X is a focally degenerate hypersurface.

Proof
There remains only to show that if X is an irreducible hypersurface, component of the algebraic set X' obtained from an inverse construction : then Σ is a component of the focal locus of X. This follows since, by 5"'), $x - O(s)$ is a normal vector to X', respectively since $O(s)$ is a normal vector to X'; moreover, Z' dominates Σ by the assumption that r be admissible. □

However, the inverse constructions, as we are going to see, work more generally also in the case where X' has smaller dimension than the expected dimension $m - 1$.

We have in fact the following

Theorem 3 *Let X be a focally degenerate variety of dimension n in $\mathbf{C}^m$ and let Σ be a component of the focal locus of dimension $a \leq m - 1$, projection of a component Z of Y_X. Then Σ determines birationally an irreducible subvariety $\Sigma^\star$ of $\Sigma \times \mathbf{C}$ corresponding to an algebraic section $r(s)$ of $\mathcal{O}_\Sigma(1)$ and, applying the appropriate inverse construction to focal degeneracy, we get an algebraic set X' which is focally degenerate, and indeed isotropically focally degenerate in the case where Σ is not contained in the hyperplane at infinity $\mathbf{P}_\infty$ and $dim Z' = m$ (in this case the fibres X'_s of $Z' \to \Sigma$ are affine spaces).*

There are seven cases :

- *1) X is isotropically focally degenerate : then $X = X'$, $dim Z' = m$ and the fibres X_s of $N_X \to \Sigma$ are affine spaces. Moreover, here Σ is not contained in $\mathbf{P}_\infty$.*

- *2) Σ is not contained in $\mathbf{P}_\infty$, Z projects onto X and X' is not isotropically focally degenerate: then X is a component of X'*

- *3) Z projects onto X, X' is isotropically focally degenerate, but X is not isotropically focally degenerate: then $X \subset X'$ is a divisor, Z is the restriction to X of the normal bundle $N_{X'}$, and Σ is the focal locus of X' (again here Σ is not contained in $\mathbf{P}_\infty$)*

- *4) Σ is not contained in $\mathbf{P}_\infty$, Z projects onto a divisor $X"\subset X$, $X"$ is a component of X', $X"$ is focally degenerate, with a component $Z"$ of the ramification locus $Y_{X"}$ which is a subbundle of $N_{X"}$: then the tangent bundle to X around $X"$ is annihilated by the given subbundle $Z"$.*

- *5) Z projects onto a divisor $X"\subset X$ which is focally degenerate, $X"$ is a divisor of X' and X' is isotropically focally degenerate (again here Σ is not contained in $\mathbf{P}_\infty$). Then X and X' are tangent along $X"$.*

- *6) Σ is contained in $\mathbf{P}_\infty$, Z projects onto some affine point of X , whence it dominates X and $X = X'$ is obtained via the asymptotic inverse focal construction.*

- *7) Σ is contained in $\mathbf{P}_\infty$, Z projects onto a component Δ of X_∞. In this case Z is the restriction of N_X to Δ, the second projection to $\mathbf{P}_\infty$ is not surjective. This case is characterized by the property that $\Delta \subset X_\infty$ be projectively isotropically degenerate, which is equivalent to the property that Δ be obtained via the isotropic projective inverse focal construction (this case will be treated separately in the next proposition).*

Conversely, start from any admissible pair of a variety Σ not contained in $\mathbf{P}_\infty$, and of an algebraic section $r(s)$. Consider the algebraic set X' obtained from the inverse construction : then X' is focally degenerate (if it has two components, this means that each of them is focally degenerate) and isotropically focally degenerate iff the fibres of $Z' \to \Sigma$ are affine spaces of dimension $m - dim\Sigma$ (then $Z' = N_{X'}$) .

All the isotropically focally degenerate varieties X are gotten by the inverse construction as such an X'.

In case 6), where Σ is a component of the strict focal locus contained in $\mathbf{P}_\infty$, all such weakly focally degenerate varieties are obtained from the asymptotic inverse focal construction.

Let us consider the remaining cases where Σ is not contained in $\mathbf{P}_\infty$.

Then the weakly focally degenerate varieties in the non vertical case (i.e., when Z dominates X) are gotten either
(i) as a component of such an X', or
(ii) as a divisor in an isotropically focally degenerate variety X', which is transversal to the general fibres X'_s of $N'_X \to \Sigma_{X'}$ and where dim $\Sigma_{X'} \leq m-2$.

Instead, in the vertical case, the weakly focally degenerate varieties are given as varieties containing a focally degenerate divisor $X"$ such that either (i) $X"$ is a component of X', with a component $Z"$ of the ramification locus $Y_{X"}$ which is a subbundle of $N_{X"}$, and such that the tangent bundle of X along $X"$ is given by the annihilator of the subbundle $Z"$ or
(ii) $X"$ is a divisor of X', X' is isotropically focally degenerate with dim $\Sigma_{X'} \leq m-2$, *$X"$ is transversal to the fibres X'_s of $N'_X \to \Sigma_{X'}$, and X and X' are tangent along $X"$.*

Proof
We discuss first of all the case where Σ is not contained in $\mathbf{P}_\infty$ (whence Z does not project to $\mathbf{P}_\infty$ under the first projection) .

Around each smooth point of X there are a Zariski open set U of $\mathbf{C}^m$ and polynomials $F_1(x), ...F_{m-n}(x)$ such that $X \cap U = \{x \in U | F_1(x) = ...F_{m-n}(x) = 0\}$ and such that $X \cap U$ consists of smooth points.

Therefore, the gradients of the polynomials $F_1(x), ...F_{m-n}(x)$ yield a framing of the Euclidean normal bundle on $X \cap U$, and the endpoint map is locally given by

$\epsilon(x, \lambda_1, ..\lambda_{m-n}) = x + \Sigma_{i=1,..m-n} \lambda_i \nabla F_i(x)$.

We choose as we did before a component Z of the ramification locus Y_X which maps onto an irreducible variety Σ of dimension $a \leq m-2$ (respectively $a \leq m-1$ in the focally isotropically degenerate case) and local coordinates

$s = (s_1, ..., s_a)$ for the points of Σ and $t = (t_1, ..t_{\nu-a})$ for the fibres of π, where ν equals $m-1$ in the non focally isotropically degenerate case, otherwise $m = \nu$ and $Z = N_X$.

Whence, we have local functions $x(s,t), \lambda(s,t)$ parametrizing the points of Z ,

and a function $O(s)$ parametrizing the image $\pi(Z) = \Sigma$ such that

1") $F_i(x(s,t)) \equiv 0 \ \forall i$

2") $x(s,t) - O(s) \equiv -\Sigma_{i=1,..m-n}\lambda_i(s,t)\nabla F_i(x(s,t))$.

Differentiating 1") with respect to both sets of variables s, t, we infer that
$< \nabla F_j(x(s,t)), (\partial x(s,t)/\partial t_i) > \equiv < \nabla F_j(x(s,t)), (\partial x(s,t)/\partial s_h) > \equiv 0 \forall i,j,h$.
By 2") $x(s,t) - O(s)$ is a normal vector, whence

3") $< x(s,t) - O(s), x(s,t) - O(s) > \equiv r(s)$.
and
4") $(\partial r(s)/\partial s_j) = -2 < x(s,t) - O(s), (\partial O(s)/\partial s_j) >$.

Therefore, for fixed s, the projection X_s of the fibre Z_s (generally a manifold of dimension $\nu - a$) is contained in the intersection X'_s of a sphere S_s of centre $O(s)$ and radius $r(s)^{1/2}$ with an affine space Π_s of codimension a (since at the general point we can assume $\partial O(s)/\partial s_1), ...\partial O(s)/\partial s_a)$ to be linearly independent).

Thus, the manifold X'_s has dimension either $m-1-a$ or $m-a$ (but in the latter case, by Lemma 3, the orthogonal to $T\Sigma_{O(s)}$ is contained in $T\Sigma_{O(s)}$).

Consider as before the locus X' given as the projection of the locus
$Z' \subset \mathbf{P}^m \times \Sigma$ defined as
3"') $\{(x,s) | < x - O(s), x - O(s) > \equiv r(s)$
4"') $(\partial r(s)/\partial s_j) = -2 < x - O(s), (\partial O(s)/\partial s_j) >\}$.

Lemma 5 $Z' \subset N_{X'}$

Proof
We must prove that the vector $x - O(s)$ is normal to X'. This follows ¿from the calculation of the tangent space to Z' at the point (x,s) that we have done above (cfr. 5"'). □

Corollary 5 *Each component of X' is focally degenerate and indeed isotropically focally degenerate iff $X'_s = \Pi_s$ (whence, in the latter case, X' is also irreducible).*

Proof
If $X'_s = \Pi_s$, then Z' is irreducible and $dim\, Z' = dim\, N_{X'} = m$ so that $Z' = N_{X'}$ and X' is irreducible and isotropically focally degenerate.

If $dim\, X'_s = m - 1 - a$ (in this case $a \leq m-2$), then Z' is a divisor in $N_{X'}$ and hence Σ is contained in $\Sigma_{X'}$. Either Σ is a component of Σ_{X^0}, for

each component X^0 of X', or there is a component X^0 of X' which is focally isotropically degenerate.

Assume that the latter holds: then, for general $O(s) \in \Sigma$, X'_s is a divisor of the fibre of $N_{X^0} \to \Sigma_{X^0}$, whence by dimension reasons $\Sigma = \Sigma_{X^0}$.

Since the direction of Π_s is the vector subspace $W = T\Sigma^{\perp}_{O(s)}$, and $\Sigma = \Sigma_{X^0}$, it follows that $N_{X^0_s} = \Pi_s$.

Moreover, being X^0 isotropically focally degenerate, by lemma 3 follows that W is totally isotropic, whence the quadratic function $< x - O(s), x - O(s) >$ is then constant on Π_s, contradicting the fact that for general s X'_s is a nonempty and proper divisor in Π_s. $\Box$

If X is focally isotropically degenerate, the projection X_s of Z_s has dimension $m - a$, whence it equals X'_s, and it follows immediately that X equals X'.

Suppose then that X is not isotropically focally degenerate, and let X'' be the projection of Z, that is the closure of $\cup_s X_s$. Thus $X" \subset X$ and $X" \subset X'$.

Assume first that $dim\, X'_s = dim\, X_s = m - 1 - a$. Therefore, $X"$ is a component of X' and Z equals a component $Z"$ of Z'. It follows that either $X" = X$ and case 2) of the theorem occurs, or $X" \subset X$ would be a divisor and Z would be the restriction to $X"$ of the normal bundle N_X, a subbundle of the normal bundle $N_{X"}$.

Whence, $X"$ is focally degenerate, with a component $Z = Z"$ of the ramification locus which is a projective subbundle of ¿$N_{X"}$, and case 4) occurs. Any variety M containing $X"$ as a divisor, and with tangent bundle annihilated by the given subbundle would be a weakly focally degenerate variety with Σ in the focal locus.

In other words, in the vertical case, the inverse focal construction can by no means reconstruct X, but only the first order neighbourhood of X along $X"$.

Finally, there remains the case where $dim\, X_s = m-1-a$, $dim X'_s = m-a$, in which case X' is isotropically focally degenerate. Then Z is a divisor in $Z' = N_{X'}$.

Assume $X" = X'$: since then $X' \subset X$, but $X' \neq X$ since X is not isotropically focally degenerate, we get that $Z = N_X|_{X'} \subset Z' = N_{X'}$, and we are again in case 4).

Thus we may consider the remaining cases where $X"$ is a divisor of X'.

Furthermore, either $X" = X$ or $X"$ is a divisor in X. If $X = X"$, then Z is the restriction of $N_{X'}$ to X, and case 3) occurs. If $X"$ is a divisor of X, we have that $Z = N_X|_{X"} = N_{X'}|_{X"}$ so that X and X' are tangent along $X"$ and case 5) occurs. Conversely, let X' be a isotropically focally degenerate variety and let X be a divisor inside X'; since $N_{X'}|_X \subset N_X$ is a divisor, it follows immediately that, setting $Z = N_{X'}|_X$, the image of Z is contained in $\Sigma_{X'}$. If moreover, as it should be, the divisor X is transversal to the fibres X'_s, then its image equals $\Sigma_{X'}$, whence Z will make X weakly focally degenerate if and only if $dim\,\Sigma_{X'} \leq m-2$. More generally, if M is any variety containing X as a divisor and such that M and X' are tangent along X, then M is weakly focally degenerate.

Let us then consider case 6) : then, analogously to the case of hypersurfaces we can find a parametrization $O(s)$ of Σ in homogeneous coordinates such that

$O(s) \equiv -\Sigma_{i=1,..m-n}\lambda_i(s,t)\nabla F_i(x(s,t))$.

Then $< \partial x(s,t)/\partial t_i, O(s) > \equiv < \partial x(s,t)/\partial s_j, O(s) > \equiv 0$.

¿From the first equalities we conclude that there exists a local function $r(s)$ such that

6) $< x(s,t), O(s) > \equiv r(s)$.

One moment's reflection, since the vector $O(s)$ gives homogeneous coordinates for Σ, shows that indeed $r(s)$ globalizes to a (possibly multivalued and with poles) section of $\mathcal{O}_\Sigma(1)$.

¿From the second equalities follows also

7) $(\partial r(s)/\partial s_j) \equiv < x(s,t), (\partial O(s)/\partial s_j) >, for j = 1, ...a.$

Thus an entirely similar argument yields that X is gotten ¿from the asymptotic inverse focal construction, and conversely if X is obtained in this way then X is weakly focally degenerate and we are in case 6).

□

Let us discuss case 7), where the whole condition of degeneracy bears on X_∞, and tells that, $O(s)$ being the V'- vector valued function giving local homogeneous coordinates around a smooth point of Σ as usual, there is a local function $\lambda(s,t)$ and a local parametrization $x(s,t)$, of X_∞ this time, and giving homogeneous coordinates, such that

$\lambda(s,t)x(s,t) - O(s)$ is a normal vector to X_∞ at the point $x(s,t)$, in the sense that

$< \lambda(s,t)x(s,t) - O(s), x(s,t) >\equiv$
$< \partial x(s,t)/\partial t_i, \lambda(s,t)x(s,t) - O(s) >\equiv$
$< \partial x(s,t)/\partial s_j, \lambda(s,t)x(s,t) - O(s) >\equiv 0.$

At the points where $\lambda(s,t)$ is not vanishing we can replace the parametrization $x(s,t)$ by $\lambda(s,t)x(s,t)$, so with these new homogeneous coordinates we have

$I) < x(s,t) - O(s), x(s,t) >\equiv$
$II) < \partial x(s,t)/\partial t_i, x(s,t) - O(s) >\equiv$
$III) < \partial x(s,t)/\partial s_j, x(s,t) - O(s) >\equiv 0.$

Deriving equation I) with respect to $\partial/\partial t_i$, and using II) we obtain

$IV) < \partial x(s,t)/\partial t_i, x(s,t) >\equiv 0$

whereas, applying $\partial/\partial s_j$ to I) and using III) we get

$V) < \partial x(s,t)/\partial s_j, x(s,t) >\equiv < \partial O(s)/\partial s_j, x(s,t) >$.

IV) yields

$A) < x(s,t), x(s,t) >\equiv < O(s), x(s,t) >\equiv r(s)$ which implies, together with V) :

$B) < \partial O(s)/\partial s_j, x(s,t) >\equiv 1/2\partial r(s)/\partial s_j.$

Since we chose a smooth point of Σ the $a+1$ vectors

$O(s), \partial O(s)/\partial s_1, ...\partial O(s)/\partial s_a$ are linearly independent, and it follows that the vectors $x(s,t)$, for s fixed, vary in an affine space $X"_s$ of dimension $m-1-a$.

Since however X_s is assumed to have dimension exactly equal to $m-1-a$, it follows that $X_s = X"_s$, where $X"_s$ is defined by the equations

$A') < O(s), x >\equiv r(s)$

$B') < \partial O(s)/\partial s_j, x >\equiv 1/2\partial r(s)/\partial s_j.$

However, also the equality $< x, x >\equiv r(s)$ must be satisfied on $X_s = X"_s$, thus by Lemma 3 we get an affine linear subspace with direction W which is totally isotropic, and is contained in the orthogonal $W^\perp$ to W.

The conclusion is that the projective tangent space to Σ at any smooth point has a totally isotropic annihilator .

Definition 8 *Let Σ be a projective subvariety of the projective space $\mathbf{P}(V') = \mathbf{P}_\infty$ associated to a vector space V' of dimension m endowed with a non degenerate quadratic form Q_∞, such that any point $O(s)$ of Σ the projective*

tangent space to Σ at $O(s)$ (a vector subspace of V') has a totally isotropic annihilator.

Let $r(s)$ be an algebraic section of $\mathcal{O}_\Sigma(1)$ and consider the developable variety X" defined by the union of the subspaces X"$_s$ defined by the equations A') and B').

Assume moreover that r be **admissible** *in the sense that the local function (constant on X"$_s$)*

$< x(s,t), x(s,t) > \equiv R(s)$ be equal to $r(s)$.

Then we shall say that X" is projectively isotropically degenerate and that X" is obtained via the isotropic projective inverse focal construction from the admissible pair (Σ, r).

Proposition 3 *Assume that X is weakly focally degenerate and that a component Σ of the focal locus is contained in $\mathbf{P}_\infty$, with the corresponding component Z of Y_X projecting onto a component Δ of X_∞ (case 7) of theorem 3). In this case Z is the restriction of N_X to Δ, Δ is projectively isotropically degenerate. Conversely, if Δ is obtained via the isotropic projective inverse focal construction , then X is weakly focally degenerate and we are in case 7) of theorem 3.*

Proof
If X is as in case 7) of theorem 3, then we have already seen that $\Delta \subset X_\infty$ is projectively isotropically degenerate.

It remains to prove the converse, which follows since A'), B') and our assumption $R(s) = r(s)$ imply A), B) by which immediately follow I), II), and III), whence $x(s,t) - O(s)$ is a normal vector to X_∞. Since $X" = \Delta$ and $X"_s$ has dimension $m-1-a$ we get a component Z of dimension $m-1$ projecting onto the a-dimensional variety Σ contained in the hyperplane at infinity and we are done.

□

Remark 12 *It follows from the previous theorem that any variety Σ is a component of some focal locus.*

Moreover, in the asymptotic inverse construction, we see immediately that the tangent space at a point of X_s depends only upon s , so that then our X is developable.

In particular, if $m = 3$ we get either a linear subspace or a developable, whence singular, surface.

Observe finally that if X is orthogonally general and projective, then only cases 6) or 7) can a priori occur.

For case 6), start choosing X_∞ as a smooth and transversal variety to Q_∞, apply then the asymptotic inverse focal construction : then we get a variety X which will be orthogonally general exactly iff X is smooth. But the smoothness of X, as we have just seen, is the main obstruction.

Example 6 *Let $m = 3$, and let Σ be the line at infinity parametrized as*

$O(s) = (0, 0, 1, s)$, and set, in these affine coordinates, $r(s) = s^2/2$.

Then an easy computation for the asymptotic inverse focal construction yields the quadric cone

$X = \{x | x_0 x_2 - x_3^2/2 = 0\}$, whose vertex lies at infinity.

If instead we choose $O(s) = (0, 1, s, s^2)$, and $r(s) \equiv 1$, X will be the quadric cone

$X = \{x | 4(x_1 - x_0)x_3 - x_2^2 = 0\}$, whose vertex does not lie at infinity.

Example 7 *Let us now consider the most classical example, namely the rotational torus X obtained rotating a circle of radius, say, 1 around the point with coordinates $(2, 0)$. This is the example of a strongly focally degenerate variety.*

The equation F of X, in affine coordinates (x, y, z) for which Q_∞ yields the standard Euclidean scalar product, is then given, setting

$q(x, y, z) = (x^2 + y^2 + z^2 + 3)$, or , in homogeneous coordinates (x, y, z, w), $q(x, y, z, w) = (x^2 + y^2 + z^2 + 3w^2)$, by

(*) $q^2 - 16(x^2 + y^2)w^2$.

The intersection with the plane at infinity is precisely our conic $Q_\infty = \{q = w = 0\}$, which is a double curve for the quartic surface X. Moreover, $Sing(X)$ consists of Q_∞ and of the two points $\{P, P'\} = \{q = x = y = 0\} = \{(z^2 + 3w^2) = x = y = 0\}$.

Now, a classical and easy formula for a rotation surface of a curve C $=r(s), z(s)$ parametrized by arclength,

$x(s, \theta) = r(s)cos(\theta)$

$y(s,\theta) = r(s)sin(\theta)$

$z(s,\theta) = z(s)$

is that the two principal curvatures equal $k(s)$, $z'(s)/r(s)$.

In this case, $r(s), z(s) = (2+cos(s), sin(s))$, whence $k \equiv 1$ and $z'(s)/r(s)$ $= 1 - 2/r(s)$.

These formulae are easily rationalized on our surface X since $q^2 = 16(x^2 + y^2)$, whence $r = q/4$. Therefore the critical points are obtained by taking the multiples of the unit normal by the opposites of their inverses, i.e., -1 and $-q/(q-8)$. Finally, the unit normal is obtained by the gradient of F

$\nabla F = (4x(q-8), 4y(q-8), 4qz)$ upon dividing by its norm, which equals

$|\nabla F| = 4((q-8)^2(x^2+y^2) + q^2z^2)^{1/2} =$

$(16(q-8)^2(x^2+y^2) + 16q^2z^2)^{1/2} = q((q-8)^2 + 16z^2)^{1/2}$. But since

$z^2 = q - 3 - q^2/16$, we get $q(64-48)^{1/2} = 4q$,

and the focal locus is obtained for the values $\lambda = -1/4q, \lambda = -1/4(q-8)$ as the image of the endpoint map $(x,y,z) + \lambda\nabla F(x,y,z)$.

For $\lambda = -1/4(q-8)$ we get the points $(0,0,z(8/q-8))$, for $\lambda = -1/4q$ we get the points $(8x/q, 8y/q, 0)$.

The conclusion is that the focal locus consists of the z-axis and of the circle $z = 0, x^2 + y^2 = 4$. That is, our surface is strongly focally degenerate, and we can indeed see geometrically the two families of circles corresponding to the two components of the focal locus.

We end this protracted example by observing that the rotation surface is clearly a rational surface. Indeed, we can say more, since a smooth model is obtained by blowing up the singular conic Q_∞ and the two points P, P'.

Let R and E, E' be the respective exceptional divisors in the blow-up $\widetilde{\mathbf{P}}$ of $\mathbf{P}^3$: the first is a ruled surface $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \bigoplus \mathcal{O}_{\mathbf{P}^1}(2))$, the other are two $\mathbf{P}^2$'s.

Let S be the strict transform of X: it belongs to the linear system $|4H - 2R - 2E - 2E'|$, whereas the canonical system of $\widetilde{\mathbf{P}}$ equals $|-4H + R + 2E + 2E'|$. Thus S belongs to $|-K-R|$, and by the exact sequence

$0 \to \mathcal{O}_{\widetilde{\mathbf{P}}}(-S) \to \mathcal{O}_{\widetilde{\mathbf{P}}} \to \mathcal{O}_S \to 0$

we infer $h^i(\mathcal{O}_S) = h^{i+1}(\mathcal{O}_{\widetilde{\mathbf{P}}}(K+R) = h^{2-i}(\mathcal{O}_{\widetilde{\mathbf{P}}}(-R) = 0$, since R is irreducible. S is clearly then rational, and the anticanonical effective divisor has self-intersection 4.

Example 8 *More generally, for a rotation surface* $(r(s)cos(\theta), r(s)sin(\theta), z(s))$ *the unit normal is given by* $(z'(s)cos(\theta), z'(s)sin(\theta), r'(s))$ *, therefore we see easily that the focal locus consists of the* z*-axis and of the rotation surface*

obtained by rotating the evolute of the plane curve $C = \{r(s), z(s)\}$ *we were starting with.*

Therefore, general rotation surfaces provide examples of weakly but not strongly focally degenerate varieties.

Example 9 *This last example shows the important role of the algebraic function* $r(s)$.

Let Σ *be the line* $\{(0,0,s) \in \mathbf{C}^3\}$ *: then if we take the function* $r(s) \equiv R$, *where* $R \in \mathbf{C}$ *is a constant, the inverse construction yields a cylinder* X'. *Instead, if we choose* $r(s) \equiv R + s^2$, *we obtain as* X' *simply a circle in the plane* $z = 0$.

7 Isotropically focally degenerate hypersurfaces

In the preceding section we gave a characterization, in terms of the inverse focal construction, of the focally isotropically degenerate varieties. However, in general such a construction yields a hypersurface, which is only weakly degenerate, and although in the next section we shall write down conditions which characterize the focally isotropically degenerate case, in the case of hypersurfaces, we can give an easier characterization for the isotropically degenerate case with a direct proof.

Let thus $F(x_1, ...x_{n+1}) = 0$ be the polynomial equation of an affine hypersurface X, which we may assume, without loss of generality, to be irreducible.

Again the gradient ∇F of F gives a map of the Normal Bundle N_X, $\pi : N_X \to \mathbf{P^{n+1}}$ which we will also call the endpoint map

$\epsilon(x, \lambda) = x + \lambda \nabla F(x)$, where x is a point of X (thus, for $\lambda = 0$ we reobtain the points of X).

Proposition 4 *Let* X *be a projective hypersurface : then* X *is focally isotropically degenerate if and only if* X *coincides with its focal locus* Σ_X.

Proof
In this case the focal locus equals the image Σ_X of the map $\pi : N_X \to \mathbf{P}^{n+1}$,

and since X may be assumed to be irreducible, N_X is irreducible, whence Σ_X is also irreducible. But X is contained in Σ_X and has not lesser dimension, thus equality holds. $\square$

Remark 13 *We derive thus the equality*

$F(x + \lambda \nabla F(x)) \equiv 0 \ \forall \lambda$.

In particular $(d/d\lambda)F(x + \lambda \nabla F(x)) \equiv 0$, *and, for* $\lambda = 0$, *we get*

(I) $< \nabla F(x), \nabla F(x) > \equiv 0$.

By the previous proposition the general fibre of π has dimension 1, and for each $x_0 \in X$, $\lambda_0 \in \mathbf{C}$ there exists a curve

(II) $x(t), \lambda(t)$ such that $x(0) = x_0, \lambda(0) = \lambda_0$, which is a fibre of π.

Since a fibre intersects a normal line $x_0 \times \mathbf{C}$ in at most one point, it follows that up to a birational transformation we can take (x_0, t) as coordinates on N_X by taking the curves $x(x_0, t), \lambda(x_0, t)$ satisfying (II) for $\lambda_0 = 0$, and assume that the curve $x(x_0, t)$ is a non constant curve in X satisfying

(III) $x(x_0, t) + \lambda(x_0, t) \nabla F(x(x_0, t)) \equiv x_0$.

We argue as in the preceding section :

$x(x_0, t) - x_0 \equiv -\lambda(x_0, t) \nabla F(x(x_0, t))$,

thus by (I) our usual function $r(x_0) \equiv 0$ and

(IV) $< x(x_0, t) - x_0, x(x_0, t) - x_0 > \equiv 0$.

In this case we also get, if $s = (s_1, ..., s_n)$ are local coordinates for $x_0 \in X$, that $(dr(s)/ds_j) \equiv 0$ and

(V) $0 = -2 < x(x_0(s), t) - x_0(s), (dx_0(s)/ds_j) >$ for each s, t.

Since the tangent space to X at x_0 has dimension n, we infer that, fixing s and varying t, we obtain a curve $x(x_0, t)$ which moves on the line through x_0 with direction $\nabla F(x_0)$.

We can thus write

(VI) $x(x_0, t) = x_0 + \mu(x_0, t) \nabla F(x_0)$,

and then (VI) and (III) combine to yield

(VII) $\lambda(x_0, t) \nabla F(x_0, t) \equiv -\mu(x_0, t) \nabla F(x_0)$.

Since the function $\lambda(x_0, t)$ is non zero, it follows that not only the line through x_0 with direction $\nabla F(x_0)$ is contained in X, but also that the normal direction stays constantly proportional to $\nabla F(x_0)$ on it.

We have thus proven the following

Theorem 4 *A hypersurface X is isotropically focally degenerate if and only if it is isotropically developable, i.e., for each point the normal line is contained in X, and along this line the tangent space to X does not vary.*

We would like now to give some examples and show where lies the difficulty in the fine classification of isotropically focally degenerate hypersurfaces.

It is classically known that in 3-space the analytical surfaces which are developable are only cones, cylinders, and tangential developable surfaces.

Proposition 5 *Assume X is an isotropically developable surface. Then , if X a cylinder then X is a plane. If X is a cone , it is the cone over Q_∞ with vertex in a point of affine space.*

Proof
If X is a cylinder, then the generatrices are the normal lines, therefore the normal direction is constant on the whole surface and the surface is a plane. If X is a cone, with vertex, say, at the origin, then the vectors x and $\nabla F(x)$ are proportional,

but the vector $\nabla F(x)$ is always isotropic, whence $< x, x >\equiv 0$ on X, q.e.d. $\square$

Let us now discuss the tangential surface X of a curve C.

We write as usual X parametrically as

$x(s,t) = \alpha(s) + t\alpha'(s)$,

so that the tangent plane is generated by the two vectors

$\alpha'(s), \alpha''(s)$.

Up to local analytic reparametrization we can assume that one and only one of the following two possibilities occurs:

(I) $< \alpha'(s), \alpha'(s) >\equiv 0$

(U) $< \alpha'(s), \alpha'(s) >\equiv 1$.

In both cases follows that

(*) $< \alpha'(s), \alpha''(s) >\equiv 0$.

In the isotropic case (I), then clearly $\alpha'(s)$ is a normal vector to X, constant on the generatrices, and our X is thus isotropically developable.

We could stop our discussion here, since the isotropic ruling, in the situation we are interested in, is obtained by fixing s and varying t, which means

that we are in principle through with our discussion. Nevertheless, for curiosity, we analyze also the unitary case which we could avoid to consider in view of the assumption that our surface is not only developable, but also isotropically developable.

Lemma 6 *The unitary case (U) occurs only if the curve C is a plane curve, thus its tangential surface is a plane.*

Proof
In the unitary case (U), the normal vector must be proportional to $\alpha''(s)$, whence X is isotropically degenerate if and only if

$(**) < \alpha''(s), \alpha''(s) >\equiv 0$. Now, by taking derivatives of (*) and (**), and using (**), we obtain

$(***) < \alpha''(s), \alpha'''(s) >\equiv 0$

$< \alpha'(s), \alpha'''(s) >\equiv 0$

¿from which it follows that $\alpha'''(s), \alpha''(s)$ are proportional vectors, whence also

$< \alpha'''(s), \alpha'''(s) >\equiv 0.$

By induction, we show that for each integer n

$(*\mathrm{n}^*) < \alpha''(s), \alpha^{(n)}(s) >\equiv 0$

$< \alpha'(s), \alpha^{(n)}(s) >\equiv 0$

whence $\alpha''(s), \alpha^{(n)}(s)$ are proportional and thus also

$< \alpha^{(n)}(s), \alpha^{(n)}(s) >\equiv 0.$

Consider now the Taylor development of $\alpha(s)$ at any point : from the fact that all higher derivative vectors are proportional follows that $\alpha(s)$ yields a plane curve.

But this means that its tangential surface is a plane.

□

It is now clear that in order to classify the non-trivial isotropically developable surfaces in 3-space we would need to classify the isotropic space curves C (i.e., those whose tangent vector is always an isotropic vector,that is, (I) holds).

Now, the condition that C is algebraic is an obstacle!

Indeed, C will be the birational image of a smooth curve B , given through 4 sections (s_0, s_1, s_2, s_3) of a line bundle on B: the isotropicity condition

amounts to the following equation (where $'$ represents the derivative with respect to a local parameter on B)

$(\mathcal{E})$ $\Sigma_{i=1,2,3}(s_i' s_0 - s_0' s_i)^2 \equiv 0$.

Example 10 *It is rather easy to give examples of rational curves which are isotropic.*

It suffices, chosen an affine coordinate t on $\mathbf{C}$, to set $s_0 \equiv 1$ and let (s_1, s_2, s_3) be polynomials in t such that their derivatives satisfy

$\Sigma_{i=1,2,3}(s_i')^2 \equiv 0$.

In other words ,(s_1', s_2', s_3') give a rational parametrization of the conic Q_∞ and (s_1, s_2, s_3) are taken to be the integrals of the three polynomials (s_1', s_2', s_3').

In this way we see more generally that, up to translation, our curve C is determined by our map $B \to Q_\infty$.

Using in our particular case of the rational curves fixed isomorphisms of $\mathbf{P^1}$ with B and with Q_∞, we obtain that our isotropic rational curves are parametrized by a pair of polynomials $f_0(t), f_1(t)$.

In concrete terms , we may take

$(s_1', s_2', s_3') = (f_0^2 + f_1^2, if_1^2 - if_0^2, -2if_0 f_1)$.

Assume now not only that the map $f_0(t), f_1(t)$ is of positive degree and is primitive (does not factor through an intermediate cover), e.g. it could be a cyclic Galois cover of prime order p.

If the map (s_1, s_2, s_3) would not be birational onto its image, then the tangent map from C to Q_∞ would be a birational isomorphism.

But, in the example we gave above, $f_0(t) = 1, f_1(t) = t^p$, we see immediately that (s_1, s_2, s_3) are not polynomials in t^p.

8 Isotropically focally degenerate varieties and further examples

In the previous section we have given a classification, and concrete examples of isotropically focally degenerate hypersurfaces.

It is easy to obtain concrete examples in higher codimension by the following simple device : consider varieties $M \subset \mathbf{C^m}$, $W \subset \mathbf{C^w}$ and consider the product variety $X = M \times W$ in the orthogonal direct sum $\mathbf{C^m} \bigoplus^{\perp} \mathbf{C^w} = \mathbf{C^{m+w}}$.

It is immediate to see that in this case the normal bundle of X is a product, likewise the endpoint map.

Remark 14 *If thus $M \subset \mathbf{C^m}$ and $W \subset \mathbf{C^w}$ are isotropically focally degenerate, then $X = M \times W \subset \mathbf{C^{m+w}}$ is also isotropically focally degenerate, and $\Sigma_X = \Sigma_M \times \Sigma_W$. In particular, we obtain in this way Σ_X of arbitrary codimension.*

We obtain also, by letting M be an isotropically developable hypersurface, and W general, an example of a variety X of arbitrary codimension which is isotropically focally degenerate, and whose Σ_X is a hypersurface.

We now finally observe that the inverse focal construction gives a characterization of the isotropically focally degenerate varieties in terms of their focal variety Σ_X.

Theorem 5 *Let Σ be a projective variety of dimension a, and let $r(s)$ be an algebraic function on its affine part. Assume moreover that*

1) at any point $O(s)$ of Σ the vector tangent space to Σ at $O(s)$ (a vector subspace of V') has a totally isotropic annihilator.

Then, if X is gotten from (Σ, r) via the inverse focal construction, and moreover

2) the algebraic function $r(s)$ satisfies the conditions

2.1) $dr(s) \in Im(T_{\Sigma,s} \rightarrow^{Q_\infty} \rightarrow T^\vee_{\Sigma,s})$

2.2) given ξ with $Im(\xi) = df$, then $1/4 < \xi, \xi >= r(s)$

then X is isotropically focally degenerate and $\Sigma = \Sigma_X$.

Proof

This follows immediately from Lemma 3, since conditions 1) and 2.1) imply that on the affine space given by equations 4"") the quadratic function Q_∞ is constant, and 2.2) guarantees that this constant equals $r(s)$, whence also 3"") is satisfied and thus the sphere X_s, fibre over the point $O(s)$, is then an affine space of dimension $m - a$.

□

Remark 15 *The above theorem immediately implies the characterization given in the previous section of the isotropically focally degenerate hypersurfaces. Because in the case of hypersurfaces we noticed that $X = \Sigma_X$,*

and then the tangential condition on Σ_X reads out as the condition that the normal vector is isotropic, moreover by the inverse focal construction X is developable, and the fibre dimension equals $m - a = m - (m-1) = 1$. Thus X is developable with the ruling by lines given by the normal direction.

We end by showing an explicit example of the situation considered in case 3) of Theorem 3.

Example 11 *Consider first $X' \subset \mathbf{C}^6$ obtained as the product $X' = M \times W$ of ¿two isotropically developable surfaces :*

thus X' has a parametrization

$(\alpha(s) + t\alpha'(s), \beta(\sigma) + \tau\beta'(\sigma))$.

Inside X' we consider the divisor X obtained by setting $\tau = t$.

Whence X has a parametrization

$(\alpha(s) + t\alpha'(s), \beta(\sigma) + t\beta'(\sigma))$,

and, remembering that $\alpha'(s), \beta'(\sigma)$ are isotropic vectors it follows that the normal space to X is spanned, at the smooth points of X , by the three vectors $(\alpha'(s), 0)$, $(0, \beta'(\sigma))$ *and*

$(- < \beta'''(\sigma), \beta'(\sigma) > [\alpha''(s) + t\alpha'''(s)], < \alpha'''(s), \alpha'(s) > [\beta''(\sigma) + t\beta'''(\sigma)])$.

The endpoint map is given by

$(\alpha(s) + t\alpha'(s) + \lambda_1\alpha'(s) - \lambda_3 < \beta'''(\sigma), \beta'(\sigma) > [\alpha''(s) + t\alpha'''(s)], \beta(\sigma) + t\beta'(\sigma) + \lambda_2\beta'(\sigma) + \lambda_3 < \alpha'''(s), \alpha'(s) > [\beta''(\sigma) + t\beta'''(\sigma)])$

thus its image equals the image of the map

$(\alpha(s) + \lambda_1\alpha'(s) - \lambda_3 < \beta'''(\sigma), \beta'(\sigma) > [\alpha''(s) + t\alpha'''(s)], \beta(\sigma) + \lambda_2\beta'(\sigma) + \lambda_3 < \alpha'''(s), \alpha'(s) > [\beta''(\sigma) + t\beta'''(\sigma)])$.

To simplify the discussion we may assume $< \alpha''(s), \alpha''(s) > \equiv -1$, and similarly $< \beta''(\sigma), \beta''(\sigma) > \equiv -1$, therefore our formula simplifies to

$(\alpha(s) + \lambda_1\alpha'(s) - \lambda_3[\alpha''(s) + t\alpha'''(s)], \beta(\sigma) + \lambda_2\beta'(\sigma) + \lambda_3[\beta''(\sigma) + t\beta'''(\sigma)])$

and we see that the image of the normal bundle N_X will in general be dominant.

Therefore X is weakly focally degenerate, not focally isotropically degenerate, but the inverse focal construction reconstructs only the isotropically focally degenerate fourfold X'.

References

[*Arnoldetal.*] V. Arnold, A. Varchenko, S. Goussein-Zade', *Singularites des Applications Differentiables. I : Classification des points critiques, des caustiques et des fronts d' onde*, Editions MIR, Moscow (1986), translation of

the Russian 1982 edition.
[*Coolidge*] J.L. Coolidge, *A Treatise on Algebraic Plane Curves*, Oxford University press (1931), reedited by Dover, New York (1959).
[$D-F-N$] B.A. Dubrovin, A.T. Fomenko, S.P. Novikov *Modern Geometry - Methods and Applications. Part 2: The geometry and topology of manifolds*, reedited in Springer GTM 104, (1985).
[*Fantechi*] B. Fantechi, *The Evolute of a Plane Algebraic Curve*, UTM 408, University of Trento, (1992).
[*Hartshorne*] R. Hartshorne, *Algebraic Geometry*, Springer GTM 52 (1977).
[*Milnor*] J. Milnor, *Morse Theory*, Princeton, Princeton Univ. Press, (1973).
[$Salmon-Fiedler$] G. Salmon, W. Fiedler, *Analytische Geometrie des Raumes*, 2 vols., Leipzig, Teubner-Verlag, (1879-1880).
[*Trifogli*] C. Trifogli, *Focal Loci of Algebraic Hypersurfaces: a General Theory*, Geom. Dedicata 70: 1-26 (1998).
[*Trif2*] C. Trifogli, *The Geometry of Focal Loci*, Ph. D. Thesis, Universita' di Milano (in preparation).

Address of the authors:
Fabrizio Catanese,
Mathematisches Institut der Georg August Universität Göttingen,
Bunsenstrasse 3-5, D 37073 Göttingen,
catanese@uni-math.gwdg.de

Cecilia Trifogli,
Dipartimento di Matematica F. Enriques, Universita' di Milano,
via C. Saldini 50,I 20133 Milano
trifogli@vmimat.mat.unimi.it and : All Souls College, Oxford OX1 4AL
cecilia.trifogli@all-souls.ox.ac.uk