WELL-APPROXIMABLE POINTS FOR JULIA SETS WITH PARABOLIC AND CRITICAL POINTS

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ABSTRACT. In this paper we consider rational functions $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with parabolic and critical points contained in their Julia sets $J(f)$ such that $\lim_{n\to\infty} |(f^n)'(f(c))| = \infty$ for each critical point $c \in J(f)$. We calculate the Hausdorff dimensions of subsets of $J(f)$ consisting of elements z for which $\inf\{\mathrm{dist}(f^n(z), \mathrm{Crit}(f)) : n \geq 0\} > 0$ and which are well-approximable by backward iterates of the parabolic periodic points of f.

1. Introduction and Statement of Main Results

In [S1] [S2] [SU1] [SU3] we developed a certain type of Diophantine analysis for geometrically finite Kleinian groups with parabolic elements, parabolic rational maps and tame parabolic iterated function systems. In this paper we extend this analysis to rational maps f with parabolic points and with Julia sets $J(f)$ containing critical points such that for each $c \in J(f)$ of these critical points we have that

$$\lim_{n\to\infty} |(f^n)'(f(c))| = \infty.$$

The idea is to perform this Diophantine analysis on the set of points $z \in J(f)$ for which we have, with $\mathrm{Crit}(f)$ denoting the set of critical points of f, that

$$\inf\{\mathrm{dist}(f^n(z), \mathrm{Crit}(f)) : n \geq 0\} > 0.$$

We give a further development of the techniques elaborated in the afore-mentioned papers, mainly by using results obtained in [DU] and [P] (see also [PU]). In particular, a crucial quantity in our main theorem (Theorem 1.1) is the so-called dynamical dimension $\mathrm{DD}(J(f))$ of $J(f)$, in the literature occasionally also referred to as the essential dimension, which is given by ([DU]; see also [P] [PU])

$$\mathrm{DD}(J(f)) := \sup\{\mathrm{HD}(\mu)\}.$$

Here, the supremum is taken with respect to all f-invariant, ergodic Borel probability measures μ of positive entropy, and $\mathrm{HD}(\mu)$ denotes the Hausdorff dimension of μ. In [DU] it was shown that $\mathrm{DD}(J(f))$ coincides with the least possible exponent for conformal measures associated with f. Also, in [PU] we obtained that $\mathrm{DD}(J(f))$ coincides with the hyperbolic dimension of f in the sense of Shishikura ([Sh]).

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In order to state our main result, we first have to introduce some notation. Let Ω be the set of all parabolic periodic points of f, that is

$$\Omega := \{\omega \in \overline{\mathbb{C}} \colon f^q(\omega) = \omega \ \text{ and } \ (f^q)'(\omega) = 1 \text{ for some } \ q \geq 1\}.$$

It is well-known that Ω is a finite subset of $J(f)$. For fixed $\omega \in \Omega$ and for every pre-parabolic point $x \in \bigcup_{n\geq 0} f^{-n}(\omega)$, let $n(x)$ denote the least integer such that $f^{n(x)}(x) = \omega$. Since the set of critical points of f is finite and since critical points are not periodic, there exists a number $q(\omega) \geq 1$ such that for every pre-parabolic point x for which $n(x) \geq q(\omega)$, we have that

$$\mathrm{Crit}(f) \cap \bigcup_{n\geq 0} f^{-n}(x) = \emptyset.$$

In the following let x be some fixed element of this type. Then for every $y \in \bigcup_{n\geq 0} f^{-n}(x)$ there exists a unique integer $k(y)$ such that $f^{k(y)}(y) = x$. Now, for some fixed $R > 0$ and every $\delta > 0$ we define the closed ball

$$B_y^\delta := \overline{B}\Big(y, (R\,|(f^{k(y)})'(y)|^{-1})^{1+\delta}\Big).$$

Then, for $\epsilon > 0$ let

$$J_{x,\epsilon}^\delta := \bigcap_{q\geq 1} \bigcup_{n\geq q} \bigcup_{*} B_z^\delta,$$

where the union $\bigcup_*$ is taken with respect to all elements $z \in \bigcup_{l\geq 0} f^{-l}(x) \backslash \bigcup_{j\geq 0} f^{-j}(B(\mathrm{Crit}(f), \epsilon))$ for which $k(z) = n$. Our main interest in this paper will be focused on the sets

$$J_x^\delta := \bigcup_{\epsilon>0} J_{x,\epsilon}^\delta, \ J_\omega^\delta := \bigcup_{\{z : n(z) = q(\omega)\}} J_z^\delta \quad \text{and} \quad J^\delta := \bigcup_{\omega\in\Omega} J_\omega^\delta.$$

Note that these sets represent natural analogues of the set of well-approximable irrationals in number theory.

We are now in the position to state our main result in this paper. Here, we have used the common notation $p(\omega)$ to denote the number of petals of $\omega \in \Omega$.

Theorem 1.1. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map with parabolic periodic points and critical points such that for each critical point $c \in J(f)$ we have that $\lim_{n\to\infty} |(f^n)'(f(c))| = \infty$. Then for every parabolic point $\omega \in \Omega$ and each $\delta > 0$ it holds that*

$$\mathrm{HD}(J_\omega^\delta) = \begin{cases} \frac{\mathrm{DD}(J(f))}{1+\delta} & \text{if } \ \delta \geq \mathrm{DD}(J(f)) - 1 \\ \frac{\mathrm{DD}(J(f))+\delta p(\omega)}{1+\delta(1+p(\omega))} & \text{if } \ \delta \leq \mathrm{DD}(J(f)) - 1. \end{cases}$$

Hence in particular, we have with $p_{\min} := \min\{p(\omega) : \omega \in \Omega\}$ that

$$\mathrm{HD}(J^\delta) = \begin{cases} \frac{\mathrm{DD}(J(f))}{1+\delta} & \text{if } \delta \geq \mathrm{DD}(J(f)) - 1 \\ \frac{\mathrm{DD}(J(f))+\delta p_{\min}}{1+\delta(1+p_{\min})} & \text{if } \delta \leq \mathrm{DD}(J(f)) - 1. \end{cases}$$

We remark that a similar type of Diophantine analysis could also be given for periodic repelling fixed points rather than parabolic periodic points. In fact, in that case the arguments and also the results would be far less involved, mainly due to the lack of 'conformal fluctuation' of the conformal measure at periodic repelling fixed points.

2. Holomorphic Inverse Branches

For $z \in \overline{\mathbb{C}}$, $n \geq 0$ and $\theta > 0$, let $C_n(z,\theta)$ denote the connected component of $f^{-n}(B(f^n(z),\theta))$ which contains the point z, and define

$$\hat{\theta} := \sup\{\operatorname{diam}(C_1(x,\theta)) : x \in J(f)\}.$$

By the behaviour of the forward orbits of the critical points in the Fatou set of f, it follows that for $\theta > 0$ small enough there exists $0 < R_1(\theta) < \theta/4$ such that for every $x \in J(f) \setminus B(\Omega,\theta)$ we have that

$$\bigcup_{\omega\in\Omega} \bigcup_{y\in f^{-1}(\Omega)\setminus\Omega} C_1(y,2\theta) \cap \bigcup_{n\geq 0} f^n\Big(\operatorname{Crit}(f) \setminus J(f)\Big) = \emptyset, \tag{2.1}$$

$$B(x,2R_1(\theta)) \cap \bigcup_{n\geq 0} f^n\Big(\operatorname{Crit}(f) \setminus J(f)\Big) = \emptyset. \tag{2.2}$$

For the same reason, f acts faithfully in a neighbourhood of a parabolic point, that is we have that

$$\text{if } z \in B(y,2\hat{\theta}) \text{ and } f(z) \in B(f(y),2\theta) \text{ for some } y \in f^{-1}(\Omega)\setminus\Omega, \text{ then } z \in C_1(y,2\theta). \tag{2.3}$$

Finally, for $\epsilon > 0$ we define the set

$$K_\epsilon = J(f) \setminus \bigcup_{n\geq 0} f^{-n}(B(\operatorname{Crit}(f),\epsilon)).$$

The following lemma gives the main result of this section.

Lemma 2.1. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map with parabolic periodic points and critical points such that for each critical point $c \in J(f)$ we have $\lim_{n\to\infty} |(f^n)'(f(c))| = \infty$. Then for every sufficiently small $\theta > 0$ and every $\epsilon > 0$ there exists $0 < R(\theta,\epsilon) \leq R_1(\theta)$ such that the following holds. If $y \in J(f)$, $x \in K_\epsilon \cup f^{-n}(y)$ for some $n \geq 1$, and if either $y \notin B(\Omega,\theta)$ or $y \in B(\Omega,\theta)$ and $f^{n-1}(x) \in C_1(\omega_1,\theta)$ for some $\omega_1 \in f^{-1}(\Omega)\setminus\Omega$, then we have that $C_n(x,2R(\theta,\epsilon)) \cap Crit(f^n) = \emptyset$.*

Proof. Fix a sequence $\{b_j\}_{j=1}^\infty$ of positive numbers b_j such that $\sum_{j=1}^\infty b_j = 1/2$, and let $b_0 = 0$. Since $K(\epsilon) \cap \operatorname{Crit}(f) = \emptyset$, there exists $0 < R_2 \leq R_1(\theta)$ such that $C_1(x,R_2) \cap \operatorname{Crit}(f) = \emptyset$ for all $x \in K_\epsilon$. For fixed $y \in J(f)$ we now prove by induction that

$$\operatorname{Crit}(f^n) \cap C_n\Big(x, R_3 \prod_{j=0}^{n-1}(1-b_j)\Big) = \emptyset \tag{2.4}$$

for all $n \geq 1$, all $x \in K_\epsilon \cup f^{-n}(y)$ and for some constant $0 < R_3 \leq R_2$. First note that the assertion in (2.4) is in fact true for any positive choice of $R_3 \leq R_2$. Now, suppose that (2.4) is true for some $n \geq 1$ and for all $x \in K_\epsilon \cup f^{-n}(y)$. Fix some $x \in K_\epsilon \cup f^{-(n+1)}(y)$ and suppose that

$$\operatorname{Crit}(f^{n+1}) \cap C_{n+1}\Big(x, R_3 \prod_{j=0}^{n}(1-b_j)\Big) \neq \emptyset.$$

Since $f(x) \in K_\epsilon \cup f^{-n}(y)$ and $f\Big(C_{n+1}\Big(x, R_3 \prod_{j=0}^{n}(1-b_j)\Big)\Big) \subset C_n\Big(f(x), R_3 \prod_{j=0}^{n-1}(1-b_j)\Big)$, the inductive assumption implies that there exists a critical point

$$c \in \mathrm{Crit}(f) \cap C_{n+1}\Big(x, R_3 \prod_{j=0}^{n}(1-b_j)\Big)\Big).$$

If $y \notin B(\Omega, \theta)$, then we have that

$$|f^{n+1}(c) - y| \leq R_3 \prod_{j=0}^{n}(1-b_j) \leq R_1(\theta).$$

Here we were assuming without loss of generality that $R_3 \leq R_1(\theta)$. Hence using (2.3), it follows that $c \in J(f)$.

If $y \in B(\Omega, \theta)$, then $f^n(x) \in C_1(\eta, \theta)$ for some $\eta \in f^{-1}(\Omega) \setminus \Omega$. Hence, we have that

$$f(f^n(c)) \in B\Big(f(\eta), \theta + R_3 \prod_{j=0}^{n}(1-b_j)\Big) \subset B\Big(f(\eta), \theta + R_3\Big) \subset B\Big(f(\eta), 2\theta\Big) \tag{2.5}$$

and

$$\begin{aligned} f^n(c) &\in B\Big(\eta, |f^n(c) - f^n(x)| + |f^n(x) - \eta|\Big) \\ &\subset B\Big(\eta, \operatorname{diam} C_1\Big(f^n(x), R_3 \prod_{j=0}^{n}(1-b_j) + \operatorname{diam}(C_1(\eta, \theta))\Big)\Big) \subset B(\eta, 2\hat{\theta}). \end{aligned}$$

Combining these observations with (2.5) and (2.3), we conclude that $f^n(c) \in C_1(\eta, 2\theta)$. Thus, (2.1) implies that $c \in J(f)$.

Hence, we have now shown that $c \in J(f)$ in both cases, and we proceed as follows. Since $f(c) \in C_n\Big(f(x), R_3 \prod_{j=0}^{n}(1-b_j)\Big)$ and since by the inductive assumption we have that $\mathrm{Crit}(f^n) \cap C_n\Big(x, R_3 \prod_{j=0}^{n-1}(1-b_j)\Big) = \emptyset$, Koebe's distortion theorem gives $|f(c)-f(x)| \leq Cb_n^3 |(f^n)'(f(c)|^{-1}$ for some universal constant $C > 0$. Therefore, there exists a universal constant $A > 0$ and an integer $q \geq 2$ (the maximal order of a critical point of f) such that

$$|c - x| \leq A\left(Cb_n^3 |(f^n)'(f(c))|^{-1}\right)^{1/q}.$$

Hence, since $\lim_{n\to\infty} |(f^n)'(f(c))| = \infty$, and by choosing b_n sufficiently small, we conclude that $|c-x| < \epsilon$. This clearly contradicts the fact that $x \in K_\epsilon$, and therefore the proof of (2.4) is complete. Now, by defining $R(\theta, \epsilon) := R_3 \prod_{j=0}^{\infty}(1-b_j)$, the lemma follows. ∎

As an immediate consequence of the preceding lemma we have the following corollary.

Corollary 2.2. *For each $\epsilon > 0$ there exists $\sigma > 0$ such that for all $z \in K_\epsilon$ and all $n \geq 1$ we have that $C_n\Big(z, \sigma \operatorname{dist}(f^n(z), \Omega)\Big) \cap Crit(f^n) = \emptyset$.*

3. **Proof of the Main Theorem**

In this section we give the proof of Theorem 1.1. For this let $\theta > 0$ be a fixed sufficiently small number such that the formulae (2.1), (2.2) and (2.3) hold. The first step in the proof is to verify the following upper bound for the Hausdorff dimension.

Lemma 3.1. *For each $\omega \in \Omega$, $y \in \bigcup_{n\geq 0} f^{-n}(\omega)$ such that $n(y) \geq q(\omega)$, and for every $\epsilon, \delta > 0$ we have that*

$$\mathrm{HD}(J^{\delta}_{y,\epsilon}) \leq \min\left\{\frac{\mathrm{DD}(J(f))}{1+\delta}, \frac{\mathrm{DD}(J(f)) + \delta p(\omega)}{1+\delta(1+p(\omega))}\right\}.$$

Proof. Without loss of generality we can assume that $\theta > 0$ is chosen to be small enough such that $y \notin B(\Omega, \theta)$. For $t > \mathrm{DD}(J(f))$ it is known (see [Pr]) that

$$\mathrm{HD}\left(\left\{z \in J(f) : \sum_{n\geq 1}\sum_{x\in f^{-n}(z)} |(f^n)'(x)|^{-t} = \infty\right\}\right) = 0.$$

Hence,for such t there exists $z \in B(y, R(\theta, \epsilon))$ such that

$$\sum_{n\geq 1}\sum_{x\in f^{-n}(z)} |(f^n)'(x)|^{-t} < \infty.$$

By Koebe's distortion theorem and Lemma 2.1, it then follows that

$$S := \sum_{n\geq 1}\sum_{x\in K_\epsilon \cap f^{-n}(y)} |(f^n)'(x)|^{-t} < \infty. \tag{3.1}$$

Thus, for every $k \geq 1$ we have that

$$\sum_{n\geq k}\sum_{x\in K_\epsilon\cap f^{-n}(y)} \left((R|(f^n)'(x)|^{-1})^{1+\delta}\right)^{\frac{t}{1+\delta}} = (2R)^t \sum_{n\geq k}\sum_{x\in K_\epsilon\cap f^{-n}(y)} |(f^n)'(x)|^{-t} \leq (2R)^t S.$$

Since $\{B^{\delta}_x : x \in K_\epsilon \cap f^{-n}(y), n \geq k\}$ is a covering of $J^{\delta}_{y,\epsilon}$ for every $k \geq 1$, and since $\lim_{n\to\infty} \max\{|(f^n)'(x)| : x \in K_\epsilon \cap f^{-n}(y)\} = \infty$, we have that $\mathrm{H}^{\frac{t}{1+\delta}}(J^{\delta}_{y,\epsilon}) \leq (2R)^t S < \infty$. This implies that $\mathrm{HD}(J^{\delta}_{y,\epsilon}) \leq \frac{t}{1+\delta}$. By letting $t \searrow \mathrm{DD}(J(f))$, we obtain that

$$\mathrm{HD}(J^{\delta}_{y,\epsilon}) \leq \frac{\mathrm{DD}(J(f))}{1+\delta}. \tag{3.2}$$

In order to obtain the second upper estimate in the lemma, note that Fatou's flower theorem implies in particular that for each $n \geq 1$ and $x \in K_\epsilon \cap f^{-n}(y)$ the ball $\overline{B}(\omega, (R|(f^n)'(x)|^{-1})^{\delta})$ admits a covering by balls of radii $(R|(f^n)'(x)|^{-1})^{\delta(p(\omega)+1)}$ such that the cardinality of this covering is comparable to $|(f^n)'(x)|^{\delta p(\omega)}$. Consequently, by Lemma 2.1 and Koebe's distortion theorem, each ball B^{δ}_x with $x \in K_\epsilon \cap f^{-n}(y)$ can be covered by at most a number comparable to $|(f^n)'(x)|^{\delta p(\omega)}$ of balls with radii comparable to $(R|(f^n)'(x)|^{-1})^{1+\delta(1+p(\omega))}$. Hence by using

(3.1), we have for each $\eta > 0$ that

$$\mathrm{H}^{\frac{\mathrm{DD}(J(f))+\delta p(\omega)}{1+\delta(1+p(\omega))}+\eta}(J^{\delta}_{y,\epsilon}) \ll \lim_{q\to\infty}\sum_{n\geq q}\sum_{x\in K_\epsilon\cap f^{-n}(y)}(R|(f^n)'(x)|^{-1})^{(1+\delta(1+p(\omega)))\left(\frac{\mathrm{DD}(J(f))+\delta p(\omega)}{1+\delta(1+p(\omega))}+\eta\right)}|(f^n)'(x)|^{\delta p(\omega)}$$
$$\asymp \lim_{q\to\infty}\sum_{n\geq q}\sum_{x\in K_\epsilon\cap f^{-n}(y)}|(f^n)'(x)|^{-(\mathrm{DD}(J(f))+\eta(1+\delta(1+p(\omega))))} = 0.$$

From this it follows that $\mathrm{HD}(J^{\delta}_{y,\epsilon}) \leq \frac{\mathrm{DD}(J(f))+\delta p(\omega)}{1+\delta(1+p(\omega))} + \eta$. Hence, letting $\eta \searrow 0$, we conclude that

$$\mathrm{HD}(J^{\delta}_{y,\epsilon}) \leq \frac{\mathrm{DD}(J(f))+\delta p(\omega)}{1+\delta(1+p(\omega))}.$$

■

As an immediate consequence of this lemma we have the following corollary.

Corollary 3.2. *For each $\omega \in \Omega$ and for every $\delta > 0$ we have that*

$$\mathrm{HD}(J^{\delta}_{\omega}) \leq \min\left\{\frac{\mathrm{DD}(J(f))}{1+\delta}, \frac{\mathrm{DD}(J(f))+\delta p(\omega)}{1+\delta(1+p(\omega))}\right\}$$

and

$$\mathrm{HD}(J_{\delta}) \leq \min\left\{\frac{\mathrm{DD}(J(f))}{1+\delta}, \frac{\mathrm{DD}(J(f))+\delta p_{\min}}{1+\delta(1+p_{\min})}\right\}.$$

For the remaining part of the proof assume additionally that $\theta > 0$ is so small such that for every $z \in (J(f)\setminus\Omega)\cap B(\Omega,\theta)$ there exists $n \geq 1$ with $f^n(z) \notin B(\Omega,\theta)$. For fixed $\epsilon > 0$ and for $z \in K_\epsilon \setminus \bigcup_{n\geq 0} f^{-n}(\Omega)$ let $\{n_j\}_{j=1}^{\infty}$ denote the sequence which is given by $f^{n_j}(z) \notin B(\Omega,\theta)$ for all j, and $f^k(z) \in B(\Omega,\theta)$ for all $k \notin \{n_1, n_2, \cdots\}$. Clearly, such a sequence is infinite. Also, for every $j \geq 1$ we let $r_j(z) := |(f^{n_j})'(z)|^{-1}$, and for every $r > 0$ sufficiently small we define

$$r^*_j(z) := r_j(z) = \min\{r_k(z) : r_k(z) > r\} \quad \text{and} \quad r_{j*}(z) := \max\{r_k(z) : r_k(z) \leq r\}.$$

If we choose $\epsilon > 0$ small enough such that $K_\epsilon \neq \emptyset$, then it was shown in [DU] that there exists a Borel probability measure m_ϵ supported on K_ϵ and a number $h_\epsilon \leq \mathrm{DD}(J(f))$ such that the following properties are satisfied:-

$$\lim_{\epsilon\searrow 0} h_\epsilon = \mathrm{DD}(J(f)); \tag{3.3}$$

for every Borel set $E \subset J(f)$ such that $f|_E$ is injective, we have that

$$m_\epsilon(f(E)) \geq \int_E |f'(z)|^{h_\epsilon} dm_\epsilon(z); \tag{3.4}$$

for every Borel set $F \subset K_\epsilon \setminus \partial B(\mathrm{Crit}(f),\epsilon)$ such that $f|_F$ is injective, we have that

$$m_\epsilon(f(F)) = \int_F |f'(z)|^{h_\epsilon} dm_\epsilon(z).$$

Finally, we define the conformal fluctuation function ζ_ϵ for $z \in K_\epsilon$ and $r > 0$ by

$$\zeta_\epsilon(z,r) := \frac{m_\epsilon(B(z,r))}{r^h}.$$

Now, the remaining part of the proof is completely analogous to the constructions in [SU1] [SU2] [SU3], and we refer to these papers for the details. Namely, in order to obtain a uniform upper estimate for the conformal fluctuation function, we can now proceed as in [SU2] and [SU3]. Note that the proof of this estimate uses (3.4), Lemma 2.1, Corollary 2.2 and the local properties of f and m_ϵ around parabolic points.

Lemma 3.3. *For each $z \in K_\epsilon$ we have that*

$$\zeta_\epsilon(z,r) \ll \begin{cases} \left(\frac{r}{r_j^*(z)}\right)^{(h_\epsilon-1)p(\omega)} & \text{for } \; r_j^*(z) \geq r \geq r_j^*(z)\left(\frac{r_{j*}(z)}{r_j^*(z)}\right)^{\frac{1}{p(\omega)+1}}, f^{n_j+1}(z) \in B(\omega,\theta) \\ \left(\frac{r_{j*}(z)}{r}\right)^{h_\epsilon-1} & \text{for } \; r_{j*}(z) \leq r \leq r_j^*(z)\left(\frac{r_{j*}(z)}{r_j^*(z)}\right)^{\frac{1}{p(\omega)+1}}, f^{n_j+1}(z) \in B(\omega,\theta). \end{cases}$$

Furthermore, on the basis of the latter lemma, Lemma 2.1 and the local properties of f and m_ϵ around parabolic points, we can proceed once more exactly as in [SU1] and [SU3] to obtain the following result. Note that in [SU1] and [SU3], in order to derive this lower estimates for the Hausdorff dimension, we exclusively require the upper estimate for the conformal fluctuation function.

Lemma 3.4. *For each $\omega \in \Omega$ and every $\epsilon > 0$ we have that*

$$\mathrm{HD}(J_\omega^\delta) \geq \begin{cases} \frac{h_\epsilon}{1+\delta} & \text{if } \; \delta \geq h_\epsilon - 1 \\ \frac{h_\epsilon + \delta p(\omega)}{1+\delta(1+p(\omega))} & \text{if } \; \delta \leq h_\epsilon - 1. \end{cases}$$

Consequently, with $p_{\min} := \min\{p(\omega) : \omega \in \Omega\}$, it holds that

$$\mathrm{HD}(J^\delta) \geq \begin{cases} \frac{h_\epsilon}{1+\delta} & \text{if } \delta \geq h_\epsilon - 1 \\ \frac{h_\epsilon + \delta p_{\min}}{1+\delta(1+p_{\min})} & \text{if } \delta \leq h_\epsilon - 1. \end{cases}$$

The proof of Theorem 1.1 now follows from Lemma 3.4, and from Lemma 3.1 where we let $\epsilon \searrow 0$ and make use of the limit behaviour stated in (3.3). ∎

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