On Gibbs Families for Fibrewise Expansive Systems

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Abstract

We prove that compact fibred systems, whose fibre maps are uniformly open, expansive and topologically exact, and whose fibres are continuous in the Hausdorff metric, possess, for every globally continuous, uniformly fibrewise locally constant potential, a unique Gibbs family of conditional measures on its fibres. If the system upon observation has only countably many fibres, then we associate to any continuous potential a Gibbs family, though, this time, this family may not be unique.

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Contents

Abstract **iii**

Acknowledgements **iv**

Introduction **2**

1 Preliminaries **7**

1.1 Fibred Systems 7

1.1.1 Definition 7

1.1.2 Morphisms 8

1.1.3 Composition, Invertibility and Quasi Invertibility 9

1.2 Fibrewise Expansiveness 10

1.3 Fibrewise Uniform Expansiveness 11

1.4 Fibred Generators 13

1.5 Fibrewise Expanding Systems 18

1.6 Fibrewise Ruelle Expanding Systems 19

1.7 Fibrewise Openness 20

1.8 The Uniform FULC Property 25

1.9 Fibrewise Topological Exactness 32

2 On the Relations between Different Fibrewise Expansion Behaviors **37**

2.1 Fibrewise Expansive vs. Fibrewise Expanding Systems 37

2.2 The Invertible and Quasi Invertible Cases 50

2.3 Fibrewise Expanding vs. Fibrewise Ruelle Expanding Systems 58

2.4 Some Interesting Implications for the Families of Skew-products and Triangular Maps 66

3 On the Existence of a Gibbs Family for Fibrewise Expansive Systems **70**

Introduction

A fibred system is a dynamical system $T : Y \to Y$ on a metric space Y that forms a fibre bundle over another dynamical system $S : X \to X$, bundle which is preserved by the dynamics of T by means of a factor map $\pi : Y \to X$ (that is, $\pi \circ T = S \circ \pi$).

In 1999, M. Denker and M. Gordin [8] (see also [9]) showed that compact fibred systems, whose fibre maps are Ruelle expanding and topologically exact, admit, for every globally continuous, uniformly fibrewise Hölder continuous potential, a unique Gibbs family of conditional measures on its fibres. This result reduces to the celebrated Ruelle-Perron-Frobenius theorem in the non-fibred case (or, in other words, in the one-fibre case, that is, when X consists of only one point).

It is here considered whether such a family exists for systems having fibre maps that are expansive rather than Ruelle expanding. We will show that, whenever X is countable, compact fibred systems, whose fibre maps are uniformly open, expansive and topologically exact, and whose fibres are continuous in the Hausdorff metric, possess, for every continuous potential, a Gibbs family. Moreover, we will prove that, whatever the cardinality of X is, this class of systems admits a unique Gibbs family for every globally continuous, uniformly fibrewise locally constant potential.

Denker and Gordin derived the existence of a Gibbs family by means of relative transfer operators (also called Ruelle-Perron-Frobenius operators) and Birkhoff's theory of cones. They associated a transfer operator to each fibre. Each of these operators was defined on a dense cone of functions, in such a way that this bundle of cones contracted with respect to Hilbert's projective metrics under the application of the transfer operators. They then used the relative transfer operators to construct a sequence of globally defined operators that converged, thanks to the cones contraction, to an operator of conditional expectation. They subsequently projected this latter operator on each fibre and showed that the corresponding system of conditional probabilities constitutes the desired Gibbs family.

The approach we will use here is based upon Denker and Gordin's results. The main idea is the following: Knowing that Borel measures have no consciousness of the

metric the underlying space Y is equipped with, we make a change of metric on Y, making sure that the topology be preserved, and show that the initially fibre expansive system is expanding along its fibres under the reign of the new metric. We then give a necessary and sufficient condition for the fibre maps to be Ruelle expanding. Finally, we find a family of potentials that retain their uniform fibrewise Hölder continuity under any compatible change of metric and apply Denker and Gordin's results, hence obtaining the wished Gibbs family.

The method of changing the metric for a more appropriate one was originally developed by E. Coven and W. Reddy [5] and relies on Frink's Metrization Lemma [10, 13]. They showed that positively expansive maps of compact manifolds are Ruelle expanding. The notion of Ruelle expandability is evidently due to D. Ruelle [19]. In his now famous book on thermodynamic formalism, Ruelle extracted the properties of expanding endomorphisms that are sufficient for topological dynamical systems to retain the ergodic theory of these maps. An expanding endomorphism of a compact differentiable manifold Y is, as introduced by M. Shub [22], a C^1 map $f : Y \to Y$ for which there is a compatible Riemannian metric and a constant $\Lambda > 1$ such that the induced norm $\| \cdot \|$ satisfies $\|Df^n(v)\| \geq \Lambda^n \|v\|$ for all $n \in \mathbb{N}$ and all tangent vectors v. Ruelle defined a continuous map $f : Y \to Y$ of a compact metrizable topological space Y as expanding (henceforth called Ruelle expanding) if there is a compatible metric d_Y for Y and constants $a > 0$ and $\Lambda > 1$ such that if $y_1, y_2' \in Y$ are such that $d_Y(f(y_1), y_2') < 2a$, then there is a unique $y_2 \in Y$ with $f(y_2) = y_2'$, $d_Y(y_1, y_2) < 2a$ and, furthermore, $d_Y(f(y_1), f(y_2)) \geq \Lambda\, d_Y(y_1, y_2)$. This notion is obviously stronger than the usual concept of expandability that is described by the existence of constants $a > 0$ and $\Lambda > 1$ such that $d_Y(f(y_1), f(y_2)) \geq \Lambda\, d_Y(y_1, y_2)$ whenever $d_Y(y_1, y_2) < 2a$. However, it was clear to Coven and Reddy (see [5]) that these two notions of expandability coincide exactly when f is an open map. They could thereby reduce their proof to the establishment of the existence of a compatible metric under which a positively expansive map of a compact manifold is expanding. They based their proof on the fact that such a map is a covering map and hence has a universal covering space. They could therefore lift the map to a homeomorphism.

Reddy [16] later remarked that the main ideas of the proof could be almost integrally transported to the case of a surjective map of a compact metric space. This time, though, he did not use the fact that such a map is a covering map.

Finally, M. Denker and M. Urbański [6] brought up a slightly different argument. Despite the fact that the chief ideas are the same as Reddy's ones and that their argument is lengthier and heavier, it has the advantage of being expressed in terms of the metrics $d_{n,Y}(y_1, y_2) = \max_{0 \leq m < n} d_Y(f^m(y_1), f^m(y_2))$, where $n \in \mathbb{N}$. These metrics are

sometimes called Bowen's metrics, since R. Bowen [2] used them extensively in defining topological entropy for non-compact dynamical systems. Using these metrics, we will extend Reddy's result. For this, we will, in analogy to Denker and Gordin's definition of Ruelle expandability along the fibres of fibred systems, introduce 'fibrewised' notions of (positive) expansiveness and expandability. These notions will be respectively called fibrewise expansiveness and fibrewise expandability, whereas Denker and Gordin's notion of expandability will be accordingly called fibrewise Ruelle expandability. We will show that fibrewise uniformly expansive uniformly continuous systems, whose fibres are uniformly continuous in the Hausdorff metric, are fibrewise expanding under a certain compatible metric. Our result generalizes Coven and Reddy's work in two different manners. Firstly, it applies not only to (positively) expansive maps of compact metric spaces, but also to the larger family of uniformly expansive uniformly continuous maps of non-compact spaces. In this regard, we further show that the new metric is uniformly topologically equivalent to the old one, so that the map retains its uniform continuity. Secondly, it applies to the larger class of systems whose expansiveness property lies along the fibres and is therefore weaker than global expansiveness. However, it is not known whether all fibrewise expansive systems are subject to this change of metric.

An analogous result holds for expansive homeomorphisms. Indeed, Reddy [17] and K. Sakai's [20] works revealed that every expansive homeomorphism of a compact space admits a hyperbolic metric. This means that there exists a compatible metric with respect to which the homeomorphism is contracting on sufficiently small local stable sets, whereas its inverse is contracting on the corresponding local unstable sets. Here again we will show that this result can be extended to the theory of invertible fibred systems and, in particular, that a uniformly expansive uniform homeomorphism of a non-compact space admits a hyperbolic metric. Moreover, in this latter case, the metric is, as in the non-invertible case, uniformly topologically equivalent to the original metric, hence preserving the uniform character of the homeomorphism. On the other hand, we will show that it suffices, whenever π is uniformly continuous, that the system be quasi invertible, that is, that the base transformation S be locally injective, for a change of metric to take place. This follows from the fact that one may always assume for purely topological problems that the map π is a strong contraction.

This right to pass from expansiveness (respectively, fibrewise expansiveness) to expandability (respectively, fibrewise expandability) at the cost of a change of metric has some interesting implications. In the non-fibred case, it was showed that an infinite compact metric space does not accept any positively expansive homeomorphism (see [1]). It was also proved, as a simple corollary to this result, that the closed unit

interval does not support any expansive map. As a consequence of our result, we will show that no fibrewise surjective triangular map can be fibrewise expansive, much less expansive. This discloses another similarity between maps of the unit interval and triangular maps.

We will also shed some light on the relation between fibrewise expanding systems and fibrewise Ruelle expanding ones. As mentioned before, Coven and Reddy were aware that a compact dynamical system is Ruelle expanding if and only if it is open and expanding. However, we will show that a system which is fibrewise open and fibrewise expanding may not be fibrewise Ruelle expanding. This is due to the fact that fibrewise openness and uniform fibrewise openness are two distinct notions for fibred systems, even for compact ones. Indeed, we will show that a fibrewise expanding system is fibrewise Ruelle expanding if and only if it is uniformly fibrewise open. Since it turns out to be usually quite difficult to prove directly that a system is uniformly fibrewise open, we will give some conditions under which this property is fulfilled. In particular, it appears that it is not sufficient for the map T of a compact system $\mathcal{Y} = (Y, T, X, S, \pi)$ to be open to ensure that the system be uniformly fibrewise open. In fact, it does not even ensure that $\mathcal{Y}$ be fibrewise open. We will show, though, that if, in addition, S is locally injective, then the system is uniformly fibrewise open. We will also provide some examples of systems, for which S is locally injective everywhere except at one point, where it is locally two-to-one, that are not uniformly fibrewise open. We also investigate what properties the fibre maps must have for the system to be uniformly fibrewise open. We will prove that this is the case whenever each fibre map is a homeomorphism. This is a very strong requirement, but it turns out that there are some systems for which every fibre map is a homeomorphism, but one, which is a local homeomorphism, systems that are not uniformly fibrewise open. Finally, we impose some topological conditions on the fibres. We will look more closely to uniform fulc spaces, that is, those spaces whose fibres are uniformly locally connected have some uniformity between them. We show that every fibrewise open, fibrewise expanding uniform fulc compact system is fibrewise Ruelle expanding. Nevertheless, we provide some examples of fibrewise open, fibrewise expanding fulc compact systems that are not fibrewise Ruelle expanding. We also show that a compact skew-product is uniformly fulc if and only if one of its fibres (that are essentially all identical) is locally connected.

Having done that, we derive the existence of a Gibbs family for an important class of fibrewise expansive compact systems. The case X countable is completely tackled by noticing that the Lipschitz potentials are dense in the space of continuous potentials (this is a simple application of Stone-Weierstrass theorem). We can therefore

approximate the potential imposed on Y by a sequence of Lipschitz potentials. This sequence gives rise, according to Denker and Gordin, to a sequence of Gibbs families. We will show that the 'accumulation points' of this latter form Gibbs families for the potential under consideration. Unfortunately, the uniqueness of the Gibbs family may be lost in the process. The case X uncountable proves to be much trickier and we adopt another strategy to partially solve it. We extract a family of potentials that keep their uniform fibrewise Hölder continuity under any compatible change of metric. This family regroups all uniformly fibrewise locally constant potentials. For each such potential, we deduce the existence of a unique Gibbs family.

Chapter 1

Preliminaries

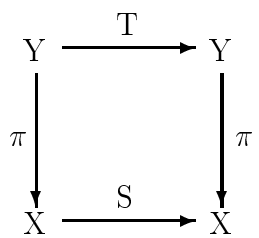
In this chapter, we introduce, among others, the basic concepts of *fibred systems, fibrewise expansiveness, fibrewise expandability and fibrewise Ruelle expandability.* These definitions are direct generalizations of well-known notions in the theory of dynamical systems.

1.1 Fibred Systems

1.1.1 Definition

Fibred systems constitute a special class of dynamical systems. Loosely speaking, they can be described as systems consisting of a fibre bundle (over another system) which is preserved under iteration.

Throughout this dissertation, a fibred system is a collection $\mathcal{Y} = (Y, T, X, S, \pi)$, where Y and X are metric spaces, $T : Y \to Y$ and $S : X \to X$ are continuous maps and $\pi : Y \to X$ is a continuous surjective map which satisfies $\pi \circ T = S \circ \pi$, that is, the following diagram commutes:



Thus T preserves the fibres $Y_x := \pi^{-1}(\{x\})$. The restriction of T^n $(n \in \mathbb{N})$ to the fibre Y_x will be denoted by T_x^n, so $T_x^n : Y_x \to Y_{S^n(x)}$. The metric on Y will be denoted by d_Y, while the metric imposed upon X will be denoted by d_X.

We will often be in the presence of two different metrics for Y and be willing to 'compare' them. Let us recall that two metrics d and ρ for a space Y are topologically

equivalent (or compatible) if they induce the same topology on Y. This is equivalent to the exigence that the identity map $Id : (Y, d) \to (Y, \rho)$ be a homeomorphism. Recall also that d and ρ are said to be uniformly topologically equivalent (or uniformly compatible) if the identity map Id is a uniform homeomorphism, that is, if both Id and its inverse are uniformly continuous maps.

A fibred system $\mathcal{Y}$ is a skew-product if $Y = X \times Z$, where Z is a metric space and $T(y) = T(x, z) = (S(x), T_x(z))$. Skew-products form an important subclass of fibred systems, as the reader will see later.

A fibred system is said to be compact if Y is a compact space (note that this also implies that X is compact). It is also said to be uniformly continuous if T has this property with respect to d_Y.

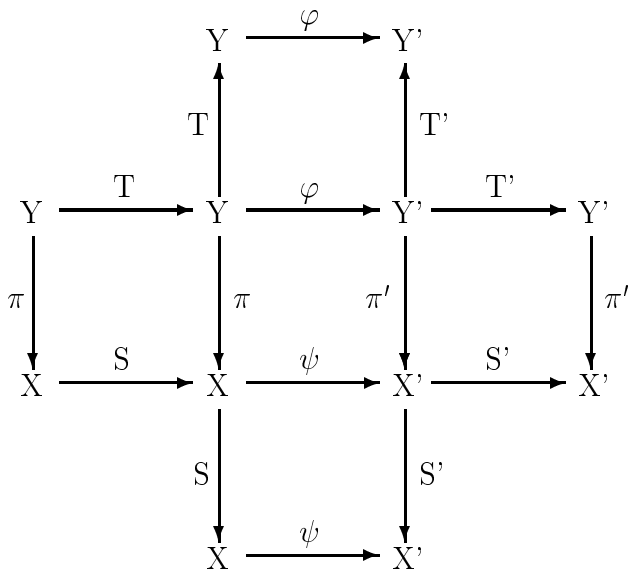
We will also say that a system $\mathcal{Y}' = (Y' \subseteq Y, T|_{Y'}, \pi(Y'), S|_{\pi(Y')}, \pi|_{Y'})$ is a subsystem of a system $\mathcal{Y} = (Y, T, X, S, \pi)$ if $T(Y') \subseteq Y'$. In the compact case, we will additionally require that Y' be closed in order to ensure properties inheritance.

Closely related to fibred systems are fibred spaces. Fibred spaces constitute the underlying space structures of fibred systems, in the same manner as topological spaces or metric spaces form the space structures of topological or metric dynamical systems. Strictly speaking, a fibred space is a triple (Y, X, π), where Y and X are metric spaces and $\pi : Y \to X$ is a continuous surjective map. In analogy to fibred systems, a fibred space (Y, X, π) will be called compact if Y has this property.

A fibred space can be viewed as a space that consists of a fibre bundle over another space. One can also interpret it as a fibred system from which the dynamics has been wiped out.

1.1.2 Morphisms

A fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ is homomorphic to another fibred system $\mathcal{Y}' = (Y', T', X', S', \pi')$ if there exist two continuous maps $\varphi : Y \to Y'$ and $\psi : X \to X'$ such that $\varphi \circ T = T' \circ \varphi$ and $\psi \circ \pi = \pi' \circ \varphi$ (notice that this implies $\psi \circ S = S' \circ \psi$). The pair of maps (φ, ψ) is therefore called a homomorphism from $\mathcal{Y}$ into $\mathcal{Y}'$. The situation may be represented schematically by the commutative diagram:



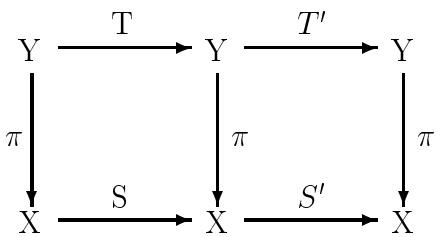
In such circumstances, each fibre of $\mathcal{Y}$ is mapped by φ into a fibre of $\mathcal{Y}'$.

A fibred system $\mathcal{Y}'$ is a factor of a system $\mathcal{Y}$ if there exists a homomorphism (φ, ψ) from $\mathcal{Y}$ to $\mathcal{Y}'$ such that φ is a surjective map (note that this also ensures that ψ is surjective). We thereby say that (φ, ψ) is a factor pair of maps from $\mathcal{Y}$ onto $\mathcal{Y}'$.

Finally, two fibred systems $\mathcal{Y}$ and $\mathcal{Y}'$ are called isomorphic (or fibrewise conjugate) if there exists a factor pair of maps (φ, ψ) from $\mathcal{Y}$ onto $\mathcal{Y}'$ such that both maps φ and ψ are homeomorphisms. The couple (φ, ψ) is therefore called a isomorphism (or, alternatively, a fibrewise conjugacy) from $\mathcal{Y}$ onto $\mathcal{Y}'$. Remark also that $(\varphi^{-1}, \psi^{-1})$ is an isomorphism from $\mathcal{Y}'$ onto $\mathcal{Y}$. Moreover, note that the fibres of $\mathcal{Y}$ are then in a one-to-one correspondence with and homeomorphic to the fibres of $\mathcal{Y}'$.

1.1.3 Composition, Invertibility and Quasi Invertibility

The composition of two fibred systems of the form $\mathcal{Y} = (Y, T, X, S, \pi)$ and $\mathcal{Y}' = (Y, T', X, S', \pi)$ is defined by $\mathcal{Y}' \circ \mathcal{Y} = (Y, T' \circ T, X, S' \circ S, \pi)$. This operation is depicted by the following commutative diagram:



A fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ will be referred to as invertible if the maps T and S are homeomorphisms. This implies in particular that $\mathcal{Y}^{-1} := (Y, T^{-1}, X, S^{-1}, \pi)$

forms a fibred system. For an invertible fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$, it is easy to see that $\mathcal{Y}^{-1} \circ \mathcal{Y} = Id = \mathcal{Y} \circ \mathcal{Y}^{-1}$, where $Id = (Y, Id_Y, X, Id_X, \pi)$. In fact, a fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ is invertible if and only if there exists a system $\mathcal{Y}' = (Y, T', X, S', \pi)$ satisfying $\mathcal{Y}' \circ \mathcal{Y} = Id = \mathcal{Y} \circ \mathcal{Y}'$.

Finally, we will say that a fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ is quasi invertible if T is a homeomorphism and S locally injective.

1.2 Fibrewise Expansiveness

The concept of expansion has drawn the attention and aroused the interest of a great many mathematicians. Amidst the large variety of such dynamical behaviors, expansiveness has turned out to be a rather weak but useful notion. It has hence been extensively studied.

This notion was introduced by W.R. Utz [23] in 1950. He used the term *unstable homeomorphisms* to describe these maps but Gottschalk and Hedlund [11] later suggested the term *expansive homeomorphisms*, which has been used ever since. Five years after Utz, R.F. Williams [26] investigated positive expansiveness of maps. Among all the other early contributors, we would like to mention the works realized by B.F. Bryant [3] and H.B. Keynes and J.B. Robertson [14].

We now introduce a concept of expansiveness along the fibres of a system.

Definition 1.1 *A fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ is said to be (uniformly) fibrewise expansive if there exists a constant $\epsilon > 0$ such that the following holds:*
If $y_1, y_2 \in Y$, $\pi(y_1) = \pi(y_2)$ and $y_1 \neq y_2$, then

$$\sup_{n \in \mathbb{Z}_+} d_Y\Big(T^n(y_1), T^n(y_2)\Big) > \epsilon,$$

that is,

$$\sup_{n \in \mathbb{N}} d_{n,Y}(y_1, y_2) > \epsilon,$$

where

$$d_{n,Y}(y_1, y_2) = \max_{0 \leq m < n} d_Y\Big(T^m(y_1), T^m(y_2)\Big)$$

are new metrics defined on Y (though the notation does not show the dependence on T, it is crucial to remember that these metrics arise from the dynamics of the system (Y, T)). The open ball centred at y and of radius r in the metric $d_{n,Y}$ is $\cap_{m=0}^{n-1} T^{-m}(B_Y(T^m(y), r))$ and will be denoted by $B_{n,Y}(y, r)$ (remark that $d_{1,Y} = d_Y$ and hence $B_{1,Y}(y, r) = B_Y(y, r)$).

Equivalently, $\mathcal{Y}$ is fibrewise expansive if there is $\epsilon > 0$ such that $y_1, y_2 \in Y$, $\pi(y_1) = \pi(y_2)$ and $\sup_{n\in\mathbb{Z}_+} d_Y(T^n(y_1), T^n(y_2)) \leq \epsilon$ implies that $y_1 = y_2$. We call ϵ a fibrewise expansive constant. We will also say under these circumstances that $\mathcal{Y}$ is fibrewise ϵ-expansive.

It is clear that an expansive fibred system is fibrewise expansive. Though the converse is not true, we now show that if S is expansive, so is a fibrewise expansive system $\mathcal{Y} = (Y, T, X, S, \pi)$.

Theorem 1.2 *Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a fibrewise expansive system for which π is uniformly continuous. If S is expansive, so is T.*

Proof. Let $\epsilon > 0$ be a fibrewise expansive constant for the system and let $0 < \delta < \epsilon$. By uniform continuity of π, there exists $0 < \zeta < \epsilon$ such that

$$d_Y(y_1, y_2) < \zeta \Rightarrow d_X\Big(\pi(y_1), \pi(y_2)\Big) < \delta.$$

If T were not expansive, there would exist points $y_\delta, \tilde{y}_\delta \in Y, \pi(y_\delta) \neq \pi(\tilde{y}_\delta)$ with

$$\sup_{n\in\mathbb{Z}_+} d_Y\Big(T^n(y_\delta), T^n(\tilde{y}_\delta)\Big) < \zeta.$$

This would imply that

$$\sup_{n\in\mathbb{Z}_+} d_X\Big(S^n(\pi(y_\delta)), S^n(\pi(\tilde{y}_\delta))\Big) = \sup_{n\in\mathbb{Z}_+} d_X\Big(\pi(T^n(y_\delta)), \pi(T^n(\tilde{y}_\delta))\Big) \leq \delta.$$

Hence there would be $x_\delta (:= \pi(y_\delta)) \neq \tilde{x}_\delta (:= \pi(\tilde{y}_\delta)) \in X$ such that

$$\sup_{n\in\mathbb{Z}_+} d_X\Big(S^n(x_\delta), S^n(\tilde{x}_\delta)\Big) \leq \delta.$$

Since this would be possible for any $0 < \delta < \epsilon$, one would conclude that S is not expansive. This would lead to a contradiction. □

Note that the expansiveness of S does not in general constrain T to be expansive. Nor does the expansiveness of T force S to be expansive.

We will later investigate in detail the properties of fibrewise expansiveness. For now, let us define the notion of fibrewise uniform expansiveness.

1.3 Fibrewise Uniform Expansiveness

In 1962, B.F. Bryant [4] remarked that there is a certain type of uniformity associated with the expansiveness of compact dynamical systems. The uniformity lies in the minimum rate at which the points of the system leave their mutual neighbourhood of expansiveness. This rate depends solely on the distance separating points. This behavior was formalized and studied by M. Sears [21] eleven years later. Its generalization to fibred systems can be stated as follows.

Definition 1.3 *A fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ is called (uniformly) fibrewise uniformly expansive if there is $\epsilon > 0$ with the property that for all $0 < \tilde{\epsilon} \leq \epsilon$ there exists $N = N(\tilde{\epsilon}) \in \mathbb{N}$ such that the following holds:*
If $y_1, y_2 \in Y$, $\pi(y_1) = \pi(y_2)$ and $d_Y(y_1, y_2) > \tilde{\epsilon}$, then $d_{N,Y}(y_1, y_2) > \epsilon$.
The constant ϵ is then called a fibrewise uniform expansive constant for the system and this latter is said to be fibrewise uniformly ϵ-expansive.

One possible characterization of fibrewise uniform expansiveness is the following.

Lemma 1.4 *A fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ is fibrewise uniformly ϵ-expansive if and only if*

$$\lim_{n\to\infty} \sup_{y\in Y} \operatorname{diam}_Y \left(B_{n,Y}(y, \epsilon) \cap Y_{\pi(y)} \right) = 0.$$

Proof. This can be easily seen from Definition 1.3. □

Corollary 1.5 *Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a fibrewise uniformly ϵ-expansive fibred system. Then there exists $M \in \mathbb{N}$ such that*

$$\sup_{y\in Y} \operatorname{diam}_Y \left(B_{M+1,Y}(y, \epsilon) \cap Y_{\pi(y)} \right) < \frac{\epsilon}{8},$$

that is, for any $y \in Y$,

$$B_{M+1,Y}(y, \epsilon) \cap Y_{\pi(y)} \subseteq B_Y\left(y, \frac{\epsilon}{8}\right).$$

It is evident that a fibrewise uniformly expansive system is fibrewise expansive. As in the non-fibred case or, in other words, the one-fibre case, the compactness of the system guarantees that these two notions are equivalent (see [21]).

Lemma 1.6 *Every compact fibrewise expansive system is fibrewise uniformly expansive.*

Proof. Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a fibrewise expansive compact fibred system with fibrewise expansive constant $\epsilon > 0$.

First, notice that for any $y \in Y$ and $\delta > 0$

$$B_{n_1,Y}(y, \delta) \subseteq B_{n_2,Y}(y, \delta)$$

whenever $n_1 \geq n_2$ (since $d_{n_1,Y} \geq d_{n_2,Y}$). Then

$$B_{n_1,Y}(y, \delta) \cap Y_{\pi(y)} \subseteq B_{n_2,Y}(y, \delta) \cap Y_{\pi(y)}.$$

This implies that

$$\operatorname{diam}_Y \left(B_{n_1,Y}(y, \delta) \cap Y_{\pi(y)} \right) \leq \operatorname{diam}_Y \left(B_{n_2,Y}(y, \delta) \cap Y_{\pi(y)} \right),$$

and hence

$$\sup_{y\in Y} \mathrm{diam}_Y \left(B_{n_1,Y}(y,\delta)\cap Y_{\pi(y)}\right) \leq \sup_{y\in Y} \mathrm{diam}_Y \left(B_{n_2,Y}(y,\delta)\cap Y_{\pi(y)}\right).$$

The sequence $(\sup_{y\in Y} \mathrm{diam}_Y(B_{n,Y}(y,\delta)\cap Y_{\pi(y)}))_{n\in\mathbb{N}}$ is therefore decreasing and hence converges to the infimum of its set of values.

Let

$$\eta := \lim_{n\to\infty} \sup_{y\in Y} \mathrm{diam}_Y \left(B_{n,Y}(y,\epsilon)\cap Y_{\pi(y)}\right) = \inf_{n\in\mathbb{N}} \sup_{y\in Y} \mathrm{diam}_Y \left(B_{n,Y}(y,\epsilon)\cap Y_{\pi(y)}\right).$$

If $\eta > 0$, then for each $n \in \mathbb{N}$, there would exist $y_n, \tilde{y}_n \in Y$ such that $\tilde{y}_n \in B_{n,Y}(y_n,\epsilon)\cap Y_{\pi(y_n)}$ and $d_Y(\tilde{y}_n, y_n) \geq \eta/2$. By taking a subsequence if necessary, we may assume, without loss of generality, that the sequences $(\tilde{y}_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ converge to, say, $\tilde{y}$ and y respectively. Then

$$d_Y(\tilde{y}, y) = \lim_{n\to\infty} d_Y(\tilde{y}_n, y_n) \geq \frac{\eta}{2} > 0.$$

Hence $\tilde{y} \neq y$. But $\pi(\tilde{y}_n) = \pi(y_n)$. The continuity of π then ensures that $\pi(\tilde{y}) = \pi(y)$.

Moreover, for $m \in \mathbb{Z}_+$, the continuity of T^m implies that

$$\begin{aligned} d_Y\left(T^m(\tilde{y}), T^m(y)\right) &= \lim_{n\to\infty} d_Y\left(T^m(\tilde{y}_n), T^m(y_n)\right) \\ &\leq \sup_{n\in\mathbb{N}} d_{n,Y}(\tilde{y}_n, y_n) \\ &\leq \epsilon. \end{aligned}$$

Since $m \in \mathbb{Z}_+$ was chosen arbitrarily, we deduce that

$$\sup_{m\in\mathbb{Z}_+} d_Y\left(T^m(\tilde{y}), T^m(y)\right) \leq \epsilon.$$

The fibrewise expansiveness of the system would then imply that $\tilde{y} = y$. But we know that this is not the case. Hence $\eta = 0$ and the result follows from Lemma 1.4. □

Nonetheless, note that these two notions may differ for non-compact systems (again, see [21]).

1.4 Fibred Generators

The original concept of generator was introduced by V.A. Rokhlin [18] in ergodic theory. Its topological analogue is due to Keynes and Robertson [14]. They characterized expansive homeomorphisms as being those homeomorphisms that can be associated with a generator. As the concept of generator has proved to be useful, we are naturally led to generalize it.

Definition 1.7 *Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a compact fibred system (respectively a compact invertible fibred system).*

A finite open cover $\mathcal{U}$ of Y is called one-sided (respectively two-sided) fibred generator if for every sequence (resp. bisequence) $(U_i)_{i\in I}$ (with I being equal to $\mathbb{Z}_+$ and $\mathbb{Z}$ respectively) of atoms of $\mathcal{U}$, the set $(\cap_{i\in I} T^{-i}(\overline{U_i})) \cap Y_x = \cap_{i\in I} T^{-i}(\overline{U_i} \cap Y_{S^i(x)})$ consists of at most one point for every $x \in X$.

$\mathcal{U}$ is said to be a weak one-sided (respectively two-sided) fibred generator if for every sequence (resp. bisequence) $(U_i)_{i\in I}$ (with I chosen as above) of atoms of $\mathcal{U}$, the set $(\cap_{i\in I} T^{-i}(U_i)) \cap Y_x$ consists of at most one point for every $x \in X$.

Clearly, a fibred generator is a weak fibred generator. Though the converse is not true, one can show, as expected from Keynes and Robertson's work, that a fibred system admitting a weak fibred generator also has a fibred generator.

Unless otherwise stated, $\mathcal{Y}$ is assumed to be a compact fibred system (resp. a compact invertible fibred system) for the remainder of this section. It is also understood that the terms *one-sided* (resp. *two-sided*) and *sequence* (resp. *bisequence*) relate to a fibred system (resp. an invertible fibred system). Moreover, for an invertible fibred system, fibrewise expansiveness is defined in terms of all (backward and forward) iterates, whereas we will refer to positive fibrewise expansiveness if we just want to look at the forward iterates. In other words, an invertible fibred system $\mathcal{Y}$ is said to be fibrewise expansive if there exists $\epsilon > 0$ such that if $y_1, y_2 \in Y$, $\pi(y_1) = \pi(y_2)$ and $y_1 \neq y_2$, then $\sup_{n\in\mathbb{N}} d^{\pm}_{n,Y}(y_1, y_2) > \epsilon$, where $d^{\pm}_{n,Y}(y_1, y_2) = \max_{-n<m<n} d_Y(T^m(y_1), T^m(y_2))$. We will specify that $\mathcal{Y}$ is positively fibrewise expansive if it is fibrewise expansive when considered as a non-invertible system, that is, satisfies Definition 1.1.

The proofs of the following results go along the same lines as Keynes and Robertson [14] and Walters [24].

Lemma 1.8 *If $\mathcal{Y}$ has a weak fibred generator, then it has a fibred generator.*

Proof. Suppose that $\mathcal{U}$ is a weak fibred generator for $\mathcal{Y}$ and let $\delta > 0$ be a Lebesgue number for $\mathcal{U}$. Let $\mathcal{V}$ be a finite open cover of sets V_i such that $\mathrm{diam}_Y(\overline{V_i}) < \delta$. If $(V_{i_n})_{n\in I}$ is a sequence (resp. a bisequence) in $\mathcal{V}$, then for all $n \in I$ (I being either $\mathbb{Z}_+$ or $\mathbb{Z}$), there exists j_n such that $\overline{V_{i_n}} \subseteq U_{j_n}$. Hence

$$\cap_{n\in I} T^{-n}(\overline{V_{i_n}}) \bigcap Y_x \subseteq \cap_{n\in I} T^{-n}(\overline{U_{j_n}}) \bigcap Y_x,$$

which is either empty or a single point for any $x \in X$. So $\mathcal{V}$ is a fibred generator. □

Theorem 1.9 *$\mathcal{Y}$ is fibrewise expansive if and only if it has a weak fibred generator and hence if and only if it admits a fibred generator.*

Proof. By Lemma 1.8, it suffices to show that $\mathcal{Y}$ is fibrewise expansive if and only if it admits a fibred generator.

Let $\delta > 0$ be a fibrewise expansive constant for $\mathcal{Y}$ and $\mathcal{U}$ be a finite cover by open balls of radius $\delta/2$. Suppose that $y_1, y_2 \in \cap_{n\in I} T^{-n}(\overline{U_n}) \cap Y_x$ for some $U_n \in \mathcal{U}$ and some $x \in X$. Then $d_Y(T^n(y_1), T^n(y_2)) < \delta$ for all $n \in I$ and $\pi(y_1) = \pi(y_2)$. The fibrewise expansiveness of the system therefore implies that $y_1 = y_2$. Hence $\mathcal{U}$ is a fibred generator.

Conversely, suppose that $\mathcal{U}$ is a fibred generator. Let $\delta > 0$ be a Lebesgue number for $\mathcal{U}$. If $y_1, y_2 \in Y$ are such that $\pi(y_1) = \pi(y_2)$ and $d_Y(T^n(y_1), T^n(y_2)) \leq \delta$ for all $n \in I$, then there exists for each n an atom $U_n \in \mathcal{U}$ with $T^n(y_1), T^n(y_2) \in U_n$ and so $y_1, y_2 \in \cap_{n\in I} T^{-n}(U_n) \cap Y_{\pi(y_i)}$. Since $\mathcal{U}$ is a fibred generator, we deduce that $y_1 = y_2$. Hence $\mathcal{Y}$ is fibrewise δ-expansive. □

As usual, the notion of generator can be used to deduce the following properties (again, compare with [24], page 139).

Corollary 1.10 *The following assertions hold:*

(i) Fibrewise expansiveness is independent of the metric on Y as long as the metric induces the topology of Y. However, the fibrewise expansive constant may change. Furthermore, fibrewise expansiveness does in general depend on the metric on Y when $\mathcal{Y}$ is non-compact.

(ii) More generally, fibrewise expansiveness is a fibrewise conjugacy invariant, that is, if $\mathcal{Y}$ and $\mathcal{Y}'$ are two isomorphic (or fibrewise conjugate) fibred systems, then $\mathcal{Y}$ is fibrewise expansive if and only if $\mathcal{Y}'$ is.

(iii) If $k \neq 0$, then $\mathcal{Y}$ is fibrewise expansive if and only if all of its iterates $\mathcal{Y}^k = (Y, T^k, X, S^k, \pi)$ are fibrewise expansive (the case $k < 0$ is only considered when $\mathcal{Y}$ is invertible).

(iv) A subsystem $\mathcal{Y}'$ of a fibrewise expansive system $\mathcal{Y}$ is fibrewise expansive.

Proof.

(i) This follows from the fact that the concept of generator does not depend on the metric.

(ii) $\mathcal{U}$ is a fibred generator for $\mathcal{Y}$ if and only if $\varphi(\mathcal{U})$ is a fibred generator for $\mathcal{Y}'$.

(iii) If $\mathcal{U}$ is a fibred generator for $\mathcal{Y}$, then $\vee_{j=0}^{k-1} T^{-j}(\mathcal{U})$ is a fibred generator for $\mathcal{Y}^k$. Moreover, any fibred generator for $\mathcal{Y}^k$ is a fibred generator for $\mathcal{Y}$.

(iv) This follows directly from the definition of fibrewise expansiveness. One can also argue that the restriction of a fibred generator for the system to any of its subsystems is a fibred generator for the chosen subsystem.

Keynes and Robertson showed that generators completely determine the topology of the underlying space upon which a dynamical system lives (see [14]). Let us now investigate what information fibred generators give.

Lemma 1.11 *Let $\mathcal{Y}$ be a compact fibred system (respectively a compact invertible fibred system) and let $\mathcal{U}$ be a one-sided (resp. two-sided) fibred generator for $\mathcal{Y}$. Then for every $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$ such that each set of the form $\vee_{n=0}^{N} T^{-n}(\mathcal{U}) \cap V_{1/N}(Y_x)$ (respectively $\vee_{n=-N}^{N} T^{-n}(\mathcal{U}) \cap V_{1/N}(Y_x)$) has diameter less than ϵ.*

Conversely, for each $N \in \mathbb{N}$, there exists $\epsilon = \epsilon(N) > 0$ such that $d_Y(y_1, y_2) < \epsilon$ implies that

$$y_1, y_2 \in \cap_{n=0}^{N} T^{-n}(U_n) \bigcap V_{1/N}(Y_x)$$

for some $U_0, \ldots, U_N \in \mathcal{U}$ and some $x \in X$ (n varying from $-N$ to N in the invertible case).

Here $V_\delta(A)$ stands for the open δ-neighbourhood of the set A with respect to the metric d_Y imposed on Y.

Proof. We only give a proof for the non-invertible case. The reader may supply a similar argument for invertible systems.

Suppose that the first statement is not true. Then there is $\epsilon > 0$ such that for each $m \in \mathbb{N}$, there are $x_m \in X$, $U_m^{(k)} \in \mathcal{U}$ $(0 \leq k \leq m)$ and $y_m^{(1)}, y_m^{(2)} \in \cap_{k=0}^{m} T^{-k}(U_m^{(k)}) \cap V_{1/m}(Y_{x_m})$ with $d_Y(y_m^{(1)}, y_m^{(2)}) \geq \epsilon$. Since X and Y are compact spaces, there exist subsequences (for clarity, we will identify these subsequences with the original sequences) of $(x_m)_{m\in\mathbb{N}}$, $(y_m^{(1)})_{m\in\mathbb{N}}$ and $(y_m^{(2)})_{m\in\mathbb{N}}$ that converge to, say, $x, y^{(1)}$ and $y^{(2)}$ respectively. We then have that $d_Y(y^{(1)}, y^{(2)}) \geq \epsilon$ and hence $y^{(1)} \neq y^{(2)}$.

Furthermore, since $y_m^{(1)}, y_m^{(2)} \in V_{1/m}(Y_{x_m})$, there exist $\tilde{y}_m^{(1)}, \tilde{y}_m^{(2)} \in Y_{x_m}$ such that $d_Y(\tilde{y}_m^{(i)}, y_m^{(i)}) < 1/m$. Hence

$$\pi(y^{(i)}) = \pi(\lim_{m\to\infty} y_m^{(i)}) = \pi(\lim_{m\to\infty} \tilde{y}_m^{(i)}) = \lim_{m\to\infty} \pi(\tilde{y}_m^{(i)}) = \lim_{m\to\infty} x_m = x.$$

Thus, $y^{(1)} \neq y^{(2)}$ and $y^{(1)}, y^{(2)} \in Y_x$.

Now, consider the sequence of sets $(U_m^{(0)})_{m\in\mathbb{N}}$. Infinitely many of them coincide since $\mathcal{U}$ is finite. Thus, there is $U_0 \in \mathcal{U}$ such that $y_m^{(1)}, y_m^{(2)} \in U_0$ for infinitely many m's and hence $y^{(1)}, y^{(2)} \in \overline{U_0}$. Similarly, for each k, infinitely many $U_m^{(k)}$ coincide and we obtain $U_k \in \mathcal{U}$ with $y^{(1)}, y^{(2)} \in T^{-k}(\overline{U_k})$. Therefore $y^{(1)}, y^{(2)} \in \cap_{k=0}^{\infty} T^{-k}(\overline{U_k}) \cap Y_x$, hence contradicting the fact that $\mathcal{U}$ is a fibred generator.

To prove the converse, let $N \in \mathbb{N}$ and $\delta > 0$ be a Lebesgue number for $\mathcal{U}$. Choose $0 < \epsilon < 1/N$ such that $d_Y(y_1, y_2) < \epsilon$ implies that $d_{N+1,Y}(y_1, y_2) < \delta$. If $y_1, y_2 \in Y$ are such that $d_Y(y_1, y_2) < \epsilon$, then for every $0 \leq n < N+1$, there is some $U_n \in \mathcal{U}$ with $T^n(y_1), T^n(y_2) \in U_n$. Therefore $y_1, y_2 \in \cap_{n=0}^{N} T^{-n}(U_n) \cap V_{1/N}(Y_{\pi(y_i)})$ for $i = 1, 2$. $\square$

This shows that one can generate the topology of Y by projecting a fibred generator on neighbourhoods of the fibres of the system upon observation.

Keynes and Robertson also showed that expansive homeomorphisms are factors of subshifts of finite shifts. This family of subshifts therefore plays a special rôle in the theory of dynamical systems.

We now obtain an analogue for fibred systems. The following result states that a system $\mathcal{Y} = (Y, T, X, S, \pi)$ which has a fibred generator (or, in other words, that is fibrewise expansive) is a factor of a subsystem of a skew-product. This skew-product consists of the dynamical system (X, S) that forms the base of $\mathcal{Y}$ and of a full finite shift space.

Theorem 1.12 *Suppose that a compact fibred system (respectively a compact invertible fibred system) $\mathcal{Y} = (Y, T, X, S, \pi)$ has a fibred generator. Then $\mathcal{Y}$ is a factor of a closed subsystem W of the skew-product $(X \times \{1, \ldots, d\}^I, S \times \sigma, X, S, \pi_X)$, where I is equal to $\mathbb{Z}_+$ (respectively $\mathbb{Z}$), $\{1, \ldots, d\}^I$ is the one-sided (respectively two-sided) full d-shift space, σ is the one-sided (respectively two-sided) left shift map and π_X is the projection on X.*

Proof. We will here give a proof for the invertible case. The non-invertible situation can be treated in a similar manner.

Let $\mathcal{U} = \{U_1, \ldots, U_d\}$ be a two-sided fibred generator for the compact invertible system $\mathcal{Y}$. Let $x \in X$ and define

$$Z_x := \left\{ z \in \mathcal{A}^{\mathbb{Z}} \;\middle|\; \exists\, (U_{z_i})_{i \in \mathbb{Z}}, U_{z_i} \in \mathcal{U} \text{ such that } \cap_{i=-\infty}^{\infty} T^{-i}(\overline{U_{z_i}}) \bigcap Y_x \neq \emptyset \right\},$$

where $\mathcal{A} = \{1, \ldots, d\}$. Note that Z_x is not necessarily invariant, although $\sigma(Z_x) = Z_{S(x)}$. Define also

$$W := \bigcup_{x \in X} \{x\} \times Z_x$$

and notice that $(S \times \sigma)(W) = W$. Hence W is an invertible subsystem of the invertible skew-product $(X \times \{1, \ldots, d\}^{\mathbb{Z}}, S \times \sigma, X, S, \pi_X)$.

Now let $\varphi : W \to Y$ be defined by

$$\varphi(x, z) = y \text{ if } y \in \cap_{i=-\infty}^{\infty} T^{-i}(\overline{U_{z_i}}) \bigcap Y_x.$$

One can easily see that φ is well-defined, is surjective and, if ψ is the identity map on X, then $\pi \circ \varphi = \psi \circ \pi_X$ and $T \circ \varphi = \varphi \circ (S \times \sigma)$.

We now want to show that W is closed and φ continuous. For this, suppose $((x^j, z^j))_{j\in\mathbb{N}}$ is a sequence in W converging to a point (x, z). For each $i \in \mathbb{Z}$, we have $\lim_{n\to\infty} z_i^j = z_i$, and hence there exists $j_i \in \mathbb{N}$ such that $z_i^j = z_i$ for all $j \geq j_i$. Thus, $\varphi(x^j, z^j) \in T^{-i}(\overline{U_{z_i}})$ for all $j \geq j_i$. Since $T^{-i}(\overline{U_{z_i}})$ is compact, by considering a subsequence if necessary, we may assume that $\lim_{j\to\infty} \varphi(x^j, z^j) = y$, where $y \in T^{-i}(\overline{U_{z_i}})$ for every $i \in \mathbb{Z}$. Thus, $y \in \cap_{i=-\infty}^{\infty} T^{-i}(\overline{U_{z_i}}) \neq \emptyset$. Furthermore, as $\pi(\varphi(x^j, z^j)) = x^j$ for all j, the continuity of π implies that

$$\pi(y) = \pi\Big(\lim_{j\to\infty} \varphi(x^j, z^j)\Big) = \lim_{j\to\infty} \pi(\varphi(x^j, z^j)) = \lim_{j\to\infty} x^j = x.$$

Hence, $y \in \cap_{i=-\infty}^{\infty} T^{-i}(\overline{U_{z_i}}) \cap Y_x \neq \emptyset$ and $\varphi(x, z) = y$. Therefore $(x, z) \in W$ and W is closed.

Remark also that every convergent subsequence of $(\varphi(x^j, z^j))_{j\in\mathbb{N}}$ must converge to y, as otherwise $\cap_{i=-\infty}^{\infty} T^{-i}(\overline{U_{z_i}}) \cap Y_x$ would contain more than one point. We conclude that $\lim_{j\to\infty} \varphi(x^j, z^j) = \varphi(x, z)$, so that φ is continuous.

This shows that $\mathcal{Y}$ is a factor of W. □

Corollary 1.13 *Suppose that a compact fibred system (respectively a compact invertible fibred system) $\mathcal{Y} = (Y, T, X, S, \pi)$, with Y a zero-dimensional space, has a fibred generator. Then $\mathcal{Y}$ is embedded in a skew-product of the form $(X \times \{1, \ldots, d\}^I, S \times \sigma, X, S, \pi_X)$, where I is equal to $\mathbb{Z}_+$ (respectively $\mathbb{Z}$), $\{1, \ldots, d\}^I$ is the one-sided (respectively two-sided) full d-shift space, σ is the one-sided (respectively two-sided) left shift map and π_X is the natural projection on X.*

Proof. Since Y is zero-dimensional, we can choose a clopen (that is, closed-open) generator $\mathcal{V}$. Let $\mathcal{U}$ be the partition generated by $\mathcal{V}$. Then $\mathcal{U}$ is a clopen generator too. Now apply Theorem 1.12 with the generator $\mathcal{U}$ and remark that the map φ is injective due to the $\mathcal{U}$-partitioning of the space. Hence φ is a continuous bijective map from a compact metric space onto another one. Therefore φ is a homeomorphism and (φ, ψ) is an isomorphism from an invertible subsystem $(X \times \{1, \ldots, d\}^I, S \times \sigma, X, S, \pi_X)$ onto $\mathcal{Y} = (Y, T, X, S, \pi)$. This completes the proof. □

The foregoing two results show that skew-products play a particular rôle in the theory of fibred systems. In particular, they show that a fibrewise expansive system $\mathcal{Y} = (Y, T, X, S, \pi)$ is a factor of a subsystem of the skew-product of its base (X, S) with a full finite shift space. This family of skew-products plays in the theory of fibred systems the rôle performed by the shift spaces in the theory of dynamical systems.

1.5 Fibrewise Expanding Systems

We now define a stronger notion of expansion along the fibres of a system.

Definition 1.14 *A fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ is said to be (uniformly) fibrewise expanding if there exist constants $a > 0$ and $\Lambda > 1$ such that the following holds:*
If $y_1, y_2 \in Y, \pi(y_1) = \pi(y_2)$ and $d_Y(y_1, y_2) < 2a$, then

$$d_Y\Big(T(y_1), T(y_2)\Big) \geq \Lambda\, d_Y(y_1, y_2).$$

Such a system is also referred to as being fibrewise (a, Λ)-expanding under the metric d_Y.

Evidently, an expanding fibred system (under some metric) is fibrewise expanding (with respect to that same metric) but the converse is obviously false.

It is also easy to see that a fibrewise expanding system is fibrewise uniformly expansive. We will show in the next chapter that these two notions coincide, up to a change of metric, for a huge class of fibred systems.

1.6 Fibrewise Ruelle Expanding Systems

In 1978, D. Ruelle [19] introduced a notion of expansion that allows the recovery of the ergodic properties of expanding endomorphisms of differentiable manifolds. In 1999, M. Denker and M. Gordin [8] initiated the investigation of systems that present such a behavior along their fibres.

Definition 1.15 *A fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ is called (uniformly) fibrewise Ruelle expanding if there exist constants $a > 0$ and $\Lambda > 1$ such that the following condition holds:*
If $y_1, y_2' \in Y$, $\pi(T(y_1)) = \pi(y_2')$ and $d_Y(T(y_1), y_2') < 2a$, then there exists a unique $y_2 \in Y$ such that $T(y_2) = y_2', \pi(y_2) = \pi(y_1)$ and $d_Y(y_1, y_2) < 2a$. Furthermore,

$$d_Y\Big(T(y_1), T(y_2)\Big) \geq \Lambda\, d_Y(y_1, y_2).$$

Such a system is called fibrewise Ruelle (a, Λ)-expanding under the metric d_Y.

In contradistinction to the notions of expansivity and expandability, it is not true, in general, that a Ruelle expanding fibred system is fibrewise Ruelle expanding. We will later give some examples of such situations.

Evidently, a fibrewise Ruelle (a, Λ)-expanding system is fibrewise (a, Λ)-expanding. We will show in the next chapter that the converse does not hold and will find general conditions under which it does.

1.7 Fibrewise Openness

We now review some concepts of topology and introduce new notions that will play a preponderant rôle in our work.

Let Y and Y' be topological spaces and $T : Y \to Y'$ a continuous map from Y into Y'. The function T is said to be open if it maps every open set U of Y to an open set $T(U)$ of Y'. When Y and Y' are both metric spaces, the openness of T can be rephrased as follows. A function $T : Y \to Y'$ from a metric space Y into a metric space Y' is open if for every $y \in Y$ and $\delta > 0$, there exists $\epsilon_y(\delta) > 0$ such that $T(B_Y(y, \delta)) \supseteq B_{Y'}(T(y), \epsilon_y(\delta))$.

If, in addition, one can choose the $\epsilon_y(\delta)$'s in such a way that $\inf_{y \in Y} \epsilon_y(\delta) > 0$ for every $\delta > 0$, then one says that T is uniformly open. In other words, a function $T : Y \to Y'$ is uniformly open if for every $\delta > 0$ there exists $\epsilon(\delta) > 0$ such that $T(B_Y(y, \delta)) \supseteq B_{Y'}(T(y), \epsilon(\delta))$ for all $y \in Y$.

Evidently, a uniformly open map is open. The converse is not true though, but the two notions coincide whenever the domain of T, namely Y, is compact.

Lemma 1.16 *Let Y and Y' be metric spaces, Y being compact. Then any continuous open map $T : Y \to Y'$ is uniformly open.*

Proof. Suppose this is not the case. Then there exist $\delta > 0$ and a sequence $(y_n)_{n \in \mathbb{N}}$ in Y such that $T(B_Y(y_n, \delta)) \not\supseteq B_{Y'}(T(y_n), 1/n)$.

Without loss of generality, we may assume that this sequence converges to, say, y. The continuity of T then ensures that the sequence $(T(y_n))_{n \in \mathbb{N}}$ converges to $T(y)$ in Y'.

Let $\epsilon > 0$ be such that $T(B_Y(y, \delta/2)) \supseteq B_{Y'}(T(y), \epsilon)$. Let also $N \geq 2/\epsilon$ be such that $d_Y(y_n, y) < \delta/2$ and $d_{Y'}\Big(T(y_n), T(y)\Big) < \epsilon/2$ for all $n \geq N$.

Then

$$T\Big(B_Y(y_n, \delta)\Big) \supseteq T\Big(B_Y(y, \delta/2)\Big) \supseteq B_{Y'}\Big(T(y), \epsilon\Big) \supseteq B_{Y'}\Big(T(y_n), \epsilon/2\Big) \supseteq B_{Y'}\Big(T(y_n), 1/n\Big)$$

for all $n \geq N$. This contradicts the definition of the y_n's. □

We now define some notions of openness along the fibres of a system.

Definition 1.17 *A fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ is called fibrewise open if the fibre map $T_x : Y_x \to Y_{S(x)}$ is open for every $x \in X$.*

The system $\mathcal{Y}$ is said to be fibrewise uniformly open if each fibre map T_x is uniformly open.

Clearly, a fibrewise uniformly open system is fibrewise open, but the converse does not hold in general.

The following is an immediate consequence of Lemma 1.16 and the fact that a compact fibred system has compact fibres.

Lemma 1.18 *Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a fibrewise open compact fibred system. Then $\mathcal{Y}$ is fibrewise uniformly open, that is, for each $x \in X$ and every $\delta > 0$, there exists $\epsilon_x(\delta) > 0$ such that for all $y \in Y_x$*

$$T\Big(B_Y(y, \delta) \cap Y_x\Big) \supseteq B_Y\Big(T(y), \epsilon_x(\delta)\Big) \cap Y_{S(x)}.$$

Unfortunately, it may happen that the infimum of the supremum of the constants $\{\epsilon_x(\delta)\}_{x \in X}$ that can be chosen be 0 for some δ. Indeed, consider the following example of a fibred system which should be regarded to as being embedded in $\mathbb{R}^2$. (All fibred systems presented here as examples should be looked upon in this way.)

Example 1.19 *Let $X = [0, 1]$, $Y = \cup_{x \in X} Y_x$, where $Y_x = \{(x, 0), (x, x)\}$, and $\pi : Y \to X$ with $\pi(x, y) = x$, the projection along the y-axis in $\mathbb{R}^2$.*

Define also $S : X \to X$ by

$$S(x) = \begin{cases} 2x & \text{if } x \leq 1/2 \\ 2 - 2x & \text{if } x \geq 1/2 \end{cases}$$

and $T : Y \to Y$ by

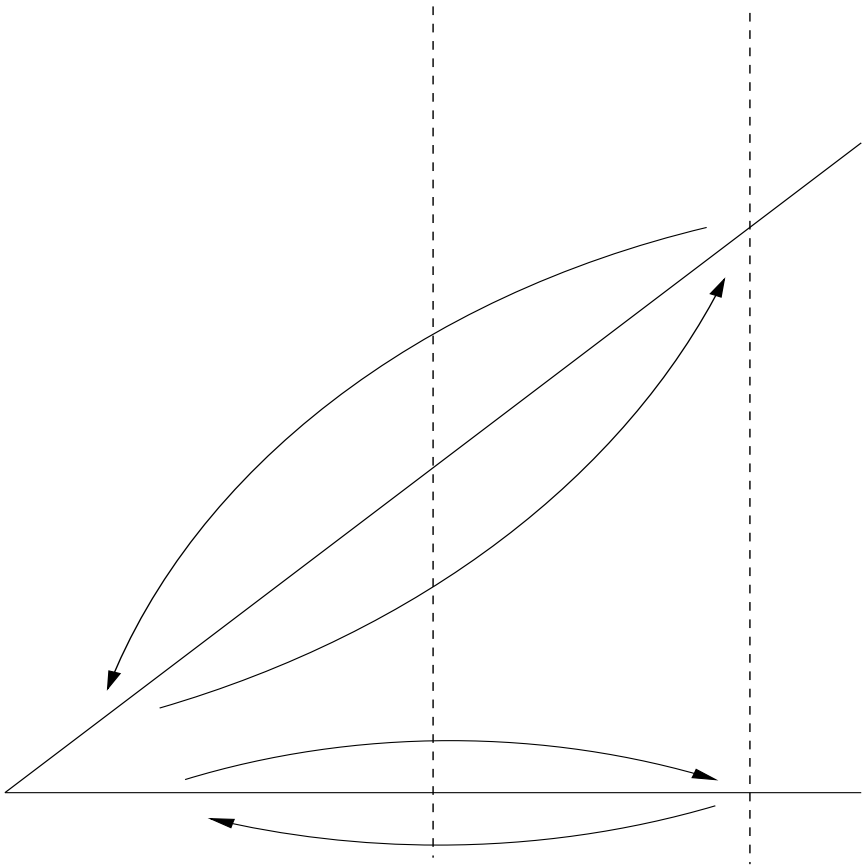
$$\begin{array}{rcrl} T(x, y) & = & (2x, 2y) & \text{if } x \leq 1/2 \\ T(x, 0) & = & (2 - 2x, 0) & \text{if } x \geq 1/2 \\ T(x, x) & = & (2 - 2x, 2 - 2x) & \text{if } x \geq 1/2. \end{array}$$

Note that the fibres of the system coincide with the intersection of vertical lines (represented by dashed lines in the forthcoming figure) with Y.

We now verify that $\mathcal{Y}$ is a fibrewise open compact fibred system (and hence fibrewise uniformly open). We leave it to the reader to supply the arguments showing that $\mathcal{Y}$ really forms a compact fibred system and we concentrate on the fibrewise openness.

It is readily seen that $\mathcal{Y}$ is fibrewise open since the fibres are finite discrete subspaces. More precisely, given $\delta > 0$, we obtain that $\epsilon_x^{\sup}(\delta) := \sup \epsilon_x(\delta) = S(x)$ whenever $\delta \leq x < 1$, whereas $\epsilon_x^{\sup}(\delta) = \infty$ whenever $x < \delta$ or $x = 1$. (Remark that this only happens when the image of a relative ball of radius δ covers the image fiber). Therefore, given $0 < \delta < 1$, we deduce that $\inf_{x \in X} \epsilon_x^{\sup}(\delta) = \inf_{\delta \leq x < 1} S(x) = 0$.

Moreover, the reader may show that this system is fibrewise (a, Λ)-expanding with $1 < \Lambda \leq 2$ and $0 < 2a \leq 2/(\Lambda + 2)$. However, this system is not fibrewise Ruelle



expanding. Indeed, given $0 < a < 1/2$, choose $x > \max(2a, 1-a)$, $y_1 = (x, x)$ and $y_2' = (2-2x, 0)$. Then $\pi(T(y_1)) = 2-2x = \pi(y_2')$ and $d_Y(T(y_1), y_2') = 2-2x < 2a$. The only point $y_2 \in Y$ satisfying $T(y_2) = y_2' = (2-2x, 0)$ and $\pi(y_2) = \pi(y_1) = x$ is the point $y = (x, 0)$. But this point does not satisfy $d_Y(y_1, y_2) < 2a$, for $d_Y(y_1, y) = x > 2a$. On the other hand, if $a \geq 1/2$, then choose $x > 2/3$ and y_1, y_2' as above. Then $\pi(T(y_1)) = \pi(y_2')$ and $d_Y(T(y_1), y_2') < 2a$. This time there is a unique point $y_2 \in Y$ that satisfies $T(y_2) = y_2'$, $\pi(y_2) = \pi(y_1)$ and $d_Y(y_1, y_2) < 2a$. This point is $y_2 = (x, 0)$. Nevertheless, $d_Y(T(y_1), T(y_2)) = 2 - 2x < x = d_Y(y_1, y_2)$. This proves that the system is not fibrewise Ruelle expanding.

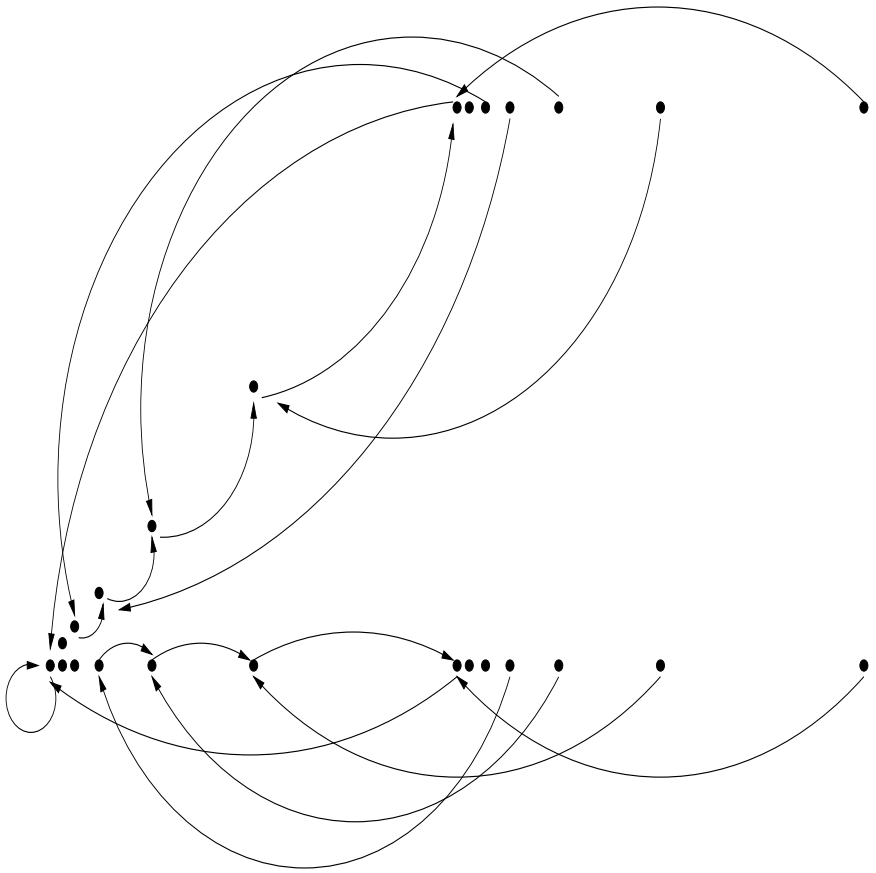
A similar study reveals that the following somehow similar system constitutes another example of a fibrewise uniformly open, fibrewise expanding, but not fibrewise Ruelle expanding, compact fibred system. As in the previous example, the fibres lie on vertical lines of $\mathbb{R}^2$.

Example 1.20 *Set*

$$X = \left\{1/2^s, 1 + 1/2^s\right\}_{s \in \mathbb{Z}_+ \cup \{\infty\}}$$

and $Y = \cup_{x \in X} Y_x$, *where*

$$Y_x = \begin{cases} \{(x,0), (x,x)\} & \text{if} \quad x \leq 1 \\ \{(x,0), (x,1)\} & \text{if} \quad x \geq 1. \end{cases}$$



Once again, let $\pi : Y \to X$ be the projection along the y-axis. Define also $S : X \to X$ by

$$S(x) = \begin{cases} 2x & \text{if } x \leq 1/2 \\ x - 1 & \text{if } x \geq 1 \end{cases}$$

and $T : Y \to Y$ by

$$\begin{array}{rcrl} T(x,y) & = & (2x, 2y) & \text{if } x \leq 1/2 \\ T(x,0) & = & (x-1, 0) & \text{if } x \geq 1 \\ T(x,1) & = & (x-1, x-1) & \text{if } x \geq 1. \end{array}$$

The dynamics associated to this fibred system is depicted by the above graphical representation.

Hence the situation of openness may differ considerably from one fibre to another. Thus we feel the necessity of defining a notion of uniform fibrewise openness or, equivalently, uniform fibrewise uniform openness.

Definition 1.21 *Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a fibred system. $\mathcal{Y}$ is called uniformly fibrewise (uniformly) open if for every $\delta > 0$, there exists $\epsilon = \epsilon(\delta) > 0$ such that for all $y \in Y$*

$$T\Big(B_Y(y, \delta) \cap Y_{\pi(y)}\Big) \supseteq B_Y\Big(T(y), \epsilon\Big) \cap Y_{S(\pi(y))}.$$

Equivalently, $\mathcal{Y}$ is uniformly fibrewise open if it is fibrewise uniformly open and one can choose the $\epsilon_x(\delta)$'s (appearing in Lemma 1.18) in such a way that $\inf_{x\in X}\epsilon_x(\delta) > 0$ for every $\delta > 0$.

This means in particular that the systems considered in Examples 1.19 and 1.20 are not uniformly fibrewise open.

Theorem 1.22 *Fibrewise (uniform) openness and uniform fibrewise openness are fibrewise conjugacy invariants for the class of compact fibred systems, that is, if two compact fibred systems are isomorphic to each other, then one of them is fibrewise (uniformly) open (resp. uniformly fibrewise open) if and only if the other one is.*

Proof. We will give an argument for the preservation of uniform fibrewise openness. It can be proved in a similar way that fibrewise (uniform) openness is preserved under an isomorphism. Let $\mathcal{Y} = (Y, T, X, S, \pi)$ and $\mathcal{Y}' = (Y', T', X', S', \pi')$ be compact fibred systems that are fibrewise conjugate to each other. This means that there exist two homeomorphisms $\varphi : Y \to Y'$ and $\psi : X \to X'$ such that $\varphi\circ T = T'\circ\varphi$ and $\psi\circ\pi = \pi'\circ\varphi$ (recall that this implies $\psi \circ S = S' \circ \psi$). Assume also that $\mathcal{Y}$ is uniformly fibrewise open and let $\delta > 0$.

As φ is a homeomorphism on a compact metric space, φ and φ^{-1} are uniformly open. Hence there exists $\zeta > 0$ such that for all $y' \in Y'$

$$\varphi^{-1}\Big(B_{Y'}(y',\delta)\Big) \supseteq B_Y\Big(\varphi^{-1}(y'),\zeta\Big).$$

Consequently,

$$\begin{aligned}\varphi^{-1}\Big(B_{Y'}(y',\delta)\cap Y'_{\pi'(y')}\Big) &= \varphi^{-1}\Big(B_{Y'}(y',\delta)\Big)\cap\varphi^{-1}(Y'_{\pi'(y')})\\ &= \varphi^{-1}\Big(B_{Y'}(y',\delta)\Big)\cap Y_{\pi(\varphi^{-1}(y'))}\\ &\supseteq B_Y\Big(\varphi^{-1}(y'),\zeta\Big)\cap Y_{\pi(\varphi^{-1}(y'))}.\end{aligned}$$

Since $\mathcal{Y}$ is uniformly fibrewise open, there exists $\xi > 0$ such that for any $y' \in Y'$

$$\begin{aligned}T\Big(B_Y(\varphi^{-1}(y'),\zeta)\cap Y_{\pi(\varphi^{-1}(y'))}\Big) &\supseteq B_Y\Big(T(\varphi^{-1}(y')),\xi\Big)\cap Y_{S(\pi(\varphi^{-1}(y')))}\\ &= B_Y\Big(T(\varphi^{-1}(y')),\xi\Big)\cap Y_{\pi(T(\varphi^{-1}(y')))}.\end{aligned}$$

As φ is uniformly open, there also exists $\epsilon > 0$ such that for all $y \in Y$

$$\varphi\Big(B_Y(y,\xi)\Big) \supseteq B_{Y'}\Big(\varphi(y),\epsilon\Big).$$

Therefore, for all $y' \in Y'$,

$$\begin{aligned}\varphi\Big(B_Y(T(\varphi^{-1}(y')),\xi)\cap Y_{\pi(T(\varphi^{-1}(y')))}\Big) &= \varphi\Big(B_Y(T(\varphi^{-1}(y')),\xi)\Big)\cap\varphi\Big(Y_{\pi(T(\varphi^{-1}(y')))}\Big)\\ &\supseteq B_{Y'}\Big(\varphi(T(\varphi^{-1}(y'))),\epsilon\Big)\cap Y'_{\pi'(\varphi(T(\varphi^{-1}(y'))))}\\ &= B_{Y'}\Big(T'(y'),\epsilon\Big)\cap Y'_{\pi'(T'(y'))}\\ &= B_{Y'}\Big(T'(y'),\epsilon\Big)\cap Y'_{S'(\pi'(y'))}.\end{aligned}$$

It follows that for all $y' \in Y'$

$$\begin{aligned} T'\Big(B_{Y'}(y',\delta) \cap Y'_{\pi'(y')}\Big) &= \varphi\Big(T\Big(\varphi^{-1}\Big(B_{Y'}(y',\delta) \cap Y'_{\pi'(y')}\Big)\Big)\Big) \\ &\supseteq \varphi\Big(T\Big(B_Y(\varphi^{-1}(y'),\zeta) \cap Y_{\pi(\varphi^{-1}(y'))}\Big)\Big) \\ &\supseteq \varphi\Big(B_Y(T(\varphi^{-1}(y')),\xi) \cap Y_{\pi(T(\varphi^{-1}(y')))}\Big) \\ &\supseteq B_{Y'}\Big(T'(y'),\epsilon\Big) \cap Y'_{S'(\pi'(y'))}. \end{aligned}$$

This completes the proof. □

Thus, these notions are, within the family of compact fibred systems, independent of the metric considered (as long as the metric is compatible with the original topology).

Remark that the assumption of compactness of the system is crucial. This notion generally depends on the metric the space is equipped with in the non-compact case.

1.8 The Uniform FULC Property

We proceed on with our revision of topological concepts and introduction of new notions that will later play an important rôle.

A topological space Y is said to be locally connected at a point y of Y if y has a neighbourhood base consisting of connected sets or, in other words, a connected neighbourhood base. The space Y is locally connected if it is locally connected at each of its points. Note that a metric space Y is locally connected at y if and only if for every $\epsilon > 0$, there is a $\delta = \delta(y,\epsilon) > 0$ such that any point $y' \in Y$ with $d_Y(y',y) < \delta$ lies together with y in a connected set of diameter less than ϵ.

The concept of local path-connectedness is also well-known. A space Y is called locally path-connected at a point y of Y if y admits a path-connected neighbourhood base. It is locally path-connected if it is locally path-connected at each of its points. For metric spaces, this definition can be rephrased as follows. A metric space Y is locally path-connected at a point y if for every $\epsilon > 0$, there is a $\delta = \delta(y,\epsilon) > 0$ such that any point $y' \in Y$ with $d_Y(y',y) < \delta$ can be joined to y via a path of diameter less than ϵ.

A frequently used abbreviation for the phrase *locally connected* are the letters 'lc', whereas the letters 'lpc' are often used to refer to the phrase *locally path-connected*.

Evidently, a lpc space is lc. Though the converse is not true, a lc compact metric space is lpc and, perhaps most surprisingly, even more. We will come back to this nebulous statement in a moment.

These two concepts of 'local connectivity' can be readily altered into uniform notions analogous to uniform continuity, uniform openness or any other similar notion that arises in the theory of uniform spaces. One says that a metric space Y is uniformly

locally connected (abbreviated 'ulc') if for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that any pair of points y, y' of Y with $d_Y(y, y') < \delta$ lie together in a connected subset of Y of diameter less than ϵ.

Obviously, a ulc space is lc. Though the converse does not hold in general, a lc compact metric space is ulc.

In analogy to the definition of ulc spaces, one says that a metric space Y is uniformly locally path-connected (abbreviated 'ulpc') provided that for every $\epsilon > 0$, there exists a $\delta > 0$ such that any pair of points y, y' of Y with $d_Y(y, y') < \delta$ can be joined together by means of a path of diameter less than ϵ.

A ulpc space is indubitably a lpc space. Furthermore, a lpc compact metric space is a ulpc space.

On the other hand, a ulpc space is ulc. The converse is certainly not true in general, but a ulc compact metric space is ulpc. In fact, a lc compact metric space is ulpc.

Theorem 1.23 *Let Y be a compact metric space. If Y is locally connected, then it is uniformly locally path-connected.*

Proof. For convenience, we reproduce the proof (and correct a small mistake) that appears in Aoki and Hiraide [1].

Let $\epsilon > 0$ be given. Since Y is assumed to be a locally connected compact metric space, it is uniformly locally connected. This implies that there exists a sequence of numbers $(\delta_n)_{n\in\mathbb{N}}$, $\delta_n > 0$, such that any two points $y_1, y_2 \in Y$ with $d_Y(y_1, y_2) < \delta_n$ lie together in a connected set of diameter less than $\epsilon/2^n$.

We show that any two points $y(0), y(1) \in Y$ with $d_Y(y(0), y(1)) < \delta_0$ can be joined by means of a path of diameter at most 4ϵ.

To do this we choose a sequence of denominators $1 = k_0 < k_1 < k_2 < \ldots$, each of which divides the next. For each fraction of the form i/k_n between 0 and 1 we choose an intermediate point $y(i/k_n)$ satisfying two conditions:

(1) Any two consecutive fractions i/k_n and $(i+1)/k_n$ correspond to points $y(i/k_n)$ and $y((i+1)/k_n)$ which have a distance less than δ_n.

Thus any two such points are contained in a connected set $C(i, k_n)$ which has diameter less than $\epsilon/2^n$. The second condition is the following:

(2) Each point of the form $y(j/k_{n+1})$, where j/k_{n+1} lies between i/k_n and $(i+1)/k_n$, belongs to the set $C(i, k_n)$, and thus

$$d_Y\Big(y(j/k_{n+1}), y(i/k_n)\Big) < \epsilon/2^n \text{ and } d_Y\Big(y(j/k_{n+1}), y((i+1)/k_n)\Big) < \epsilon/2^n.$$

By induction on n the construction of such denominators k_n and intermediate points $y(i/k_n)$ is straightforward. Thus, we may suppose that $y(r)$ has been defined for a dense set of rational numbers r in $[0, 1]$.

Next we prove that the correspondence $r \mapsto y(r)$ is uniformly continuous. Let r and r' be any two rational numbers such that $y(r)$ and $y(r')$ are defined. Suppose that $|r - r'| < 1/(2k_n)$. We can hence choose i/k_n as close as possible to both r and r'. Then we have

$$d_Y\Big(y(r), y(i/k_n)\Big) < 2\epsilon/2^n \text{ and } d_Y\Big(y(r'), y(i/k_n)\Big) < 2\epsilon/2^n,$$

and so

$$d_Y\Big(y(r), y(r')\Big) < 4\epsilon/2^n.$$

This proves uniform continuity. Consequently, there is a unique continuous extension $t \mapsto y(t)$ which is defined for all $t \in [0, 1]$. We have constructed the required path of diameter at most 4ϵ from $y(0)$ to $y(1)$. Therefore Y is uniformly locally path-connected. □

We may now go on with the definition of new concepts that specifically pertain to the space structure of fibred systems. In a similar manner as one can uniformize a given property on a space, one can 'fibrewise' it over a fibred space.

Definition 1.24 *A fibred space (Y, X, π) is said to be fibrewise locally connected (abbreviated 'flc') if each of its fibres is locally connected (lc).*

Similarly, the space (Y, X, π) is called fibrewise locally path-connected (abbreviated 'flpc') if each of its fibres is locally path-connected (lpc).

Clearly, a flpc space is flc. From what we said before, it is not really astonishing that a flc compact fibred space be flpc. In fact, it is even fulpc. But, for this to make sense, we must establish the following definition.

Definition 1.25 *A fibred space (Y, X, π) is called fibrewise uniformly locally connected (resp. fibrewise uniformly locally path-connected) (abbreviated 'fulc', resp. 'fulpc') if each fibre of the space is uniformly locally connected (ulc) (resp. uniformly locally path-connected (ulpc)).*

Similar remarks as for the other properties so far presented can be made about fulc and fulpc spaces. A fulpc space is fulc and a fulc compact space is fulpc.

Both definitions can also be restated in the following way. A fibred space (Y, X, π) is fulc provided that for every $x \in X$ and $\epsilon > 0$, there is $\delta_x(\epsilon) > 0$ such that any pair of points $y_1, y_2 \in Y_x$ with $d_Y(y_1, y_2) < \delta_x(\epsilon)$ lies in a connected subset of Y_x whose diameter is less than ϵ.

Similarly, a fibred space (Y, X, π) is fulpc if for every $x \in X$ and $\epsilon > 0$, there exists a $\delta_x(\epsilon) > 0$ such that any pair of points $y_1, y_2 \in Y_x$ with $d_Y(y_1, y_2) < \delta_x(\epsilon)$ can be joined via a path in Y_x of diameter less than ϵ.

Finally, the fibres of a space may even have a stronger topological structure.

Definition 1.26 *A fibred space (Y, X, π) is referred to as being uniformly fulc (resp. uniformly fulpc) if for every $\epsilon > 0$, there is a $\delta > 0$ such that any two points $y_1, y_2 \in Y$ with $\pi(y_1) = \pi(y_2)$ and $d_Y(y_1, y_2) < \delta$ lie together in a connected subset of (resp. can be joined via a path in) $Y_{\pi(y_i)}$ of diameter less than ϵ.*

Equivalently, a space is uniformly fulc (resp. uniformly fulpc) if it is fulc (resp. fulpc) and for all $\epsilon > 0$, one can choose the $\delta_x(\epsilon)$'s (that appeared previously) in such a way that $\inf_{x \in X} \delta_x(\epsilon) > 0$.

It is unequivocal that a uniformly fulpc space is uniformly fulc and that a uniformly fulc compact space is uniformly fulpc. As this fact will later have some importance, we state it as a theorem.

Theorem 1.27 *Let (Y, X, π) be a compact fibred space. If (Y, X, π) is uniformly fulc, then it is uniformly fulpc.*

Proof. The watershed of the proof of Theorem 1.23 is that whenever two points are close enough, one can join them via a path whose diameter is small enough and independent of the two points considered. This path is constructed via the existence of sufficiently uniformly small connected sets. The theorem is an immediate corollary to that proof. □

A uniformly fulc (resp. uniformly fulpc) space is undoubtedly fulc (resp. fulpc). However, the converse does not hold in general. There are some fulpc spaces which are not uniformly fulc.

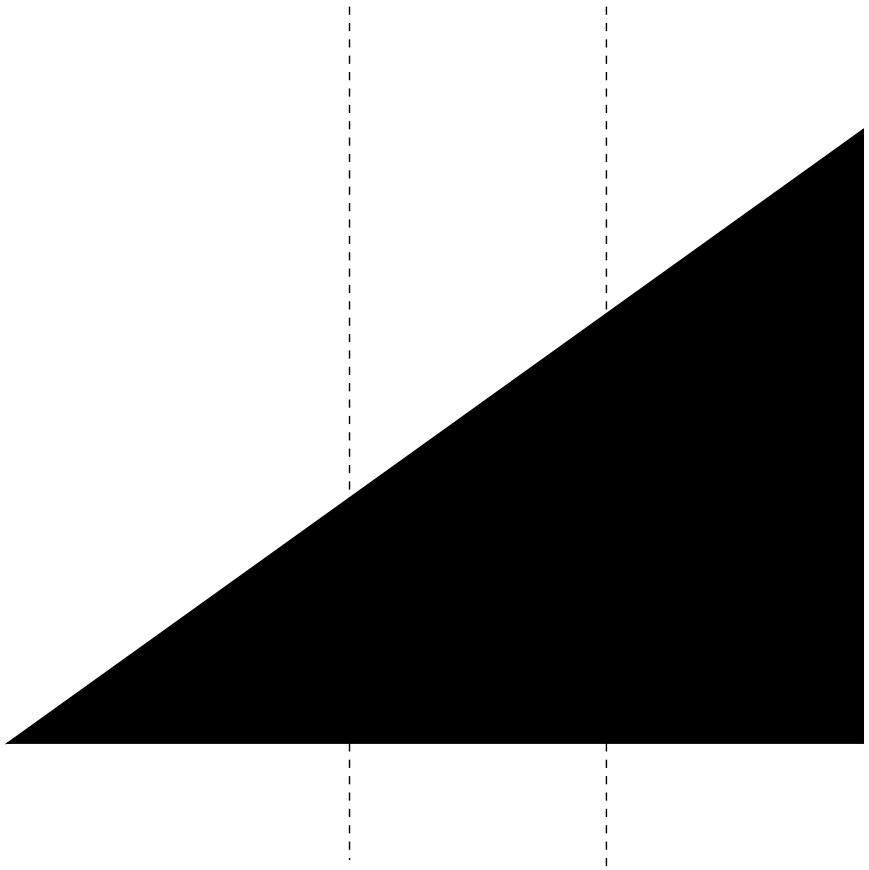
Example 1.28 *Let $X = [0, 1]$, $Y = \cup_{x \in X} Y_x$, where $Y_x = \{(x, 0), (x, x)\}$ and define $\pi : Y \to X$ by $\pi(x, y) = x$, the projection along the y-axis in $\mathbb{R}^2$. Thus the fibres of the space coincide with the non-empty intersections of the vertical lines of $\mathbb{R}^2$ with Y.*

The reader may seem familiar with this fibred space. It is indeed the underlying space structure of the fibred system considered in Example 1.19.

This space is obviously fulpc. Furthermore, for every $0 < \epsilon < 1$ and each $0 < x < \epsilon$, the supremum of the $\delta_x(\epsilon)$'s that can be chosen is $\delta_x^{\sup}(\epsilon) = x$. Hence $\inf_{x \in X} \delta_x^{\sup}(\epsilon) \leq \inf_{0 < x < \epsilon} x = 0$ and this space is not uniformly fulpc.

But if each of the fibres of this space is 'filled in', then the resulting space is uniformly fulpc.

Example 1.29 *Let $X = [0, 1]$, $Y = \{(x, y) \in \mathbb{R}^2 \,|\, x \in X, y \in [0, x]\}$ or, more explicitly, $Y = \cup_{x \in X} Y_x$, where $Y_x = \{(x, y) \in \mathbb{R}^2 \,|\, y \in [0, x]\}$. As usual, $\pi : Y \to X$ is the projection along the y-axis in $\mathbb{R}^2$. Thus the fibres of the space are, as before, the non-empty intersections of the vertical lines of $\mathbb{R}^2$ with Y (see figure at the top of the next page).*



Again, it is clear that this space is fulpc. Moreover, the reader may verify that, given $\epsilon > 0$, one can choose $\delta_x(\epsilon) = \epsilon$ for every $x \in X$. Hence $\inf_{x \in X} \delta_x(\epsilon) = \epsilon$. Therefore (Y, X, π) is a uniformly fulpc space.

There are some important cases in which a fulc (resp. fulpc) space is uniformly fulc (resp. uniformly fulpc). To describe one of those situations, let us define an equivalence relation on the fibres of the system (or, equivalently, on X), saying that two fibres Y_{x_1} and Y_{x_2} are *identical* if there is an isometry between them. This equivalence relation generates a partition of the space X into equivalence classes. We then have the following result.

Theorem 1.30 *Let (Y, X, π) be a fibred space for which the corresponding set $X/\sim$ is finite. Then (Y, X, π) is uniformly fulc (resp. uniformly fulpc) if and only if it is fulc (resp. fulpc).*

In particular, a skew-product space $(Y = X \times Z, X, \pi)$ is uniformly fulc (resp. uniformly fulpc) if and only if Z is ulc (resp. ulpc).

Proof. We give a proof for the fulc case and leave to the reader the task of convincing himself that a similar argument applies to the fulpc situation.

Let $[x]$ denote the equivalence class of x with respect to the equivalence relation $\sim$. By hypothesis, the fibre Y_x is ulc. Hence, given $\epsilon > 0$, there exists $\delta_x(\epsilon) > 0$ such that

any pair of points $y_1, y_2 \in Y_x$ with $d_Y(y_1, y_2) < \delta_x(\epsilon)$ lies in a connected subset of Y_x whose diameter is less than ϵ. Let $x' \in X$ be such that $[x'] = [x]$, that is, the fibres $Y_{x'}$ and Y_x are identical. Let also $y'_1, y'_2 \in Y_{x'}$ be such that $d_Y(y'_1, y'_2) < \delta_x(\epsilon)$. Since $Y_{x'}$ and Y_x are identical, there is an isometry $I_{x',x} : Y_{x'} \to Y_x$ between them. This implies that $d_Y(I_{x',x}(y'_1), I_{x',x}(y'_2)) = d_Y(y'_1, y'_2) < \delta_x(\epsilon)$. Therefore there exists a connected subset C of Y_x of diameter less than ϵ that contains both $I_{x',x}(y'_1)$ and $I_{x',x}(y'_2)$. It follows that $I^{-1}_{x',x}(C)$ is a connected subset of $Y_{x'}$ of diameter less than ϵ that contains y'_1 and y'_2. One hence obtains that one can choose $\delta_{x'}(\epsilon) = \delta_x(\epsilon)$. This means that the $\delta_.(\epsilon)$'s can be chosen so that they are constant on each equivalence class. One can therewith define a map $\tilde{\delta}_.(\epsilon)$ from $X/\!\sim$ into $\mathbb{R}^*_+$. If $X/\!\sim$ is finite, then the minimum of $\tilde{\delta}_.(\epsilon)$ is a strictly positive number and by taking δ to be this minimum, it follows immediately that the space is uniformly fulc.

The second part of the statement follows from the simple observation that $X/\!\sim$ reduces to one point whenever (Y, X, π) is a skew-product space. □

Remarks.

(1) In the compact case, the theorem can be rephrased as follows. Let (Y, X, π) be a compact fibred space for which the corresponding set $X/\!\sim$ is finite. Then (Y, X, π) is uniformly fulc (or, equivalently, uniformly fulpc) if and only if it is flc (or, equivalently, flpc). In particular, a compact skew-product space $(Y = X \times Z, X, \pi)$ is uniformly flc if and only if Z is lc.

(2) The result does not generally hold when $X/\!\sim$ is infinite. Examples 1.28 and 1.29 prove it.

(3) One could define a weaker relation on the fibres of a space by requiring that two fibres be equivalent if there exists a bi-Lipshitz map between them. One may then ask whether the previous theorem holds or not. The answer to this question must be negative. For, if it were holding, the spaces considered in Examples 1.28 and 1.29, that consist of only two such equivalence classes, would be uniformly fulc. But we know this is not the case.

Finally, we show an invariance theorem.

Theorem 1.31 *Fulc-ness and uniform fulc-ness are isomorphism invariants for the category of compact fibred spaces, that is, if two compact fibred spaces are isomorphic to each other, then one of them has one of the above cited properties if and only if the other one has it too.*

Proof. We give a proof of the invariance of uniform fulc-ness and leave it to the reader to persuade himself that fulc-ness is, under these circumstances, invariant under an isomorphism too.

Let (Y, X, π) and (Y', X', π') be compact fibred spaces that are isomorphic to each other. This means that there exist two homeomorphisms $\varphi : Y \to Y'$ and $\psi : X \to X'$ such that $\psi \circ \pi = \pi' \circ \varphi$ (recall that this implies that the fibers of the two spaces are homeomorphic in pairs and that they are in a one-to-one correspondence). Assume also that $\mathcal{Y}$ is uniformly fulc and let $\epsilon > 0$.

As φ is a homeomorphism on a compact metric space, φ and φ^{-1} are uniformly open. Hence there exists $\delta > 0$ such that for all $y' \in Y'$

$$\varphi^{-1}\Big(B_{Y'}(y', \epsilon)\Big) \supseteq B_Y\Big(\varphi^{-1}(y'), \delta\Big).$$

Consequently

$$\begin{aligned} \varphi^{-1}\Big(B_{Y'}(y', \epsilon) \cap Y'_{\pi'(y')}\Big) &= \varphi^{-1}\Big(B_{Y'}(y', \epsilon)\Big) \cap \varphi^{-1}(Y'_{\pi'(y')}) \\ &= \varphi^{-1}\Big(B_{Y'}(y', \epsilon)\Big) \cap Y_{\pi(\varphi^{-1}(y'))} \\ &\supseteq B_Y\Big(\varphi^{-1}(y'), \delta\Big) \cap Y_{\pi(\varphi^{-1}(y'))}. \end{aligned}$$

Since $\mathcal{Y}$ is uniformly fulc, there exist $\xi > 0$ and a relative connected neighbourhood $V(\varphi^{-1}(y'))$ of $\varphi^{-1}(y')$ in the fibre $Y_{\pi(\varphi^{-1}(y'))}$ that satisfies

$$\begin{aligned} B_Y\Big(\varphi^{-1}(y'), \xi\Big) \cap Y_{\pi(\varphi^{-1}(y'))} &\subseteq V(\varphi^{-1}(y')) \\ &\subseteq B_Y\Big(\varphi^{-1}(y'), \delta\Big) \cap Y_{\pi(\varphi^{-1}(y'))} \\ &\subseteq \varphi^{-1}\Big(B_{Y'}(y', \epsilon) \cap Y'_{\pi'(y')}\Big). \end{aligned}$$

As φ is uniformly open, there also exists $\zeta > 0$ such that for all $y \in Y$

$$\varphi\Big(B_Y(y, \xi)\Big) \supseteq B_{Y'}\Big(\varphi(y), \zeta\Big).$$

It follows that

$$\begin{aligned} B_{Y'}(y', \zeta) \cap Y'_{\pi'(y')} &= B_{Y'}\Big(\varphi(\varphi^{-1}(y')), \zeta\Big) \cap Y'_{\pi'(\varphi(\varphi^{-1}(y')))} \\ &\subseteq \varphi\Big(B_Y(\varphi^{-1}(y'), \xi) \cap Y_{\pi(\varphi^{-1}(y'))}\Big) \\ &\subseteq \varphi(V(\varphi^{-1}(y'))) \\ &\subseteq \varphi\Big(\varphi^{-1}\Big(B_{Y'}(y', \epsilon) \cap Y'_{\pi'(y')}\Big)\Big) \\ &= B_{Y'}(y', \epsilon) \cap Y'_{\pi'(y')}. \end{aligned}$$

We have hence found a relative connected neighbourhood $\varphi(V(\varphi^{-1}(y')))$ of y' in the fibre $Y'_{\pi'(y')}$. $\square$

1.9 Fibrewise Topological Exactness

The concept of exactness is well-known in ergodic theory. There are few mathematicians, though, that are familiar with its topological counterpart.

Definition 1.32 *A topological dynamical system (Y, T) is said to be topologically exact if for every open set U of Y, there exists $M = M(U) \in \mathbb{Z}_+$ such that for every $m \geq M$*

$$T^m(U) = Y.$$

Remarks.

(1) Topological exactness of a system is a conjugacy invariant, in that it is preserved by any topological conjugacy.

(2) A topologically exact system is surjective.

(3) Since topological exactness implies surjectivity of the map, it is sufficient that the condition be satisfied for some m.

(4) Topological exactness can be characterized as follows.

A system is topologically exact if and only if there exists a base $\mathcal{B}$ for Y and for each member B of $\mathcal{B}$, there is $M = M(B) \in \mathbb{Z}_+$ such that for every $m \geq M$

$$T^m(B) = Y.$$

(5) The topological exactness of a compact metric dynamical system can also be defined as follows.

A compact metric dynamical system is topologically exact if for every $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{Z}_+$ such that for every $y \in Y$ and $n \geq N$

$$T^n\Big(B_Y(y, \epsilon)\Big) = Y.$$

Note again that it suffices that the latter equation hold for N for the condition to be fulfilled. Notice also that this characterization does not hold (in general) for a non-compact system. It may also depend on the metric in this latter case and is a stronger requirement than topological exactness.

A generalization of this notion was introduced by Denker and Gordin [8] for fibred systems. They defined it in the following way.

Definition 1.33 *A fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ is topologically exact along fibres if for every $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{Z}_+$ such that for every $y \in Y$ and $n \geq N$*

$$T^n\Big(B_Y(y, \epsilon) \cap Y_{\pi(y)}\Big) = Y_{S^n(\pi(y))}.$$

Remarks.

(1) It is obvious that this notion reduces to topological exactness for systems that have only one fibre (that is, non-fibred systems or, in other words, metric dynamical systems).

(2) Topological exactness along fibres of a system does not necessarily imply that this system is fibrewise surjective (meaning that each fibre map $T_x : Y_x \to Y_{S(x)}$ is surjective). Nor does it imply that the system is surjective. But it does imply that there exists $N \in \mathbb{N}$ such that $\mathcal{Y}$ is fibrewise surjective on $\cup_{n \geq N} S^n(X)$. It further implies that the fibred systems $\mathcal{Y}^n = (Y, T^n, X, S^n, \pi)$ are fibrewise surjective for $n \geq N$. This means in particular that the requirement $n \geq N$ can make, by contradistinction with the non-fibred case, a difference. However, if S is surjective, then a system that is topologically exact along fibres is both surjective and fibrewise surjective.

Denker and Gordin [8] were interested in fibred systems that are topologically exact along fibres but did not study the property itself. It seems natural to extend to fibred systems the original definition of topological exactness in the way they did. But is this property metric independent? Or, even better, is it a fibrewise conjugacy invariant?

Theorem 1.34 *Topological exactness along fibres is a fibrewise conjugacy invariant for the class of compact systems, that is, if two compact fibred systems are isomorphic to each other, then one of them has this property if and only if the other one does.*

In particular, topological exactness along fibres of a compact system is independent of the metric considered as far as the metric is compatible with the topology.

Proof. Let $\mathcal{Y} = (Y, T, X, S, \pi)$ and $\mathcal{Y}' = (Y', T', X', S', \pi')$ be compact fibred systems that are fibrewise conjugate to each other. This means that there exist two homeomorphisms $\varphi : Y \to Y'$ and $\psi : X \to X'$ such that $\varphi \circ T = T' \circ \varphi$ and $\psi \circ \pi = \pi' \circ \varphi$ (recall that this implies $\psi \circ S = S' \circ \psi$). Assume that $\mathcal{Y}$ is topologically exact along fibres and let $\epsilon > 0$.

As φ is a homeomorphism on a compact metric space, φ and φ^{-1} are uniformly open. Hence there exists $\delta > 0$ such that for all $y' \in Y'$

$$\varphi^{-1}\Big(B_{Y'}(y', \epsilon)\Big) \supseteq B_Y\Big(\varphi^{-1}(y'), \delta\Big).$$

Consequently,

$$\begin{aligned}\varphi^{-1}\Big(B_{Y'}(y',\epsilon)\cap Y'_{\pi'(y')}\Big) &= \varphi^{-1}\Big(B_{Y'}(y',\epsilon)\Big)\cap\varphi^{-1}(Y'_{\pi'(y')})\\ &= \varphi^{-1}\Big(B_{Y'}(y',\epsilon)\Big)\cap Y_{\pi(\varphi^{-1}(y'))}\\ &\supseteq B_Y\Big(\varphi^{-1}(y'),\delta\Big)\cap Y_{\pi(\varphi^{-1}(y'))}.\end{aligned}$$

Since $\mathcal{Y}$ is topologically exact along fibres, there exists $N\in\mathbb{Z}_+$ such that for all $y'\in Y'$

$$T^n\Big(B_Y(\varphi^{-1}(y'),\delta)\cap Y_{\pi(\varphi^{-1}(y'))}\Big)=Y_{S^n(\pi(\varphi^{-1}(y')))}.$$

It follows that

$$T^n\Big(\varphi^{-1}\Big(B_{Y'}(y',\epsilon)\cap Y'_{\pi'(y')}\Big)\Big)=Y_{S^n(\pi(\varphi^{-1}(y')))}$$

and therefore

$$\begin{aligned}(T')^n\Big(B_{Y'}(y',\epsilon)\cap Y'_{\pi'(y')}\Big) &= \varphi\Big(T^n\Big(\varphi^{-1}\Big(B_{Y'}(y',\epsilon)\cap Y'_{\pi'(y')}\Big)\Big)\Big)\\ &= \varphi\Big(Y_{S^n(\pi(\varphi^{-1}(y')))}\Big)\\ &= \varphi\Big(Y_{\pi(T^n(\varphi^{-1}(y')))}\Big)\\ &= Y'_{\pi'(\varphi(T^n(\varphi^{-1}(y'))))}\\ &= Y'_{\pi'((T')^n(y'))}\\ &= Y'_{(S')^n(\pi'(y'))}.\end{aligned}$$

This completes the proof. □

Remark. The assumption of compactness of the system is crucial. Topological exactness along fibres may depend on the metric a non-compact system is endowed with.

Now, we would like to address the problem of defining a more general notion that we will call fibrewise topological exactness. We would like this notion to be expressed in terms of open sets only.

First, observe that topological exactness along fibres implies that for every open set U of Y and every $x\in X$ such that $U\cap Y_x\neq\emptyset$, there exists $N=N(U,x)\in\mathbb{Z}_+$ such that for every $n\geq N$

$$T^n(U\cap Y_x)=Y_{S^n(x)}.$$

However, one can see that there is not necessarily any uniformity over the fibres. This leads us to the following definition.

Definition 1.35 *A fibred system $\mathcal{Y}=(Y,T,X,S,\pi)$ is called (uniformly) fibrewise topologically exact if there exists a base $\mathcal{B}$ for Y and for each member B of $\mathcal{B}$, there is $M=M(B)\in\mathbb{Z}_+$ such that for every $m\geq M$ and every $x\in X$ for which $B\cap Y_x\neq\emptyset$*

$$T^m(B\cap Y_x)=Y_{S^m(x)}.$$

Remarks.

(1) Fibrewise topological exactness of a system does not necessarily imply that the system is fibrewise surjective. Nor does it imply that the system is surjective. But it does imply, when the system is compact, that there exists $N \in \mathbb{N}$ such that $\mathcal{Y}$ is fibrewise surjective on $\cup_{n \geq N} S^n(X)$ and that the fibred systems $\mathcal{Y}^n = (Y, T^n, X, S^n, \pi)$ are fibrewise surjective for $n \geq N$. Hence, if S is surjective, then a fibrewise topologically exact compact system is surjective and fibrewise surjective.

(2) Note that the requirement that a base of the space has a given property is much weaker than requiring that all open sets share this property.

(3) It is easy to see that this definition reduces to the usual definition of topological exactness in the non-fibred case.

We now establish various relations between the two notions of 'fibred exactness' previously defined.

On the one hand, we show that fibrewise topological exactness is a stronger requirement than topological exactness along fibres for the class of compact systems.

Proposition 1.36 *A fibrewise topologically exact compact system is topologically exact along fibres.*

Proof. Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a fibrewise topologically exact compact system. Let also $\epsilon > 0$ and $y \in Y$. Since $\mathcal{Y}$ is fibrewise topologically exact, there exist an open neighbourhood U_y of y with $B_Y(y, \delta_y) \subseteq U_y \subseteq B_Y(y, \epsilon/2)$ (for some $0 < \delta_y \leq \epsilon/2$) and $M_y \in \mathbb{Z}_+$ such that for every $x \in X$ for which $U_y \cap Y_x \neq \emptyset$ and every $m \geq M_y$

$$T^m(U_y \cap Y_x) = Y_{S^m(x)}.$$

This implies that, for any point $\tilde{y} \in B_Y(y, \delta_y)$,

$$\emptyset \neq U_y \cap Y_{\pi(\tilde{y})} \subseteq B_Y(y, \epsilon/2) \cap Y_{\pi(\tilde{y})} \subseteq B_Y(\tilde{y}, \epsilon) \cap Y_{\pi(\tilde{y})}$$

and hence

$$T^m\Big(B_Y(\tilde{y}, \epsilon) \cap Y_{\pi(\tilde{y})}\Big) = Y_{S^m(\pi(\tilde{y}))}$$

for every point $\tilde{y} \in B_Y(y, \delta_y)$ and every $m \geq M_y$.

We carry on this procedure at each point $y \in Y$. As Y is compact, we can extract from the open cover $\{B_Y(y, \delta_y)\}_{y \in Y}$ a finite subcover $\{B_Y(y_i, \delta_{y_i})\}_{i=1}^k$. Choosing $M = \max_{1 \leq i \leq k} M_{y_i}$, we conclude that for every $y \in Y$ and every $m \geq M$

$$T^m\Big(B_Y(y, \epsilon) \cap Y_{\pi(y)}\Big) = Y_{S^m(\pi(y))}.$$

This proves that $\mathcal{Y}$ is topologically exact along fibres. $\square$

On the other hand, we prove that topological exactness along fibres is stronger an exigency than fibrewise topological exactness when π is an open map.

Proposition 1.37 *A system $\mathcal{Y} = (Y, T, X, S, \pi)$ that is topologically exact along fibres and for which π is an open map is fibrewise topologically exact.*

Proof. Let $\epsilon > 0$ and $y \in Y$. Then the set

$$\begin{aligned} U(y,\epsilon) &:= \pi^{-1}\left[\pi\Big(B_Y(y,\epsilon/2)\Big)\right] \cap B_Y(y,\epsilon) \\ &= \Big(\bigcup_{\tilde{y}\in B_Y(y,\epsilon/2)} Y_{\pi(\tilde{y})}\Big) \cap B_Y(y,\epsilon) \\ &= \bigcup_{\tilde{y}\in B_Y(y,\epsilon/2)} \Big(Y_{\pi(\tilde{y})} \cap B_Y(y,\epsilon)\Big) \\ &\supseteq \bigcup_{\tilde{y}\in B_Y(y,\epsilon/2)} \Big(B_Y(\tilde{y},\epsilon/2) \cap Y_{\pi(\tilde{y})}\Big) \end{aligned}$$

is an open neighbourhood of y, which is smaller than $B_Y(y,\epsilon)$ and contains an open ball of radius $\epsilon/2$ in each fibre that it intersects. Indeed, let $x \in X$ be such that $U(y,\epsilon) \cap Y_x \neq \emptyset$. Then $x \in \pi(B_Y(y,\epsilon/2))$. Now choose $y' \in B_Y(y,\epsilon/2)$ such that $\pi(y') = x$. Then

$$U(y,\epsilon) \cap Y_x \supseteq B_Y(y',\epsilon/2) \cap Y_{\pi(y')}.$$

The result follows directly from the definition of topological exactness along fibres. $\square$