Remarks on the Generator-Problem

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Für Euch, Kinder der Wissenschaft und der Weisheit, haben wir dieses geschrieben. Erforschet das Buch und suchet euch unsere Ansicht zusammen, die wir verstreut und an mehreren Orten dargetan haben; was euch an einem Orte verborgen bleibt, das haben wir an einem anderen offengelegt, damit es faßbar werde für eure Weisheit.

Heinrich Cornelius Agrippa von Nettesheim
De occulta philosophia, 3, 65

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Kaiserslautern, im Juli 2001

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A Guide to Notation

$\mathbb{N} := \{1, 2, \dots\}$
$\mathbb{N}_0 := \{0, 1, 2, \dots\}$
$\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}$
$\mathbb{Q} := \{p/q : \ p \in \mathbb{Z}, q \in \mathbb{N}\}$
$\mathbb{R}_+ := [0, \infty)$

α 6
α^- 11
$\alpha^\sharp$ 68
α_T^- 60
$\alpha_T^\mp$ 74
α_T^n 60

$\mathfrak{B}$ 1
$\mathfrak{B}_\alpha$ 6
$\mathfrak{B}(M)$ 3

$d_\eta(\alpha, \beta)$ 67
$d_\mu(\alpha, \beta)$ 10

$\mathcal{E}$ 6
$\mathbb{E}(f \mid \mathfrak{C})$ 7
ϵ 6
ϵ_X 60

$H_\mu(\alpha)$ 9
$H_\mu(\alpha \mid \beta)$ 9
$h_\mu(T)$ 12
$h_\mu(T, \alpha)$ 11
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k_T 17

$(\hat{M}, \hat{\mathfrak{M}}, \hat{\nu}, \hat{U})$ 39
μ 2
μ_α 6
μ_M 3, 7

$\mathcal{P}$ 7
$\mathcal{P}^I$ 49
$\mathcal{P}_X$ 60

$\mathcal{R}_\eta$ 67
$\mathcal{R}_\eta^\sharp$ 68

$\sigma(\alpha)$ 6

T_α 59
$T^{-i}\alpha^-$ 11
ϑ 6
ϑ_X 60

X_α 6
$(X, \mathfrak{B})$ 1
$(X, \mathfrak{B}, \mu, T)$ 3
$(X, \mathfrak{B}, \mu)$ 2
$\hat{x}_{\alpha,n}$ 30
$(X, T; Y, S; \pi)$ 65
$(\hat{X}, \hat{T}; \hat{Y}, \hat{S}; \hat{\pi})$ 65

$\mathfrak{Y}$ 65
$\hat{\mathfrak{Y}}$ 65

$\mathcal{Z}$ 10
$\mathcal{Z}_\eta$ 67

$\alpha_n \nearrow \alpha$ 9
$\bigvee_{j=1}^n \alpha_{i_j}$ 8
$\alpha_n \searrow \alpha$ 9
$\bigwedge_{j=1}^n \alpha_{i_j}$ 8
$\lfloor r \rfloor$ 44
$\leq$ 8
$\leq_\varepsilon$ 49
Δ 2
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$\biguplus$ 48

Introduction

The mathematical branch of measure-theoretical ergodic theory traces back to the studies of Birkhoff and von Neumann in the early 1930's (cf. [Bir31]). In its broadest interpretation, measure-theoretical ergodic theory is the study of the qualitative properties of actions on measure spaces. According to its historical origin in the kinetic theory of gases, (measure-theoretical) ergodic theory mainly deals with one generic type of problem, the so-called *isomorphism problem*. This problem can be described as the problem of deciding whether or when, respectively, two actions show the same behaviour. The usual way to tackle such a problem is to look for properties which two actions with the same behaviour either both have or both do not have. However, the isomorphism problem kept mathematicians busy for quite some time without any progress worth mentioning. A first major step forward in the isomorphism problem was the introduction of Kolmogorov's concept of entropy in 1958 (cf. [Kol58]). Unfortunately, this concept was not easy to handle. For this reason, the mathematical interest has focused on methods which allow to simplify the application of Kolmogorov's idea of entropy. The importance of such methods was again emphasised by Ornstein's proof in 1970 that entropy is a complete invariant for Bernoulli-shifts (cf. [Orn70]). The strongest method which has been found so far is based on a special measurable structure which is called a *generator*. However, in those days the existence and the nature of generators were yet an almost completely open problem (cf. [Rok67]), the so-called *generator-problem*. In the following two decades, there has been a lot of efforts and some proceedings in the generator-problem, e.g. [Kri70], [Smo71], [Den74], [Kow84], [Kow88], [Kam89], [Kow95], or [KW00]. But in the case of endomorphisms the generator-problem[1] is an open problem till this very day.

This thesis is concerned with the generator-problem for endomorphisms in classical dynamical systems as well as in relative dynamical systems. After briefly recalling the necessary terminology and relations from ergodic theory, we shall consider in part I of this thesis the generator-problem in the classical setup. More precisely, we shall turn our attention to the question of a construction of finite one-sided generators. The main result in this part is an explicit construction

[1] A detailed description of the generator-problem and the difficulties related to it in the context of endomorphisms is given in the introduction to part I on pages 15-18.

of a finite one-sided generator for aperiodic automorphisms with zero entropy. We also provide a sketch of a straightforward proof of Rokhlin's Lemma for endomorphisms which is an indispensable ingredient for direct coding (for the complete proof cf. [HS00]). Moreover, a simple example will demonstrate the complexity of the generator-problem for endomorphisms in the case of positive entropy. Afterwards, in part II we shall treat the generator-problem in the setup of fibred dynamical systems. For this purpose, we shall first introduce a suitable and generic concept of relative entropy which does not rely on the assumption, that at least one of the underlying dynamics is an automorphism. By these means, we will obtain an approach which is more powerful than all concepts invented before (cf. [AR66], [LW77], and [Kam89]). In particular, our concept includes both Rokhlin's classical concept of entropy and Abramov and Rokhlin's idea of mixed entropy. Based on the idea of relative entropy, we finally shall be concerned with the question of the use and the existence of relative generators. The main results in part II are the Abramov-Rokhlin formula for fibred dynamical systems and the existence of relative unilateral generators with finite entropy. The first mentioned result is a generalisation of the well-known Abramov-Rokhlin formula to relative dynamical systems which do not base on an invertible factor; the second constitutes a progress in the generator-problem for relative dynamics.

Chapter 0

Foundations

The purpose of this chapter is to give a short review of the basic mathematical ideas and notations that will be used throughout this thesis. For the reader's convenience, we start with a brief overview of the necessary definitions and relations of classical (measure-theoretical) ergodic theory in section 0.1. Readers not familiar with this type of material or readers more interested in these topics are referred to [Coh80], [Bil65], [CFS82], [DGS76], [Fri70], or [Wal82]. Subsequently, section 0.2 gives an introduction to the powerful concept of Lebesgue spaces. A detailed treatment of this subject can be found in [Rok66] and [Rud90]. Finally, we complete the chapter by a glimpse at entropy theory in section 0.3. An excellent introduction to this mathematical area is given by [Rok67] and [Par69].

0.1 Basic Ergodic Theory

Ergodic Theory was invented in order to have a mathematically satisfying model for kinetic theory of gases. This modeling was done in the language of measure theory. For this reason, we begin with some basic definitions and relations borrowed from measure theory. Based on this, we turn towards ergodic theory by presenting the measure-theoretical viewpoint of dynamics. Finally, we shall complete this section with some selected results in ergodic theory which will be needed later in this thesis.

For completeness' sake, we state some basic pieces of notations.

Definition 0.1.1. Let X be a set and let $\mathfrak{B}$ be a family of subsets of the set X, which is closed under complement and countable union and which contains the set X itself. $\mathfrak{B}$ is then called a *σ-algebra of X* and the elements of $\mathfrak{B}$ are called *$\mathfrak{B}$-measurable*.
Moreover, a pair $(X, \mathfrak{B})$ consisting of a set X and a σ-algebra $\mathfrak{B}$ of X is called a *measurable space*.

Definition 0.1.2. Let $(X, \mathfrak{B})$ and $(X', \mathfrak{B}')$ be two measurable spaces and let

$g : X \to X'$ be a map. Then we call g *measurable* if

$$g^{-1}(B') \in \mathfrak{B} \qquad \text{for all } B' \in \mathfrak{B}' .$$

To complete the usual setup in measure theory, we have yet to explain what we understand by a measure.

Definition 0.1.3. Let $(X, \mathfrak{B})$ be a measurable space and let $\mu : \mathfrak{B} \to [0, \infty]$ be a non-negative set function. If the function μ is countably additive, i.e. $\mu(\emptyset) = 0$ and, for any sequence $(F_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets with $F := \bigcup_{n\in\mathbb{N}} F_n$, the equality

$$\mu(F) = \sum_{n\in\mathbb{N}} \mu(F_n)$$

is valid, then μ is called a *measure on* $(X, \mathfrak{B})$. The triple $(X, \mathfrak{B}, \mu)$ is said to be a *measure space*.

Due to this definition, a measure μ need not be finite, e.g. the space $(\mathbb{R}, \mathfrak{B}, \lambda)$ consisting of the real line endowed with the Borel-σ-algebra $\mathfrak{B}$ and the Lebesgue measure λ is a measure space with an infinite measure. We will only work with the following class of measures.

Definition 0.1.4. Let $(X, \mathfrak{B}, \mu)$ be a measure space with $\mu(X) < \infty$. Then μ is called *finite* and $(X, \mathfrak{B}, \mu)$ is called *a measure space with finite measure*. In particular, if $\mu(X) = 1$, we say that μ is a *probability measure* and that $(X, \mathfrak{B}, \mu)$ is a *probability space*.

Since classical ergodic theory only deals with probability spaces, from now on we shall exclusively work with probability measures even though some of the following definitions and results are still valid for a wider class of measure spaces.

In ergodic theory, one usually works 'mod 0', i.e. one does not draw any distinction between sets which differ only by a null-set. More precisely, we use the following kind of identification.

Definition 0.1.5. Let A and B be two measurable sets. Then we say that A and B are (measure-theoretically) equal, if

$$\mu(A \triangle B) := \mu(A \setminus B) + \mu(B \setminus A) = 0 .$$

In particular, we say that a statement $\mathcal{S}$ holds *almost everywhere* (a.e.), *almost surely* (a.s.), respectively, if

$$F := \{x \in X : \mathcal{S} \text{ is false}\} \in \mathfrak{B} \qquad \text{and} \qquad \mu(F) = 0$$

holds.

As mentioned at the beginning of this section, classical ergodic theory can be viewed as a qualitative measure-theoretical modeling of kinetics of gas particles. For that purpose we have to introduce a so-called *transformation* $T : X \to X$ whose iteration will serve as a model for the physical dynamics.

Definition 0.1.6. Let $(X, \mathfrak{B}, \mu)$ and $(X', \mathfrak{B}', \mu')$ be two probability spaces and let $T : X \to X'$ be a measurable map. We then say T is *measure-preserving*, if the relation

$$\mu\left(T^{-1}(B')\right) = \mu'(B')$$

is satisfied for all $B' \in \mathfrak{B}'$.

Definition 0.1.7. Let $(X, \mathfrak{B}, \mu)$ be a probability space and let $T : X \to X$ be a measure-preserving map. Then T is called a *transformation* or an *endomorphism*. If T is even invertible and T^{-1} is also a transformation, then we say that T is an *automorphism*.

In classical ergodic theory, the majority of results refers to automorphisms even though a lot of these statements (sometimes under just a few additional assumptions) are valid for endomorphisms as well. As already mentioned in the introduction, we do not suppose in this thesis that a given endomorphism T is invertible or bi-measurable.

Definition 0.1.8. Let $(X, \mathfrak{B}, \mu)$ be a probability space and let T be an endomorphism. The quadruple $(X, \mathfrak{B}, \mu, T)$ is then called a *(measure-theoretical) dynamical system*.

The physical interpretation of a dynamical system is given by the following correspondence. The set X represents a system of gas particles x. Their movements per time unit are described by the transition $x \to Tx$. Then the measure can be interpreted as the probability for a certain configuration. Following this interpretation, we can ask when we should consider two dynamical system as 'being equal' (describing the same dynamics). A satisfying answer to this question is given by the following concept.

Definition 0.1.9. Let $(X, \mathfrak{B}, \mu)$ and $(X', \mathfrak{B}', \mu')$ be two probability spaces. Then we say that $(X, \mathfrak{B}, \mu)$ and $(X', \mathfrak{B}', \mu')$ are *isomorphic*, if there are two sets $M \in \mathfrak{B}$, $M' \in \mathfrak{B}'$ with $\mu(M) = \mu'(M') = 1$ and an invertible map $\Phi : M \to M'$ with the property that Φ and Φ^{-1} are measure-preserving with respect to the σ-algebras $\mathfrak{B}(M) := \{M \cap B \,|\, B \in \mathfrak{B}\}$ and $\mathfrak{B}'(M') := \{M' \cap B' \,|\, B' \in \mathfrak{B}'\}$ and the corresponding *restricted measures* $\mu_M := \mu(M \cap \cdot)$ and $\mu'_{M'} := \mu'(M' \cap \cdot)$.

Definition 0.1.10. Two dynamical systems $(X, \mathfrak{B}, \mu, T)$ and $(X', \mathfrak{B}', \mu', T')$ are *isomorphic*, if there are sets $M \in \mathfrak{B}$ and $M' \in \mathfrak{B}'$ and a map $\Phi : M \to M'$ such that

i) $T(M) \subseteq M$, $T'(M') \subseteq M'$;

ii) $(X, \mathfrak{B}, \mu)$ and $(X', \mathfrak{B}', \mu')$ are isomorphic in the sense of definition 0.1.9 with $M \subseteq X$, $M' \subseteq X'$, and $\Phi : M \to M'$;

iii) $\Phi \circ T = T' \circ \Phi$ on M.

One of the most important and fundamental problems in ergodic theory is to understand endomorphisms and try to decide when two of them are isomorphic. We complete this section with some selected basic properties of endomorphisms.

Theorem 0.1.11 (Poincaré's Recurrence Theorem). *Let $T : X \to X$ be an endomorphism of a probability space $(X, \mathfrak{B}, \mu)$ and let E be a measurable set with $\mu(E) > 0$. Then almost all points of E return infinitely often to E under positive iteration of T, i.e. there is a measurable set $F \subseteq E$ with $\mu(F) = \mu(E)$ and the property that for each $x \in F$ there is a sequence $n_1 < n_2 < n_3 < \cdots$ of non-negative integers such that $T^{n_i}x \in F$ for each $i \in \mathbb{N}$.*

Proof. See theorem 1.4 in [Wal82]. □

A second, much more stronger property is the indecomposability of an endomorphism. This means, if there was a set $B \in \mathfrak{B}$ with $T^{-1}B = B$, we could study T by studying the two simpler endomorphisms $T_{|B}$ and $T_{|X\setminus B}$ which we get by restricting T to B, to $X \setminus B$, respectively. If $0 < \mu(B) < 1$ this would simplify the study of T. So we could try to express every endomorphism in terms of indecomposable ones, which are called ergodic.

Definition 0.1.12. Let $(X, \mathfrak{B}, \mu, T)$ be a dynamical system. T is then called *ergodic* if the only sets $B \in \mathfrak{B}$ with $T^{-1}B = B$ satisfy $\mu(B) = 0$ or $\mu(B) = 1$.

There are several other ways of stating the ergodicity condition. Readers interested in equivalent characterisations of ergodicity are referred to [Wal82].
A very important consequence of ergodicity was discovered by G.D. Birkhoff in 1931 (cf. [Bir31]).

Theorem 0.1.13 (Birkhoff's Ergodic Theorem). *Let $(X, \mathfrak{B}, \mu, T)$ be a dynamical system and let $f : X \to \mathbb{R}$ be an integrable, real valued function $(f \in L^1(\mu))$, i.e. $\int_X |f|\, d\mu < \infty$. Then $(1/n)\sum_{i=0}^{n-1} f(T^i(x))$ converges a.e. to a function $f^\star \in L^1(\mu)$ with $f^\star \circ T = f^\star$ and $\int_X f^\star\, d\mu = \int_X f\, d\mu$.*

Proof. See pp. 29-45 in [Rud90], or [Bir31]. □

Finally, the following property of an endomorphism plays an important role for our purpose.

Definition 0.1.14. Let $(X, \mathfrak{B}, \mu)$ be a dynamical system. Then we say that T is *aperiodic* if

$$\mu(\{x \in X : \exists n \in \mathbb{N} :\, T^n x = x\}) = 0\,.$$

0.2 Lebesgue Spaces and Canonical Measures

In the late forties of the last century, V.A. Rokhlin developed the axiomatic theory of Lebesgue spaces, partitions and canonical systems of measures. In doing so, Rokhlin gave a stronger formulation than closely related theories invented before by Halmos and von Neumann in [HvN42]. We shall use the following section for recalling those parts of this theory which will be necessary for our purpose. A comprehensive introduction to Rokhlin's approach can be found in [Rok66]. Readers more interested in some application of this theory to ergodic theory are referred to [Rud90].

We start with a definition of Lebesgue spaces which is different from but equivalent to Rokhlin's original one.

Definition 0.2.1. A probability space $(X, \mathfrak{B}, \mu)$ is called a *Lebesgue space* if it is isomorphic to a probability space which is the disjoint union of a countable (or finite) number of points $\{x_1, x_2, \ldots\}$, each of which has positive measure $\{p_1, p_2, \ldots\}$, and the space $([0, s), \mathcal{L}([0, s)), \lambda)$ where $s = 1 - \sum_{i \geq 1} p_i$, $\mathcal{L}([0, s))$ is the σ-algebra of the Lebesgue measurable subsets of $[0, s]$, and λ is the Lebesgue measure.

In what follows we will only consider Lebesgue spaces. In order to get a feeling for this special kind of probability spaces we recall some of its features.

Proposition 0.2.2. *Every Lebesgue measure μ of a Lebesgue space $(X, \mathfrak{B}, \mu)$ is complete, that means all subsets of null-sets are measurable and have measure zero.*

Proof. This follows immediately from the definition and the fact that the spaces $([0, s), \mathcal{L}([0, s)), \lambda)$ are complete (cf. 1.5.2 in [Coh80]). □

A very useful property of Lebesgue spaces is the existence of a *basis*.

Theorem 0.2.3. *Let $(X, \mathfrak{B}, \mu)$ be a Lebesgue space. Then there is a basis of X, i.e. there exists a countable system $\{B_i\}_{i \in I} \subseteq \mathfrak{B}$ such that*

i) for any measurable set $A \subseteq X$ there is a set D in the σ-algebra $\sigma(\{B_i\}_{i \in I})$ generated by the sets $\{B_i\}_{i \in I}$ such that $A \subseteq D$ and $\mu(D \setminus A) = 0$;

ii) for any two points $x, y \in X$ there is an index $j \in I$ such that $x \in B_j, y \notin B_j$ or $x \notin B_j, y \in B_j$.

Proof. See №1 of §2 in [Rok66]. □

From definition 0.2.1 we deduce the following topological property.

Proposition 0.2.4. *Every Lebesgue space $(X, \mathfrak{B}, \mu)$ is a separable metrisable space.*

In the next proposition, we see that the property of being a Lebesgue space is dominant for all measurable subsets of positive measure.

Proposition 0.2.5. *Let $(X, \mathfrak{B}, \mu)$ be a Lebesgue space and let $A \in \mathfrak{B}$ be a measurable set of positive measure. Then $(A, \mathfrak{B}(A), \mu_A)$ is also a Lebesgue space.*

Proof. See №3 of §2 in [Rok66]. □

A further notion which will accompany us throughout the whole thesis is the notion of a *partition*.

Definition 0.2.6. Let $(X, \mathfrak{B}, \mu)$ be a Lebesgue space and let $\alpha = \{A_i\}_{i \in I}$ be a decomposition of X into pairwise disjoint subsets, where I is an index set. If the sets A_i are measurable we then say that α is a *partition of* X. The elements A_i are called *atoms*.

Unless mentioned otherwise, we use small Greek letters for denoting partitions of the space X.

Example 0.2.1. Among all partitions of X there are two partitions which will be used often enough to deserve special symbols. These are the *trivial partition* ϑ and the *point partition* ϵ where

$$\vartheta := \{\emptyset, X\} \qquad \text{and} \qquad \epsilon := \{\{x\} : x \in X\}\,.$$

Moreover, we denote the set of all finite partitions of X by $\mathcal{E}$.

Definition 0.2.7. Let α be a partition of X. Then a measurable set A is called an *α-set*, if A is a union of atoms of α.

With the aid of a partition α, we are able to construct a new Lebesgue space $(X_\alpha, \mathfrak{B}_\alpha, \mu_\alpha)$. For this, we first define X_α to be given by the partition α itself and construct a map $f_\alpha : X \to X_\alpha$ by sending $x \in X$ to the unique atom of α of which x is an element. The $\sigma-$algebra $\mathfrak{B}_\alpha$ and the measure μ_α are then defined by

$$\mathfrak{B}_\alpha := f_\alpha\left(\sigma(\alpha)\right) \qquad \text{and} \qquad \mu_\alpha := \mu \circ f_\alpha^{-1}\,,$$

where $\sigma(\alpha)$ denotes the $\sigma-$algebra generated by α.

Definition 0.2.8. Let $(X, \mathfrak{B}, \mu)$ be a Lebesgue space and let α be a partition of X, then $(X_\alpha, \mathfrak{B}_\alpha, \mu_\alpha)$ defined as above is called the *factor space of X with respect to α*.

Now by using the measure μ_α, it is possible to speak of almost all (a.a.) atoms of a partition α.

Factor spaces will often be used to simplify the work with more than one partition. The reason for this comes from the following proposition.

Proposition 0.2.9. *For each partition α of X, the corresponding factor space $(X_\alpha, \mathfrak{B}_\alpha, \mu_\alpha)$ is also a Lebesgue space.*

Proof. See №2 of §3 in [Rok66]. □

We close our brief excursion on Lebesgue spaces with the important theorem about the canonical measures which is based on an idea of A.N. Kolmogorov.

Theorem 0.2.10. *Let $(X, \mathfrak{B}, \mu)$ be a Lebesgue space and let α be a partition of X. Then for almost all $A \in \alpha$, there is a σ-algebra $\mathfrak{B}_A$ and a measure μ_A defined on $\mathfrak{B}_A$ such that $(A, \mathfrak{B}_A, \mu_A)$ is a Lebesgue space, and moreover we have that*

i) for any set $B \in \mathfrak{B}$, the set $B \cap A$ is an element of $\mathfrak{B}_A$;

ii) for any set $G \in \mathfrak{B}$, the function $\mu_\cdot(\cdot \cap G)$ is $\mathfrak{B}_\alpha$-measurable;

iii) for any $B \in \mathfrak{B}$, the measures μ_A satisfy the relation

$$\mu(B) = \int_{X_\alpha} \mu_A(A \cap B)\, d\mu_\alpha\,. \tag{0.1}$$

Furthermore, the spaces $(A, \mathfrak{B}_A, \mu_A)$ are uniquely (up to μ_α-null-sets) defined.

Proof. See №1 of §3 in [Rok66]. □

The system of measures so defined is called a *canonical system of measures with respect to α*, or shortly *canonical measures w.r.t. α*.

Remark 0.1. *i*) If $A \in \alpha$ is an atom of positive measure, then μ_A coincides with the measure $\mu(A)^{-1} \cdot \mu$ defined in definition 0.1.9.
ii) Let $f \in L^1(\mu)$, let $\mathfrak{C} \subseteq \mathfrak{B}$ be a sub-σ-algebra, and let us denote the conditional expectation of f with respect to $\mathfrak{C}$ by $\mathbb{E}(f \,|\, \mathfrak{C})$. Then, for μ_α-almost all A, we have that

$$\mu_A(A \cap B) = \mathbb{E}(1_B \,|\, \sigma(\alpha))_{|_A}\,.$$

0.3 Fundamentals of Entropy Theory

We shall use this section to recall the most basic definitions of Rokhlin's approach to entropy and a few selected relations which are fundamental for all further considerations. Readers not familiar with this theory are referred to Rokhlin's wonderful opus on entropy in [Rok67], or to [Wal82, Par69, DGS76].

First we start with a few remarks concerning partitions and sub-σ-algebras of $\mathfrak{B}$. For this, let us consider the set $\mathcal{P}$ of all partitions of X.

Definition 0.3.1. Let α and β be two elements of $\mathcal{P}$. If there is a measurable subset $X' \subseteq X$ such that $\mu(X') = 1$ and each atom of the induced partition $\beta \cap X'$ of X' can be represented as a union of atoms of the induced partition $\alpha \cap X'$ of X', then we say that α is *finer* than β. We write $\beta \leq \alpha$ and sometimes we also say that β is *coarser* than α. Furthermore, we write that $\alpha = \beta$ if $\alpha \leq \beta$ and $\beta \leq \alpha$.

It is easy to show that definition 0.3.1 introduces a half-ordering '$\leq$' on the set $\mathcal{P}$. Moreover, it immediately follows that two partitions which are equal in the sense of this half-ordering have the same atoms up to null-sets. We will soon see that the reverse statement is also true. Thus, we can say that this half-ordering is well adapted to our measure-theoretical point of view. This means we do not draw any distinction between '$\leq$' on $\mathcal{P}$ and the set inclusion '$\subseteq$' on the system of the corresponding σ-algebras.

Lemma 0.3.1. *Let α and β be two elements in $\mathcal{P}$. Then α is finer than β if and only if $\sigma(\beta)$ is a sub-σ-algebra of $\sigma(\alpha)$.*

Proof. A comparison of the half-ordering '$\leq$' and the set-inclusion '$\subseteq$' proves the assertion. □

Now let us consider a countable family $\{\alpha_i, i \in I \subseteq \mathbb{N}\} \subseteq \mathcal{P}$ of elements of $\mathcal{P}$.

Definition 0.3.2. Let $\{\alpha_i, i \in I \subseteq \mathbb{N}\}$ be a countable family of partitions of X. Then the partition

$$\bigvee_{j=1}^{n} \alpha_{i_j} := \left\{ \bigcap_{j=1}^{n} A_{k_j} : A_{k_j} \in \alpha_{i_j} \right\}$$

is called the *(finite) refinement* of the partitions $\alpha_{i_1}, \ldots, \alpha_{i_n}$. Moreover, the partition $\bigvee_{i \in I} \alpha_i$ is then defined as the coarsest partition which is finer than any finite refinement $\bigvee_{j=1}^{n} \alpha_{i_j}$.

Example 0.3.1. Let α and β be two partitions of X. Then the refinement $\alpha \vee \beta$ is obviously given by

$$\alpha \vee \beta = \{A \cap B; A \in \alpha, B \in \beta\} \ .$$

The next definition explains what we mean by 'coarsening partitions'.

Definition 0.3.3. Let $\{\alpha_i, i \in I \subseteq \mathbb{N}\}$ be a countable family of partitions. Then the *enlargement* $\bigwedge_{j=1}^{n} \alpha_{i_j}$ of the partitions $\alpha_{i_1}, \ldots, \alpha_{i_n}$ is defined as the finest partition α such that

$$\alpha \leq \alpha_{i_j} \text{ for all } j \in \{1, \ldots, n\} .$$

Moreover, the enlargement $\bigwedge_{i \in I} \alpha_i$ of the family $\{\alpha_i, i \in I \subseteq \mathbb{N}\}$ is then defined as the finest partition α such that

$$\alpha \leq \alpha_i \text{ for all } i .$$

According to definition 0.3.2, for a sequence $(\alpha_n)_{n\in\mathbb{N}}$ of partitions in $\mathcal{P}$, we write $\alpha_n \nearrow \alpha$ in order to express that $\alpha_n \leq \alpha_{n+1}$ and that α is the coarsest partition finer than all partitions α_n. In correspondence to definition 0.3.3, we use $\alpha_n \searrow \alpha$ in an analogous fashion.

We are now in the position to recall Rokhlin's definition of entropy in four steps.

Definition 0.3.4. Let α be a partition of X and let $\{A_i, i \in I \subseteq \mathbb{N}\}$ be its atoms of positive measure. Then the *entropy* $H_\mu(\alpha)$ of the partition α is defined as

$$H_\mu(\alpha) := \begin{cases} -\sum_i \mu(A_i)\log\mu(A_i) & \text{if } \mu(X \setminus \bigcup_i A_i) = 0, \\ +\infty & \text{if } \mu(X \setminus \bigcup_i A_i) > 0\,. \end{cases}$$

It is obvious that $H_\mu(\cdot)$ is a measurable, non-negative function defined on $\mathcal{P}$ which can be infinite.

Now let β be another partition of X. According to theorem 0.2.10, we then know that for μ-a.a. atoms B the partition α induces a $\mathfrak{B}_B$ measurable partition by $\alpha \cap B$. Therefore we can transfer definition 0.3.4 to the Lebesgue spaces $(B, \mathfrak{B}_B, \mu_B)$ by setting

$$H_{\mu_B}(\alpha \cap B) := \begin{cases} -\sum_i \mu_B(A_i \cap B)\log\mu_B(A_i \cap B) & \text{if } \mu_B(B \setminus \bigcup_i A_i) = 0, \\ +\infty & \text{if } \mu_B(B \setminus \bigcup_i A_i) > 0\,. \end{cases} \tag{0.2}$$

Due to theorem 0.2.10, $H_{\mu_\cdot}(\alpha \cap \cdot)$ is a measurable, non-negative function defined on X_β.

Definition 0.3.5. Let α and β be two partitions of X. Then the integral

$$H_\mu(\alpha \,|\, \beta) := \int_{X_\beta} H_{\mu_B}(\alpha \cap B)\; d\mu_\beta$$

is called *the conditional entropy of α with respect to β.*

The following proposition presents several basic properties of the entropy of a partition and the conditional entropy. Readers interested in more special features are referred to [Rok67] and [Par69].

Proposition 0.3.6. *Let α, β and γ be three elements in $\mathcal{P}$. Then the following statements are valid.*

i) $H_\mu(\alpha) = 0$ *if and only if* $\alpha = \vartheta$;

ii) if α *is finer than* β, *then* $H_\mu(\beta) \leq H_\mu(\alpha)$;

iii) if $\alpha = \{A_1, \ldots, A_n\}$, *then* $H_\mu(\alpha) \leq \log n$;

iv) $H_\mu(\alpha \vee \beta) \leq H_\mu(\alpha) + H_\mu(\beta)$;

v) *if* β *is finer than* γ, *then* $H_\mu(\alpha \,|\, \beta) \leq H_\mu(\alpha \,|\, \gamma)$;

vi) $H_\mu(\alpha \vee \beta \,|\, \gamma) = H_\mu(\alpha \,|\, \gamma) + H_\mu(\beta \,|\, \alpha \vee \gamma)$;

vii) *if* $\alpha_n \nearrow \alpha$, *then* $H_\mu(\alpha_n \,|\, \beta) \nearrow H_\mu(\alpha \,|\, \beta)$;

viii) *if* $\alpha_n \searrow \alpha$ *and* $H_\mu(\alpha_1 \,|\, \beta) < \infty$, *then* $H_\mu(\alpha_n \,|\, \beta) \searrow H_\mu(\alpha \,|\, \beta)$;

ix) *if* $\alpha_n \nearrow \alpha$ *and* $H_\mu(\beta \,|\, \alpha_1) < \infty$, *then* $H_\mu(\beta \,|\, \alpha_n) \searrow H_\mu(\beta \,|\, \alpha)$.

Proof. See §4 & §5 in [Rok67]. □

Now, with the aid of conditional entropy, we can show that the half-ordering is well adapted to the ergodic-theoretical viewpoint.

Lemma 0.3.2. *Let* α *and* β *be two partitions of* X. *Then* $\alpha \leq \beta$ *if and only if the relation* $\mu_B(A \cap B) \in \{0, 1\}$ *holds for* μ_α*-a.a.* $A \in \alpha$ *and* μ_β*-a.a.* $B \in \beta$.

Proof. In order to prove lemma 0.3.2 we show the following two statements.

i) $H_\mu(\alpha \,|\, \beta) = 0$ if and only if for μ_α-a.a. $A \in \alpha$ and μ_β-a.a. $B \in \beta$ the relation $\mu_B(B \cap A) \in \{0, 1\}$ holds ;

ii) $H_\mu(\alpha \,|\, \beta) = 0$ if and only if $\alpha \leq \beta$.

We start with the proof of assertion i). For this, we consider the equality $H_\mu(\alpha \,|\, \beta) = 0$. Due to definition 0.3.5, it is clear that $H_\mu(\alpha \,|\, \beta) = 0$ is equivalent to $H_{\mu_B}(\alpha \cap B) = 0$ μ_β-a.s. According to (0.2), these equations are equivalent to the statement that, for μ_α-a.a. $A \in \alpha$ and μ_β-a.a. $B \in \beta$, the condition $\mu_B(A \cap B) \in \{0, 1\}$ is satisfied.

It remains to show assertion ii). First, let us suppose that $\alpha \leq \beta$. Then, due to definition 0.3.1, there is a measurable set X' such that $\mu_B(B \cap X') = 1$ μ_β-a.s. and each atom of $\alpha \cap X'$ is a $\beta \cap X'$-set. Hence, for μ_α-a.a. $A \in \alpha$ and μ_β-a.a. $B \in \beta$, we have $\mu_B(A \cap B) \in \{0, 1\}$. But this directly implies $H_\mu(\alpha \,|\, \beta) = 0$. Secondly, let us suppose that the condition $H_\mu(\alpha \,|\, \beta) = 0$ is fulfilled. Then, for μ_β-a.a. $B \in \beta$, the partition $\alpha \cap B$ has at most countably many atoms. Thus, according to the Cross-Section theorem (cf. theorem 1.2.1), there is a partition γ with at most countably many atoms and the property that $\alpha \vee \beta = \gamma \vee \beta$. Moreover, since $H_\mu(\alpha \,|\, \beta) = 0$, the entropy of γ is equal to zero. Consequently, due to proposition 0.3.6 i), we obtain that $\gamma = \vartheta$ which directly implies that $\beta = \beta \vee \gamma = \beta \vee \alpha \geq \alpha$. □

Let us consider the set $\mathcal{Z} \subseteq \mathcal{P}$ of all partitions with finite entropy. On the product space $\mathcal{Z} \times \mathcal{Z}$ we then define a non-negative function d_μ by

$$d_\mu(\alpha, \beta) := H_\mu(\alpha \,|\, \beta) + H_\mu(\beta \,|\, \alpha) .$$

Lemma 0.3.3. *The function d_μ is a metric on $\mathcal{Z}$.*

Proof. This follows directly from proposition 0.3.6. □

Furthermore, $\mathcal{Z}$ has the following topological features.

Proposition 0.3.7. *The set of finite partitions is dense in $\mathcal{Z}$.*

Proof. See 6.2 in [Rok67]. □

Proposition 0.3.8. *Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of partitions with $\alpha_n \nearrow \epsilon$. Then the set of all partitions $\beta \in \mathcal{Z}$ with $\beta \leq \alpha_n$ for at least one $n \in \mathbb{N}$ is dense in $\mathcal{Z}$.*

Proof. See 6.3 in [Rok67]. □

These two features lead to the following topological structure on $\mathcal{Z}$.

Theorem 0.3.9. *The set $\mathcal{Z}$ is a separable and complete metric space with respect to the metric d_μ.*

Proof. See 6.1-6.3 in [Rok67]. □

An easy calculation provides both the continuity of $H_\mu(\cdot)$ on $\mathcal{Z}$ and the continuity of $H_\mu(\cdot\,|\,\cdot)$ on $\mathcal{Z}\times\mathcal{Z}$. More precisely, the following relations are valid.

Proposition 0.3.10. *Let α, β and γ be partitions in $\mathcal{Z}$. Then we have that*

$$|H_\mu(\alpha\,|\,\beta) - H_\mu(\gamma\,|\,\beta)| \leq d_\mu(\alpha,\gamma) \quad \text{and} \quad |H_\mu(\alpha\,|\,\gamma) - H_\mu(\alpha\,|\,\beta)| \leq d_\mu(\gamma,\beta)\ .$$

Proof. See 6.5 in [Rok67]. □

Now, let $T : X \to X$ be an endomorphism on X and let α be an element of $\mathcal{P}$. It is clear that, for each $j \in \mathbb{N}_0$, the family of sets $T^{-j}\alpha$ is also an element of $\mathcal{P}$.

Proposition 0.3.11. *Let $(X,\ \mathfrak{B},\ \mu,\ T)$ be a dynamical system and let α be an element of $\mathcal{P}$. Then, for each $j \in \mathbb{N}_0$ the equality $H_\mu(T^{-j}\alpha) = H_\mu(\alpha)$ holds.*

Proof. This follows directly from the fact that T is measure-preserving. □

Definition 0.3.12. Let $(X,\ \mathfrak{B},\ \mu,\ T)$ be a dynamical system and α be a partition of X. Then *the entropy of T with respect to α* is defined by

$$h_\mu(T,\alpha) := H_\mu\left(\alpha \,\middle|\, \bigvee_{j=1}^{\infty} T^{-j}\alpha\right)\ .$$

In the following, we usually denote the partition $\bigvee_{j=0}^{\infty} T^{-j}\alpha$ by α^-, and, for any $i \in \mathbb{N}$ the partition $\bigvee_{j=i}^{\infty} T^{-j}\alpha$ by $T^{-i}\alpha^-$. The next proposition states some properties of the function $h_\mu(T,\cdot)$.

Proposition 0.3.13. *Let $(X,\, \mathfrak{B},\, \mu,\, T)$ be a dynamical system and let α and β be two elements of $\mathcal{P}$. Then*

i) $h_\mu(T,\alpha) = 0$ *if and only if* $\alpha \leq \bigvee_{j=1}^{\infty} T^{-j}\alpha$;

ii) if $\beta \leq \alpha^-$ *and* $\alpha \in \mathcal{Z}$, *then* $h_\mu(T,\beta) \leq h_\mu(T,\alpha)$;

iii) the function $h_\mu(T,\cdot)$ *is continuous on* $\mathcal{Z}$. *More precisely, we have that*

$$|h_\mu(T,\alpha) - h_\mu(T,\beta)| \leq d_\mu(\alpha,\beta) \ ;$$

iv) if $\alpha \in \mathcal{Z}$, *then* $h_\mu(T,\alpha) = \lim_{n\to\infty} 1/n\ H_\mu\left(\bigvee_{j=0}^{n-1} T^{-j}\alpha\right)$.

Proof. See §7 & §8 in [Rok67]. □

Finally, we can recall Rokhlin's definition for the entropy of an endomorphism.

Definition 0.3.14. Let $(X,\, \mathfrak{B},\, \mu,\, T)$ be a dynamical system. We call

$$h_\mu(T) := \sup\{h_\mu(T,\alpha)\,, \alpha \in \mathcal{P}\}$$

the entropy of T.

We complete this section with some features of the entropy of T.

Proposition 0.3.15. *Let $(X,\, \mathfrak{B},\, \mu,\, T)$ be a dynamical system. Then the following identities are satisfied*

i) $h_\mu(T) = \sup\{h_\mu(T,\alpha)\,, T^{-1}\alpha \leq \alpha\}$;

ii) $h_\mu(T) = \sup\{h_\mu(T,\alpha)\,, \alpha \in \mathcal{Z}\}$.

Proof. See 7.1 and 9.1 in [Rok67]. □

Theorem 0.3.16. *Let $(X,\, \mathfrak{B},\, \mu,\, T)$ and $(X',\, \mathfrak{B}',\, \mu',\, T')$ be two isomorphic dynamical systems. Then the entropies of T and T' are equal.*

Proof. See 9.2 in [Rok67]. □

Proposition 0.3.17. *Let $(M, \mathfrak{M}, \nu, U)$ be the direct product of two dynamical systems $(X,\, \mathfrak{B},\, \mu,\, T)$ and $(X',\, \mathfrak{B}',\, \mu',\, T')$. Then the entropy of U satisfies the relation*

$$h_\nu(U) = h_\mu(T) + h_{\mu'}(T')\,.$$

Proof. See theorem 4.23 in [Wal82]. □

Part I

The Generator-Problem for Non-Invertible Maps

In 1958 A.N. Kolmogorov introduced the notion of entropy to ergodic theory (cf. [Kol58]). With this concept, he made a first major step forward in the isomorphism problem, on which there had been no progress for quite some time. As a first application of entropy (cf. [Kol59]), Kolmogorov proved that the $(1/2, 1/2)$-shift and the $(1/3, 1/3, 1/3)$-shift are not isomorphic which had been an open problem for a long time. A few years later, V.A. Rokhlin extended Kolmogorov's classical concept to a powerful theory which became a lively field of mathematical research in the following thirty years. However, Rokhlin's approach has a blemish common with the original one, namely that it is often very difficult to calculate $h_\mu(T)$ following its definition. For this reason, methods which simplify the calculation of $h_\mu(T)$ have played an important role in the research on entropy theory. The main method for doing so is connected with the notion of a generator.

Definition I.1. Let $(X, \mathfrak{B}, \mu, T)$ be a dynamical system and let γ be a partition of X. Then γ is called a one-sided generator for T, if

$$\bigvee_{j=0}^{\infty} T^{-j}\gamma = \epsilon\,.$$

Moreover, if T is even an automorphism and $\bigvee_{j\in\mathbb{Z}} T^j\gamma = \epsilon$, then γ is called a two-sided generator for T.

Sinai proved in [Sin62] that, for automorphisms, this type of partitions can simplify the calculation of $h_\mu(T)$. Just a few years later, Rokhlin extended this so-called Kolmogorov-Sinai theorem to the following form.

Theorem I.1 (Kolmogorov-Sinai-Theorem). *Let $(X, \mathfrak{B}, \mu, T)$ be a dynamical system and let γ be a partition of X with finite entropy. If γ is a one-sided generator for an endomorphism or a two-sided generator for an automorphism, then the relation*

$$h_\mu(T) = h_\mu(T, \gamma) \tag{0.3}$$

holds.

Proof. See 9.4 in [Rok67] □

The Kolmogorov-Sinai theorem obviously simplifies the calculation of $h_\mu(T)$, but it also raises two new questions. Namely, what are the conditions under which generators exist and how generators can be found, be constructed, respectively. These two problems were called the *generator-problem*.

According to definition I.1, a necessary condition for the existence of generators with finite entropy clearly is that T must be aperiodic for continuous μ. Moreover, the existence of a one-sided generator with finite entropy implies that the inverse image of points is finite or at most countable.

A first sufficient condition for the existence of generators was already found by Rokhlin. He proved the following two statements.

Theorem I.2. *Let $(X, \mathfrak{B}, \mu, T)$ be a dynamical system endowed with an aperiodic automorphism T of finite entropy. Then there exists a two-sided generator γ with finite entropy.*

Proof. See 10.7 in [Rok67]. □

Theorem I.3. *Let $(X, \mathfrak{B}, \mu, T)$ be a dynamical system endowed with an aperiodic endomorphism T such that $h_\mu(T, \epsilon) = h_\mu(T) < \infty$. Then there exists a one-sided generator with finite entropy.*

Proof. See 10.11 in [Rok67] □

Rokhlin's results marked a first progress in the generator-problem. But neither are these theorems able to exactly answer the question how many atoms a generator has nor do their proofs give a way to construct generators. However, in order to simplify the calculation of $h_\mu(T, \gamma)$ and thereby also the calculation of $h_\mu(T)$, exactly these answers are essential.

The next proceedings in the generator-problem were made by W. Krieger's works in [Kri70] and [Kri71]. He was able to fix the number of atoms of two-sided generators.

Theorem I.4. *Let $(X, \mathfrak{B}, \mu, T)$ be a dynamical system endowed with an aperiodic automorphism T. Furthermore, let $h_\mu(T)$ be finite. Then every exhaustive σ-algebra with respect to T contains a two-sided generator with k atoms where* $\exp(h_\mu(T)) \leq k \leq \exp(h_\mu(T)) + 1$.

Proof. See [Kri70] and [Kri71]. □

Since Krieger's proofs strongly rely on the assumption that T is an automorphism, they do, however, not allow to state an analogous result in the non-invertible case. Moreover, Krieger's proofs do not include any instruction how this two-sided generator can be constructed. In 1971, the latter of these both problems was solved for the first time by M. Smorodinsky (cf. [Smo71]). He discovered a construction of a two-sided generator for ergodic and aperiodic automorphisms which is based on Ornstein's ideas in [Orn70]. Three years later, by his work in [Den74], M. Denker introduced a different, more powerful method to construct a two-sided generator for ergodic and aperiodic automorphisms. His idea also made it possible to acquire stronger statements about additional properties of the obtained generators, e.g. finitary generators or generators with adjustable marginals (cf. [DGS76] and [GK76]). Furthermore, Denker's construction could even be extended by Grillenberger (cf. [DGS76]) to all aperiodic automorphisms. With these works, the generator-problem was solved in a satisfying manner for automorphisms. However, since neither Ornstein's nor Denker's ideas do allow any conclusions for the non-invertible case, the generator-problem was still open for endomorphisms.

A first advance in the generator-problem for non-invertible endomorphisms was made in 1984. Z.S. Kowalski realised that a combination of Krieger's main theorem in [Kri71] and several of Rokhlin's results in [Rok67] provide the existence of finite one-sided generators in the ergodic, non-invertible case. More precisely, Kowalski proved the following theorem.

Theorem I.5. *Let $(X, \mathfrak{B}, \mu, T)$ be a dynamical system endowed with an ergodic and aperiodic endomorphism T. Then T has a finite one-sided generator if and only if $h_\mu(T) = h_\mu(T, \epsilon) < \infty$ and there is a positive integer n such that $\sharp\{T^{-1}(x)\} \leq n$ a.e.*

Proof. See [Kow84] or see [Kri71] and 10.11 in [Rok67]. □

During the next eleven years, Kowalski successively improved this theorem (cf. [Kow87, Kow88, Kow95]) to the following.

Theorem I.6. *Let $(X, \mathfrak{B}, \mu, T)$ be a dynamical system endowed with an aperiodic endomorphism T. Furthermore, let k_T be defined by*

$$k_T := \min\left\{k : \sharp\left\{y : y \in T^{-1}(x) \text{ and } \mu_{T^{-1}(x)}(y) > 0\right\} \leq k \quad \mu_{T^{-1}\epsilon}\text{-a.s.}\right\}.$$

Then T has a finite one-sided generator if and only if $H_\mu(\epsilon \mid T^{-1}\epsilon) = h_\mu(T) < \infty$ and $k_T < \infty$. Moreover, if T admits a finite one-sided generator, then there exists a one-sided generator γ such that $k_T \leq \sharp\gamma \leq k_T + 1$.

Proof. See [Kow95][1]. □

Kowalski's result completely answered the question about the existence of one-sided generators and their minimal cardinality. But his approach does not include any idea or hint how a one-sided generator can be constructed. In so far, the generator-problem for endomorphisms has only been partially solved. The construction of one-sided generators which, however, is the most interesting fact for any application of generators remains an open problem till this very day.

In this first part of the thesis, we shall be concerned with this still open problem. Fortunately, the basic idea of such a construction is canonical. As in the two-sided case, the existence of a finite one-sided generator $\gamma = \{C_0, C_1, \ldots, C_m\}$ implies, that the system $(X, \mathfrak{B}, \mu, T)$ can be embedded into the dynamical system $\left(\{0, \ldots, m\}^{\mathbb{N}_0}, \bigotimes_{i\in\mathbb{N}_0} 2^{\{0,\ldots,m\}}, \bigotimes_{i\in\mathbb{N}_0}(\mu(C_0), \ldots, \mu(C_m)), S\right)$, where S denotes the ordinary shift. Therefore, constructing a finite generator means to encode the elements of X with the aid of finitely many different symbols, such that the corresponding shift space is a sup-system of $(X, \mathfrak{B}, \mu, T)$. Unfortunately, the methods of coding used hitherto both are not able to deal with different points having the same image under T and do strongly rely on the forward measurability

[1] unfortunately, the only edition available to the author contained a slight gap (cf. proof of measurability) which can be removed with the aid of theorem 2.2.2

of T. Furthermore, the main auxiliary tool in all constructions of two-sided generators, Rokhlin's Lemma which plays the role of the embedding mapping, has only been given for aperiodic automorphisms (cf. №3 of §3 in [Rok67]). For these reasons, the known constructions are only applicable in the two-sided case. Thus, in order to find a construction which is also applicable for one-sided generators, one first has to get rid of these two substantial difficulties.

In chapter 1 we shall start with the problem of separating different points having the same image under T in a measurable way. We shall present a 'constructive' method to do this, which is based upon Rokhlin's Cross-Section-theorem. We complete chapter 1 by applying this method to two examples (cf. section 1.3). Subsequently, in chapter 2, we shall treat the problem of an appropriate encoding. For that, in section 2.1 we shall investigate in the question in how far the construction from chapter 1 is also able to serve for a beginning of a construction of one-sided generators. In particular, we will demonstrate the complexity of this question by means of treating two examples. Afterwards, in section 2.2, we shall put an end to the restriction of Rokhlin's Lemma to aperiodic automorphisms. Finally, we will complete chapter 2 with a new construction of one-sided generators for aperiodic automorphisms with zero entropy.

Chapter 1

The Cross-Section Theorem

The aim of this chapter is to provide the reader with a basic result in the advanced area of entropy theory, the so-called *Cross-Section theorem*. This theorem will also play an important role for the purpose of constructing finite generators for endomorphisms. We will see that it allows to construct a partition with the property that two different points which have the same image under the endomorphism T are in two different atoms of the partition.

Since the original proof given by Rokhlin (cf. [Rok63]) is only published in Russian and other proofs usually refer for some parts to the original one (cf. [Par69]) or, unfortunately, use inexact references, we shall provide a detailed treatment of Rokhlin's Cross-Section theorem. For this, we shall first be concerned with *one-sheeted sets* and *independent complements of partitions* in section 1.1. Subsequently, in section 1.2 we shall prove the Cross-Section theorem. Moreover, we will show in corollary 1.2.2 how it allows to distinguish between two different points which have the same image under T. Finally, in section 1.3 we shall apply corollary 1.2.2 to two examples.

1.1 One-Sheeted Sets

In this first section we treat the concepts of *one-sheeted sets with respect to a partition* and *independent complements of partitions*. We shall always work in the setting of Lebesgue spaces $(X, \mathfrak{B}, \mu)$.

Definition 1.1.1. Let α be a partition of X. Then we call a measurable set S of positive measure *one-sheeted with respect to* α, if the set S intersects each atom of α in at most one point.

Concerning one-sheeted sets, we are mainly interested in conditions for their existence and statements about their masses. The second problem is solved by the following lemma.

Lemma 1.1.1. *Let α be a partition of X such that there are one-sheeted sets. Then there exists a one-sheeted set of maximal measure.*

Proof. According to our assumption, there are one-sheeted sets with respect to α. Obviously, their masses are bounded from above. Thus, we find a smallest upper bound c. The idea of the proof is to construct inductively a sequence of one-sheeted sets E_n such that the limit set exists, that it differs from a one-sheeted set only by a null-set, and that its mass is equal to c. In order to construct the sets E_n, we first choose a sequence $\{D_n\}_{n\in\mathbb{N}}$ of one-sheeted sets with

$$\lim_{n\to\infty} \mu(D_n) = c\,.$$

We set $E_1 = D_1$. In order to define E_n for $n \geq 2$, we suppose that the set E_{n-1} is already defined. By Theorem 0.2.10, we then know that, for any $G \in \mathfrak{B}$, the map $\mu_\cdot(\cdot \cap G) : X_\alpha \to [0,\,1]$ is $\mathfrak{B}_\alpha$-measurable. Thus, we obtain that the set

$$X_{n-1} := \big\{A \in X_\alpha : \mu_A(A \cap D_n) > \mu_A(A \cap E_{n-1})\big\}.$$

is $\mathfrak{B}_\alpha$-measurable. Consequently, the set $F_{n-1} := f_\alpha^{-1}(X_{n-1})$ is $\mathfrak{B}$-measurable. With the aid of this set, we then define

$$E_n := ((X \setminus F_{n-1}) \cap E_{n-1}) \cup (F_{n-1} \cap D_n)\,.$$

Obviously, E_n is a one-sheeted set. Furthermore, this construction gives that

$$A \cap E_n = A \cap E_{n-1} \qquad \text{for } \mu_\alpha\text{-a.a. } A \in X_\alpha \setminus X_{n-1}$$

and that

$$A \cap E_n = A \cap D_n \qquad \text{for } \mu_\alpha\text{-a.a. } A \in X_{n-1}\,.$$

For μ_α-a.a. $A \in X_\alpha$, these relations then imply that

$$\mu_A(A \cap E_n) \geq \mu_A(A \cap E_{n-1})\,.$$

Thus, for μ_α-a.a. $A \in X_\alpha$, we get by $(A \cap E_n)_{n\in\mathbb{N}}$ a sequence of measurable sets, each consisting of at most one point such that $(\mu_A(A \cap E_n))_{n\in\mathbb{N}}$ is an increasing sequence in $[0, 1]$. Due to theorem 0.2.10, we also know that $\mu_A(\cdot)$ is a probability measure for μ_α-a.a. $A \in \alpha$. Hence, there are at most finitely many different elements $A \cap E_k$. But this means that for μ_α-a.a. A we find a positive integer $n(A) \in \mathbb{N}$ such that

$$A \cap E_n = A \cap E_{n(A)} \qquad \text{for all } n \geq n(A)\,.$$

Therefore, the limit set of the sequence $(E_n)_{n\in\mathbb{N}}$ exists and differs from the set

$$M := \bigcup_{A\in X_\alpha} A \cap E_{n(A)}$$

only by a null-set. Hence, the set M is measurable and one-sheeted with the property that $\mu(M) = \lim_{n\to\infty} \mu(E_n)$. It remains to show that the equality

$$\lim_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} \mu(D_n)$$

is also satisfied. For this, we have to consider the sets $X_\alpha \setminus X_{n-1}$ and X_{n-1} again. According to their definition, μ_α-a.a. elements $A \in X_\alpha \setminus X_{n-1}$ fulfil the relation

$$\mu_A(A \cap E_n) = \mu_A(A \cap E_{n-1}) \geq \mu_A(A \cap D_n)$$

and μ_α-a.a. elements $A \in X_{n-1}$ satisfy

$$\mu_A(A \cap E_n) = \mu_A(A \cap D_n)\,.$$

Consequently, we obtain that

$$\mu(E_n) = \int_{X_\alpha} \mu_A(A \cap E_n)\, d\mu_\alpha \geq \int_{X_\alpha} \mu_A(A \cap D_n)\, d\mu_\alpha = \mu(D_n)\,.$$

But this relation and the fact that M is one-sheeted finally imply that

$$c = \lim_{n\to\infty} \mu(D_n) \leq \lim_{n\to\infty} \mu(E_n) = \mu(M) \leq c\,.$$

□

In the remaining part of this section, we are concerned with the existence of one-sheeted sets.

Definition 1.1.2. Let α and β be two elements in $\mathcal{P}$ such that each measurable α-set A and each measurable β-set B satisfy the relation $\mu(A \cap B) = \mu(A)\mu(B)$. Then α and β are called *independent*. Moreover, if the partitions α and β also fulfil the condition $\alpha \vee \beta = \epsilon$, then α and β are called *independent complements of each other*.

Remark 1.1. Let α and β be partitions with at most countably many atoms $\{A_i\}_{i\in I\subseteq\mathbb{N}_0}$ and $\{B_j\}_{j\in I'\subseteq\mathbb{N}_0}$. Then the partitions α and β are independent if and only if $\mu(A_i \cap B_j) = \mu(A_i)\mu(B_j)$ holds for each pair $(i,j) \in I \times I'$.

The following proposition states that the existence of one-sheeted sets is connected with the existence of an independent complement.

Proposition 1.1.3. *Let α be a partition of X without one-sheeted sets. Then α has an independent complement.*

Proof. See №3 of §4, pages $45 - 51$ in [Rok66]. □

Apart from the relation between one-sheeted sets and independent complements, proposition 1.1.3 also includes a statement about the canonical measures w.r.t. α. More precisely, we have the following theorem.

Theorem 1.1.4. *Let $(X, \mathfrak{B}, \mu)$ be a Lebesgue space and let α be a partition of X. Then there does not exist a one-sheeted set with respect to α if and only if μ_α-almost all canonical measures μ_A are continuous.*

Proof. First, we suppose that μ_α-a.a. canonical measures μ_A of α are continuous. According to theorem 0.2.10, we know that for any measurable set C the canonical measures μ_A satisfy the relation

$$\mu(C) = \int_{X_\alpha} \mu_A(A \cap C)\, d\mu_\alpha\,.$$

Using this relation and the fact that μ_α-a.a. canonical measures are continuous, we have that for any set D which intersects each atom of α in at most one point

$$\mu(D) = \int_{X_\alpha} \mu_A(A \cap D)\, d\mu_\alpha = \int_{X_\alpha} 0\, d\mu_\alpha\,.$$

It remains to show that the reverse statement is also true. In order to do so, we suppose there is no one-sheeted set with respect to α. Due to proposition 1.1.3, we then know that α has an independent complement α^*. The independence of these two partitions implies that each α-set A and each α^*-set A^* fulfil the relation

$$\mu(A \cap A^*) = \mu(A)\mu(A^*) = \int_{f_\alpha A} \mu(A^*)\, d\mu_\alpha\,.$$

But, according to theorem 0.2.10, we also have that

$$\mu(A \cap A^*) = \int_{f_\alpha A} \mu_A(A \cap A^*)\, d\mu_\alpha\,.$$

Since each atom A^* of α^* intersects each atom A of α in at most one point, our assumption provides that the atoms A^* are null-sets. Consequently, the last two relations imply that for μ_α-a.a. atoms A and for μ_A-a.a. $x \in A$ the relation

$$\mu_A(\{x\}) = \mu_A(A \cap A^*(x)) = \mu(A^*(x)) = 0$$

holds. This means that μ_α-a.a. canonical measures μ_A are continuous. □

1.2 The Cross-Section Theorem

As already mentioned, Rokhlin's Cross-Section theorem is an important auxiliary tool for advanced topics in entropy theory and it will also play an important role for our purpose. Since the only complete proof of it is published in Russian (cf. [Rok63], we use this section to present a detailed proof of this theorem. Moreover, we shall deduce a method from Rokhlin's Cross-Section theorem which enables us to distinguish between different points having the same image under an endomorphism T.

Theorem 1.2.1 (Cross-Section theorem). *Let $(X, \mathfrak{B}, \mu)$ be a Lebesgue space and let α and β be two partitions, such that, for μ_β-a.a. $B \in \beta$, the partition $\alpha \cap B$ induced by α is μ_B-a.s. countable. Then there exists an at most countable partition γ with the property that*

$$\alpha \vee \beta = \gamma \vee \beta \,. \tag{1.1}$$

Moreover, if $H_\mu(\alpha \,|\, \beta)$ is finite, then the entropy of γ is also finite. In particular, in the case $H_\mu(\alpha \,|\, \beta) < 1$, the entropy of γ satisfies the relation

$$H_\mu(\gamma) \leq 6\sqrt{H_\mu(\alpha \,|\, \beta)} \,.$$

Proof. Before beginning with the proof, let us consider two simplifications. First, without loss of generality, we can reduce the given statement about $\alpha \vee \beta$ to an equivalent statement about α by assuming that $\beta \leq \alpha$. Second, by taking the factor space X_α as the basis for our considerations if necessary, we can also suppose that $\alpha = \epsilon$. We are starting with the proof in this new setup.

We have that μ_β-a.a. atoms $B \in \beta$ are given by a union of at most countably many different points $\{\{b_i\}\}_{i \in I \subseteq \mathbb{N}}$ and a μ_B-null-set B_0, say

$$B = \bigcup_{i \in I \subseteq \mathbb{N}} \{b_i\} \cup B_0 \,.$$

Moreover, without loss of generality, we may suppose that the numbering of the points $\{\{b_i\}\}_{i \in I \subseteq \mathbb{N}}$ is chosen such that

$$\mu_B(\{b_1\}) \geq \mu_B(\{b_2\}) \geq \mu_B(\{b_3\}) \geq \ldots \,.$$

Due to our assumptions, we know that the canonical measures μ_B are not continuous for μ_β-a.a. $B \in \beta$. Hence, according to Theorem 1.1.4, there are one-sheeted sets with respect to β. Applying the construction from Lemma 1.1.1, we find a measurable one-sheeted set C_1 of maximal mass with the property that

$$C_1 = \bigcup_{B \in \beta} \{b_1\} \,.$$

Furthermore, the set $X \setminus C_1$ and the partitions $\alpha \cap (X \setminus C_1)$ and $\beta \cap (X \setminus C_1)$ also satisfy the assumptions of the theorem. Therefore, repeating the above chain of arguments for the set $X \setminus C_1$, we find a one-sheeted set $C_2 \subseteq X \setminus C_1$ with respect to the $\beta \cap (X \setminus C_1)$ by setting

$$C_2 = \bigcup_{B \in \beta} \{b_2\} \,.$$

In this fashion, we inductively find pairwise disjoint measurable one-sheeted sets $\{C_i\}_{i\in I\subseteq\mathbb{N}}$ with respect to β, such that, for any $i \in I$ we have that

$$C_i = \bigcup_{B\in\beta} \{b_i\}$$

and the set C_i is one-sheeted with maximal measure in the set $X \setminus \bigcup_{k=1}^{i-1} C_k$. Thus,

$$\gamma = \{C_1, C_2, \ldots\}\,,$$

is a partition with the desired property that

$$\gamma \vee \beta = \epsilon\,.$$

It remains to calculate the entropy of γ. For this, we first define

$$\mu_\gamma^n(B) := \mu_B(B \cap C_n) \qquad \text{and} \qquad \mu_\gamma^n := \int_{X_\beta} \mu_\gamma^n(B)\, d\mu_\beta\,. \tag{1.2}$$

Then, for any $n \in \mathbb{N}$, we have that

$$\mu_\gamma^{n+1}(B) \le \mu_\gamma^n(B) \qquad \text{and} \qquad \mu_\gamma^n(B) \le 1/n\,.$$

Moreover, according to (0.1), we get that

$$\mu_\gamma^n = \int_{X_\beta} \mu_\gamma^n(B)\, d\mu_\beta = \int_{X_\beta} \mu_B(B \cap C_n)\, d\,\mu_\beta = \mu(C_n)\,.$$

Applying this relation to the entropy of γ, we then obtain that

$$H_\mu(\gamma) = -\sum_{n\in\mathbb{N}} \mu_\gamma^n \log \mu_\gamma^n\,.$$

In order to find an upper bound for $H_\mu(\gamma)$, we use the above representation of $H_\mu(\gamma)$ and split it into three terms

$$H_\mu(\gamma) = -\mu_\gamma^1 \log \mu_\gamma^1 + \left(-\sum_{\substack{k\ge 2\\ \mu_\gamma^k \ge k^{-s}}} \mu_\gamma^k \log \mu_\gamma^k \right) + \left(-\sum_{\substack{l\ge 2\\ \mu_\gamma^l < l^{-s}}} \mu_\gamma^l \log \mu_\gamma^l \right)\,, \tag{1.3}$$

where s is a positive integer greater than 2. Now, let us consider the conditional entropy $H_\mu(\alpha \,|\, \beta)$. With the aid of the definitions (1.2), we can write

$$\begin{aligned} H_\mu(\gamma \,|\, \beta) &= \int_{X_\beta} -\sum_{i\ge 1} \mu_B(B \cap C_i) \log \mu_B(B \cap C_i)\, d\mu_\beta \\ &= \int_{X_\beta} -\sum_{n\in\mathbb{N}} \mu_\gamma^n(B) \log \mu_\gamma^n(B)\, d\mu_\beta\,. \end{aligned}$$

Since $\mu_\gamma^n(B)$ is decreasing in $n \in \mathbb{N}$, we get that

$$\begin{aligned} H_\mu(\gamma \,|\, \beta) &\geq \int_{X_\beta} -\log \mu_\gamma^1(B) \sum_{n\in\mathbb{N}} \mu_\gamma^n(B)\, d\mu_\beta \\ &= \int_{X_\beta} -\log \mu_\gamma^1(B)\, d\mu_\beta \,. \end{aligned}$$

Hence, Jensen's inequality immediately gives an upper bound for the first term in (1.3), namely

$$H_\mu(\gamma \,|\, \beta) \geq -\log \int_{X_\beta} \mu_\gamma^1(B)\, d\mu_\beta = -\log \mu_\gamma^1 \geq -\mu(C_1) \log \mu(C_1) \,.$$

Turning towards the second term on the right hand side of (1.3), we easily see that it is bounded from above by

$$-\sum_{\substack{k\geq 2 \\ \mu_\gamma^k \geq k^{-s}}} \mu_\gamma^k \log \mu_\gamma^k \leq s \cdot \sum_{n\in\mathbb{N}} \log(n) \mu(C_n) \,.$$

But for the sum on the right hand side, we also find the following upper estimate

$$\begin{aligned} \sum_{n\geq 1} \log(n) \mu(C_n) &= \int_{X_\beta} \sum_{n\geq 1} \log(n) \mu_\gamma^n(B)\, d\mu_\beta \\ &\leq \int_{X_\beta} -\sum_{n\geq 1} \mu_\gamma^n(B) \log \mu_\gamma^n(B)\, d\mu_\beta \\ &= H_\mu(\gamma \,|\, \beta) \,. \end{aligned}$$

Thus, a bound from above for the second term in (1.3) is given by

$$-\sum_{\substack{k\geq 2 \\ \mu_\gamma^k \geq k^{-s}}} \mu_\gamma^k \log \mu_\gamma^k \leq s \cdot H_\mu(\gamma \,|\, \beta) \,.$$

Finally, we consider the atoms which are collected in the third term on the right hand side of (1.3). By the choice of s, they fulfil the relation

$$\mu(C_n) < n^{-s} \leq 1/2^s < 1/e \,.$$

Thus, together with the fact that the function $\varphi(x) = -x \log x$ is increasing on $(0,\, 1/e)$, we get an upper estimate for the third term by

$$\begin{aligned} -\sum_{\substack{l\geq 2 \\ \mu_\gamma^l < l^{-s}}} \mu_\gamma^l \log \mu_\gamma^l &\leq -\sum_{\substack{l\geq 2 \\ \mu_\gamma^l < l^{-s}}} s/n^s \log n \\ &\leq \int_1^\infty s t^{-s} \log t\, dt + s/2^s \log 2 \\ &\leq s \cdot \left[\left(t^{1-s} \log t\right)/(1-s) - \left(t^{1-s}\right)/(1-s)^2 \right]_1^2 + 1/s \\ &\leq s/(1-s)^2 + 1/s \leq 5/s \,. \end{aligned}$$

Consequently, applying the three estimates to (1.3), we obtain the inequality

$$H_\mu(\gamma) \leq (1+s)H_\mu(\gamma \,|\, \beta) + 5/s\,.$$

Since $H_\mu(\gamma \,|\, \beta) = H_\mu(\gamma \vee \beta \,|\, \beta) = H_\mu(\alpha \,|\, \beta)$, this is equivalent to

$$H_\mu(\gamma) \leq (1+s)H_\mu(\alpha \,|\, \beta) + 5/s\,.$$

In particular, for $H_\mu(\alpha \,|\, \beta) < 1$ and $s = \sqrt{5}/\sqrt{H_\mu(\alpha \,|\, \beta)}$, we have that

$$H_\mu(\gamma) \leq 2\sqrt{5}\sqrt{H_\mu(\alpha \,|\, \beta)} + \sqrt{H_\mu(\alpha \,|\, \beta)} \leq 6\sqrt{H_\mu(\alpha \,|\, \beta)}\,.$$

□

The following corollary presents a tool which enables us to separate different points which have the same image under T.

Corollary 1.2.2. *Let $(X, \mathfrak{B}, \mu, T)$ be a dynamical system with the property that*

$$k_T = \min\left\{k : \sharp\left\{y : y \in T^{-1}(x) \text{ and } \mu_{T^{-1}(x)}(y) > 0\right\} \leq k \quad \mu_{T^{-1}\epsilon}\text{-a.s.}\right\} < \infty\,.$$

Then there exists a finite partition $\beta^* = \left\{B_1^*, \ldots, B_{k_T}^*\right\}$ such that $\epsilon = \beta^* \vee T^{-1}\epsilon$.

Proof. The condition $k_T < \infty$ provides that, for $\mu_{T^{-1}\epsilon}$-a.a. atoms $T^{-1}x \in T^{-1}\epsilon$ the partition $\epsilon \cap T^{-1}x$ has $\mu_{T^{-1}\epsilon}$-a.s. k_T elements. According to theorem 1.2.1, we obtain a partition $\beta =: \beta^*$ with $\epsilon = \beta^* \vee T^{-1}\epsilon$. Moreover, due to the construction in the proof of theorem 1.2.1, the partition β^* has k_T atoms. □

1.3 Examples

In the last section we have seen how a finite partition can be constructed such that different points which have the same image are in different atoms of this partition. We shall use this section to apply this result to two simple examples.

Example 1.3.1. Let $(X, \mathfrak{B}, \mu, T)$ be the dynamical system consisting of the Lebesgue space $([0,1), \mathcal{L}([0,1)), \lambda)$ and the endomorphism $T(x) = 2x(\mathrm{mod}1)$.

Obviously, we have that $\sharp\{T^{-1}x\} = 2$ for all $x \in X$. Using this fact, we define a set function $\lambda_{T^{-1}x}$ on $\mathfrak{B}(T^{-1}x)$ by setting

$$\lambda_{T^{-1}x}(A) := \frac{1}{2} \cdot \sharp\{y \in A : T(y) = x\}\,. \qquad (*)$$

It is clear that each $\lambda_{T^{-1}x}$ is a measure on the corresponding atom $T^{-1}x \in T^{-1}\epsilon$. In the following we show that the family $\{\lambda_{T^{-1}x}\}$ is the canonical system of

measures w.r.t. $T^{-1}\epsilon$. For this, we first consider measurable sets $D^* \in \mathcal{L}([0,1))$ which fulfils the conditions

$$\lambda(D^*) = \frac{1}{2} \cdot \lambda(T(D^*)) \quad \text{and} \quad \lambda\left(\left\{x \in D^* : T^{-1}(T(x)) \subseteq D^*\right\}\right) = 0 \qquad (1.4)$$

(due to our choice of T, such sets D^* exist). Then we get that

$$\lambda(D^*) = \frac{1}{2} \cdot \lambda(T(D^*)) = \int_{X_{T^{-1}\epsilon}} \frac{1}{2} \cdot \lambda(T(D^*))\, d\mu_{T^{-1}\epsilon}\,.$$

Due to the fact that T is measure-preserving, we have that

$$\lambda(D^*) = \int_{X_{T^{-1}\epsilon}} \frac{1}{2} \cdot \lambda(T(D^*))\, d\mu_{T^{-1}\epsilon} = \int_{X_{T^{-1}\epsilon}} \lambda_{T^{-1}x}(T^{-1}x \cap D^*)\, d\mu_{T^{-1}\epsilon}\,.$$

Now, let D be any set in $\mathcal{L}([0,1))$. Then we can decompose D into two disjoint and measurable sets D_1 and D_2 such that each of them fulfil the conditions (1.4). Consequently, we obtain

$$\begin{aligned}\lambda(D) &= \lambda(D_1) + \lambda(D_2) \\ &= \int_{X_{T^{-1}\epsilon}} \lambda_{T^{-1}x}(T^{-1}x \cap D_1)\, d\mu_{T^{-1}\epsilon} + \int_{X_{T^{-1}\epsilon}} \lambda_{T^{-1}x}(T^{-1}x \cap D_2)\, d\mu_{T^{-1}\epsilon} \\ &= \int_{X_{T^{-1}\epsilon}} \lambda_{T^{-1}x}(T^{-1}x \cap D)\, d\mu_{T^{-1}\epsilon}\,.\end{aligned}$$

It only remains to show that the set functions $\lambda_{T^{-1}x}$ also satisfy condition ii) in theorem 0.2.10. But this is clear by the definition of $\lambda_{T^{-1}x}$ and the form of T.

Using the canonical measures $\{\lambda_{T^{-1}x}\}$, we then see that $k_T = 2$. Applying the construction of theorem 1.2.1, where

$$T^{-1}x = \{x_0, x_1\} \quad \text{and} \quad x_i = T^{-1}(x) \cap [i/2, (i+1)/2)\,,$$

we finally get that $\beta^* = \{[0,1/2), [1/2,1)\}$.

According to $(*)$ and definition 0.3.5, it is also clear that $H_\lambda(\epsilon \,|\, T^{-1}\epsilon) = \log 2$. ⧫

Our second example is a system which is very similar to the first one. In chapter 2, however, we shall see that the corresponding partitions β^* possess a completely different rank in the context of constructing one-sided generators.

Example 1.3.2. Let $(M, \mathfrak{M}, \nu, U)$ be the dynamical system which consists of the Lebesgue space $([0,1) \times (0,1], \mathcal{L}([0,1) \times (0,1]), \lambda^2)$ and the endomorphism $U : M \to M$ given by $U(x,y) = (2x \bmod 1, y)$.

Obviously, we have again that $\sharp\{U^{-1}(x,y)\} = 2$ for all $(x,y) \in X$. Then we define on $\mathfrak{M}(U^{-1}(x,y))$ the set function

$$\lambda_{U^{-1}(x,y)}(A \times B) := \frac{1}{2} \cdot \sharp\{(x',y') \in A \times B : U(x',y') = (x,y)\}\,. \qquad (**)$$

Using an analogous argumentation as in example 1.3.1, we obtain both that the family $\left\{\lambda_{U^{-1}(x,y)}\right\}$ is the canonical system of measures w.r.t. $U^{-1}\epsilon$ and that $k_T = 2$. With the aid of the following labelling

$$U^{-1}(x,y) = \{(x_0,y_0),(x_1,y_1)\}$$

where

$$(x_i,y_i) = U^{-1}(x,y) \cap \left([i/2,(i+1)/2) \times (0,1]\right),$$

we get that $\beta^* = \{[0,1/2) \times (0,1], [1/2,1) \times (0,1]\}$.

Due to $(**)$ and definition 0.3.5, we also obtain that $H_{\lambda^2}(\epsilon \,|\, U^{-1}\epsilon) = \log 2$ holds.

♦

Chapter 2

Finite Generators for Endomorphisms

In the following chapter we shall be concerned with the fundamental question about an appropriate encoding in order to construct a one-sided generator. For that purpose, in section 2.1 we shall first refer to the partitions β^* from corollary 1.2.2. In doing so, we will direct our attention to their rank inside a potential construction of one-sided generators. The complexity of this question and its importance in the context of the generator-problem will be presented by means of considering two examples at the end of section 2.1. Subsequently, in section 2.2 we shall prove a generalisation of Rokhlin's Lemma to non-invertible situations. Finally, in section 2.3 we shall present a construction of one-sided generators for aperiodic automorphisms with zero entropy, which directly leads to the generator, thus exceeding methods known so far (cf. [Orn70], [Den74] and [Kow95]).

2.1 Dissipative Points or How Good is β^*

As already mentioned one fundamental problem in the construction of finite one-sided generators is to deal with different points which possess the same image under T. In corollary 1.2.2, we have seen that a 'constructive' solution of this problem can be given by the partition β^*. We shall use the following section in order to investigate in the question if β^* also can serve as a preliminary version of a one-sided generator. More precisely, we will deal with the problem how strong β^* has to be refined for getting a one-sided generator.

For this purpose, let us consider a dynamical system $(X, \mathfrak{B}, \mu, T)$ which has a finite one-sided generator. According to theorem I.6 and corollary 1.2.2, we then find a finite partition $\beta^* = \left\{B_0^*, \ldots, B_{k_T-1}^*\right\}$ such that $\epsilon = \beta^* \vee T^{-1}\epsilon$. Moreover, we know that β^* is not unique. For simplicity, we will, however, speak of 'the' partition β^* throughout this section.

The following lemma shows that the partition β^* already solves the problem

on which the generator-problem was originally based.

Lemma 2.1.1. *The partition β^* has full entropy, i.e. $h_\mu(T) = h_\mu(T, \beta^*)$.*

Proof. According to proposition 0.3.6 and the fact that $\epsilon = \beta^* \vee T^{-1}\epsilon$, we get that

$$h_\mu(T) = H_\mu\left(\epsilon \,|\, T^{-1}\epsilon\right) = H_\mu\left(\beta^* \,|\, \bigvee_{j=1}^{n-1} T^{-j}\beta^* \vee T^{-n}\epsilon\right) \leq H_\mu\left(\beta^* \,|\, \bigvee_{j=1}^{n-1} T^{-j}\beta^*\right)$$

holds for each integer $n \geq 2$. Moreover, by the fact that $\left(H_\mu(\beta^* | \bigvee_{j=1}^{n-1} T^{-j}\beta^*)\right)_{n\geq 2}$ is a decreasing sequence which converges to $h_\mu(T, \beta^*)$, the relation above also implies that $h_\mu(T, \beta^*) = h_\mu(T)$. □

In order to gauge the role which the partition β^* plays in the context of the generator-problem, we shall investigate its capability of separating different points. For that, we will use the standard method of encoding a point $x \in X$ with the aid of its images $T^\ell x$ and a countable partition of X.

Definition 2.1.1. Let $(X, \mathfrak{B}, \mu, T)$ be a dynamical system and let $x \in X$ be a point of X. Furthermore, let $\alpha = \{A_i,\ i \in I \subseteq \mathbb{N}_0\}$ be a partition of X with at most countably many different atoms. Then we call the sequence

$$\hat{x}_{\alpha,\,n} := (i_0, \ldots, i_{n-1}) \in I^n\ , \qquad n \in \mathbb{N}$$

with the property that $T^k x \in A_{i_k}$ holds for each $0 \leq k \leq n-1$ the (α, n)-*name of x with respect to T*. Moreover, the sequence

$$\hat{x}_\alpha := (i_\ell)_{\ell \in \mathbb{N}_0}$$

with the property that $T^\ell x \in A_{i_\ell}$ holds for all $\ell \in \mathbb{N}_0$ is called the α-*name of x with respect to T*.

Remark 2.1. *i*) Let α be a partition of X with at most countably many different atoms and let $x \in X$. Then we denote the i-th coordinate of $\hat{x}_\alpha$ by $(\hat{x}_\alpha)_i$. Analogously, for $0 \leq j \leq n$ we denote the j-th coordinate of $\hat{x}_{\alpha,\,n}$ by $(\hat{x}_{\alpha,\,n})_j$.

ii) A partition α of X is a one-sided generator for T if and only if almost all points $x \in X$ have mutually different α-names.

The following definition provides a notion which allows us to measure the ability of a partition to separate different points.

Definition 2.1.2. Let α be a partition of X with at most countably many different atoms and let $B \subseteq X$ be a measurable set. Then we call $x \in B$ a *dissipative point of B with respect to α*, if there is a point $y \in B \setminus \{x\}$ such that $\hat{x}_\alpha = \hat{y}_\alpha$.

Due to definition 2.1.2, it is clear that a partition α separates different points of a measurable set A if and only if all points $x \in A$ fulfil the relation

$$\{u \in X : \ \hat{u}_\alpha = \hat{x}_\alpha\} \cap A = \{x\} \,. \tag{2.1}$$

A necessary, but weak condition for equation (2.1) is obviously given by

$$\mu(\{u \in X : \ \hat{u}_\alpha = \hat{x}_\alpha\} \cap A) = 0 \,. \tag{2.2}$$

With the aid of condition (2.2), we now extract those subsets of X on which β^* is not able to separate different points.

Lemma 2.1.2. *Let $\hat{z} := (z_n)_{n\in\mathbb{N}_0}$ be an element of $\{0, \ldots, \ k_T - 1\}^{\mathbb{N}_0}$ with the property that $\mu(\{x \in X : \ \hat{x}_{\beta^*} = \hat{z}\}) > 0$. Then the sequence $\hat{z}$ is periodic, i.e. there is an integer $p \in \mathbb{N}$ minimal with the property that for each $k \in \mathbb{N}_0$ the relation $z_k = z_{k \bmod p}$ is satisfied.*

Proof. Let us suppose that there is a non-periodic sequence $\hat{z}$ with the property that $\mu(\{x \in X : \ \hat{x}_{\beta^*} = \hat{z}\}) > 0$. We then find an integer $m \in \mathbb{N}$ such that $\mu(\{x \in X : \ \hat{x}_{\beta^*} = \hat{z}\}) > \frac{1}{m}$. Furthermore, we define the sets

$$F_0 := \bigcap_{j=0}^{\infty} T^{-j} B^*_{z_j} \qquad \text{and} \qquad F_k^{(i_0,\ldots,i_{k-1})} := \bigcap_{j=0}^{k-1} T^{-j} B^*_{i_j} \cap \bigcap_{j=k}^{\infty} T^{-j} B^*_{z_{j-k}} \tag{§}$$

where $1 \le k \le m-1$ and $(i_0, \ldots, i_{k-1}) \in \{0, \ldots, k_T - 1\}^k$. It is clear that $F_0 = \{x \in X : \ \hat{x}_{\beta^*} = \hat{z}\}$ as well as that $T^{-j}F_0 = \bigcup_{i_0,\ldots,i_{j-1}=0}^{k_T-1} F_j^{(i_0,\ldots,i_{j-1})}$ holds for each $1 \le j \le m-1$. Due to the definitions (§), we then have that the relation

$$F_0 \cap F_k^{(i_0,\ldots,i_{k-1})} \neq \emptyset$$

is satisfied for $1 \le k \le m-1$ if and only if

$$z_\ell = z_{\ell \bmod k} \,. \tag{$*$}$$

Moreover, we get that the relation

$$F_k^{(i_0,\ldots,i_{k-1})} \cap F_\ell^{(i'_0,\ldots,i'_{\ell-1})} \neq \emptyset$$

holds for $1 \le k \le \ell \le m-1$ if and only if

$$i_j = i'_j \ \text{ for } 0 \le j \le k \qquad \text{and} \qquad z_j = z_{j+(\ell-k)} \ \text{ for } j \ge 0 \,. \tag{$**$}$$

Thus, by our assumption that $\hat{z}$ is not periodic and by the relations $(*)$ and $(**)$, we obtain that the sets

$$\left\{F_0, F_k^{(i_0,\ldots,i_{k-1})} : \ 1 \le k \le m-1, (i_0, \ldots, i_{k-1}) \in \{0, \ldots, k_T - 1\}^k\right\}$$

are pairwise disjoint. Consequently, since T is measure-preserving, we have that

$$\mu\left(\bigcup_{\ell=0}^{m-1} T^{-\ell}F_0\right) = \sum_{\ell=0}^{m-1} \mu(T^{-\ell}F_0) = m \cdot \mu(F_0) > 1$$

which contradicts the fact that μ is a probability measure. □

Let us subdivide the set X into the measurable sets

$$P := \{x \in X : \hat{x}_{\beta^*} \text{ is periodic}\} \quad \text{and} \quad X \setminus P\,.$$

According to lemma 2.1.2 and the fact that β^* is finite, we know that β^* is not able to separate the points in P if the relation $\mu(P) > 0$ holds. In order to recognise how β^* has to be refined for having this property, we first have a look at the set P and suppose that $0 < \mu(P) < 1$.

Lemma 2.1.3. *The set $P \subseteq X$ is μ-a.s. completely invariant under the endomorphism T, i.e. $\mu(T^{-1}P \setminus P) = 0$.*

Proof. According to definition 2.1.1 and the fact that for each $x \in P$ the β^*-name of x is periodic, it immediately follows that $TP \subseteq P$. But this relation also implies that $P \subseteq T^{-1}P$. Since T is measure-preserving, we finally obtain that $\mu(T^{-1}P \setminus P) = 0$. □

Due to lemma 2.1.3, we obtain a decomposition of the system $(X,\ \mathfrak{B},\ \mu,\ T)$ into the following two dynamical systems

$$\big(P,\ \mathfrak{B}(P),\ \mu_P,\ T_{|P}\big) \quad \text{and} \quad \big(X \setminus P,\ \mathfrak{B}(X \setminus P),\ \mu_{X\setminus P}, T_{|X\setminus P}\big)$$

where $\mu_P := \mu(P)^{-1}\mu$, $\mu_{X\setminus P} := \mu(X\setminus P)^{-1}\mu$, and $T_{|P}$, $T_{|X\setminus P}$ denote the restriction of T to P, the restriction of T to $X \setminus P$, respectively.

Lemma 2.1.4. *The endomorphism $T_{|P}$ is even an aperiodic automorphism.*

Proof. Since P is totally invariant under T, the set of periodic points with respect to $T_{|P}$ is a subset of the set of periodic points w.r.t. T. Thus, the aperiodicity of T implies that $T_{|P}$ is aperiodic. Now, let x and y be two elements in P such that $T(x) = T(y)$. Then we have that the relation $(\hat{x}_{\beta^*})_j = (\hat{y}_{\beta^*})_j$ holds for all $j \in \mathbb{N}$. Since $\hat{x}_{\beta^*}$ is periodic with prime period p, we get that $\hat{y}_{\beta^*}$ also has to be periodic with prime period p. However this directly gives that the first coordinates $(\hat{x}_{\beta^*})_0$ and $(\hat{y}_{\beta^*})_0$ satisfy the relation $(\hat{x}_{\beta^*})_0 = (\hat{x}_{\beta^*})_p = (\hat{y}_{\beta^*})_p = (\hat{y}_{\beta^*})_0$. Hence, we have that $\hat{x}_{\beta^*} = \hat{y}_{\beta^*}$. Since β^* separates different points having the same image under T, we even obtain that $x = y$. Finally, the forward measurability of $T_{|P}$ follows from theorem 8.3.7 in [Coh80] and the fact that there are at most countably many different periodic β^*-names. □

The following lemmata show that the question how β^* has to be refined can be subdivided into two corresponding, but slightly different questions for the sets P and $X \setminus P$.

Lemma 2.1.5. *Let $(X, \mathfrak{B}, \mu)$ be a probability space and let $\mathfrak{C} \subseteq \mathfrak{B}$ be a sub-σ-algebra of $\mathfrak{B}$. Furthermore, let $A \in \mathfrak{C}$ be a $\mathfrak{C}$-measurable set such that $\mu(A) > 0$. Then, for any $f \in L^1(\mu)$ the relation*

$$\mathbb{E}\,(f \,|\, \mathfrak{C})_{|A} = \mathbb{E}\left(f_{|A} \,|\, \mathfrak{C}(A)\right) \qquad \mu_A\text{-a.s.}$$

holds.

Proof. Due to the fact that $\mu(A) > 0$, it is clear that the triple $(A, \mathfrak{B}(A), \mu_A)$ is also a probability space. Moreover, each function $g \in L^1(\mu)$ obviously satisfies the relation

$$\int_A |g_{|A}|\, d\mu_A = \mu(A)^{-1} \int_A |g|\, d\mu \leq \int_X |g|\, d\mu < \infty\,.$$

Hence, the restriction $g_{|A}$ is an element of $L^1(\mu_A)$ for all $g \in L^1(\mu)$ which means that the conditional expectation $\mathbb{E}\left(g_{|A} \,|\, \mathfrak{D}(A)\right)$ is well-defined for every sub-σ-algebra $\mathfrak{D}(A) \subseteq \mathfrak{B}(A)$. Now let us consider a function $f \in L^1(\mu)$ and a measurable set $D \in \mathfrak{C}(A)$. Since $\mathfrak{C}(A) \subseteq \mathfrak{C}$, we then have that

$$\begin{aligned}\int_D \mathbb{E}\left(f_{|A} \,|\, \mathfrak{C}(A)\right)\, d\mu_A &= \mu(A)^{-1} \int_D f\, d\mu \\ &= \mu(A)^{-1} \int_D \mathbb{E}\,(f \,|\, \mathfrak{C})\, d\mu \\ &= \int_D \mathbb{E}\,(f \,|\, \mathfrak{C})_{|A}\, d\mu_A\,.\end{aligned}$$

Consequently, according to the definition of the conditional expectation, we finally obtain that $\mathbb{E}\,(f \,|\, \mathfrak{C})_{|A} = \mathbb{E}\left(f_{|A} \,|\, \mathfrak{C}(A)\right)$ holds μ_A-a.s. □

Lemma 2.1.6. *Let $(X, \mathfrak{B}, \mu, T)$ be a dynamical system and let G be a measurable and completely invariant set. Then $h_\mu(T)$ satisfies the relation*

$$h_\mu(T) = \mu(X \setminus G) \cdot h_{\mu_{X\setminus G}}(T) + \mu(G) \cdot h_{\mu_G}(T).$$

Proof. If $G \in \{X, \emptyset\}$, it is obvious that the assertion is true. Now let us suppose that G satisfies the relation $0 < \mu(G) < 1$. In this case G clearly provides a non-trivial completely invariant partition $\zeta = \{Z, Z^c\} := \{G, X \setminus G\}$. Moreover, according to proposition 0.3.15, we know that the entropy of T can be calculated with the aid of invariant partitions $\eta = \{E_i\}_{i\in I}$ of X, namely by $h_\mu(T) = \sup\{H_\mu(\eta \,|\, T^{-1}\eta)\,,\ T^{-1}\eta \leq \eta\}$ where, due to lemma 2.1.5,

$$H_\mu\left(\eta \,|\, T^{-1}\eta\right) = \int_{X_{T^{-1}\eta}} \sum_{E\in\eta\cap T^{-1}E'} \mathbb{E}\left(1_E \,|\, \sigma(T^{-1}\eta)\right) \log \mathbb{E}\left(1_E \,|\, \sigma(T^{-1}\eta)\right)\, d\mu_{T^{-1}\eta}\,.$$

Let us consider an invariant partition α such that $\zeta \leq \alpha$. Furthermore let α_Z and α_{Z^c} denote the partition of G, of $X \setminus G$, respectively, induced by α. Since ζ is completely invariant, we have that $G \in \sigma(T^{-1}\alpha)$. Thus, $H_\mu(\alpha \,|\, T^{-1}\alpha)$ satisfies the relation

$$H_\mu\left(\alpha \,|\, T^{-1}\alpha\right) = \int_{Z^c_{T^{-1}\alpha}} \sum_{A\in\alpha\cap T^{-1}A'} \mathbb{E}\left(1_A \,|\, \sigma(T^{-1}\alpha)\right) \log \mathbb{E}\left(1_A \,|\, \sigma(T^{-1}\alpha)\right) \, d\mu_{T^{-1}\alpha} + \int_{Z_{T^{-1}\alpha}} \sum_{A\in\alpha\cap T^{-1}A'} \mathbb{E}\left(1_A \,|\, \sigma(T^{-1}\alpha)\right) \log \mathbb{E}\left(1_A \,|\, \sigma(T^{-1}\alpha)\right) \, d\mu_{T^{-1}\alpha}\,.$$

Due to lemma 2.1.5 and the invariance of ζ, the sum on the right hand side of the equation above is equal to

$$\mu(Z^c) \int_{Z^c_{T^{-1}\alpha_{Z^c}}} \sum_{A\in\alpha_{Z^c}\cap T^{-1}A'} \mathbb{E}\left(1_A \,|\, \sigma(T^{-1}\alpha_{Z^c})\right) \log \mathbb{E}\left(1_A \,|\, \sigma(T^{-1}\alpha_{Z^c})\right) \, d\nu_{T^{-1}\alpha_{Z^c}} + \mu(Z)\cdot \int_{Z_{T^{-1}\alpha_Z}} \sum_{A\in\alpha_Z\cap T^{-1}A'} \mathbb{E}\left(1_A \,|\, \sigma(T^{-1}\alpha_Z)\right) \log \mathbb{E}\left(1_A \,|\, \sigma(T^{-1}\alpha_Z)\right) \, d\nu'_{T^{-1}\alpha_Z}$$

where ν and ν' denote the measures $\mu_{Z^c} := \mu_{X\setminus E}$ and $\mu_Z := \mu_E$. Hence, we obtain that

$$H_\mu\left(\alpha \,|\, T^{-1}\alpha\right) = \mu(Z^c)\, H_{\mu_{Z^c}}(\alpha_{Z^c}|T^{-1}_{|Z^c}\alpha_{Z^c}) + \mu(Z)\, H_{\mu_Z}(\alpha_Z|T^{-1}_{|Z}\alpha_Z)\,.$$

According to proposition 0.3.15 and statement 8.8 in [Rok67], we also get that

$$h_\mu(T) = \sup_{\substack{\zeta\leq\eta \\ T^{-1}\eta\leq\eta}} \{H_\mu\left(\eta \,|\, T^{-1}\eta\right)\}$$

which finally implies that

$$h_\mu(T) = \mu(Z^c)\, h_{\mu_{Z^c}}(T_{|Z^c}) + \mu(Z)\, h_{\mu_Z}(T_{|Z})\,.$$

□

Lemma 2.1.7. *Let $(X, \mathfrak{B}, \mu, T)$ be a dynamical system with the property that $h_\mu(T) = H_\mu(\epsilon \,|\, T^{-1}\epsilon) < \infty$. Furthermore, let G be a measurable and completely invariant set with $\mu(G) > 0$. Then, we have that*

$$h_{\mu_G}(T_{|G}) = H_{\mu_G}(\epsilon\cap G \,|\, T^{-1}_{|G}(\epsilon\cap G))\,.$$

Proof. Due to proposition 0.3.15, we know that for each invariant set E the relation

$$h_{\mu_E}(T_{|E}) \geq H_{\mu_E}(\epsilon\cap E \,|\, T^{-1}_{|E}(\epsilon\cap E))$$

is satisfied. Moreover, according to lemma 2.1.6, we obtain that

$$\begin{aligned} H_\mu\left(\epsilon \,|\, T^{-1}\epsilon\right) &= \mu(Z^c)\, H_{\mu_{Z^c}}(\epsilon_{Z^c} \,|\, T^{-1}_{|Z^c}\epsilon_{Z^c}) + \mu(Z)\, H_{\mu_Z}(\epsilon_Z \,|\, T^{-1}_{|Z}\epsilon_Z) \\ &\leq \mu(Z^c)\, h_{\mu_{Z^c}}(T_{|Z^c}) + \mu(Z)\, h_{\mu_Z}(T_{|Z}) \\ &= h_\mu\left(T\right) \end{aligned}$$

which proves the assertion. □

Corollary 2.1.3. *The dynamical system* $\left(P, \mathfrak{B}(P), \mu_P, T_{|P}\right)$ *possesses zero entropy, i.e.* $h_{\mu_P}(T_{|P}) = 0$.

Proof. The hypothesis directly follows from lemma 2.1.4 and lemma 2.1.7. □

According to lemma 2.1.2, we see that the role of β^* inside a construction of finite one-sided generators can be splited into two different parts. One part is given by the rank of β^* in the construction of finite one-sided generators for aperiodic automorphism with zero entropy. We will treat this problem in section 2.3. The other part is given by the ability of β^* of separating points in the case that condition (2.2) is satisfied for each $x \in X$ with respect to β^*. In the following subsection, we will demonstrate the complexity of this question as well as its importance by means of two examples.

2.1.1 Examples

In the remaining part of this section we consider two examples which show that β^* can play completely different roles in the context of the (one-sided) generator-problem.

Example 2.1.1. Let us consider the dynamical system $(X,\, \mathfrak{B},\, \mu,\, T)$ given by the Lebesgue space $([0,1), \mathcal{L}([0,1)), \lambda)$ and the endomorphism $T(x) = 2x \bmod 1$. Due to example 1.3.1, we already know that the partition

$$\beta^* = \left\{ \left[0, \frac{1}{2}\right), \left[\frac{1}{2}, 1\right) \right\}$$

satisfies the relation $\epsilon = \beta^* \vee T^{-1}\epsilon$. In the following we will show both that $T(x) = 2x \bmod 1$ fulfils the conditions in theorem I.6 and that β^* is even a one-sided generator for T. For that, we start with some preparatory considerations.

Definition 2.1.4. Let $k \geq 2$ be a fixed integer and let $(p_0, \ldots, p_{k-1})$ be a probability vector with non-zero entries. Furthermore, let $(Y, 2^Y, \nu)$ denote the probability space where $Y = \{0, 1, \ldots, k-1\}$ and the point i has measure p_i. Then the dynamical system $(X,\, \mathfrak{B},\, \mu,\, T)$ consisting of the probability space $(X,\, \mathfrak{B},\, \mu) = \prod_{i \in \mathbb{N}_0}(Y, 2^Y, \nu)$ and the endomorphism $T\left((x_i)_{i\in\mathbb{N}_0}\right) = (y_i)_{i\in\mathbb{N}_0}$ where $y_n = x_{n+1}$ is called the *one-sided* $(p_0, \ldots, p_{k-1})$*-shift*.

Lemma 2.1.8. *The one-sided $(p_0, \ldots, p_{k-1})$-shift is ergodic.*

Proof. Let E be a measurable set such that $T^{-1}E = E$. Furthermore, let $\varepsilon > 0$ be given and let $\mathcal{A}$ denote the algebra of all finite unions of measurable rectangles. Then we find a set $A \in \mathcal{A}$ such that $\mu(E \triangle A) < \varepsilon$ holds. Moreover, we have that

$$|\mu(E) - \mu(A)| = |\mu(E \setminus A) - \mu(A \setminus E)| \leq \mu(E \triangle A) < \varepsilon \,.$$

Since $A \in \mathcal{A}$, we also find an integer $n_0 \in \mathbb{N}$ such that the set $B = T^{-n_0}A$ depends upon different coordinates from A on the first n_0 positions. By the fact that μ is a product measure, this implies that the sets A and B satisfy the relation

$$\mu(B \cap A) = \mu(B)\mu(A) = \mu(A)^2 \,.$$

Moreover, since $E \triangle (A \cap B) \subseteq (E \triangle A) \cup (E \triangle B)$ and T is measure-preserving, we get that

$$\mu(E \triangle (A \cap B)) \leq \mu(E \triangle A) + \mu(E \triangle B) = 2\mu(E \triangle A) < 2\varepsilon \,.$$

Thus, we finally obtain that the set E satisfies the relation

$$\begin{aligned} |\mu(E) - \mu(E)^2| &\leq |\mu(E) - \mu(A \cap B)| + |\mu(A \cap B) + \mu(E)^2| \\ &\leq \mu(E \triangle (A \cap B)) + |\mu(A)^2 - \mu(E)^2| \\ &< 2\varepsilon + \mu(A)|\mu(A) - \mu(E)| + \mu(E)|\mu(A) - \mu(E)| \\ &< 4\varepsilon \,. \end{aligned}$$

□

Proposition 2.1.5. *The dynamical system $([0,1), \mathcal{L}([0,1)), \lambda, 2x \bmod 1)$ is isomorphic to the one-sided $(\frac{1}{2}, \frac{1}{2})$-shift. Moreover, the partition $\beta^* = \{[0, \frac{1}{2}), [\frac{1}{2}, 1)\}$ is a one-sided generator for $T(x) = 2x \bmod 1$.*

Proof. Let us consider the partition $\beta^* = \{B_0 := [0, \frac{1}{2}), B_1 := [\frac{1}{2}, 1)\}$. It is clear that, for each $x \in [0,1)$ the β^*-name $\hat{x}_{\beta^*}$ is an element of $\{0,1\}^{\mathbb{N}_0}$. Using this relation, we define the map $\Phi' : [0,1) \to \{0,1\}^{\mathbb{N}_0}$ by

$$\Phi'(x) := \hat{x}_{\beta^*} \,.$$

Since T is expansive with constant $\frac{1}{2}$ and $diam(B_0) = diam(B_1) = \frac{1}{2}$, we get that, for every $x \in [0,1)$ the β^*-name $\hat{x}_{\beta^*}$ of x is unique. Thus, there are no dissipative points in $[0,1)$ with respect to β^*. This implies that the map Φ' is injective as well as that β^* is a one-sided generator for T. Now let us consider a rectangle $(i_0, \ldots, i_{n-1})$ of $\{0,1\}^{\mathbb{N}_0}$. Due to the definition of Φ' and definition 2.1.1, we have that

$$\Phi'^{-1}((i_0, \ldots, i_{n-1})) = \{x \in [0,1) : \ \hat{x}_{\beta^*, n} = (i_0, \ldots, i_{n-1})\} = \bigcap_{j=0}^{n-1} T^{-j} B_{i_j} \,.$$

Moreover, a simple calculation shows that, for each integer $\ell \in \mathbb{N}_0$ and every finite sequence $(k_0, \ldots, k_{\ell-1}) \in \{0,1\}^\ell$, the intersection $\bigcap_{j=0}^{\ell-1} T^{-j} B_{k_j}$ satisfies the relations

$$\bigcap_{j=0}^{\ell-1} T^{-j} B_{k_j} \neq \emptyset \qquad (*)$$

and

$$\lambda\left(\bigcap_{j=0}^{\ell-1} T^{-j} B_{k_j}\right) = 2^{-(\ell-1)} \lambda(B_{k_{\ell-1}}) = 2^\ell = \prod_{j=0}^{\ell-1} \lambda(B_{k_j})\,. \qquad (**)$$

Now let us consider $(x_i)_{i\in\mathbb{N}_0} \in \{0,1\}^{\mathbb{N}_0} \setminus P_{\bar{1}}$ where the set $P_{\bar{1}} \subseteq \{0,1\}^{\mathbb{N}_0}$ is defined by

$$P_{\bar{1}} := \left\{(x_i)_{i\in\mathbb{N}_0} \in \{0,1\}^{\mathbb{N}_0} : \exists\, n \in \mathbb{N}_0 \text{ such that } x_j = 1 \text{ for all } j \geq n\right\}\,.$$

It is clear that $\{0,1\}^{\mathbb{N}_0} \setminus P_{\bar{1}}$ is measurable and has full measure with respect to the product measure $\bigotimes_{i\in\mathbb{N}_0}(\frac{1}{2}, \frac{1}{2})$. Furthermore, by $(*)$ and the fact that $\mathfrak{B} = \mathcal{L}([0,1))$, we find a decreasing sequence $(K_j)_{j\in\mathbb{N}_0}$ of non-empty compact sets $K_j \subseteq [0,1)$ such that

$$\bigcap_{j=0}^{\ell} T^{-j} B_{x_j} \subseteq K_{\ell-1} \subseteq \bigcap_{j=0}^{\ell-1} T^{-j} B_{x_j}\,.$$

Hence, there is a point $x \in [0,1)$ such that

$$x \in \bigcap_{j\in\mathbb{N}_0} K_j\,.$$

But this implies that $\Phi : [0,1) \to \{0,1\}^{\mathbb{N}_0} \setminus P_{\bar{1}}$ defined by $\Phi(x) := \Phi'(x)$ is even bijective. Moreover, since $P_{\bar{1}}$ is a null-set, relation $(**)$ shows that Φ and Φ^{-1} are measure-preserving. Consequently, we obtain that Φ is an isomorphism. □

Remark 2.2. According to example 1.3.1 and theorem I.1, we have that

$$\log 2 = H_\lambda\left(\epsilon \,|\, T^{-1}\epsilon\right) \leq h_\lambda(T) = h_\lambda\left(T, \beta^*\right) \leq H_\lambda\left(\beta^*\right) \leq \log 2\,.$$

Thus, the entropy of $T(x) = 2x \bmod 1$ satisfies the relation

$$h_\lambda(T) = H_\lambda\left(\epsilon \,|\, T^{-1}\epsilon\right) = \log 2\,.$$

♦

Example 2.1.2. In our second example, we shall consider the dynamical system $(M, \mathfrak{M}, \nu, U)$ which is given by

$$\left([0,1) \times (0,1], \mathcal{L}\left([0,1) \times (0,1]\right), \lambda^2, (2x \bmod 1, y)\right)\,.$$

Since $(M, \mathfrak{M}, \nu, U)$ is the direct product of the dynamical systems $(X,\ \mathfrak{B},\ \mu,\ T) = ([0,1), \mathcal{L}([0,1)), \lambda, 2x \bmod 1)$ and $(Y, \mathcal{D}, \mu', S(y)) = ((0,1], \mathcal{L}((0,1]), \lambda, y)$, proposition 0.3.17 and remark 2.2 show that

$$h_\nu(U) = h_\lambda(T) + h_\lambda(S) = \log 2$$

holds. Furthermore, due to example 1.3.2, we already know that

$$H_\nu(\epsilon \,|\, U^{-1}\epsilon) = \log 2 = \log k_U\,.$$

Consequently, according to theorem I.6, there is a finite one-sided generator for U. Moreover, theorem I.6 states that there is also a one-sided generator with minimal cardinality k_U or $k_U + 1$. In the following we will see that each partition β^* from corollary 1.2.2 is not a one-sided generator for U as well as how a one-sided generator for U with minimal cardinality can look like.

Definition 2.1.6. Let $(X,\ \mathfrak{B},\ \mu,\ T)$ be a dynamical system endowed with an automorphism $T : X \to X$. Then we call T a *Bernoulli-shift* if there is a partition α of X consisting of a finite or countable number of atoms A_i such that

i) the partitions $(T^k\alpha)_{k\in\mathbb{Z}}$ are independent;

ii) α is a two-sided generator for T.

Definition 2.1.7. Let $k \geq 2$ be a fixed integer and let $(p_0, \ldots, p_{k-1})$ be a probability vector with non-zero entries. Furthermore, let $(Y, 2^Y, \nu)$ denote the probability space where $Y = \{0, 1, \ldots, k-1\}$ and the point i has measure p_i. Then the dynamical system $(X,\ \mathfrak{B},\ \mu,\ T)$ consisting of the probability space $(X,\ \mathfrak{B},\ \mu) = \prod_{i\in\mathbb{Z}}(Y, 2^Y, \nu)$ and the automorphism $T\,((x_i)_{i\in\mathbb{Z}}) = (y_i)_{i\in\mathbb{Z}}$ where $y_n = x_{n+1}$ is called the *two-sided* $(p_0, \ldots, p_{k-1})$*-shift.*

Remark 2.3. *i.)* Let $k \geq 2$ be fixed and let $(p_0, \ldots, p_{k-1})$ be a probability vector with non-zero entries. Then the two-sided $(p_0, \ldots, p_{k-1})$-shift is obviously the natural extension (cf. §3 in [Rok64]) of the one-sided $(p_0, \ldots, p_{k-1})$-shift.
ii.) The two-sided $(p_0, \ldots, p_{k-1})$-shift is a Bernoulli-shift.

Lemma 2.1.9. *Let $(X,\ \mathfrak{B},\ \mu)$ be a probability space and let ζ, η be two finite partitions of X. Then ζ and η are independent if and only if $H_\mu(\zeta \,|\, \eta) = H_\mu(\zeta)$.*

Proof. The hypothesis follows immediately from remark 1.1 and theorem 4.4 in [Wal82]. □

The following proposition states that it is not possible to separate almost all points in $[0,1) \times (0,1]$ with the aid of their β^*-name where β^* is a partition from corollary 1.2.2.

Proposition 2.1.8. *There does not exist a one-sided generator for U which consists of two atoms.*

Proof. Let us suppose that the assertion is wrong. Then there is a one-sided generator $\gamma = \{G_0, G_1\}$ for U which only possesses two atoms. Furthermore, let us denote the natural extension of $(M, \mathfrak{M}, \nu, U)$ by $(\hat{M}, \hat{\mathfrak{M}}, \hat{\nu}, \hat{U})$. Since the natural extension $\hat{\epsilon}$ of the point partition ϵ is exhaustive, we then have that the natural extension $\hat{\gamma} = \{\hat{G}_0, \hat{G}_1\}$ of γ becomes a two-sided generator for U. Thus, due to theorem I.1, the partition $\hat{\gamma}$ satisfies the relation

$$h_{\hat{\nu}}\left(\hat{U}, \hat{\gamma}\right) = h_{\hat{\nu}}\left(\hat{U}\right) = h_{\nu}(U) = \log 2\,.$$

According to proposition 0.3.6 and definition 0.3.12, we even obtain that

$$h_{\hat{\nu}}\left(\hat{U}, \hat{\gamma}\right) = H_{\hat{\nu}}\left(\hat{\gamma}\right) = \log 2\,.$$

Since $\hat{U}$ is measure-preserving, this implies that, for each pair $(k, \ell) \in \mathbb{Z} \times \mathbb{N}$ the relation

$$H_{\hat{\nu}}\left(\hat{U}^k\hat{\gamma} \,|\, \bigvee_{j=-k+1}^{-k+\ell} \hat{U}^{-j}\hat{\gamma}\right) = H_{\hat{\nu}}\left(\hat{U}^k\hat{\gamma}\right)$$

also holds. Due to lemma 2.1.9, we then get that, for each pair $(k, \ell) \in \mathbb{Z} \times \mathbb{N}$ the partitions

$$\hat{U}^k\hat{\gamma} \qquad \text{and} \qquad \bigvee_{j=-k+1}^{-k+\ell} \hat{U}^{-j}\hat{\gamma}$$

are independent. In particular we have that for every $n \in \mathbb{N}_0$ the relation

$$\begin{aligned}
\hat{\nu}\left(\bigcap_{j=-n}^{n} \hat{U}^{-j}\hat{G}_{i_j}\right) &= \hat{\nu}\left(\hat{U}^n\hat{G}_{i_n}\right)\hat{\nu}\left(\bigcap_{j=-n+1}^{n} \hat{U}^{-j}\hat{G}_{i_j}\right) \\
&\vdots \\
&= \prod_{j=-n}^{n} \hat{\nu}\left(\hat{U}^{-j}\hat{G}_{i_j}\right) \\
&= \prod_{j=-n}^{n} \hat{\nu}\left(\hat{G}_{i_j}\right)
\end{aligned}$$

is satisfied. Hence, the partition $\hat{\gamma}$ is a two-sided generator with the property that the partitions $\left(\hat{U}^i\hat{\gamma}\right)_{i\in\mathbb{Z}}$ are independent. Thus, due to definition 2.1.6, the system $(\hat{M}, \hat{\mathfrak{M}}, \hat{\nu}, \hat{U})$ is a Bernoulli-shift with $h_{\hat{\nu}}(\hat{U}) = \log 2$. According to Ornstein's theorem (cf. [Orn70]), we then obtain that $(\hat{M}, \hat{\mathfrak{M}}, \hat{\nu}, \hat{U})$ is isomorphic to the two-sided $(\frac{1}{2}, \frac{1}{2})$-shift. However, since U is not ergodic, this is a contradiction to remark 2.3 and lemma 2.1.8. □

In the remaining part of this section we give an answer to the question how a one-sided generator for U can look like. For that purpose, let us consider the partition $\gamma^* = \{G_0^*, G_1^*, G_2^*\}$ where

$$\begin{aligned}
G_0^* := &\left\{(x,y) \in M :\ 0 < y < 1 \ \wedge\ y \le x < \min\left\{y + \frac{1}{3}, 1\right\}\right\} \\
&\cup\ \left\{(x,y) \in M :\ \frac{2}{3} < y \le 1 \ \wedge\ 0 \le x < y - \frac{2}{3}\right\} \\
G_1^* := &\left\{(x,y) \in M :\ 0 < y < \frac{2}{3} \ \wedge\ y + \frac{1}{3} \le x < \min\left\{y + \frac{2}{3}, 1\right\}\right\} \\
&\cup\ \left\{(x,y) \in M :\ \frac{1}{3} < y \le 1 \ \wedge\ \max\left\{y - \frac{2}{3}, 0\right\} \le x < y - \frac{1}{3}\right\} \\
G_2^* := &\left\{(x,y) \in M :\ 0 < y < \frac{1}{3} \ \wedge\ y + \frac{2}{3} \le x < 1\right\} \\
&\cup\ \left\{(x,y) \in M :\ 0 < y \le 1 \ \wedge\ \max\left\{y - \frac{1}{3}, 0\right\} \le x < y\right\}
\end{aligned} \tag{2.3}$$

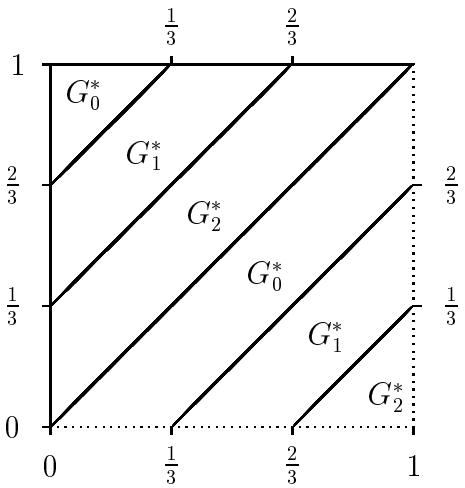

Figure 2.1: The partition $\gamma^* = \{G_0^*, G_1^*, G_2^*\}$

In the following we will show that γ^* is a one-side generator for U. We start with some preparatory considerations.

Let us go back to the dynamical system $(X, \mathfrak{B}, \mu, T)$ which is given by $([0,1), \mathcal{L}([0,1)), \lambda, T(x) = 2x \bmod 1)$. For $y \in (0,1]$ we then denote by γ_y^* the partition of $[0,1)$ into the sets

$$\gamma_y^* := \left\{\left[y \oplus 0, y \oplus \frac{1}{3}\right), \left[y \oplus \frac{1}{3}, y \oplus \frac{2}{3}\right), \left[y \oplus \frac{2}{3}, y\right)\right\}$$

where $a \oplus b := (a+b) \bmod 1$ and $[a,b) := \begin{cases} [a,b) & \text{if } a \le b \\ [a,1) \cup [0,b) & \text{if } a > b \end{cases}$ for $a, b \in [0,1]$.

Remark 2.4. Due to (2.3), it is clear that for every $v \in (0,1]$ the relation

$$\gamma^* \cap \{(x,y) \in M : \ y = v\} = \gamma_v^*$$

holds.

Moreover, let K be the unit circle and d_K be the (canonical) metric on K which is induced by the arclength. Then we denote by $\|\cdot - \cdot\|$ the metric on $[0,1)$ which is defined by

$$\|x - u\| := (2\pi)^{-1} d_K\big(\exp(2\pi i x), \exp(2\pi i u)\big)\,. \tag{2.4}$$

Lemma 2.1.10. *Let $p \in \mathbb{N}_0$ and let $z \in [0,1)$ be a periodic point with period 2^p. Furthermore, let $k \in \{1, \ldots, 2^{2^p} - 1\}$. Then, for any $y \in (\frac{k-1}{2^{2^p}-1}, \frac{k}{2^{2^p}-1}]$ the γ_y^*-name of z is constant.*

Proof. In order to prove the hypothesis, it is sufficient to show that the $(\gamma_y^*, 2^p)$-name of z is constant for $y \in (\frac{k-1}{2^{2^p}-1}, \frac{k}{2^{2^p}-1}]$. Let us choose $v, y \in (\frac{k-1}{2^{2^p}-1}, \frac{k}{2^{2^p}-1}]$ such that $\frac{k-1}{2^{2^p}-1} < v < y = \frac{k}{2^{2^p}-1}$. It is clear that $\hat{z}_{\gamma_y^*,\,2^p}$ differs from $\hat{z}_{\gamma_v^*,\,2^p}$ if and only if there is at least one $0 \le \ell \le 2^p - 1$ such that $(\hat{z}_{\gamma_y^*})_\ell \neq (\hat{z}_{\gamma_v^*})_\ell$. This means that $\hat{z}_{\gamma_y}$ is different from $\hat{z}_{\gamma_v}$ if and only if there is at least one periodic point c with period 2^p such that

$$c \in \{z, (2z) \bmod 1, \ldots, (2^{p-1}z) \bmod 1\} \quad \text{and} \quad (\hat{c}_{\gamma_y^*})_0 \neq (\hat{c}_{\gamma_v^*})_0\,.$$

Thereby, the relation $(\hat{c}_{\gamma_y^*})_0 \neq (\hat{c}_{\gamma_v^*})_0$ is satisfied if and only if

$$c \notin \left[v \oplus \frac{(\hat{c}_{\gamma_y^*})_0}{3}, v \oplus \frac{((\hat{c}_{\gamma_y^*})_0 + 1) \bmod 3}{3}\right)\,.$$

Now, let us consider a periodic point $c \in \{z, (2z) \bmod 1, \ldots, (2^{p-1}z) \bmod 1\}$. Since $\gcd(2^{2^p} - 1, 3) = 3$, we then have that the relation

$$\left\| c - y \oplus \frac{(\hat{c}_{\gamma_y^*})_0}{3} \right\| \le \frac{1}{3} - \frac{1}{2^{2^p} - 1}$$

is satisfied. Thus, by $0 < y - v < \frac{1}{2^{2^p}-1}$, we obtain that

$$\left[y \oplus \frac{(\hat{c}_{\gamma_y^*})_0}{3}, y \oplus \frac{((\hat{c}_{\gamma_y^*})_0 + 1) \bmod 3}{3}\right)$$

$$\subseteq \left[v \oplus \frac{(\hat{c}_{\gamma_y^*})_0}{3}, v \oplus \left(\frac{((\hat{c}_{\gamma_y^*})_0 + 1) \bmod 3}{3} + \frac{1}{2^{2^p} - 1}\right)\right)$$

as well as that

$$\left\| c - y \oplus \frac{(\hat{c}_{\gamma_y^*})_0}{3} \right\| < \left\| c - v \oplus \frac{(\hat{c}_{\gamma_y^*})_0}{3} \right\| < \left\| c - y \oplus \frac{(\hat{c}_{\gamma_y^*})_0}{3} \right\| + \frac{1}{2^{2^p} - 1} \le \frac{1}{3}\,.$$

Consequently, we have that $(\hat{c}_{\gamma_v^*})_0 = (\hat{c}_{\gamma_y^*})_0$ holds. □

$\left(\frac{14}{15}, 1\right]$	0000	0001	0010	0121	0100	1212	1210	1222	1000	1012	2121	2122	2101	2212	2221
$\left(\frac{13}{15}, \frac{14}{15}\right]$	0000	0011	0110	0122	1100	1212	1220	1022	2000	2012	2121	2102	2201	2210	0221
$\left(\frac{12}{15}, \frac{13}{15}\right]$	0000	0012	0120	1122	1200	1212	1221	1002	2001	2112	2121	2100	2211	0210	0021
$\left(\frac{11}{15}, \frac{12}{15}\right]$	0000	0112	1120	1102	1201	1212	1021	2002	2011	2110	2121	2200	0211	0220	0022
$\left(\frac{2}{3}, \frac{11}{15}\right]$	0000	1112	1121	1202	1211	1212	2021	2000	2111	2120	2121	0200	0212	0020	0002
$\left(\frac{9}{15}, \frac{2}{3}\right]$	1111	1112	1121	1202	1211	2020	2021	2000	2111	2120	0202	0200	0212	0020	0002
$\left(\frac{8}{15}, \frac{9}{15}\right]$	1111	1122	1221	1200	2211	2020	2001	2100	2112	0120	0202	0201	0012	0021	1002
$\left(\frac{7}{15}, \frac{8}{15}\right]$	1111	1120	1201	2200	2011	2020	2002	2110	0112	0220	0202	0211	0022	1021	1102
$\left(\frac{6}{15}, \frac{7}{15}\right]$	1111	1220	2201	2210	2012	2020	2102	0110	0122	0221	0202	0011	1022	1001	1100
$\left(\frac{1}{3}, \frac{6}{15}\right]$	1111	2220	2202	2010	2022	2020	0102	0111	0222	0201	0202	1011	1020	1101	1110
$\left(\frac{4}{15}, \frac{1}{3}\right]$	2222	2220	2202	2010	2022	0101	0102	0111	0222	0201	1010	1011	1020	1101	1110
$\left(\frac{3}{15}, \frac{4}{15}\right]$	2222	2200	2002	2011	0022	0101	0112	0211	0220	1201	1010	1021	1120	1102	2110
$\left(\frac{2}{15}, \frac{3}{15}\right]$	2222	2201	2012	0011	0122	0101	0110	0221	1220	1001	1010	1022	1100	2102	2210
$\left(\frac{1}{15}, \frac{2}{15}\right]$	2222	2001	0012	0021	0120	0101	0210	1221	1200	1002	1010	1122	2100	2112	2211
$\left(0, \frac{1}{15}\right]$	2222	0001	0010	0121	0100	0101	1210	1222	1000	1012	1010	2122	2101	2212	2221
$y \uparrow \quad x \rightarrow$	0	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{1}{3}$	$\frac{6}{15}$	$\frac{7}{15}$	$\frac{8}{15}$	$\frac{9}{15}$	$\frac{2}{3}$	$\frac{11}{15}$	$\frac{12}{15}$	$\frac{13}{15}$	$\frac{14}{15}$

Table 2.1: The $\gamma^*_{y,4}$-names of the periodic points with period 4

Lemma 2.1.11. *Let $y \in (0,1]$ and let $p \in \mathbb{N}_0$. Furthermore, let $m \geq 2$ and $x \in [0,1)$ be a point with the property that its $(\gamma_y^*, m \cdot 2^p)$-name is periodic with period 2^p. Then there is a (unique) periodic point z with period 2^p such that*

$$\|x - z\| < \frac{1}{3(2^{2^p} - 1)2^{(2^p-1)(m-1)}}$$

holds.

Proof. Due to the assumptions, we have that

$$(\hat{x}_{\gamma_y^*})_k = (\hat{x}_{\gamma_y^*})_{k \bmod 2^p}$$

holds for each $0 \leq k \leq m \cdot 2^p - 1$. In particular this implies that for each $0 \leq \ell \leq (m-1)(2^p - 1)$ the relation

$$\left\|(2^{2^p}(2^\ell x)) \bmod 1 - (2^\ell x) \bmod 1\right\| < \frac{1}{3} \qquad (\S)$$

is satisfied. Hence, we find a unique integer $\ell' \in \{0, \ldots, 2^{2^p} - 2\}$ such that

$$\ell' - \frac{1}{3} < (2^{2^p} - 1)x < \ell' + \frac{1}{3}\,.$$

Thus, by setting $z := \frac{\ell'}{2^{2^p}-1}$, we get a periodic point with period 2^p and the property that

$$\|x - z\| < \frac{1}{3(2^{2^p} - 1)}\,.$$

Since $T(x) = 2x \bmod 1$ is expansive with constant $\frac{1}{2}$, we have that x and z satisfy the relation

$$\|(2z) \bmod 1 - (2x) \bmod 1\| < \frac{2}{3(2^{2^p} - 1)}$$

as well. However, due to (§), there is also a unique $k \in \{0, \ldots, 2^{2^p} - 2\}$ such that $c = \frac{k}{2^{2^p}-1}$ satisfies the relation

$$\|c - (2x) \bmod 1\| < \frac{1}{3(2^{2^p} - 1)}\,.$$

Since, for a periodic point c' with period 2^p the relation $\|c - c'\| < \frac{1}{2^{2^p}-1}$ is satisfied if and only if $c = c'$ holds, we get that

$$(2z) \bmod 1 = c\,.$$

This implies that x and z even fulfil the relation

$$\|z - x\| < \frac{1}{3 \cdot 2(2^{2^p} - 1)}\,.$$

By applying these arguments successively to the images $((2^k x) \bmod 1)_{1 \le k \le m \cdot 2^p - 1}$ of x, we finally obtain both that the periodic point $z = \frac{\ell'}{2^{2^p}-1}$ with period 2^p fulfils the relation

$$\|z - x\| < \frac{1}{3(2^{2^p}-1)2^{(2^p-1)(m-1)}}$$

and that z is uniquely determined. □

Lemma 2.1.12. *Let $y \in (0,1] \setminus \mathbb{Q}_{(0,1]}$ where $\mathbb{Q}_{(0,1]} := \mathbb{Q} \cap (0,1]$ and let $p \in \mathbb{N}_0$. Then there is an integer $n(y,p)$ only depending on y and p such that every periodic point z with period 2^p and each $x \in [0,1)$ with $\|x - z\| < \frac{1}{n(y,p)}$ satisfy the relation*

$$(\hat{z}_{\gamma_y^*})_0 = (\hat{x}_{\gamma_y^*})_0 \,.$$

Proof. Since y is irrational, we find $\ell \in \{0, \ldots, 2^{2^p} - 2\}$ such that

$$\frac{\ell}{2^{2^p}-1} < y < \frac{\ell+1}{2^{2^p}-1} \,.$$

By using this relation, we define

$$\delta(y,p) := \min\left\{ y - \frac{\ell}{2^{2^p}-1}, \frac{\ell+1}{2^{2^p}-1} - y \right\} > 0 \,.$$

Due to the fact that $\gcd(2^{2^p}-1, 3) = 3$, we then get that each periodic point z with period 2^p satisfies the relation

$$\min\left\{ \left\| z - y \oplus \frac{k}{3} \right\|, \; k \in \{0,1,2\} \right\} \ge \delta(y,p) \,. \qquad (*)$$

Furthermore, we set

$$n(y,p) := \left(\left\lfloor \frac{1}{\delta(y,p)} \right\rfloor + 1 \right)$$

where $\lfloor r \rfloor$ denotes the integer part of $r \in \mathbb{R}_+$. Now let us consider a periodic point z with period 2^p and $x \in [0,1)$ with $\|x - z\| < \frac{1}{n(y,p)}$. Due to definition 2.1.1, we then have that

$$z \in \left[y \oplus \frac{(\hat{z}_{\gamma_y^*})_0}{3}, y \oplus \frac{((\hat{z}_{\gamma_y^*})_0 + 1) \bmod 3}{3} \right) \,.$$

Since $0 < \frac{1}{n(y,p)} < \delta(y,p)$, relation $(*)$ provides that x also fulfils the relation

$$x \in \left[y \oplus \frac{(\hat{z}_{\gamma_y^*})_0}{3}, y \oplus \frac{((\hat{z}_{\gamma_y^*})_0 + 1) \bmod 3}{3} \right) \,.$$

Consequently, we have that $(\hat{x}_{\gamma_y^*})_0 = (\hat{z}_{\gamma_y^*})_0$. □

Definition 2.1.9. Let $(X,\, \mathfrak{B},\, \mu,\, T)$ be a dynamical system and let $x \in X$ be an element of X. The set

$$O_T(x) := \{T^n(x),\ n \in \mathbb{N}_0\}$$

is then called the *orbit* of x.

Lemma 2.1.13. *Let $X = [0,1)$ and let $T : X \to X$ be given by $T(x) = 2x \bmod 1$. Then the set $X' := \{x \in [0,1) :\ \overline{O_T(x)} = [0,1)\}$ is an invariant measurable set with the property that $\lambda(X') = 1$.*

Proof. Since $[0,1)$ is a separable, metric, and bounded space, we find a countable base $\{V_n\}_{n\in\mathbb{N}}$ for the topology of X. It is clear that $x \in X'$ if and only if $x \in \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}_0} T^{-k}V_n$. Hence, we have that

$$X' = \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}_0} T^{-k}V_n\,.$$

By $\mathcal{B}([0,1)) \subseteq \mathcal{L}([0,1))$ where $\mathcal{B}([0,1))$ denotes the σ-algebra of Borel subsets of $[0,1)$, we then get that X' is a measurable subset of $[0,1)$. Moreover, since the relation

$$T^{-1}\left(\bigcup_{k\in\mathbb{N}_0} T^{-k}V_n\right) \subseteq \bigcup_{k\in\mathbb{N}_0} T^{-k}V_n\,.$$

obviously holds and since T is measure-preserving, we obtain that

$$T^{-1}\left(\bigcup_{k\in\mathbb{N}_0} T^{-k}V_n\right) = \bigcup_{k\in\mathbb{N}_0} T^{-k}V_n\,.$$

Therefore, we get both that X' is invariant and, due to lemma 2.1.8, that the relation

$$\lambda\left(\bigcup_{k\in\mathbb{N}_0} T^{-k}V_n\right) \in \{0,1\}$$

is satisfied. Since $\bigcup_{k\in\mathbb{N}_0} T^{-k}V_n$ is a non-empty open set and $\lambda(U) > 0$ holds for every non-empty open set, we finally get that $\lambda(X') = 1$. □

Now we are prepared to prove that the partition γ^* is a one-sided generator for $U((x,y)) = (2x \bmod 1, y)$.

Lemma 2.1.14. *The set*

$$M' := \left\{x \in [0,1) :\ \overline{O_T(x)} = [0,1)\right\} \times \{y \in (0,1] \setminus \mathbb{Q}_{(0,1]}\}$$

is a measurable subset of M which is invariant with respect to U and has full λ^2-mass, i.e. $\lambda^2(M') = 1$.

Proof. The hypothesis follows immediately from lemma 2.1.13 and the fact that $\lambda(\{y \in (0,1] \setminus \mathbb{Q}_{(0,1]}\}) = 1$. □

Proposition 2.1.10. *The partition γ^* defined in (2.3) is a one-sided generator for U.*

Proof. In order to prove the assertion, we shall show that there are no dissipative points in M'. According to lemma 2.1.14 and remark 2.1 *ii*), we then obtain that γ^* is a one-sided generator for U.

Let us consider $(x,y),(u,v) \in M'$ with the property that their γ^* names are identical. Without loss of generality, we can suppose that $y \leq v$. In the following we will prove that $(x,y) = (u,v)$ holds. For that purpose, we start with showing that $x = u$. Since the orbit of x is dense, for each $\varepsilon > 0$ we find an integer $n \in \mathbb{N}$ such that $\|(2^n x) \bmod 1 - 0\| < \varepsilon$. Thus, due to lemma 2.1.11 and lemma 2.1.12, for each $m \in \mathbb{N}$ we also find an integer $n(m) \in \mathbb{N}$ such that the (γ_y^*, m)-name of $(2^{n(m)}x) \bmod 1$ is identical to the (γ_y^*, m)-name of 0. Moreover, since the (γ_y^*, ℓ)-name of $(2^k x) \bmod 1$ is identical to the (γ_v^*, ℓ)-name of $(2^k u) \bmod 1$ for each pair $(k,\ell) \in \mathbb{N}_0^2$, lemma 2.1.11 and lemma 2.1.12 even provide that there is an integer $n_1 \in \mathbb{N}$ with the property that

$$(\hat{0}_{\gamma_y^*})_0 = (\widehat{(2^{n_1}x) \bmod 1}_{\gamma_y^*})_0 = (\widehat{(2^{n_1}u) \bmod 1}_{\gamma_v^*})_0 = (\hat{0}_{\gamma_v^*})_0 \,.$$

Due to table 2.1 and to lemma 2.1.10, we then have that there is a $k \in \{1,2,3\}$ such that

$$\frac{k-1}{3} < y \leq v \leq \frac{k}{3} \,. \qquad (*)$$

Now, let us consider the point $z = \frac{1}{15}$. It is clear that z is a periodic point with prime period 4. By using the same argumentation as above, we find an integer $n_4 \in \mathbb{N}$ and a periodic point c with prime period 4, such that for $0 \leq i \leq 3$ the relation

$$(\hat{z}_{\gamma_y^*})_i = (\widehat{(2^{n_4}x) \bmod 1}_{\gamma_y^*})_i = (\widehat{(2^{n_4}u) \bmod 1}_{\gamma_v^*})_i = (\hat{c}_{\gamma_v^*})_i$$

is satisfied. According to relation $(*)$, lemma 2.1.10, and table 2.1, we then obtain both that $z = c$ and that there is a $\ell \in \{1,\ldots,15\}$ with

$$\frac{\ell-1}{15} < y \leq v < \frac{\ell}{15} \,.$$

However, the last relation implies that, for each $m \in \mathbb{N}_0$ the points x and u satisfy the relation

$$\|(2^m x) \bmod 1 - (2^m u) \bmod 1\| < \frac{1}{3} + \frac{1}{15} \,. \qquad (**)$$

Since $T(a) = 2a \bmod 1$ is expansive with constant $\frac{1}{2}$ and $diam(G_i^*) = \frac{1}{3}$ holds for each $0 \leq i \leq 2$, a simple calculation shows that relation $(**)$ is satisfied if and only if $x = u$.

It remains to show that $y = v$ also holds. In order to do that, let us assume that $y \neq v$. Then we find $p \in \mathbb{N}_0$ such that

$$\frac{1}{2^{2^p}-1} < \frac{v-y}{2}\,.$$

Since $y, v \in (0,1] \setminus \mathbb{Q}_{(0,1]}$, we also find a periodic point $z = \frac{\ell(z)}{2^{2^p}-1}$ with period 2^p and the property that $y < z < v$. Hence, due to the fact that the orbit of x is dense, there is an $\varepsilon > 0$ and an integer $n(\varepsilon) \in \mathbb{N}$ such that

$$\min\{|y-z|, |u-z|\} > \varepsilon$$

as well as that

$$|(2^{n(\varepsilon)}x) \bmod 1 - z| < \varepsilon.$$

Consequently, the γ_y^*-name of x and the γ_v^*-name of x are different in the $n(\varepsilon)$-th position. But this contradicts the assumption that $\hat{x}_{\gamma_y^*} = \hat{x}_{\gamma_v^*}$ holds. Thus, we also have that $y = v$. □

♦

2.2 Rokhlin's Lemma

Rokhlin's Lemma was originally proved for aperiodic automorphisms T of Polish spaces (cf. [Rok63]). Apart from the fact that it is one of the most important and powerful tools in ergodic theory, in particular, Rokhlin's Lemma is indispensable for all constructions of finite two-sided generators invented before. In the following, we shall prove that it can also be stated for any aperiodic endomorphisms.

We start with two lemmata which make use of the aperiodicity of T.

Lemma 2.2.1. *Let $(X, \mathfrak{B}, \mu, T)$ be a dynamical system endowed with an aperiodic endomorphism T. For any $n \in \mathbb{N}$, we then find a measurable set F of positive measure such that*

$$\mu(T^{-n}(F) \cap F) = 0. \tag{2.5}$$

Proof. We assume that we cannot determine a Borel set F of positive measure such that relation (2.5) holds. Due to the fact that X is a Lebesgue space, this assumption leads in the end to $\mu(\{x : \ x = T^n(x)\}) = 1$, in contradiction to the aperiodicity of T. We omit further details and refer to [HS00]. □

Definition 2.2.1. A measurable set F of positive μ-measure which fulfils relation (2.5) for each n from 1 to $m-1$ induces an *m-chain* $(\mathcal{F})_m$, which we define to be the m-tuple

$$(\mathcal{F})_m := \left\langle F, T^{-1}(F), \ldots, T^{-(m-1)}(F) \right\rangle.$$

Lemma 2.2.2. *Let $(X, \mathfrak{B}, \mu, T)$ be a dynamical system endowed with an aperiodic endomorphism T. For any $n \in \mathbb{N}$, there exists a set F which induces an n-chain.*

Proof. With the aid of lemma 2.2.1, the hypothesis follows by induction. We omit further details and refer to [HS00]. □

Remark 2.5. If $F \in \mathfrak{B}$ induces an n-chain, then, for any $k \in \mathbb{N}$, also the inverse image $T^{-k}(F)$ induces an n-chain.

Now we are prepared to extend Rokhlin's Lemma to non-invertible situations. We use the symbol $\biguplus$ for *disjoint* unions.

Theorem 2.2.2 (Rokhlin's Lemma). *Let $(X, \mathfrak{B}, \mu, T)$ be a dynamical system endowed with an aperiodic endomorphism T. For any $n \in \mathbb{N}$ and any $\varepsilon > 0$, there exists a measurable set $E \in \mathfrak{B}$ which induces an n-chain $(\mathcal{E})_n$[1] such that*

$$\mu((\mathcal{E})_n) := \mu\left(\biguplus_{\ell=0}^{n-1} T^{-\ell}(E)\right) > 1 - \varepsilon.$$

Proof. We choose $m \in \mathbb{N}$ such that $m \geq n$ and

$$1/m < \varepsilon/(n-1).$$

According to lemma 2.2.2, the set of m-chains $(\mathcal{F})_m$ is not empty. We equip this set, the family of inducing sets, respectively, with a partial ordering, by defining

$$F <_\mu F' \iff (F \subset F') \wedge (\mu(F) < \mu(F')) .$$

Zorn's Lemma then provides maximal elements. Let F be such a maximal inducing set and denote with $(\mathcal{F})_m$ the induced m-chain. Moreover, for $k \in \mathbb{N}_0$, we denote the k-th backward image of F with $F_k := T^{-k}(F)$. Due to theorem 0.1.11, we then obtain that the complement U of $\bigcup_{k=0}^{\infty} F_k$ is a measurable set which is both forward and backward invariant. Because of the maximality of $(\mathcal{F})_m$, we conclude that

$$\bigcup_{k=0}^{\infty} F_k = X.$$

Now, we define the measurable sets

$$F^k := F_k \setminus \bigcup_{j=0}^{k-1} F_j \quad \text{for } k \in \mathbb{N}_0, \text{ and } \quad E := \biguplus_{k=1}^{\infty} F^{kn-1} .$$

Then, the set E induces the desired n-chain $(\mathcal{E})_n$. We omit further details and refer to [HS00]. □

[1] also called an (n, ε)-Rokhlin tower

As a direct conclusion from the above theorem, we obtain the classical form of Rokhlin's Lemma.

Corollary 2.2.3 (Classical Rokhlin's Lemma). *Let $(X, \mathfrak{B}, \mu, T)$ be dynamical system endowed with an aperiodic endomorphism T. For any $n \in \mathbb{N}$ and any $\varepsilon > 0$, there is a measurable set $G \in \mathfrak{B}$ such that the sets $G, T^1(G), \ldots, T^{n-1}(G)$ are measurable and pairwise disjoint, moreover, we have that*

$$\mu\left(\biguplus_{\ell=0}^{n-1} T^\ell(G)\right) > 1 - \varepsilon.$$

Proof. Setting $G := E_{n-1}$, the hypothesis follows immediately from theorem 2.2.2. □

Remark 2.6. *i*) An explicitely computation of a Rokhlin tower for classes of interval maps and Bernoulli shifts can be found in [HS00].
ii) In the following a (n, ε)-Rokhlin tower which is given by the backward images of the inducing set is called a (n, ε)-*backward-Rokhlin tower*. Analogous a (n, ε)-Rokhlin-tower which is given by the forward images of the inducing set is called a (n, ε)-*forward-Rokhlin tower*.

2.3 One-Sided Generators for Automorphisms with Zero Entropy

According to theorem I.6, it is clear that in the case of an aperiodic automorphism T, there are finite one-sided generators for T if and only if the entropy of T is equal to zero. Moreover, we know that in this situation the minimal cardinality of a one-sided generator is equal to two. In the following section, we shall present a direct construction of such a finite one-sided generators with two atoms for an aperiodic automorphism T with zero entropy.

We start with recalling just a few facts from ergodic theory.

Definition 2.3.1. Let $(X, \mathfrak{B}, \mu)$ be a Lebesgue space and let α and β be two partitions of X. For $0 < \varepsilon < 1$, we call α *finer than* β *up to* ε if there is a measurable set X_ε of mass greater or equal than $1 - \varepsilon$ such that $\beta \cap X_\varepsilon \leq \alpha \cap X_\varepsilon$. We write $\beta \leq_\varepsilon \alpha$.

Remark 2.7. Note that $\beta \leq \alpha$ if an only if $\beta \leq_\varepsilon \alpha$ for each $\varepsilon > 0$.

Definition 2.3.2. Let $(X, \mathfrak{B}, \mu)$ be a Lebesgue space, let $I \subseteq \mathbb{N}_0$ be an index set and let $\mathcal{P}^I$ denote the set of all ordered partitions of X with index set I. We then define a real-valued map $\|\cdot, \cdot\|$ on $\mathcal{P}^I \times \mathcal{P}^I$ by setting

$$\|\alpha\,, \beta\| := \frac{1}{2} \sum_{i \in I} \mu(A_i \,\triangle\, B_i)\,.$$

Lemma 2.3.1. *i) For any index set $I \subseteq \mathbb{N}_0$, the map $\|\cdot\,,\cdot\|$ is a complete metric on $\mathcal{P}^I$ and fulfils the relation*

$$\|\alpha \vee \beta\,, \zeta \vee \eta\| \leq \|\alpha\,, \zeta\| + \|\beta\,, \eta\| \tag{2.6}$$

where α, β, ζ, and η are elements of $\mathcal{P}^I$.
ii) Let α and β two elements of $\mathcal{P}^I$ such that $\|\alpha\,, \beta\| < \varepsilon$. Then the partitions α and β also satisfy the relation $\beta \leq_{2\varepsilon} \alpha$.

Proof. i) Since we neglect null sets, it is clear that $\|\cdot\,,\cdot\|$ is positively definite and symmetric. Moreover, the triangle inequality directly follows from the relation

$$A \cap B \cap (C \cup X \setminus C) \subseteq (A \cap C) \cup (A \setminus C)$$

where A, B, and C are measurable subsets of X. Analogously, we obtain relation (2.6) by means of the equality

$$(A \cap C) \setminus (B \cap D) = ((A \setminus B) \cap C) \cup (A \cap (C \setminus D))$$

where A, B, C, and D are measurable subsets of X. It remains to prove the completeness of the metric $\|\cdot\,,\cdot\|$. For that, let $(\alpha_j)_{j\in\mathbb{N}}$ be a Cauchy sequence with respect to $\|\cdot\,,\cdot\|$ in $\mathcal{P}^I$. Without loss of generality, we can suppose that $\sum_{j\in\mathbb{N}} \|\alpha_j\,, \alpha_{j+1}\| < \infty$ holds. For each $j \in \mathbb{N}$, we then define the set

$$G_j := \{x \in X :\ (\hat{x}_{\alpha_j})_0 \neq (\hat{x}_{\alpha_{j+1}})_0\}\,.$$

It is clear that G_j is a measurable subset of $\bigcup_{i\in I}(A_{j,i} \triangle A_{j+1,i})$ where $A_{\ell,i}$ denotes the i-th atom of the partition α_ℓ. Consequently, by applying the Borel-Cantelli lemma, we obtain a partition $\alpha \in \mathcal{P}^I$ which has the property that the sequence $\big(\|\alpha\,, \alpha_j\|\big)_{j\in\mathbb{N}}$ converges to zero.
ii) Let $\alpha = \{A_i\}_{i\in I}$ and $\beta = \{B_j\}_{j\in I}$ be two elements of $\mathcal{P}^I$ such that $\|\alpha\,, \beta\| < \varepsilon$. Due to definition 2.3.2, we then have that $X_{2\varepsilon} := \bigcup_{i\in I}(A_i \cap B_i)$ is a measurable set which satisfies the relation $\mu(X_{2\varepsilon}) > 1 - 2\varepsilon$. Moreover, we obviously obtain that $\beta \cap X_{2\varepsilon} \leq \alpha \cap X_{2\varepsilon}$. □

After these preparations, we will devote ourself fully to the construction of a finite one-sided generator for an aperiodic automorphism T with zero entropy. In order to emphasise the main idea of our construction, we shall present it under the additional assumption that the automorphism is also ergodic.

Theorem 2.3.3. *Let $(X, \mathfrak{B}, \mu, T)$ be a dynamical system endowed with an ergodic and aperiodic automorphism T with $h_\mu(T) = 0$. Then there exists a finite one-sided generator $\gamma = \{G_0, G_1\}$ with two atoms.*

Proof. In order to show the hypothesis, we shall construct a converging sequence of partitions with two atoms such that the limit is a one-sided generator. More precisely, based on an integer $p_0 > 2$, the partition $\beta_0 := \{B_0^0 := X, B_1^0 := \emptyset\}$, and an increasing sequence $(a_j)_{j\in\mathbb{N}}$ of finite partitions with $\alpha_j \nearrow \epsilon$, we inductively build a sequence $(p_n)_{n\in\mathbb{N}}$ of positive integers, a sequence $(\delta_n)_{n\in\mathbb{N}}$ of positive real numbers, and a sequence $(\beta_n)_{n\in\mathbb{N}}$ of partitions of X such that, for each $n \in \mathbb{N}$ the conditions

(γ_1) $\quad p_{n-1} < p_n$,

(γ_2) $\quad \delta_n \leq 2^{-2(p_{n-1}+1)}$,

(γ_3) $\quad \beta_n$ possesses two atoms,

(γ_4) $\quad \alpha_n \leq_{\delta_n} \bigvee_{j=0}^{p_n} T^{-j}\beta_n$,

(γ_5) $\quad \|\beta_{n-1}\,,\beta_n\| \leq \delta_n$

are fulfilled. According to the conditions (γ_1) and (γ_2), we then have that

$$\sum_{n\in\mathbb{N}} \delta_n < \infty$$

holds. Hence, due to the conditions (γ_3) and (γ_5) and to lemma 2.3.1 *i*), the sequence $(\beta_n)_{n\in\mathbb{N}}$ converges to a partition β which has two atoms. Moreover, according to relation (2.6) and the conditions (γ_2) and (γ_5), we obtain that

$$\left\| \bigvee_{j=0}^{p_n} T^{-j}\beta_n\,, \bigvee_{j=0}^{p_n} T^{-j}\beta \right\| \leq p_n \cdot \|\beta_n\,,\beta\| \leq p_n \cdot \sum_{k\geq n} \|\beta_k\,,\beta_{k+1}\| \leq p_n \cdot 2^{-2p_n-1}$$

holds for each $n \in \mathbb{N}$. Due to lemma 2.3.1 *ii*) and condition (γ_4), this implies that, for all $n \in \mathbb{N}$ the relation

$$\alpha_n \leq_{\delta_n + p_n\cdot 2^{-2p_n}} \bigvee_{j=0}^{\infty} T^{-j}\beta$$

is satisfied. However, since $(\alpha_n)_{n\in\mathbb{N}}$ is an increasing sequence of partitions and the sequences $(\delta_n)_{n\in\mathbb{N}}$ and $(p_n \cdot 2^{-2p_n})$ both converge to zero, remark 2.7 yields that

$$\alpha_k \leq \bigvee_{j=0}^{\infty} T^{-j}\beta$$

even holds for all $k \in \mathbb{N}$. Consequently, we finally obtain that the partition $\gamma := \beta$ satisfies the relation

$$\epsilon \leq \bigvee_{j=0}^{\infty} T^{-j}\gamma\,.$$

In the following, we shall show by induction how the sequence $(\beta_n)_{n\in\mathbb{N}}$ can be constructed. For technical reasons, we will build the partitions $(\beta_n)_{n\in\mathbb{N}}$ in such a way that, for all $n \in \mathbb{N}$ the condition

(γ_6) $\quad \hat{x}_{\beta_n,n+2} \in \{0,1\}^{n+2} \setminus \{1\}^{n+2}$ for μ-a.a. $x \in X$

is also fulfilled. Moreover, in order to avoid unnecessary technical difficulties, we suppose that the logarithm with base 2 is used in the definition of entropy instead of the natural logarithm. This assumption may lead to different values of entropy, but it does not change any statement of those results which we shall use in the construction. In particular, the entropy of T is still equal to zero.

Let us start with the construction of β_1. For that purpose, let α_1 be given and let δ_1 and δ_1' be defined as

$$\delta_1 := 2^{-2(p_0+1)} \qquad \text{and} \qquad \delta_1' := \frac{\delta_1}{14}\,.$$

It is obvious that δ_1 satisfies condition (γ_2). According to the Shannon-McMillan-Breiman theorem, we then find a positive integer m_1 and a $\bigvee_{j=0}^{m_1-1} T^{-j}\alpha_1$-set S_1 such that

$$\mu(S_1) > 1 - \delta_1'$$

as well as that each atom $A_i \in \bigvee_{j=0}^{m_1-1} T^{-j}\alpha_1$ with $A_i \subseteq S_1$ satisfies the relation

$$\mu(A_i) > 2^{-\lfloor m_1\cdot\delta_1'\rfloor - 1}\,. \tag{§}$$

Without loss of generality, we can assume that

$$m_1 \geq 7\cdot\left(\left\lfloor\frac{1}{\delta_1'}\right\rfloor + 1\right) + 1\,. \tag{§§}$$

Moreover, according to corollary 2.2.3, we also find an $(m_1, 2\delta_1')$-forward-Rokhlin tower which is induced by a measurable set $F_1 \subseteq S_1$. Due to relation (§), we then have as well that the induced partition

$$\alpha_1^{m_1} := \bigvee_{j=0}^{m_1-1} T^j\alpha_1 \cap T^{m_1-1}F_1$$

of the set $T^{m_1-1}F_1$ has at most $2^{\lfloor m_1\cdot\delta_1'\rfloor+1}$ atoms A_i^1. By using the first m_1 backward images of $\alpha_1^{m_1}$, we assign to each of these atoms A_i^1 the uniquely determined sequence

$$s_1(A_i^1) := (A_i^1, T^{-1}A_i^1, \ldots, T^{-m_1+1}A_i^1)\,.$$

Since T is an automorphism, it is clear that, for all atoms A_i^1 the sequence $s_1(A_i^1)$ consists of m_1 mutually disjoint and measurable sets with the property that the set $T^{-k}A_i^1$ is an atom of the induced partition

$$\bigvee_{j=-k}^{m_1-(k+1)} T^j\alpha_1 \cap T^{m_1-(k+1)}F_1$$

of the set $T^{m_1-(k+1)}F_1$ for each $k \in \{0,\dots,m_1-1\}$. With the aid of the sequences $s_1(A_i^1)$, we now construct a partition β_1 and determine a positive integer p_1 such that the conditions (γ_1) and (γ_3)-(γ_6) are fulfilled. For that, we first encode the sequences $s_1(A_i^1)$ by the coding map

$$e_0 : \left\{T^{-k}A_i^1,\ k \in \{0,\dots,m_1-1\},\ A_i^1 \in \alpha_1^{m_1}\right\} \to \{0,\,1\},$$

which maps the set $T^{-k}A_i^1$ to the index of that atom of $\beta_0 = \{B_0^0 = X, B_1^0 = \emptyset\}$ which covers it. More precisely, for each atom $A_i^1 \in \alpha_1^{m_1}$, we assign the word

$$w_1'(A_i^1) := (e_0(A_i^1), e_0(T^{-1}A_i^1), \dots, e_0(T^{-m_1+1}A_i^1))$$

to the sequence $s_1(A_i^1)$. Based on this encoding, we then define a new encoding c_1 by changing every word $w_1'(A_i^1)$ on its first $1+\ell_1+3$ positions where ℓ_1 is given by $\ell_1 := 3(\lfloor m_1 \cdot \delta_1' \rfloor + 1)$. These changes occur according to the following rules:

$$\begin{aligned}
&f_1 : \{A_i^1,\ A_i^1 \in \alpha_1^{m_1}\} \to \{0\};\\
&g_1 : \{(T^{-3k-1}A_i^1, \dots, T^{-3(k+1)}A_i^1),\ 0 \le k \le \frac{\ell_1}{3} - 1, A_i^1 \in \alpha_1^{m_1}\} \to \{010, 100\};\\
&h_1 : \{(T^{-\ell_1-1}A_i^1, \dots, T^{-\ell_1-3}A_i^1),\ A_i^1 \in \alpha_1^{m_1}\} \to \{110\}\,.
\end{aligned}$$

Since there are at most $2^{\lfloor m_1 \cdot \delta_1' \rfloor + 1}$ different sequences $s_1(A_i^1)$, we can choose g_1 in such a way that, for each atom A_i^1 the word

$$w_1(A_i^1) := (c_1(A_i^1), c_1(T^{-1}A_i^1), \dots, c_1(T^{-m_1+1}A_i^1))$$

which is given by the new encoding c_1 is uniquely assigned to the sequence $s_1(A_i^1)$. Because all positions of the words $w_1(A_i^1)$ correspond to a measurable subset of the Rokhlin tower, we then obtain a partition $\beta_1 = \{B_0^1, B_1^1\}$ by setting

$$B_1^1 := {c_1}^{-1}(\{1\}) \cup \left(B_1^0 \cap \left(X \setminus \bigcup_{j=0}^{m_1-1} T^j F_1\right)\right) \quad \text{and} \quad B_0^1 := X \setminus B_1^1\,. \qquad (*)$$

According to this construction, the partition β_1 satisfies the relation

$$\bigvee_{j=-m_1+(k+1)}^{k} T^j\alpha_1 \ \cap\ T^kF_1 \le \bigvee_{j=0}^{m_1-(k+1)} T^{-j}\beta_1 \ \cap\ T^kF_1 \qquad (**)$$

for each $0 \le k \le k_1 := m_1 - (1 + \ell_1 + 3)$. Furthermore, due to the definition of f_1, g_1, and h_1 and the fact that the β_0-names of μ-a.a. $x \in X$ are identical to the word $(000\cdots)$, we also obtain that β_1 fulfils the condition (γ_6). In particular, for μ-a.a. $x \in X$ the 'entrance' of x into the set $T^{k_1}F_1$ is uniquely determined by the occurrence of the event 011 in its β_1-name $\hat{x}_{\beta_1}$. But this means that two different points $x, y \in \bigcup_{j=0}^{k_1} T^jF_1$ with identical (β_1, m_1)-names have to be elements of

the same set $T^j F_1$ where $0 \leq j \leq k_1$. Due to relation $(**)$, we then have that $(\hat{x}_\alpha)_0 = (\hat{y}_\alpha)_0$. Consequently, the partition β_1 even satisfies the relation

$$\alpha_1 \cap \bigcup_{j=0}^{k_1} T^j F_1 \leq \bigvee_{j=0}^{m_1-1} T^{-j}\beta_1 \cap \bigcup_{j=0}^{k_1} T^j F_1 \,.$$

This, however, implies that the partition $\bigvee_{j=0}^{m_1-1} T^{-j}\beta_1$ is finer than the partition α_1 on a measurable set whose mass is greater than

$$\begin{aligned} 1 - 2\cdot\delta_1' - \frac{1+3(\lfloor m_1\cdot\delta_1'\rfloor+1)+3}{m_1-1} &> 1 - 2\cdot\delta_1' - \frac{7}{m_1-1} - 3\frac{m_1}{m_1-1}\delta_1 \\ &> 1 - 9\cdot\delta_1' \\ &> 1 - \delta_1 \,. \end{aligned}$$

Moreover, we have built the partition β_1 by changing the partition β_0 on a measurable set whose mass is less than

$$\frac{1+3(\lfloor m_1\cdot\delta_1'\rfloor+1)+3}{m_1-1} < 7\cdot\delta_1' \leq \frac{1}{2}\cdot\delta_1 \,.$$

Consequently, according to $(\S\S)$ and lemma 2.3.1 *ii*), we obtain that the partition β_1 and the integer $p_1 := m_1 - 1$ satisfy the conditions (γ_1) and (γ_3)-(γ_6).

For the induction step, let us assume that, for an $n \in \mathbb{N}$ we already have determined the numbers $\{p_0, \ldots, p_n\}$, $\{\delta_1, \ldots, \delta_n\}$, and have constructed partitions $\beta_1 = \{B_0^1, B_1^1\}, \ldots, \beta_n = \{B_0^n, B_1^n\}$ such that the conditions (γ_1)-(γ_6) are fulfilled. Without loss of generality, we can also assume that $\alpha_n \vee \beta_n \leq \alpha_{n+1}$. We then define

$$\delta_{n+1} := 2^{-2(p_n+1)} \qquad \text{and} \qquad \delta_{n+1}' := \frac{\delta_{n+1}}{14} \,.$$

It is clear that δ_{n+1} satisfies condition (γ_2). In the same way as in the beginning of the induction, a combination of the Shannon-McMillan-Breiman theorem and corollary 2.2.3 yields an integer m_{n+1} and an $(m_{n+1}, 2\delta_{n+1}')$-forward-Rokhlin tower which is induced by a set F_{n+1} with the property that the induced partition

$$\alpha_{n+1}^{m_{n+1}} := \bigvee_{j=0}^{m_{n+1}-1} T^j \alpha_{n+1} \cap T^{m_{n+1}-1} F_{n+1}$$

of $T^{m_{n+1}-1}F_{n+1}$ possesses at most $2^{\lfloor m_{n+1}\cdot\delta_{n+1}'\rfloor+1}$ atoms A_i^{n+1}. Moreover, without loss of generality, we can assume as well that

$$m_{n+1} \geq (n+7)\left(\left\lfloor\frac{1}{\delta_{n+1}'}\right\rfloor + 1\right) + 1 \,. \qquad (\$)$$

Due to the choice of δ_{n+1}, we then have that $p_n < m_{n+1}$. By using the first m_{n+1} backward images of $\alpha_{n+1}^{m_{n+1}}$, we assign to each of these atoms A_i^{n+1} a unique sequence

$$s_{n+1}(A_i^{n+1}) := (A_i^{n+1}, T^{-1}A_i^{n+1}, \ldots, T^{-m_{n+1}+1}A_i^{n+1}) .$$

Since T is an automorphism, we have that the sequence $s_{n+1}(A_i^{n+1})$ consists of m_{n+1} mutually disjoint and measurable sets with the property that, for every $0 \le k \le m_{n+1}-1$ the set $T^{-k}A_i^{n+1}$ is an atom of the induced partition

$$\bigvee_{j=-k}^{m_{n+1}-(k+1)} T^j\alpha_{n+1} \quad \cap T^{m_{n+1}-(k+1)}F_{n+1}$$

of the set $T^{m_{n+1}-(k+1)}F_{n+1}$. In order to determine an integer p_{n+1} and to construct a partition β_{n+1} such that the conditions (γ_1) and (γ_3)-(γ_6) are satisfied, we first encode the sequences $s_{n+1}(A_i^{n+1})$ with the aid of the partition β_n. For that, we define the map

$$e_n := \{T^{-k}A_i^{n+1},\ 0 \le k \le m_{n+1}-1, A_i^{n+1} \in \alpha_{n+1}^{m_{n+1}}\} \to \{0,1\}$$

which maps the set $T^{-k}A_i^{n+1}$ to the index of that atom of β_n which covers it. Since $\beta_n \le \alpha_{n+1}$, the map e_n is well-defined and yields a non-injective encoding of the sequences $s_{n+1}(A_i^{n+1})$ by words

$$w'_{n+1}(A_i^{n+1}) := (e_n(A_i^{n+1}), \ldots, e_n(T^{-m_{n+1}+1}A_i^{n+1}))$$

of length m_{n+1}. Based on this encoding, we define a new encoding c_{n+1} by changing the words $w'_{n+1}(A_i^{n+1})$ on their first $1+\ell_{n+1}+(n+3)$ positions where $\ell_{n+1} := 3(\lfloor m_{n+1}\cdot\delta'_{n+1}\rfloor+1)$. These changes occur by means of the following three rules:

$$f_{n+1}\colon \{A_i^{n+1},\ A_i^{n+1} \in \alpha_{n+1}^{m_{n+1}}\} \to \{0\};$$
$$g_{n+1}\colon \{(T^{-3k-1}A_i^{n+1}, \ldots, T^{-3(k+1)}A_i^{n+1}), 0 \le k \le \ell'_{n+1}, A_i^{n+1} \in \alpha_{n+1}^{m_1}\} \to \{010, 100\};$$
$$h_{n+1}\colon \{(T^{-\ell_{n+1}-1}A_i^{n+1}, \ldots, T^{-\ell_{n+1}-(n+3)}A_i^{n+1}),\ A_i^1 \in \alpha_1^{m_1}\} \to \{\underbrace{111\cdots1}_{n+1}10\},$$

where $\ell'_{n+1} := \frac{\ell_{n+1}}{3} - 1$. Since there are at most $2^{\lfloor m_{n+1}\cdot\delta'_{n+1}\rfloor+1}$ different sequences $s_{n+1}(A_i^{n+1})$, we can choose the encoding g_{n+1} in such a way that, for each atom A_i^{n+1} the new word

$$w_{n+1}(A_i^{n+1}) := (c_{n+1}(A_i^{n+1}), \ldots, c_{n+1}(T^{-m_{n+1}+1}A_i^{n+1}))$$

which is given by the encoding c_{n+1} is uniquely assigned to $s_{n+1}(A_i^{n+1})$. Because all positions of the words $w_{n+1}(A_i^{n+1})$ corresponds to a measurable subset of the Rokhlin tower, we then get a partition $\beta_{n+1} = \{B_0^{n+1}, B_1^{n+1}\}$ by setting

$$B_1^{n+1} := {c_{n+1}}^{-1}(\{1\}) \cup \left(B_1^n \cap \left(X \setminus \bigcup_{j=0}^{m_{n+1}-1} T^jF_{n+1}\right)\right) \quad \text{and} \quad B_0^{n+1} := X \setminus B_1^{n+1} .$$

According to our construction, the partition β_{n+1} satisfies the relation

$$\bigvee_{j=-m_{n+1}+(k+1)}^{k} T^j\alpha_{n+1} \;\cap\; T^kF_{n+1} \le \bigvee_{j=0}^{m_{n+1}-(k+1)} T^{-j}\beta_{n+1} \;\cap\; T^kF_{n+1} \qquad (\$\$)$$

for all $0 \le k \le k_{n+1} := m_{n+1} - (1 + \ell_{n+1} + n + 3)$. Moreover, due to its definition, the encoding c_{n+1} results for μ-a.a. $x \in X$ in a replacement of finite subsequences in the β_n-name $\hat{x}_{\beta_n}$ by finite sequences of the same length which start and end with the number 0. Together with the assumption of the induction step - for μ-a.a. $x \in X$ the β_n-name $\hat{x}_{\beta_n}$ does not contain a subsequence of length $n+2$ which exclusively consists of the symbol 1 -, this implies that β_{n+1} fulfils condition (γ_6). In particular, the 'entrance' into the set $T^{k_{n+1}}F_{n+1}$ is uniquely determined by the occurrence of the subsequence

$$01\underbrace{11\cdots11}_{n+1}\,.$$

But this means that two different points $x, y \in \bigcup_{j=0}^{k_{n+1}} T^jF_{n+1}$ with identical (β_{n+1}, m)-names must be elements of the same set T^jF_{n+1} where $0 \le j \le k_{n+1}$. Due to ($\$\$$), we can deduce that $(\hat{x}_{\alpha_{n+1}})_0 = (\hat{y}_{\alpha_{n+1}})_0$. Consequently, the partition β_{n+1} fulfils the relation

$$\alpha_{n+1} \cap \bigcup_{j=0}^{k_{n+1}} T^jF_{n+1} \le \bigvee_{j=0}^{m_{n+1}-1} T^{-j}\beta_{n+1} \;\cap\; \bigcup_{j=0}^{k_{n+1}} T^jF_{n+1}\,.$$

Therefore, the partition $\bigvee_{j=0}^{p_{n+1}} T^{-j}\beta_{n+1}$ is finer than the partition α_{n+1} on a measurable set whose mass is greater than

$$1 - 2\cdot\delta'_{n+1} - \frac{1 + 3(\lfloor m_{n+1}\cdot\delta'_1\rfloor + 1) + n + 3}{m_{n+1} - 1} > 1 - 9\cdot\delta'_{n+1} \;>\; 1 - \delta_{n+1}\,.$$

Moreover, we have built the partition β_{n+1} by changing the partition β_n on a measurable set whose mass is less than

$$\frac{1 + 3(\lfloor m_{n+1}\cdot\delta'_{n+1}\rfloor + 1) + n + 3}{m_{n+1} - 1} < 7\cdot\delta'_{n+1} \le \frac{1}{2}\cdot\delta_{n+1}\,.$$

Consequently, according to ($\$$) and lemma 2.3.1 *ii*), the numbers $p_{n+1} := m_{n+1} - 1$, δ_{n+1}, and the partition β_{n+1} fulfil the conditions (γ_1)-(γ_6). □

Remark 2.8. In order to obtain a construction which is also applicable in the non-ergodic case, it is sufficient to apply the ideas above to the ergodic decomposition of a non-ergodic T. For more details we refer the interested reader to §30 in [DGS76].