Part II

The Theory of Relative Entropy

Apart from the problem of deciding if two systems represent the same dynamics, a second main problem in ergodic theory is to exactly understand the dynamics of a system at all. A canonical approach to this question is to consider systems whose dynamics are 'closely related' to the original one. For example, the action of one coordinate in the direct product of two dynamical systems coincides with one action of the underlying dynamical systems. Therefore, in order to completely understand the action of the direct product it is sufficient to concentrate on the dynamics of the underlying spaces. However, a product structure is not typical for dynamical systems. An appropriate description of 'closely related dynamics' has been given by the notion of *factors* which is a weakening of the notion of isomorphism.

Definition II.1. Let $(X, \mathfrak{B}, \mu, T)$ and $(Y, \mathfrak{D}, \nu, S)$ be two dynamical systems. We say that S is a *factor of* T if there are sets $X' \in \mathfrak{B}$ and $Y' \in \mathfrak{D}$ with $\mu(X') = \nu(Y') = 1$, and there exists a measure-preserving map $\pi : X' \to Y'$ such that $\pi \circ T = S \circ \pi$.

The benefit of factors for the problem of understanding the dynamics of a system is well recognisable at that type of factors which are given by invariant partitions. Namely, if α is an *invariant partition of* X, i.e. $T^{-1}\alpha \leq \alpha$, then T induces a map $T_\alpha : X_\alpha \to X_\alpha$ on the factor space X_α by

$$T_\alpha(C) := A, \text{ if } C \subseteq T^{-1}A\,.$$

Due to the projection $f_\alpha : X \to X_\alpha$, the map T_α is obviously both an endomorphism on the Lebesgue space $(X_\alpha, \mathfrak{B}_\alpha, \mu_\alpha)$ and a factor of T. Moreover, it is clear that the endomorphism T_α represents the original dynamics of T w.r.t. the partition α. Since, for a Lebesgue space $(X, \mathfrak{B}, \mu)$, each factor S of T is isomorphic to a factor T_α (cf. §2.3 in [Wal82]), we can even say that a factor S renders the dynamics of T up to an invariant sub-σ-algebra of $\mathfrak{B}$. However, it is also clear that pure factors S, i.e. $S \neq T$, exclusively provide sub-σ-algebras which are different from $\mathfrak{B}$. Therefore, in order to completely understand the action of T under the boundary condition that the dynamics of a factor S is known, it remains to investigate the dynamics of T relative to the dynamics of S. Considerations basing on this so-called *relative setup* are known in ergodic theory as *the ergodic theory of relative dynamical systems* or, shortly, *the relative ergodic theory*. Since relative dynamical systems generalise the classical setup, an increasing interest in these setups can be observed.

In the last two decades the main attention in the relative ergodic theory has focused on those problems which are connected with a thermodynamic formalism for relative setups. By their joint work in [LW77], F. Ledrappier and P. Walters started with these activities. Under the assumption that the underlying dynamics are given by automorphisms, they proved a relative variational principle which generalised both Bowen's work in [Bow71] and Walters' classical

variational principle (cf. [Wal76]). In order to do that, Ledrappier and Walters invented a definition of *relative entropy* which partially extended Rokhlin's classical concept of entropy as well as Abramov and Rokhlin's idea of mixed entropy. Twelve years later, similar to Ledrappier and Walters' definition, Kaminski formulated a version of relative entropy for group actions (cf. [Kam89]). However, Kaminski's definitions of relative entropy also strongly relied on the assumption that the underlying dynamics are invertible. The only formulation of relative entropy which was not completely based on this strong assumption was introduced by Bogenschütz in 1992 (cf. [Bog92]). But Bogenschütz's version still claimed that at least the factor is invertible. Irrespectively of these different versions of relative entropy, the generator-problem in a relative setup was only considered for Kaminski's definition by himself. The main result of Kaminski's considerations states the existence of *relative two-sided generators* with finite entropy.

In the following second part of this thesis, we shall introduce a concept of relative entropy which neither assumes that T is an automorphism nor supposes that its factor S is invertible. Moreover, we will treat the generator-problem for fibred dynamical systems. For that, we shall start in chapter 3 with a brief survey of the usual setups in the theory of relative dynamical systems, namely *skew products* and *fibred dynamical systems*. Subsequently, based on the classical conditional entropy, we shall formulate the canonical concept of *η-finite entropy* in section 4.1. Founded on this, section 4.2 will serve to give a comprehensive introduction to our generic version of relative entropy. We shall see that it extends Ledrappier and Walters' definition as well as Abramov and Rokhlin's mixed entropy. Consistently, the main result of this section will be a generalisation of the Abramov-Rokhlin formula to fibred dynamical systems. In order to obtain this result, we shall also prove that the relative entropy of a fibred dynamical system and the relative entropy of the corresponding natural extension possess the same value. Finally, in section 4.3, we will be concerned with the (relative) generator-problem for fibred dynamical systems. In theorem 4.3.7 we shall show that unilateral relative generators with finite entropy even exists. In the case of fibred dynamical systems, this theorem generalises Kaminski's main result in [Kam89].

Since in the remaining part of this thesis two dynamical systems will often be considered at the same time, we slightly modify some of our notations in order to avoid confusions. For a Lebesgue space $(X, \mathfrak{B}, \mu)$ we denote the point partition and the trivial partition of X, respectively, by ϵ_X and ϑ_X. In the same fashion, $\mathcal{P}_X$ denotes the set of all partitions of X. Furthermore, we write $\alpha_T^n := \bigvee_{j=0}^{n-1} T^{-j}\alpha$ and $\alpha_T^- := \bigvee_{j=0}^{\infty} T^{-j}\alpha$ in order to emphasise the endomorphism that is used in the refinement.

Chapter 3

Relative Dynamics

The aim of this chapter is to briefly recall those two setups which are usually meant by speaking about relative dynamics, namely skew products of endomorphisms and fibred dynamical systems. For this purpose, in section 3.1 we shall be first concerned with skew products. Readers not familiar with this type of material or readers more interested in this mathematical area are referred to [AR66], [Kow87], or [DH99]. Afterwards, section 3.2 will be devoted to the topical and more general setup of fibred dynamical systems. Readers more interested in this topic, especially interested in the thermodynamic formalism for fibred dynamical systems are referred to [BG95], [DG99], [DG98], and [DGH00].

3.1 Skew Products

For the first time the idea of skew products was formulated for a special class of dynamical systems by H. Anzai (cf. [Anz51]) in 1951. Eight years later L.M. Abramov and V.A. Rokhlin extended Anzai's ideas to more general situations (cf. [AR66]). By this work, they established a first non-trivial and non-special class of coupled dynamical systems. We use this first section to recall the definition of skew products following the ideas and terminology of Abramov and Rokhlin.

Let us consider a Lebesgue space $(X,\, \mathfrak{B},\, \mu)$ where the Lebesgue-σ-algebra $\mathfrak{B}$ is equipped with the usual metric ρ given by

$$\rho(A,B) := \mu(A \,\triangle\, B)\,.$$

Furthermore, let us subdivide the family of all endomorphisms of the space X into classes by identifying those endomorphisms which differ only on a set of μ-measure zero. We denote the set of all these classes by $E(X)$. With the aid of the metric ρ, the set $E(X)$ then also becomes a metric space. More precisely, for

instance, a metric is given by

$$\varrho(T, T') := \sum_{k=1}^{\infty} 1/2^k \cdot \rho(T^{-1}(A_k), T'^{-1}(A_k)),$$

where $A_1, A_2, \ldots$ is any sequence of sets dense in $\mathfrak{B}$. Due to this consideration, we are able to speak about measurable mappings into the space $E(X)$.

Definition 3.1.1. Let $(X, \mathfrak{B}, \mu)$ and $(Y, \mathfrak{D}, \nu)$ be to Lebesgue spaces. Then we call a measurable mapping of the space Y into $E(X)$ a *measurable family of endomorphisms of the space* X (defined on Y).

Now let us consider two Lebesgue spaces $(X, \mathfrak{B}, \mu)$ and $(Y, \mathfrak{D}, \nu)$. Furthermore let S be an endomorphism of the space Y and let $\{T_y,\ y \in Y\}$ be a measurable family of endomorphisms of the space X. We then know that the direct product $(M, \mathfrak{M}, \lambda) := (X \times Y, \mathfrak{B} \otimes \mathfrak{D}, \mu \otimes \nu)$ of the spaces $(X, \mathfrak{B}, \mu)$ and $(Y, \mathfrak{D}, \nu)$ is again a Lebesgue space (cf. [Rok66]). Moreover, if for each $y \in Y$ we understand by T_y not a class of endomorphism, but an arbitrary individual endomorphism of the corresponding class, the formula

$$U(x,\ y) := (T_y(x),\ S(y))$$

defines a certain mapping U of the space M. But the mapping U obviously depends on the choice of the endomorphism T_y in its class. Hence, this definition actually leads to several different mappings U which are in general not endomorphisms as well as they can differ on non-measurable sets. Nevertheless, the following theorem states that for 'one' representation $\mathbb{T} := \{T_y,\ y \in Y\}$ the formula above gives an endomorphism $U : M \to M$.

Theorem 3.1.2. *Let $(X, \mathfrak{B}, \mu)$ and $(Y, \mathfrak{D}, \nu)$ be two Lebesgue spaces, let S be an endomorphism of Y, and let $\{T_y,\ y \in Y\}$ be a measurable family of endomorphisms of the space X. Then amongst all representations of $\{T_y,\ y \in Y\}$ there exists a unique (up to sets of μ-measure zero) representation $\mathbb{T} := \{T_y,\ y \in Y\}$ such that the map U defined as above is an endomorphism on the direct product $(M, \mathfrak{M}, \lambda)$.*

Proof. See section 1.2 in [AR66]. □

According to this result we are now able to introduce the definition of a skew product.

Definition 3.1.3. Let $(M, \mathfrak{M}, \lambda)$ the direct product of two Lebesgue spaces $(X, \mathfrak{B}, \mu)$ and $(Y, \mathfrak{D}, \nu)$. Furthermore let S be an endomorphism of Y and let $\mathbb{T} = \{T_y,\ y \in Y\}$ be a measurable family of endomorphisms of the space X. Then the endomorphism U defined by

$$U(x,\ y) := (T_y(x),\ S(y))$$

is called *the skew product with base S and fibres $\mathbb{T} = \{T_y,\ y \in Y\}$*.

Following its definition, skew products seem to be very special dynamical systems. However, the following result states that under some weak additional assumptions each endomorphisms can be decomposed into a skew product.

Theorem 3.1.4. *Let $(M, \mathfrak{M}, \lambda, U)$ be a dynamical system and let ζ be a partition of M which possesses an independent complement ζ' and is completely invariant, i.e. $U^{-1}\zeta = \zeta$. Then the endomorphism U decomposes into a skew product with base U_ζ and fibres $\{T_C := W_{U_\zeta C} \circ U \circ W_C^{-1},\ C \in M_\zeta\}$, where W_C is the isomorphic mapping of the space C onto $M_{\zeta'}$.*

Proof. See section 1.3 in [AR66]. □

In the remaining part of this section we will be concerned with the entropy of a skew product. For that, let $(M, \mathfrak{M}, \lambda)$ be a direct product of spaces X and Y with measures μ and ν. Furthermore, let U be a skew product on M with base $S : Y \to Y$ and fibres $\mathbb{T} = \{T_y : X \to X,\ y \in Y\}$.

The most important property of the entropy of a skew product is that, apart from the usual calculation of entropy by following its definition, it also can be calculated in a similar way to the entropy of a direct product of endomorphisms (cf. proposition 0.3.17). More precisely, the entropy of U can be splited into the entropy of the base S and the so-called mixed entropy of $\mathbb{T}$ with respect to S. The mixed entropy is a notion of entropy which depends on the base S as well as on the fibres $\mathbb{T}$. In the following we have a look at the idea of mixed entropy.

In order to briefly recall the terminology of mixed entropy, let us consider a partition α of X with finite entropy. With the aid of $\mathbb{T}$, we define a *finite refinement of α with respect to the measurable family* $\mathbb{T}$, and an *infinite refinement of α w.r.t.* $\mathbb{T}$, respectively, by setting

$$\alpha_y^n := \bigvee_{j=0}^{n-1} T_y^{-1} \circ T_{Sy}^{-1} \circ \cdots \circ T_{S^{j-1}y}^{-1} \alpha \quad \text{and} \quad \alpha_y^- := \bigvee_{j=0}^{\infty} T_y^{-1} \circ T_{Sy}^{-1} \circ \cdots \circ T_{S^{j-1}y}^{-1} \alpha\,.$$

Due to our choice of α and the fact that each fibre $T_{S^k y}$ is measure-preserving, proposition 0.3.6 gives us that, for all $n \in \mathbb{N}$ and all $y \in Y$, the entropy of α_y^n satisfies the relation

$$H_\mu(\alpha_y^n) < \infty\,.$$

Moreover, we even obtain that for all $n \in \mathbb{N}$

$$E_n := 1/n \int_Y H_\mu(\alpha_y^n)\, d\nu < \infty$$

holds. Using these facts, we define the so-called mixed entropy of a partition $\alpha \in \mathcal{Z}(X)$.

Definition 3.1.5. Let U be a skew product on the direct product of $(X, \mathfrak{B}, \mu)$ and $(Y, \mathfrak{D}, \nu)$ with base $S : Y \to Y$ and fibres $\mathbb{T} = \{T_y : X \to X,\ y \in Y\}$. Furthermore let α be a partition of X with finite entropy. Then we call

$$h(\mathbb{T} \,|\, S, \alpha) := \inf_n 1/n \int_Y H_\mu(\alpha_y^n)\, d\nu$$

the *mixed entropy of the partition α with respect to the fibres $\mathbb{T}$ and the base S.*

The following proposition shows that the mixed entropy of α has properties which are analogous to those of the classical entropy of α.

Proposition 3.1.6. *Let U be a skew product with base $S : Y \to Y$ and fibres $\mathbb{T} = \{T_y : X \to X,\ y \in Y\}$ and let α be a partition of X with finite entropy. Then the mixed entropy of α satisfies the relations*

$$h(\mathbb{T} \,|\, S, \alpha) = \lim_{n\to\infty} 1/n \int_Y H_\mu(\alpha_y^n)\, d\nu$$

and

$$h(\mathbb{T} \,|\, S, \alpha) = \int_Y H_\mu(\alpha \,|\, T_y^{-1}\alpha_{Sy}^-)\, d\nu\,.$$

Proof. See section 2.2 in [AR66]. □

Based on definition 3.1.5, we are now able to define the mixed entropy of $\mathbb{T}$ with respect to S.

Definition 3.1.7. Let U be a skew product on the direct product of $(X, \mathfrak{B}, \mu)$ and $(Y, \mathfrak{D}, \nu)$ with base $S : Y \to Y$ and fibres $\mathbb{T} = \{T_y : X \to X,\ y \in Y\}$. Then we call

$$h(\mathbb{T} \,|\, S) := \sup\{h(\mathbb{T} \,|\, S, \alpha),\ \alpha \in \mathcal{Z}(X)\}$$

the mixed entropy of the fibres $\mathbb{T}$ with respect to the base S.

Remark 3.1. The idea of mixed entropy was the first notion of entropy in mathematics which takes a coupling of two systems into account.

Finally, the following theorem states the most important property of mixed entropy.

Theorem 3.1.8 (Abramov-Rokhlin Formula). *Let U be a skew product on the direct product of $(X, \mathfrak{B}, \mu)$ and $(Y, \mathfrak{D}, \nu)$ with base $S : Y \to Y$ and fibres $\mathbb{T} = \{T_y : X \to X,\ y \in Y\}$. Then the entropy $h_{\mu\otimes\nu}(U)$ of U satisfies the relation*

$$h_{\mu\otimes\nu}(U) = h(\mathbb{T} \,|\, S) + h_\nu(S)\,.$$

Proof. See section 2.3 in [AR66]. □

3.2 Fibred Dynamical Systems

The most general setup which relative ergodic theory deals with is the setup of fibred dynamical systems. In the following section we shall give a brief survey of the terminology and the most important results in the study of these systems.

Definition 3.2.1. Let $(X, \mathfrak{B}, \mu, T)$ and $(Y, \mathfrak{D}, \nu, S)$ be two dynamical systems such that S is a factor of T where $\pi : X \to Y$ denotes the factor map. Then the system $\mathfrak{Y} = (X, T; Y, S; \pi)$ is called a *(measure-theoretical) fibred dynamical system*. Furthermore, we say that S is the *base (transformation) of* $\mathfrak{Y}$ and that $\pi^{-1}(y)$ is a *fibre in* X. If S is even an automorphism, then $\mathfrak{Y}$ is called a *fibred dynamical system with an invertible base* S.

Remark 3.2. *i*) According to the definition 3.2.1, each skew product $U : M \to M$ is a fibred dynamical system $\mathfrak{Y} = (M, U; Y, S, \pi)$. The base transformation S and the factor map $\pi : M \to Y$ are given by the base S of the skew product and the projection of the second coordinate, respectively.
ii) By setting $(Y, \mathfrak{D}, \nu, S) := (\{z\}, \{z, \emptyset\}, \delta_z, Id)$ where δ_z and Id denote the normalised point mass at z and the identity map, respectively, each dynamical system $(X, \mathfrak{B}, \mu, T)$ can be understood as a fibred dynamical system.

Remark 3.3. Let $\mathfrak{Y} = (X, T; Y, S; \pi)$ be a fibred dynamical system. The natural extension $\hat{\mathfrak{Y}} = (\hat{X}, \hat{T}; \hat{Y}, \hat{S}; \hat{\pi})$ of $\mathfrak{Y}$ is then given by the natural extensions $(\hat{X}, \hat{\mathfrak{B}}, \hat{\mu}, \hat{T})$ and $(\hat{Y}, \hat{\mathfrak{D}}, \hat{\nu}, \hat{S})$ and the factor map $\hat{\pi} : \hat{X} \to \hat{Y}$ defined by $\hat{\pi}((x_i)_{i \in \mathbb{N}_0}) = (\pi(x_i))_{i \in \mathbb{N}_0}$. Moreover, the natural extensions $\widehat{\epsilon_X}$ and $\widehat{\epsilon_Y}$ of the corresponding point partitions are both exhaustive partitions with the property that

$$\hat{\pi}^{-1}(\widehat{\epsilon_Y}) \leq \widehat{\epsilon_X} \, .$$

A fruitful and still lively field of the study of fibred dynamical systems is given by the adaption of the thermodynamic formalism to relative ergodic theory. For that, one has taken the following more topological definition of fibred dynamical systems as the basis for all considerations.

Definition 3.2.2. Let X and Y be two compact metric spaces. Furthermore, let $T : X \to X$, $S : Y \to Y$, and $\pi : X \to Y$ be continuous and surjective maps such that $\pi \circ T = S \circ \pi$. We then say that the system $\mathfrak{Y} = (X, T; Y, S; \pi)$ is a *compact fibred dynamical system*, or, shortly, *compact fibred system*.

Chronologically, the first important result in this context was discovered by W. Ledrappier and P. Walters (cf. [LW77]) who established a relative variational principle for compact fibred systems.

Theorem 3.2.3. *Let $\mathfrak{Y} = (X, T; Y, S; \pi)$ be a compact fibred system endowed with homeomorphisms T and S. Furthermore let ν be a measure on Y such that*

S is measure-preserving with respect to ν. Then, for any real-valued continuous function $f \in C(X)$, the relation

$$\sup_{\substack{\mu\circ\pi^{-1}=\nu \\ \mu\circ T^{-1}=\mu}} \left\{ h_\mu(T|S) + \int_X f\, d\mu \right\} = \int_Y P(T, f, \pi^{-1}(y))\, d\nu$$

holds.

Proof. See theorem 2.1 in [LW77]. □

A main part of the theory of thermodynamic formalism is concerned with the existence of equilibrium measures and their properties. Uniqueness and the Gibbs property are the most important amongst all these properties. In the relative setup, these questions has been considered in [BG95], [DG99], [DG98], and [DGH00].

Theorem 3.2.4. *Let $\mathfrak{Y} = (X, T; Y, S; \pi)$ be a (compact) fibred system with metric d on X. Furthermore let $\mathfrak{Y}$ be fibrewise expanding and topologically exact along fibres. Then, for every Hölder continuous function $\varphi : X \to \mathbb{R}$, there exists a unique Gibbs family $\{\mu_y : \ y \in Y\}$ for φ. The function A_φ is also unique and one has that $supp\{\mu_y\} = X_y$.*

Proof. See theorem 2.6 in [DG99] □

Moreover, M. Denker, M. Gordin, and St.-M. Heinemann discovered the following relation to the relative variational principle.

Theorem 3.2.5. *Let $\varphi : X \to \mathbb{R}$ be a Hölder continuous function. Then, for any S-invariant measure ν on Y, the relation*

$$\sup_{\substack{\mu\circ\pi^{-1}=\nu \\ \mu\circ T^{-1}=\mu}} \left\{ h_\mu(T|S) + \int_X \varphi\, d\mu \right\} = \int_Y \log A_\varphi\, d\nu$$

is satisfied.

Proof. See theorem 4.4 in [DGH00]. □

Chapter 4

Relative Entropy

In the following chapter we shall present a generic concept of relative entropy for fibred dynamical systems. Furthermore, we will treat the generator-problem in this setup. For that purpose, in section 4.1 we shall start with a generalisation of Rokhlin's classical idea of partitions with finite entropy. Based on this first section, we will subsequently develop in section 4.2 our concept of relative entropy. In contrary to the concepts invented before, our approach do not suppose that at least one of the underlying dynamics is invertible. The main result of this second section will be an extension of the Abramov-Rokhlin formula (cf. theorem 3.1.8) to fibred dynamical systems. Finally in section 4.3, we complete this chapter by considering the (corresponding) generator-problem for fibred dynamical systems. We shall prove the existence of unilateral relative generators with finite entropy.

4.1 Partitions with η-Finite Entropy

In the following, we shall change Rokhlin's concept of partitions with finite entropy in such a way that it will respect given measurable structures.

Throughout this section, we base our considerations on a Lebesgue space $(X,\, \mathfrak{B},\, \mu)$ and a partition η of X.

Definition 4.1.1. Let α be a partition of X such that $\eta \leq \alpha$ and $H_\mu(\alpha \,|\, \eta) < \infty$. Then, α is called a *partition of η-finite entropy*. Moreover, we denote the set of all partitions with η-finite entropy by

$$\mathcal{R}_\eta := \{\alpha \in \mathcal{P} : \eta \leq \alpha \text{ and } H_\mu(\alpha \,|\, \eta) < \infty\}.$$

Remark 4.1. Note that neither η nor any $\alpha \in \mathcal{R}_\eta$ have to have finite entropy. Moreover, it is obvious that the set $\mathcal{Z}_\eta := \{\alpha \vee \eta,\ \alpha \in \mathcal{Z}\}$ is a subset of $\mathcal{R}_\eta$.

Now let us consider the space $\mathcal{R}_\eta \times \mathcal{R}_\eta$. With the aid of conditional entropy, we define a non-negative function d_η on it by setting

$$d_\eta(\alpha, \beta) := H_\mu(\alpha \,|\, \beta) + H_\mu(\beta \,|\, \alpha).$$

Due to proposition 0.3.6, the function d_η is well-defined. Moreover, we are able to prove the following property.

Lemma 4.1.1. *The function d_η is a metric on $\mathcal{R}_\eta$.*

Proof. Due to lemma 0.3.2 and its definition, d_η is positively definite and obviously symmetric. Moreover, by proposition 0.3.6, d_η also fulfils the triangle inequality. □

Thus, we have that $(\mathcal{R}_\eta, d_\eta)$ is a metric space.

Remark 4.2. Note that on the space $\mathcal{Z}_\eta$ the metric d_η satisfies the relation

$$d_\eta\left(\alpha \vee \eta, \beta \vee \eta\right) \leq d_\mu\left(\alpha, \beta\right).$$

In the remaining part of this section, we shall show that $(\mathcal{R}_\eta, d_\eta)$ is even a complete and separable space. For this purpose, we first show the following two lemmata.

Lemma 4.1.2. *The set $\mathcal{R}_\eta^\sharp := \left\{\alpha \in \mathcal{R}_\eta : \alpha = \alpha^\sharp \vee \eta \text{ where } \alpha^\sharp \in \mathcal{E}\right\}$ is a dense subset of $\mathcal{R}_\eta$.*

Proof. Let β be an element of $\mathcal{R}_\eta$. Due to theorem 1.2.1, we then find a partition β' with finite entropy and the property that $\beta = \beta' \vee \eta$. Furthermore, since $\mathcal{E}$ is dense in $\mathcal{Z}$, we also find a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of finite partitions with $\lim_{n\to\infty} d_\mu\left(\gamma_n, \beta'\right) = 0$. Consequently, by remark 4.2, we obtain that $(\gamma_n \vee \eta)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{R}_\eta^\sharp$ which converges to β in the sense of d_η. □

Lemma 4.1.3. *Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}$ such that $\alpha_n \nearrow \epsilon$ and each α_n is finer than η. Then the set of all partitions $\beta \in \mathcal{R}_\eta^\sharp$ with the property that $\beta \leq \alpha_n$ for some $n \in \mathbb{N}$ is a dense subset of $\mathcal{R}_\eta$.*

Proof. According to lemma 4.1.2, it is sufficient to show the hypothesis for the set $\mathcal{R}_\eta^\sharp$. Thus, let γ be an element of $\mathcal{R}_\eta^\sharp$. Since $\alpha_n \nearrow \epsilon$, for each $\delta' > 0$ we find an integer $m \in \mathbb{N}$ and α_m-sets A_i such that the relation $\mu(A_i \triangle G_i) < \delta'$ is satisfied for each atom G_i of $\gamma^\sharp = \{G_0, \ldots, G_r\}$. Therefore, for each $\delta > 0$ we obtain a finite partition $\beta^\sharp = \{B_0, \ldots, B_r\}$ by setting

$$B_i := A_i \setminus \bigcup_{j=0}^{i-1} B_j \quad \text{for } 1 \leq i \leq r-1 \qquad \text{and} \qquad B_r := X \setminus \bigcup_{j=1}^{r-1} B_j$$

such that, for δ' small enough, $d_\mu\left(\gamma^\sharp, \beta^\sharp\right) < \delta$ holds. Consequently, we get that the partition $\beta = \beta^\sharp \vee \eta$ is an element of $\mathcal{R}_\eta^\sharp$ which is coarser than α_m and fulfils the condition $d_\eta\left(\beta, \gamma\right) < \delta$. □

After these preparations, we are able to prove the main result of this section.

Theorem 4.1.2. *The metric space $(\mathcal{R}_\eta, d_\eta)$ is separable and complete.*

Proof. First, we show that $\mathcal{R}_\eta$ is separable. For this, we choose a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that $\alpha_n \nearrow \epsilon$. Moreover, we define

$$\mathcal{R}^\sharp_{\eta,\alpha_n} := \left\{\beta \in \mathcal{R}^\sharp_\eta : \beta^\sharp \leq \alpha_m \text{ for some } m \in \mathbb{N}\right\}.$$

According to the proof of lemma 4.1.3 and the fact that all α_n are finite, the set $\mathcal{R}^\sharp_{\eta,\alpha_n}$ is a dense and countable subset of $\mathcal{R}_\eta$.

It remains to show that $\mathcal{R}_\eta$ is complete. In order to do so we consider a Cauchy sequence $(\gamma_n)_{n\in\mathbb{N}}$ in $\mathcal{R}_\eta$. Without loss of generality, we can suppose that for all $n, m \in \mathbb{N}$ the partitions γ_n satisfy the relation $d_\eta(\gamma_n, \gamma_{n+m}) < 2^{-n}$. With the aid of the sequence (γ_n), we then define the partition

$$\gamma := \bigwedge_{\ell=1}^{\infty} \bigvee_{k=\ell}^{\infty} \gamma_k.$$

In the following we shall see that γ is an element of $\mathcal{R}_\eta$ and is equal to the limit of the sequence $(\gamma_n)_{n\in\mathbb{N}}$. Let us have a look at the partition γ. Obviously, γ is finer than η. Due to proposition 0.3.6, we also have that for all $n \in \mathbb{N}$ the relation

$$\sup_{\ell\in\mathbb{N}} H_\mu\left(\gamma_n \,|\, \bigvee_{k=\ell}^{\infty} \gamma_k\right) = H_\mu(\gamma_n \,|\, \gamma) \leq H_\mu(\gamma_n \,|\, \eta) < \infty$$

holds. Therefore, for each $n \in \mathbb{N}$, we find a non-negative integer $m \in \mathbb{N}$ such that

$$H_\mu(\gamma_n \,|\, \gamma) \leq H_\mu\left(\gamma_n \,|\, \bigvee_{k=m}^{\infty} \gamma_k\right) + 2^{-n} < H_\mu(\gamma_n \,|\, \gamma_m) + 2^{-n} < 2^{-n+1}.$$

Furthermore, according to proposition 0.3.6, we have that for every $n \in \mathbb{N}$ and all $\ell > n$ the relation

$$\begin{aligned} H_\mu\left(\bigvee_{k=\ell}^{\infty} \gamma_k \,|\, \bigvee_{j=n}^{\ell-1} \gamma_j\right) &= H_\mu\left(\gamma_\ell \,|\, \bigvee_{j=n}^{\ell-1} \gamma_j\right) + H_\mu\left(\bigvee_{k=\ell+1}^{\infty} \gamma_k \,|\, \bigvee_{j=n}^{\ell-1} \gamma_j\right) \\ &= \sum_{k=\ell}^{\infty} H_\mu\left(\gamma_k \,|\, \bigvee_{j=n}^{k-1} \gamma_j\right) \end{aligned}$$

is satisfied. Hence, by proposition 0.3.6, we get that

$$H_\mu\left(\bigvee_{k=n+1}^{\infty} \gamma_k \,|\, \gamma_n\right) = \sum_{k=n+1}^{\infty} H_\mu\left(\gamma_k \,|\, \bigvee_{j=n}^{k-1} \gamma_j\right) \leq \sum_{k=n+1}^{\infty} H_\mu(\gamma_k \,|\, \gamma_{k-1}) \leq 2^{-n}$$

holds for each $n \in \mathbb{N}$. Since $\gamma \leq \bigvee_{k=n+1}^{\infty} \gamma_k$ for all $n \in \mathbb{N}$, the last relation implies that

$$H_\mu(\gamma \,|\, \gamma_n) \leq H_\mu\left(\bigvee_{k=n+1}^{\infty} \gamma_k \,|\, \gamma_n\right) \leq 2^{-n}\,.$$

Consequently, we finally obtain both that

$$H_\mu(\gamma \,|\, \eta) \leq H_\mu(\gamma_1 \,|\, \eta) + H_\mu(\gamma \,|\, \gamma_1) < \infty$$

and that, for every $n \in \mathbb{N}$

$$d_\eta(\gamma_n, \gamma) < 2^{-n+2}\,.$$

□

Remark 4.3. The conditional entropy $H_\mu(\cdot \,|\, \eta)$ with respect to η can be considered as a function on $\mathcal{R}_\eta$. In particular, $H_\mu(\cdot \,|\, \eta)$ is continuous and satisfies the relation

$$|H_\mu(\alpha \,|\, \eta) - H_\mu(\beta \,|\, \eta)| \leq d_\eta(\alpha, \beta)$$

for all $\alpha, \beta \in \mathcal{R}_\eta$.

Proof. First let us suppose that $H_\mu(\alpha \,|\, \eta) \geq H_\mu(\beta \,|\, \eta)$. In this case we have that

$$|H_\mu(\alpha \,|\, \eta) - H_\mu(\beta \,|\, \eta)| \leq H_\mu(\alpha \vee \beta \,|\, \eta) - H_\mu(\beta \,|\, \eta) \leq H_\mu(\alpha \,|\, \beta)\,.$$

Now let us suppose that $H_\mu(\alpha \,|\, \eta) < H_\mu(\beta \,|\, \eta)$. Then, we get that

$$|H_\mu(\alpha \,|\, \eta) - H_\mu(\beta \,|\, \eta)| \leq H_\mu(\alpha \vee \beta \,|\, \eta) - H_\mu(\alpha \,|\, \eta) \leq H_\mu(\beta \,|\, \alpha)\,.$$

□

4.2 Relative Entropy of a Fibred Dynamical System

Our aim in this section is to present a concept of relative entropy which extends Abramov and Rokhlin's original one to fibred dynamical systems without assuming that at least one of the underlying dynamics is invertible. For this purpose, we consider a fibred dynamical system $\mathfrak{Y} = (X, T; Y, S; \pi)$, where neither T nor S is supposed to be an automorphism.

In the first part of this section, we shall introduce the relative entropy of a fibred dynamical system. For that, we proceed in two steps.

In the first step, we define the relative entropy of a partition with respect to T and the base transformation S. We start with a preparatory consideration.

Lemma 4.2.1. *Let α be an element of $\mathcal{R}_{\pi^{-1}\epsilon_Y}$. Then the limit of the sequence $(\frac{1}{n} \cdot H_\mu(\alpha_T^n \,|\, \pi^{-1}\epsilon_Y))_{n\in\mathbb{N}}$ exists and the equality*

$$\lim_{n\to\infty} \frac{1}{n} \cdot H_\mu\left(\alpha_T^n \,|\, \pi^{-1}\epsilon_Y\right) = \inf_{n\in\mathbb{N}} \frac{1}{n} \cdot H_\mu\left(\alpha_T^n \,|\, \pi^{-1}\epsilon_Y\right)$$

holds.

Proof. Let us consider an element α of $\mathcal{R}_{\pi^{-1}\epsilon_Y}$. According to proposition 0.3.6, for every $k, r \in \mathbb{N}$ we then have that

$$H_\mu\left(\alpha_T^{k+r} \,|\, \pi^{-1}\epsilon_Y\right) \leq H_\mu\left(\alpha_T^{k} \,|\, \pi^{-1}\epsilon_Y\right) + H_\mu\left(\alpha_T^{r} \,|\, \pi^{-1}\epsilon_Y\right) .$$

Moreover, each integer $n \in \mathbb{N}$ can be represented in the form $n = k \cdot p + i$, where $p \leq n$, $k = \lfloor n/p \rfloor$, and $0 \leq i \leq p-1$. Thus, by the invariance of $\pi^{-1}\epsilon_Y$, for all $n \in \mathbb{N}$ and each $p \leq n \in \mathbb{N}$ we obtain that

$$\begin{aligned}\frac{1}{n} H_\mu\left(\alpha_T^n \,|\, \pi^{-1}\epsilon_Y\right) &\leq \frac{1}{k\cdot p} \cdot \left[H_\mu\left(\alpha_T^{k\cdot p} \,|\, \pi^{-1}\epsilon_Y\right) + H_\mu\left(\alpha_T^{i} \,|\, \pi^{-1}\epsilon_Y\right)\right] \\ &\leq \frac{1}{p} \cdot H_\mu\left(\alpha_T^{p} \,|\, \pi^{-1}\epsilon_Y\right) + \frac{1}{k\cdot p} \cdot H_\mu\left(\alpha_T^{p-1} \,|\, \pi^{-1}\epsilon_Y\right) .\end{aligned}$$

However, this relation immediately provides a lower and an upper bound for the sequence $(\frac{1}{n} \cdot H_\mu(\alpha_T^n \,|\, \pi^{-1}\epsilon_Y))_{n\in\mathbb{N}}$, namely

$$\begin{aligned}0 \leq \inf_{n\in\mathbb{N}} \frac{1}{n} \cdot H_\mu\left(\alpha_T^n \,|\, \pi^{-1}\epsilon_Y\right) &\leq \liminf_{n\in\mathbb{N}} \frac{1}{n} \cdot H_\mu\left(\alpha_T^n \,|\, \pi^{-1}\epsilon_Y\right) \\ &\leq \limsup_{n\in\mathbb{N}} \frac{1}{n} \cdot H_\mu\left(\alpha_T^n \,|\, \pi^{-1}\epsilon_Y\right) \\ &\leq \inf_{n\in\mathbb{N}} \frac{1}{n} \cdot H_\mu\left(\alpha_T^n \,|\, \pi^{-1}\epsilon_Y\right) < \infty .\end{aligned}$$

□

Using the statement from above, we define the relative entropy of a partition with respect to a fibred dynamical system as follows.

Definition 4.2.1. Let $\mathfrak{Y} = (X, T; Y, S; \pi)$ be a fibred dynamical system and let α be an element of $\mathcal{R}_{\pi^{-1}\epsilon_Y}$. Then we call

$$h_\mu(T|S, \alpha) := \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_{j=0}^{n-1} T^{-j}\alpha \,|\, \pi^{-1}\epsilon_Y\right)$$

the relative entropy of α with respect to T and the base S, or *w.r.t. $\mathfrak{Y}$*, respectively. Moreover, we call $\mathcal{R}_{\pi^{-1}\epsilon_Y}$ *the set of partitions with finite relative entropy w.r.t. $\mathfrak{Y}$.*

Before we turn to the final step of the definition of relative entropy, we briefly consider several properties of $h_\mu(T|S,\cdot)$.

The following proposition presents two equivalent representations for the relative entropy of a partition with respect to $\mathfrak{Y}$.

Proposition 4.2.2. *Let α be a partition with finite relative entropy w.r.t. $\mathfrak{Y}$. Then the relative entropy of α w.r.t. $\mathfrak{Y}$ satisfies the relations*

$$h_\mu(T|S,\alpha) = \lim_{n\to\infty} H_\mu\left(T^{-n}\alpha \,|\, T^{-(n+1)}\alpha_T^- \vee \pi^{-1}\epsilon_Y\right) \tag{4.1}$$

and

$$h_\mu(T|S,\alpha) = \lim_{n\to\infty} 1/n\, H_\mu\left(\alpha_T^n \,|\, T^{-n}\alpha_T^- \vee \pi^{-1}\epsilon_Y\right) . \tag{4.2}$$

Proof. Let α be an element of $\mathcal{R}_{\pi^{-1}\epsilon_Y}$. Then, for a fixed integer $m \in \mathbb{N}$ and all $n \in \mathbb{N}$ proposition 0.3.6 and the invariance of $\pi^{-1}\epsilon_Y$ provide that

$$H_\mu\left(\alpha_T^{n-1} \,|\, T^{-n}\alpha_T^m \vee \pi^{-1}\epsilon_Y\right) \geq H_\mu\left(\alpha_T^n \,|\, T^{-(n+1)}\alpha_T^m \vee \pi^{-1}\epsilon_Y\right) .$$

Moreover, for fixed $n \in \mathbb{N}$ and all $m \in \mathbb{N}$, proposition 0.3.6 also gives that

$$H_\mu\left(\alpha_T^n \,|\, T^{-(n+1)}\alpha_T^m \vee \pi^{-1}\epsilon_Y\right) \geq H_\mu\left(\alpha_T^n \,|\, T^{-(n+1)}\alpha_T^{m+1} \vee \pi^{-1}\epsilon_Y\right) .$$

Hence, the sequence $\left(H_\mu\left(\alpha_T^\ell \,|\, T^{-(\ell+1)}\alpha_T^m \vee \pi^{-1}\epsilon_Y\right)\right)_{m,\ell\in\mathbb{N}}$ decreases in both m and ℓ, and tends to

$$\lim_{\ell,m\to\infty} H_\mu\left(\alpha_T^\ell \,|\, T^{-(\ell+1)}\alpha_T^m \vee \pi^{-1}\epsilon_Y\right) = \inf_{\ell\in\mathbb{N}} H_\mu\left(\alpha_T^\ell \,|\, T^{-(\ell+1)}\alpha_T^- \vee \pi^{-1}\epsilon_Y\right) . \tag{§}$$

Now let us consider $H_\mu(\alpha_T^n \,|\, \pi^{-1}\epsilon_Y)$. According to proposition 0.3.6, we then get that, for every $n \in \mathbb{N}$

$$H_\mu\left(\alpha_T^n \,|\, \pi^{-1}\epsilon_Y\right) = \sum_{k=0}^{n-1} H_\mu\left(T^{-k}\alpha \,|\, T^{-(k+1)}\alpha_T^{n-1-k} \vee \pi^{-1}\epsilon_Y\right)$$

holds. Thus, applying Cesàro's limit theorem, relation (§) implies equation (4.1). Finally, let us consider $H_\mu\left(\alpha_T^n \,|\, T^{-n}\alpha_T^- \vee \pi^{-1}\epsilon_Y\right)$. Due to proposition 0.3.6 again, for all $n \in \mathbb{N}$ we obtain that

$$H_\mu\left(\alpha_T^n \,|\, T^{-n}\alpha_T^- \vee \pi^{-1}\epsilon_Y\right) = \sum_{k=0}^{n-1} H_\mu\left(T^{-k}\alpha \,|\, T^{-(k+1)}\alpha_T^- \vee \pi^{-1}\epsilon_Y\right) .$$

Consequently, Cesàro's limit theorem yields that

$$\lim_{n\to\infty} \frac{1}{n} \cdot H_\mu\left(\alpha_T^n \,|\, T^{-(n)}\alpha_T^- \vee \pi^{-1}\epsilon_Y\right) = \lim_{n\to\infty} H_\mu\left(T^{-n}\alpha \,|\, T^{-(n+1)}\alpha_T^- \vee \pi^{-1}\epsilon_Y\right) .$$

□

Corollary 4.2.3. *Let $\mathfrak{Y} = (X, T; Y, S; \pi)$ be a fibred dynamical system with an invertible base S and let α be an element of $\mathcal{R}_{\pi^{-1}\epsilon_Y}$. Then the relation*

$$h_\mu(T|S,\alpha) = H_\mu\left(\alpha \,|\, T^{-1}\alpha_T^- \vee \pi^{-1}\epsilon_Y\right)$$

is satisfied.

Proof. Since the base transformation S is an automorphism, relation (4.1) immediately gives that

$$\begin{aligned} h_\mu(T|S,\alpha) &= \lim_{n\to\infty} H_\mu\left(T^{-n}\alpha \,|\, T^{-(n+1)}\alpha_T^- \vee \pi^{-1}S^{-n}\epsilon_Y\right) \\ &= H_\mu\left(\alpha \,|\, T^{-1}\alpha_T^- \vee \pi^{-1}\epsilon_Y\right) . \end{aligned}$$

□

According to its definition, we can consider $h_\mu(T|S,\cdot)$ as a non-negative function on $\mathcal{R}_{\pi^{-1}\epsilon_Y}$. The next proposition states that $h_\mu(T|S,\cdot)$ is even a continuous function on the metric space $\mathcal{R}_{\pi^{-1}\epsilon_Y}$.

Proposition 4.2.4. *Let α and β be two elements of $\mathcal{R}_{\pi^{-1}\epsilon_Y}$. Then the relation*

$$|h_\mu(T|S,\alpha) - h_\mu(T|S,\beta)| \le d_{\pi^{-1}\epsilon_Y}(\alpha,\beta)$$

holds.

Proof. First, let us suppose that $h_\mu(T|S,\alpha) \ge h_\mu(T|S,\beta)$. According to definition 4.2.1 and proposition 0.3.6, we then know that

$$|h_\mu(T|S,\alpha) - h_\mu(T|S,\beta)| \le \lim_{n\to\infty} \frac{1}{n} \cdot H_\mu(\alpha_T^n \,|\, \beta_T^n) \le H_\mu(\alpha \,|\, \beta) .$$

In the case that $h_\mu(T|S,\alpha) < h_\mu(T|S,\beta)$, we obtain in an analogous fashion that

$$|h_\mu(T|S,\alpha) - h_\mu(T|S,\beta)| \le H_\mu(\beta \,|\, \alpha) .$$

□

Now in the second step, based on definition 4.2.1, we define the relative entropy of a fibred dynamical system.

Definition 4.2.5. Let $\mathfrak{Y} = (X, T; Y, S; \pi)$ be a fibred dynamical system. Then we call

$$h_\mu(T|S) := \sup\left\{h_\mu(T|S,\alpha),\ \alpha \in \mathcal{R}_{\pi^{-1}\epsilon_Y}\right\}$$

the relative entropy of the fibred dynamical system $\mathfrak{Y}$.

Remark 4.4. According to proposition 4.2.4 and lemma 4.1.2, we can also write

$$h_\mu(T|S) = \sup\left\{h_\mu(T|S,\alpha),\ \alpha \in \mathcal{R}^\sharp_{\pi^{-1}\epsilon_Y}\right\} .$$

In the remaining part of this section, we shall consider the relation between the relative entropy $h_\mu(T|S)$ of a fibred dynamical system and the entropies of the underlying dynamics T and S. We will see that all these three values are directly connected via an Abramov-Rokhlin formula (cf. theorem 3.1.8) for fibred dynamical systems.

But first, we will make some preparatory considerations. We start with two technical tools.

Lemma 4.2.2. *Let $(X, \mathfrak{B}, \mu, T)$ be a dynamical system endowed with an automorphism T. Furthermore, let α and β be two partitions of X such that $H_\mu(\alpha \vee \beta \,|\, T^{-1}\alpha^-) < \infty$. Then we have that*

$$H_\mu\left(\alpha \vee \beta \,|\, T^{-1}(\alpha \vee \beta)^-\right) = H_\mu\left(\alpha \,|\, T^{-1}\alpha^{-1}\right) + H_\mu\left(\beta \,|\, T^{-1}\beta^{-1} \vee \alpha_T^{\mp}\right),$$

where $\alpha_T^{\mp}$ denotes the partition $\bigvee_{k\in\mathbb{Z}} T^k\alpha$.

Proof. See 7.7 in [Rok67]. □

The second technical tool will follow from the following lemma.

Lemma 4.2.3. *Let $(X, \mathfrak{B}, \mu)$ and $(Y, \mathfrak{D}, \nu)$ be two Lebesgue spaces. Furthermore, let $g : X \to Y$ be a surjective and measure-preserving map. Then, for any partition γ of Y and any set $A \in \mathfrak{D}$, the canonical measures $\{\nu_G\}_{G\in\gamma}$ w.r.t. γ and $\left\{\mu_{g^{-1}(G)}\right\}_{G\in\gamma}$ w.r.t. $g^{-1}(\gamma)$ satisfy the relation*

$$\mu_{g^{-1}(G)}\left(g^{-1}(G) \cap g^{-1}(A)\right) = \nu_G(A \cap G)$$

for ν_γ-a.a. atoms $G \in \gamma$.

Proof. Let us consider a measurable subset A of Y. With the aid of the corresponding characteristic function 1_A, we then define a measurable function by setting

$$\mathbb{E}_\nu\left(1_A \,|\, \sigma(\gamma)\right) \circ g : X \to \mathbb{R}$$

Due to its definition, the function $\mathbb{E}_\nu\left(1_A \,|\, \sigma(\gamma)\right) \circ g$ is measurable w.r.t. the σ-algebra $\sigma(g^{-1}(\gamma))$. Moreover, a simple calculation shows that, for every $B \in \sigma(\gamma)$ the relation

$$\int_{g^{-1}(B)} \mathbb{E}_\nu\left(1_A \,|\, \sigma(\gamma)\right) \circ g \, d\mu = \int_B 1_A \, d\nu = \int_{g^{-1}(B)} \mathbb{E}_\mu\left(1_{g^{-1}(A)} \,|\, \sigma(g^{-1}(\gamma))\right) d\mu$$

holds. Hence, we obtain that the function $\mathbb{E}_\nu\left(1_A \,|\, \sigma(\gamma)\right) \circ g$ is the conditional expectation of $1_{g^{-1}(A)}$ with respect to the σ-algebra $\sigma(g^{-1}(\gamma))$. Consequently, according to remark 0.1, we have that

$$\begin{aligned}
\mu_{g^{-1}(G)}\left(g^{-1}(G) \cap g^{-1}(A)\right) &= \mathbb{E}_\mu\left(1_{g^{-1}(A)} \,|\, \sigma(g^{-1}\gamma)\right)_{|g^{-1}(G)} \\
&= \left(\mathbb{E}_\nu\left(1_A \,|\, \sigma(\gamma)\right) \circ g\right)_{|g^{-1}(G)} \\
&= \mathbb{E}_\nu\left(1_A \,|\, \sigma(\gamma)\right)_{|G} = \nu_G(A \cap G).
\end{aligned}$$

□

Corollary 4.2.6. *Let $(X, \mathfrak{B}, \mu, T)$ be a dynamical system and let ζ be an invariant partition of X, i.e. $T^{-1}\zeta \leq \zeta$. Furthermore, let η be a partition of the factor space X_ζ. Then, for any partition α of X_ζ with η-finite entropy the relative entropy $H_{\mu_\zeta}(\alpha \,|\, \eta)$ satisfies the relation*

$$H_{\mu_\zeta}(\alpha \,|\, \eta) = H_\mu \left(f_\zeta^{-1}\alpha \,|\, f_\zeta^{-1}\eta \right).$$

Proof. We set $(Y, \mathfrak{D}, \nu) := (X_\zeta, f_\zeta(\sigma(\zeta)), \mu_\zeta)$ and $g := f_\zeta$. Then the hypothesis follows directly from lemma 4.2.3. □

We complete our preparations by several remarks on the relative entropy of fibred dynamical systems which are connected in a measurable way.

Proposition 4.2.7. *Let $\mathfrak{Y} = (X, T; Y, S; \pi)$ and $\tilde{\mathfrak{Y}} = (\tilde{X}, \tilde{T}; \tilde{Y}, \tilde{S}; \tilde{\pi})$ be two isomorphic fibred dynamical systems, i.e. there are isomorphisms $\Phi : X \to \tilde{X}$ and $\Psi : Y \to \tilde{Y}$ such that*

$$\Phi \circ T = \tilde{T} \circ \Phi \quad \text{and} \quad \tilde{\pi} \circ \Phi = \Psi \circ \pi \,.$$

Then the relative entropy of $\mathfrak{Y}$ and of $\tilde{\mathfrak{Y}}$ have the same value, i.e.

$$h_\mu\left(T|S\right) = h_{\tilde{\mu}}\left(\tilde{T}|\tilde{S}\right) \,.$$

Proof. First we show that there is a bijection between the set $\mathcal{R}_{\pi^{-1}\epsilon_Y}$ and the set $\mathcal{R}_{\tilde{\pi}^{-1}\epsilon_{\tilde{Y}}}$. In order to do this, we have a look on a partition $\alpha \in \mathcal{R}_{\pi^{-1}\epsilon_Y}$ and on a partition $\tilde{\alpha} \in \mathcal{R}_{\tilde{\pi}^{-1}\epsilon_{\tilde{Y}}}$. Since Φ is an isomorphism, we then have that $\Phi^{-1}\tilde{\alpha}$ and $\Phi\alpha$ are partitions. By the fact that $\mathfrak{Y}$ and $\tilde{\mathfrak{Y}}$ are isomorphic, these partitions satisfy the relations

$$\Phi\alpha \geq \Phi\pi^{-1}\epsilon_Y \geq \tilde{\pi}^{-1}\Psi\epsilon_Y = \tilde{\pi}^{-1}\epsilon_{\tilde{Y}}$$

and

$$\Phi^{-1}\tilde{\alpha} \geq \Phi^{-1}\tilde{\pi}^{-1}\epsilon_{\tilde{Y}} = \pi^{-1}\epsilon_Y.$$

Furthermore, due to corollary 4.2.6, we have that

$$H_{\tilde{\mu}}\left(\Phi\alpha \,|\, \tilde{\pi}^{-1}\epsilon_{\tilde{Y}}\right) = H_\mu\left(\alpha \,|\, \Phi^{-1}\tilde{\pi}^{-1}\epsilon_{\tilde{Y}}\right) = H_\mu\left(\alpha \,|\, \pi^{-1}\epsilon_Y\right)$$

and

$$H_\mu\left(\Phi^{-1}\tilde{\alpha} \,|\, \pi^{-1}\epsilon_Y\right) = H_{\tilde{\mu}}\left(\tilde{\alpha} \,|\, \tilde{\pi}^{-1}\epsilon_{\tilde{Y}}\right).$$

Thus, we obtain both that, for every $\alpha \in \mathcal{R}_{\pi^{-1}\epsilon_Y}$ the partition $\Phi\alpha$ is an element of $\mathcal{R}_{\tilde{\pi}^{-1}\epsilon_{\tilde{Y}}}$ and that, for each $\tilde{\alpha} \in \mathcal{R}_{\tilde{\pi}^{-1}\epsilon_{\tilde{Y}}}$ the partition $\Phi^{-1}\tilde{\alpha}$ is an element of $\mathcal{R}_{\pi^{-1}\epsilon_Y}$.

It remains to show that, for all $\alpha \in \mathcal{R}_{\pi^{-1}\epsilon_Y}$ the relation

$$h_\mu(T|S,\alpha) = h_{\tilde{\mu}}(\tilde{T}|\tilde{S},\, \Phi\alpha)$$

holds. For this, let us consider again a partition $\alpha \in \mathcal{R}_{\pi^{-1}\epsilon_Y}$. By the fact that $\mathfrak{Y}$ and $\tilde{\mathfrak{Y}}$ are isomorphic, we immediately get that, for every $n \in \mathbb{N}$ the relation

$$H_{\tilde{\mu}}\left((\Phi\alpha)_{\tilde{T}}^n \,|\, \tilde{\pi}^{-1}\epsilon_{\tilde{Y}}\right) = H_{\tilde{\mu}}\left(\Phi((\alpha)_T^n) \,|\, \tilde{\pi}^{-1}\epsilon_{\tilde{Y}}\right) = H_\mu\left(\alpha_T^n \,|\, \pi^{-1}\epsilon_Y\right)$$

is satisfied. Consequently, the value of $h_\mu(T|S,\alpha)$ coincides with the value of $h_{\tilde{\mu}}(\tilde{T}|\tilde{S},\, \Phi\alpha)$. □

Corollary 4.2.8. *Let $\mathfrak{Y} = (X,\, T;\, Y,\, S;\, \pi)$ be a fibred dynamical system endowed with two automorphisms $T : X \to X$ and $S : Y \to Y$. Furthermore, let ζ be an invariant partition of X and let η be an invariant partition of Y such that*

$$\pi^{-1}\eta \leq \zeta\,.$$

Then, for any $m \in \mathbb{N}_0$ we have that

$$h_{\mu_\zeta}(T_\zeta|S_\eta) = h_{\mu_{T^m\zeta}}(T_{T^m\zeta}|S_{S^m\eta})\,.$$

Proof. It is clear that the hypothesis holds for $m = 0$. Now let us choose $m \in \mathbb{N}$. With the aid of the factor maps $f_\zeta, f_\eta, f_{T^m\zeta}, f_{S^m\eta}$, we then define the following two fibred dynamical systems

$$\mathfrak{Y} := (X_\zeta, T_\zeta; Y_\eta, S_\eta; \pi_{\zeta,\eta}) \quad \text{and} \quad \tilde{\mathfrak{Y}} := (X_{T^m\zeta}, T_{T^m\zeta}; Y_{S^m\eta}, S_{S^m\eta}; \pi_{T^m\zeta,S^m\eta})$$

where

$$\pi_{\zeta,\eta} := f_\eta \circ \pi \circ {f_\zeta}^{-1} : X_\zeta \to Y_\eta$$

and

$$\pi_{T^m\zeta,S^m\eta} := f_{S^m\eta} \circ \pi \circ {f_{T^m\zeta}}^{-1} : X_{T^m\zeta} \to Y_{S^m\eta}\,.$$

Moreover, we set

$$\Phi := f_{T^m\zeta} \circ T^m \circ f_\zeta^{-1} : X_\zeta \to X_{T^m\zeta}$$

and

$$\Psi := f_{S^m\eta} \circ S^m \circ f_\eta^{-1} : Y_\eta \to Y_{S^m\eta}\,.$$

Then the hypothesis directly follows from lemma 4.2.7. □

A second situation in which factors of fibred dynamical systems owns the same value of relative entropy is stated in the following proposition.

Proposition 4.2.9. *Let $\mathfrak{Y} = (X,\, T;\, Y,\, S;\, \pi)$ be a fibred dynamical system endowed with two automorphisms $T : X \to X$ and $S : Y \to Y$. Furthermore, let $\zeta \in \mathcal{P}_X$ and $\eta \in \mathcal{P}_Y$ be two exhaustive partitions such that $\pi^{-1}\eta \leq \zeta$. Then, the relation*

$$h_\mu(T|S) = h_{\mu_\zeta}(T_\zeta|S_\eta)$$

holds.

Proof. First we shall show that the inequality

$$h_\mu\,(T|S) \le h_{\mu_\zeta}(T_\zeta|S_\eta)$$

is satisfied. In order to prove this, we suppose for the present that $h_\mu\,(T|S)$ is finite. Then, for any $\varepsilon > 0$ we find a partition α_ε of X with $\pi^{-1}\epsilon_Y$-finite entropy and the property that

$$h_\mu\,(T|S,\alpha_\varepsilon) > h_\mu\,(T|S) - \varepsilon/3\,.$$

According to lemma 4.1.3 and the fact that ζ is exhaustive, we also find a second partition $\beta_\varepsilon = \beta_\varepsilon^\sharp \vee \pi^{-1}\epsilon_Y$ of X with $\pi^{-1}\epsilon_Y$-finite entropy and a positive integer $n(\varepsilon) \in \mathbb{N}$ such that

$$d_{\pi^{-1}\epsilon_Y}\,(\alpha_\varepsilon,\beta_\varepsilon) < \varepsilon/3 \qquad \text{and} \qquad \beta_\varepsilon^\sharp \le T^m\zeta \quad \text{for all } m \ge n(\varepsilon)\,.$$

With the aid of β_ε, for $m \ge n(\varepsilon)$ we define a partition $\beta_{T^m\zeta,\varepsilon}$ of $X_{T^m\zeta}$ by

$$\beta_{T^m\zeta,\varepsilon} := f_{T^m\zeta}(\beta_\varepsilon^\sharp) \vee (f_{T^m\zeta} \circ (f_{S^m\eta} \circ \pi)^{-1}\epsilon_{Y_{S^m\eta}})\,.$$

According to corollary 4.2.6 the partition $\beta_{T^m\zeta,\varepsilon}$ satisfies the relation

$$H_{\mu_{T^m\zeta}}\left((\beta_{T^m\zeta,\varepsilon})^n_{T_{T^m\zeta}}\,|f_{T^m\zeta} \circ (f_{S^m\eta} \circ \pi)^{-1}\epsilon_{Y_{S^m\eta}}\right) = H_\mu\left((\beta_\varepsilon^\sharp)^n_T\,|\,(f_{S^m\eta} \circ \pi)^{-1}\epsilon_{Y_{S^m\eta}}\right)$$

for all $n \in \mathbb{N}_0$. Since the partition η is invariant, the relation above implies both that $\beta_{T^m\zeta,\varepsilon}$ is a $f_{X,T^m\zeta} \circ (f_{S^m\eta} \circ \pi)^{-1}\epsilon_{Y_{S^m\eta}}$-finite partition and that the sequence

$$\left(\frac{1}{n} \cdot H_{\mu_{T^m\zeta}}\left((\beta_{T^m\zeta,\varepsilon})^n_{T_{T^m\zeta}}\,|f_{T^m\zeta} \circ (f_{S^m\eta} \circ \pi)^{-1}\epsilon_{Y_{S^m\eta}}\right)\right)_{n\in\mathbb{N}}$$

decreases in $n \in \mathbb{N}$ with limit

$$\inf_{n\in\mathbb{N}}\ \frac{1}{n} \cdot H_\mu\left((\beta_\varepsilon^\sharp)^n_T\,|\,(f_{S^m\eta} \circ \pi)^{-1}\epsilon_{Y_{S^m\eta}}\right)\,.$$

Since η is even exhaustive, we also obtain that, for all $m \ge n(\varepsilon)$ and all $n \in \mathbb{N}$ the sequence above decreases in $n \in \mathbb{N}$ as well as in $m \in \mathbb{N}$ with limit

$$h_\mu\,(T|S,\beta_\varepsilon)\,.$$

Thus, we find a positive integer $m \ge n(\varepsilon)$ such that

$$h_{\mu_{T^m\zeta}}(T_{T^m\zeta}|S_{S^m\eta}) \ge h_{\mu_{T^m\zeta}}(T_{T^m\zeta}|S_{S^m\eta},\,\beta_{T^m\zeta,\varepsilon}) \ge h_\mu\,(T|S,\beta_\varepsilon) - \varepsilon/3\,.$$

By our choice of β_ε and corollary 4.2.8, the relation above then provides that

$$h_{\mu_\zeta}(T_\zeta|S_\eta) > h_\mu\,(T|S) - \varepsilon\,.$$

Now let us suppose that $h_\mu(T|S) = \infty$. For each $N \in \mathbb{N}$ we then find a partition $\alpha_N \in \mathcal{R}_{\pi^{-1}\epsilon_Y}$ such that

$$h_\mu(T|S, \alpha_N) > N\,.$$

Moreover, by the same chain of arguments as before, we directly obtain that, for all $\varepsilon > 0$ the relation

$$h_{\mu_\zeta}(T_\zeta|S_\eta) > h_\mu(T|S, \alpha_N) - \varepsilon$$

is satisfied. Thus, we also have that $h_{\mu_\zeta}(T_\zeta|S_\eta) = \infty$.

It remains to show that the inequality

$$h_\mu(T|S) \geq h_{\mu_\zeta}(T_\zeta|S_\eta)$$

is valid as well. For this, it is sufficient to prove the relation above in the case when $h_\mu(T|S) < \infty$. We start with a partition α of X_ζ with $f_\zeta \circ (f_\eta \circ \pi)^{-1}\epsilon_Y$-finite entropy. According to corollary 4.2.6 and proposition 0.3.6, we then get that

$$\begin{aligned} &H_{\mu_\zeta}(\alpha^n_{T_\zeta} \,|\, f_\zeta \circ (f_\eta \circ \pi)^{-1}\epsilon_Y) \\ &\quad = H_\mu\left(f_\zeta^{-1}\alpha \,|\, \pi^{-1}S^{n-1}\eta\right) + \sum_{k=0}^{n-2} H_\mu\left(f_\zeta^{-1}\alpha \,|\, T^{-1}(f_\zeta^{-1}\alpha)_T^{n-2-k} \vee \pi^{-1}S^k\eta\right)\,. \end{aligned}$$

For each positive integer $2 \leq m \leq n$ the sum on the right hand side of the equation above can also be written in the following form

$$\begin{aligned} &\sum_{k=0}^{n-2} H_\mu\left(f_\zeta^{-1}\alpha \,|\, T^{-1}(f_\zeta^{-1}\alpha)_T^{n-2-k} \vee \pi^{-1}S^k\eta\right) \\ &\qquad = \sum_{k=0}^{n-m} H_\mu\left(f_\zeta^{-1}\alpha \,|\, T^{-1}(f_\zeta^{-1}\alpha)_T^{n-2-k} \vee \pi^{-1}S^k\eta\right) \\ &\qquad\qquad + \sum_{\ell=0}^{m-2} H_\mu\left(f_\zeta^{-1}\alpha \,|\, T^{-1}(f_\zeta^{-1}\alpha)_T^{m-2-\ell} \vee \pi^{-1}S^{n-m+\ell}\eta\right)\,. \end{aligned}$$

Thus, for each $2 \leq n \in \mathbb{N}$ and every $2 \leq m \leq n$, we finally obtain that

$$\begin{aligned} &H_{\mu_\zeta}(\alpha^n_{T_\zeta} \,|\, f_\zeta \circ (f_\eta \circ \pi)^{-1}\epsilon_Y) \\ &\leq \sum_{k=0}^{n-1} H_\mu\left(f_\zeta^{-1}\alpha \,|\, T^{-1}(f_\zeta^{-1}\alpha)_T^{m-2} \vee \pi^{-1}S^k\eta\right) + (m-1)H_\mu\left(f_\zeta^{-1}\alpha \,|\, \pi^{-1}S^{n-m+1}\eta\right)\,. \end{aligned}$$

Now, let us consider the sequences $\left(H_\mu\left(f_\zeta\alpha \,|\, T^{-1}(f_\zeta^{-1}\alpha)_T^{m-2} \vee \pi^{-1}S^k\eta\right)\right)_{k\in\mathbb{N}}$ and $\left(H_\mu\left(f_\zeta^{-1}\alpha \,|\, \pi^{-1}S^k\eta\right)\right)_{k\in\mathbb{N}}$. Since η is exhaustive, both sequences are decreasing and obviously tend to

$$H_\mu\left(f_\zeta^{-1}\alpha \,|\, T^{-1}(f_\zeta^{-1}\alpha)_T^{m-2} \vee \pi^{-1}\epsilon_Y\right) \quad \text{and} \quad H_\mu\left(f_\zeta^{-1}\alpha \,|\, \pi^{-1}\epsilon_Y\right)\,.$$

By Cesàro's limit theorem, we then have that, for each $m \geq 2$ the relation

$$h_{\mu_\zeta}(T_\zeta|S_\eta, \alpha) \leq H_\mu\left(f_\zeta^{-1}\alpha \,|\, T^{-1}(f_\zeta^{-1}\alpha)_T^{m-2} \vee \pi^{-1}\epsilon_Y\right)$$

holds. Consequently, according to corollary 4.2.3 we finally obtain that

$$h_{\mu_\zeta}(T_\zeta|S_\eta, \alpha) \leq \inf_{m\in\mathbb{N}} H_\mu\left(f_\zeta^{-1}\alpha \,|\, T^{-1}(f_\zeta^{-1}\alpha)_T^{m-2} \vee \pi^{-1}\epsilon_Y\right) = h_\mu\left(T|S, f_\zeta^{-1}\alpha\right) .$$

□

Proposition 4.2.9 leads to the following two important conclusions.

Corollary 4.2.10. *The relative entropy $h_\mu(T|S)$ of a fibred dynamical system $\mathfrak{Y} = (X, T; Y, S; \pi)$ and the relative entropy $h_{\hat{\mu}}(\hat{T}|\hat{S})$ of its natural extension $\hat{\mathfrak{Y}} = (\hat{X}, \hat{T}; \hat{Y}, \hat{S}; \hat{\pi})$ have the same value.*

Proof. According to remark 3.3, the hypothesis follows directly from proposition 4.2.9. □

Corollary 4.2.11. *Let $(X, \mathfrak{B}, \mu, T)$ be a dynamical system and let $(\hat{X}, \hat{\mathfrak{B}}, \hat{\mu}, \hat{T})$ be its natural extension. Then, T and $\hat{T}$ have the same entropy.*

Proof. According to remark 3.2, the hypothesis follows directly from corollary 4.2.10. □

After all these preparations, we are now able to show how the relative entropy of a fibred dynamical system and the entropies of the underlying dynamics are connected. We will see that this connection is given in the form of an Abramov-Rokhlin formula for fibred dynamical systems.

Theorem 4.2.12. *Let $\mathfrak{Y} = (X, T; Y, S; \pi)$ be a fibred dynamical system. Then, we have that*

$$h_\mu(T) = h_{\mu\circ\pi^{-1}}(S) + h_\mu(T|S) .$$

Proof. According to corollary 4.2.11 and 4.2.10, it is sufficient to show the hypothesis under the additional assumption that T and S are automorphisms. We start with the proof in this new setup.

Let $(\alpha_m)_{m\in\mathbb{N}}$ be a sequence of finite partitions in $\mathcal{P}_X$ with $\alpha_m \nearrow \epsilon_X$ and let $(\beta_n)_{n\in\mathbb{N}}$ be a sequence of finite partitions in $\mathcal{P}_Y$ with $\beta_n \nearrow \epsilon_Y$. Then each refinement $\alpha_m \vee \pi^{-1}\beta_n$ is again a partition of X with finite entropy. In particular, due to lemma 4.2.2, we have that

$$\begin{aligned} & h_\mu\left(T, \alpha_m \vee \pi^{-1}\beta_n\right) \\ &= H_\mu\left(\alpha_m \vee \pi^{-1}\beta_n \,|\, T^{-1}(\alpha_m \vee \pi^{-1}\beta_n)_T^-\right) \\ &= H_\mu\left(\pi^{-1}\beta_n \,|\, T^{-1}(\pi^{-1}\beta_n)_T^-\right) + H_\mu\left(\alpha_m \,|\, T^{-1}(\alpha_m)_T^- \vee (\pi^{-1}\beta_n)_T^\mp\right) \\ &= h_{\mu\circ\pi^{-1}}(S, \beta_n) + H_\mu\left(\alpha_m \,|\, T^{-1}(\alpha_m)_T^- \vee \pi^{-1}((\beta_n)_S^\mp)\right) . \end{aligned} \qquad (*)$$

However, this relation implies that, for all $m \in \mathbb{N}$

$$\begin{aligned} h_\mu(T) &\geq \sup_{n\in\mathbb{N}} \left\{ h_{\mu\circ\pi^{-1}}(S,\beta_n) + H_\mu\left(\alpha_m \,|\, T^{-1}(\alpha_m)_T^- \vee \pi^{-1}((\beta_n)_S^\mp)\right)\right\} \\ &\geq \sup_{n\in\mathbb{N}} h_{\mu\circ\pi^{-1}}(S,\beta_n) + \inf_{n\in\mathbb{N}} H_\mu\left(\alpha_m \,|\, T^{-1}(\alpha_m)_T^- \vee \pi^{-1}((\beta_n)_S^\mp)\right) \end{aligned}$$

holds. According to corollary 4.2.3 and the fact that $\beta_n \nearrow \epsilon_Y$, we then get that

$$h_\mu(T) \geq h_{\mu\circ\pi^{-1}}(S) + \sup_{m\in\mathbb{N}} h_\mu(T|S,\alpha_m)\ .$$

Consequently, due to lemma 4.1.3 and proposition 4.2.4, the above relation implies that

$$h_\mu(T) \geq\ h_{\mu\circ\pi^{-1}}(S) + h_\mu(T|S)\ .$$

Furthermore, since $\alpha_m \nearrow \epsilon$, relation $(*)$ also gives that for all $n \in \mathbb{N}$

$$\begin{aligned} h_\mu(T) &= \sup_{m\in\mathbb{N}} h_\mu\left(T, \alpha_m \vee \pi^{-1}\beta_n\right) \\ &\leq\ h_{\mu\circ\pi^{-1}}(S,\beta_n) + \sup_{m\in\mathbb{N}} H_\mu\left(\alpha_m \,|\, T^{-1}(\alpha_m)_T^- \vee \pi^{-1}((\beta_n)_S^\mp)\right) \end{aligned}$$

holds. Hence, we obtain that the relation

$$\begin{aligned} h_\mu(T) &\leq\ h_{\mu\circ\pi^{-1}}(S) + \sup_{m\in\mathbb{N}} H_\mu\left(\alpha_m \,|\, T^{-1}(\alpha_m)_T^- \vee \pi^{-1}\epsilon_Y\right) \\ &=\ h_{\mu\circ\pi^{-1}}(S) + h_\mu(T|S) \end{aligned}$$

is satisfied as well. □

4.3 Relative Generators

Analogous to the classical case, it is also difficult to calculate the relative entropy of a fibred dynamical system following its definition. By this crux, methods which simplify the calculation of $h_\mu(T|S)$ are also of special interest. In the following section, we shall be concerned with this problem and present such a method.

The idea of the following method is based on the classical situation. That means that in a first step we shall define a suitable notion of a generator for relative dynamics. Afterwards in a second step we shall show that this notion simplify the calculation of $h_\mu(T|S)$.

Definition 4.3.1. Let $\mathfrak{Y} = (X, T; Y, S; \pi)$ be a fibred dynamical system and let γ be a partition of X. Then the partition γ is called a *one-sided* or *unilateral relative generator for* T, if

$$\gamma_T^- \vee \pi^{-1}\epsilon_Y = \epsilon_X\ .$$

Moreover, if both T and the base transformation S is an automorphism and γ fits the relation

$$\gamma_T^\mp \vee \pi^{-1}\epsilon_Y = \epsilon_X,$$

then we call γ a *two-sided relative generator*.

Remark 4.5. Obviously, a one-sided generator is an unilateral relative generator as well as a two-sided generator is a two-sided relative generator.

The following proposition proves that relative generators seriously simplify the calculation of $h_\mu(T|S)$.

Proposition 4.3.2. *Let $\mathfrak{Y} = (X, T; Y, S; \pi)$ be a fibred dynamical system and let γ be a partition of X with finite conditional entropy w.r.t. $\pi^{-1}\epsilon_Y$. If γ is also an unilateral relative generator for an endomorphism T, or a two-sided relative generator for an automorphism T and an invertible base S, then the relation*

$$h_\mu(T|S) = h_\mu\left(T|S, \gamma \vee \pi^{-1}\epsilon_Y\right)$$

holds.

Proof. First, let T be an endomorphism and let γ be an unilateral relative generator for T. We then have that

$$\left(\bigvee_{j=0}^{n-1} T^{-j}\gamma \vee \pi^{-1}\epsilon_Y\right)_{n\in\mathbb{N}}$$

is a sequence of partitions finer than $\pi^{-1}\epsilon_Y$ which grows up to ϵ_X. Thus, due to lemma 4.1.3, we find a sequence $(\alpha_k)_{k\in\mathbb{N}}$ of partitions with $\pi^{-1}\epsilon_Y$-finite entropy and a sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that

$$\alpha_k \leq \bigvee_{j=0}^{n_k-1} T^{-j}\gamma \vee \pi^{-1}\epsilon_Y \quad \text{and} \quad \lim_{k\to\infty} h_\mu(T|S,\alpha_k) = h_\mu(T|S)\ .$$

Moreover, since γ is of finite conditional entropy with respect to $\pi^{-1}\epsilon_Y$, we also have that for every $k \in \mathbb{N}$

$$\begin{aligned}\inf_{n\in\mathbb{N}} \frac{1}{n}\cdot H_\mu\left(\bigvee_{j=0}^{n-1} T^{-j}\alpha_k \,|\, \pi^{-1}\epsilon_Y\right) &\leq \inf_{n\in\mathbb{N}} \frac{1}{n}\cdot H_\mu\left(\bigvee_{j=0}^{n+n_k-2} T^{-j}\gamma \,|\, \pi^{-1}\epsilon_Y\right)\\ &= \inf_{n\in\mathbb{N}} \frac{1}{n}\cdot H_\mu\left(\bigvee_{j=0}^{n-1} T^{-j}\gamma \,|\, \pi^{-1}\epsilon_Y\right)\end{aligned}$$

holds. Consequently, we obtain that $h_\mu(T|S) = h_\mu(T|S, \gamma \vee \pi^{-1}\epsilon_Y)$.

Now, let us suppose that T and S are automorphisms and that γ is a two-sided relative generator with finite entropy w.r.t. $\pi^{-1}\epsilon_Y$. We then have that

$$\left(\gamma_T^m \vee T^{m-1}\gamma_T^m \vee \pi^{-1}\epsilon_Y\right)_{m\in\mathbb{N}}$$

is a sequence of partitions with $\pi^{-1}\epsilon_Y$-finite entropy which grows up to ϵ_X. Hence, by lemma 4.1.3, we find again a sequence $(\alpha_k)_{k\in\mathbb{N}}$ and a sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that

$$\alpha_k \leq \gamma_T^{n_k} \vee T^{n_k-1}\gamma_T^{n_k} \vee \pi^{-1}\epsilon_Y \quad \text{and} \quad \lim_{k\to\infty} h_\mu(T|S,\alpha_k) = h_\mu(T|S)\ .$$

Consequently, we also get in this situation that

$$\begin{aligned} h_\mu(T|S) = \lim_{k\to\infty} h_\mu(T|S,\alpha_k) &\leq \lim_{k\to\infty} h_\mu\left(T|S,\gamma_T^{n_k} \vee T^{n_k-1}\gamma_T^{n_k} \vee \pi^{-1}\epsilon_Y\right) \\ &= h_\mu\left(T|S,\gamma \vee \pi^{-1}\epsilon_Y\right) . \end{aligned}$$

□

Corollary 4.3.3. *Let $\mathfrak{Y} = (X, T; Y, S; \pi)$ be a fibred dynamical system and let γ be an unilateral generator of X with finite conditional entropy w.r.t. $\pi^{-1}\epsilon_Y$. Then the relation*

$$h_\mu(T|S) = \lim_{n\to\infty} H_\mu\left(T^{-n}\epsilon_X \,|\, T^{-(n+1)}\epsilon_X \vee \pi^{-1}\epsilon_Y\right)$$

holds.

Proof. According to the assumptions, $\gamma \vee \pi^{-1}\epsilon_Y$ is an element of $\mathcal{R}_{\pi^{-1}\epsilon_Y}$. Moreover, due to proposition 4.2.2, we also have that

$$\begin{aligned} h_\mu\left(T|S,\gamma \vee \pi^{-1}\epsilon_Y\right) &= \lim_{n\to\infty} H_\mu\left(T^{-n}(\gamma \vee \pi^{-1}\epsilon_Y) \,|\, T^{-(n+1)}(\gamma \vee \pi^{-1}\epsilon_Y)_T^- \vee \pi^{-1}\epsilon_Y\right) \\ &= \lim_{n\to\infty} H_\mu\left(T^{-n}(\gamma \vee \pi^{-1}\epsilon_Y) \,|\, T^{-(n+1)}(\gamma_T^- \vee \pi^{-1}\epsilon_Y) \vee \pi^{-1}\epsilon_Y\right) \\ &= \lim_{n\to\infty} H_\mu\left(T^{-n}\epsilon_X \,|\, T^{-(n+1)}\epsilon_X \vee \pi^{-1}\epsilon_Y\right) . \end{aligned}$$

Consequently, proposition 4.3.2 finally leads to

$$h_\mu(T|S) = \lim_{n\to\infty} H_\mu\left(T^{-n}\epsilon_X \,|\, T^{-(n+1)}\epsilon_X \vee \pi^{-1}\epsilon_Y\right) .$$

□

Analogous to the classical case, proposition 4.3.2 raises the questions about the existence of relative generators and their construction. In the case of two-sided relative generators for fibred dynamical systems, their existence is already proved by Kaminski's main result in [Kam89].

Theorem 4.3.4. *Let $\mathfrak{Y} = (X, T; Y, S; \pi)$ be a fibred dynamical system endowed with an aperiodic automorphism T and an invertible base S. If $h_\mu(T|S) < \infty$, then there is a two-sided relative generator with finite entropy.*

Proof. By setting $G := \mathbb{Z}$ and $\sigma := \pi^{-1}\epsilon_Y$, the hypothesis follows directly from section 2 in [Kam89] and corollary 4.2.3. □

In the following, we see that with the aid of Rokhlin's methods we can also find sufficient conditions for the existence of unilateral relative generators. For this purpose, we start with the following two preparatory results.

Proposition 4.3.5. *Let $\mathfrak{Y} = (X, T; Y, S; \pi)$ be a fibred dynamical system endowed with an aperiodic automorphism T and an invertible base S. Furthermore, let ζ be an exhaustive partition of X. If $h_\mu(T|S) < \infty$, then there is a two-sided relative generator γ with finite entropy and the property that $\gamma \leq \zeta$.*

Proof. According to theorem 4.3.4, we find a two-sided relative generator γ' with finite entropy. Moreover, by the exhaustiveness of ζ and lemma 4.1.3, we also find a sequence $(\beta_k)_{k\in\mathbb{N}}$ of finite partitions and an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of non-negative integers such that

$$\beta_k \leq T^{n_k}\zeta \quad \text{and} \quad d_\mu(\gamma', \beta_k) < 1/(2^{2^k+5}) .$$

Now let us consider the partitions $\{\beta_k\}_{k\in\mathbb{N}}$. Obviously, we then have that for each $k \in \mathbb{N}$

$$H_\mu(\beta_{k+1} \,|\, \beta_k) \leq d_\mu(\beta_{k+1}, \gamma') + d_\mu(\beta_k, \gamma') < 1/(2^{2^k+4})$$

holds. Thus, due to theorem 1.2.1, we find a second sequence $(\gamma_k)_{k\in\mathbb{N}}$ of finite partitions such that for any $k \in \mathbb{N}$ the relations

$$\beta_{k+1} \vee \beta_k = \gamma_k \vee \beta_k \quad \text{and} \quad H_\mu(\gamma_k) < 2^{-k}$$

are satisfied. With the aid of these two sequences, we define a new partition of X by

$$\gamma := T^{-n_1}\beta_1 \vee \bigvee_{k=1}^{\infty} T^{-n_{k+1}}\gamma_k .$$

In the following we show that the partition γ is a two-sided relative generator which possesses the desired property. First, we prove that γ is coarser than ζ and has finite entropy. Due to our construction, we know that

$$T^{-n_1}\beta_1 \leq \zeta \quad \text{as well as} \quad T^{-n_{k+1}}\gamma_k \leq T^{-n_{k+1}}(\beta_{k+1} \vee \beta_k) \leq \zeta .$$

Hence, γ is coarser than ζ by definition. Furthermore, due to the choice of the partitions $\{\gamma_k\}_{k\in\mathbb{N}}$, we obtain that

$$H_\mu(\gamma) \leq H_\mu(\beta_1) + \sum_{k=1}^{\infty} H_\mu(\gamma_k) \leq H_\mu(\beta_1) + 1 < \infty .$$

It remains to show that γ is also a two-sided relative generator. For this, let us consider the partition $\gamma_T^{\mp}$. Due to the definition of γ, we then have that

$$\gamma_T^{\mp} = \left(T^{-n_1}\beta_1 \vee \bigvee_{k=1}^{\infty} T^{-n_{k+1}}\gamma_k\right)_T^{\mp} = \left(\beta_1 \vee \bigvee_{k=1}^{\infty} \gamma_k\right)_T^{\mp} \geq \left(\bigvee_{k=1}^{\infty} \beta_k\right)_T^{\mp} .$$

Moreover, by the choice of the sequence $(\beta_k)_{k\in\mathbb{N}}$, we also have that for each $\ell \in \mathbb{N}$

$$H_\mu\left(\gamma' \,|\, \bigvee_{k=1}^{\infty} \beta_k\right) \leq H_\mu\left(\gamma' \,|\, \beta_\ell\right) \leq d_\mu\left(\gamma', \beta_\ell\right) < 1/(2^{2^\ell+5})$$

holds. Thus, according to proposition 0.3.6, we get that the sequence $(\beta_k)_{k\in\mathbb{N}}$ satisfies the relation

$$\gamma' \leq \bigvee_{k=1}^{\infty} \beta_k \,.$$

Consequently, we finally obtain that $(\gamma')_T^{\mp} \vee \pi^{-1}\epsilon_Y \leq \gamma_T^{\mp} \vee \pi^{-1}\epsilon_Y$. □

Proposition 4.3.6. *Let $\mathfrak{Y} = (X, T; Y, S; \pi)$ be a fibred dynamical system endowed with an automorphism T and an invertible base S. Furthermore, let η be an exhaustive partition of Y and let ζ be an invariant partition of X with the properties that*

$$\pi^{-1}\eta \leq \zeta \qquad \textit{and} \qquad h_\mu(T|S_\eta, \zeta) = h_\mu(T|S_\eta) = h_\mu\left(T|S\right) < \infty \,.$$

If there is a two-sided relative generator γ with the property that

$$\gamma \vee T^{-1}\zeta \vee \pi^{-1}\epsilon_Y = \zeta \vee \pi^{-1}\epsilon_Y,$$

then γ also fits the relation

$$\gamma_T^- \vee \pi^{-1}\epsilon_Y = \zeta \vee \pi^{-1}\epsilon_Y \,.$$

Proof. By lemma 4.1.2, lemma 4.1.3 and remark 4.2, it is sufficient to show that for each $\alpha \in \mathcal{R}^{\sharp}_{\pi^{-1}\epsilon_Y}$ with $\alpha \leq T^n\gamma_T^- \vee \pi^{-1}\epsilon_Y$ the relation

$$H_\mu\left(\alpha \,|\, \gamma_T^- \vee \pi^{-1}\epsilon_Y\right) = H_\mu\left(\alpha \,|\, \zeta \vee \pi^{-1}\epsilon_Y\right)$$

holds. In order to prove the identity above, let us first consider the partitions η and ζ. Since $S^n\eta \nearrow \epsilon_Y$, proposition 4.2.2 and proposition 0.3.6 provide that

$$\begin{aligned} h_\mu(T|S_\eta, \zeta) &= \lim_{n\to\infty} H_\mu\left(T^{-n}\zeta \,|\, T^{-(n+1)}\zeta_T^- \vee \pi^{-1}\eta\right) \\ &= \lim_{n\to\infty} H_\mu\left(\zeta \,|\, T^{-1}\zeta_T^- \vee \pi^{-1}S^n\eta\right) \\ &= h_\mu\left(T|S, \zeta \vee \pi^{-1}\epsilon_Y\right) \,. \end{aligned}$$

Moreover, by the invariance of ζ, for every $n \in \mathbb{N}$ we get that

$$\begin{aligned} H_\mu\left(\zeta \,|\, T^{-n}\zeta \vee \pi^{-1}\epsilon_Y\right) &= H_\mu\left((\zeta \vee \pi^{-1}\epsilon_Y)_T^n \,|\, T^{-n}(\zeta \vee \pi^{-1}\epsilon_Y)_T^-\right) \\ &= \sum_{k=0}^{n-1} H_\mu\left(\zeta \,|\, T^{-1}\zeta_T^- \vee \pi^{-1}\epsilon_Y\right) \\ &= n \cdot h_\mu\left(T|S, \zeta \vee \pi^{-1}\epsilon_Y\right) \,. \end{aligned}$$

Thus, according to $h_\mu(T|S_\eta,\zeta) = h_\mu(T|S)$, we obtain that the relation

$$\begin{aligned} H_\mu\left(\zeta \,|\, T^{-n}\zeta \vee \pi^{-1}\epsilon_Y\right) = n \cdot h_\mu(T|S) &= n \cdot h_\mu\left(T|S, \gamma \vee \pi^{-1}\epsilon_Y\right) \\ &= \sum_{k=0}^{n-1} H_\mu\left(\gamma \,|\, T^{-(k+1)}\gamma_T^- \vee \pi^{-1}\epsilon_Y \vee T^{-1}\gamma_T^k\right) \\ &= H_\mu\left(\gamma_T^n \,|\, T^{-n}\gamma_T^- \vee \pi^{-1}\epsilon_Y\right) \\ &= H_\mu\left(T^n\gamma_T^- \,|\, \gamma_T^- \vee \pi^{-1}\epsilon_Y\right) \end{aligned}$$

is satisfied for each $n \in \mathbb{N}$.

Furthermore, since $\gamma \vee T^{-1}\zeta \vee \pi^{-1}\epsilon_Y = \zeta \vee \pi^{-1}\epsilon_Y$, we also get that for all $n \in \mathbb{N}$

$$T^n\zeta \vee \pi^{-1}\epsilon_Y = T^n\gamma_T^- \vee \pi^{-1}\epsilon_Y \vee \zeta$$

holds. All these relations then imply that every partition $\alpha = T^n\gamma_T^- \vee \pi^{-1}\epsilon_Y$ satisfies the relation

$$\begin{aligned} H_\mu\left(\alpha \,|\, \gamma_T^- \vee \pi^{-1}\epsilon_Y\right) &= H_\mu\left(T^n\gamma_T^- \vee \pi^{-1}\epsilon_Y \,|\, \gamma_T^- \vee \pi^{-1}\epsilon_Y\right) \\ &= H_\mu\left(\zeta \,|\, T^{-n}\zeta \vee \pi^{-1}\epsilon_Y\right) \\ &= H_\mu\left(T^n\gamma_T^- \vee \pi^{-1}\epsilon_Y \,|\, \zeta \vee \pi^{-1}\epsilon_Y\right) . \end{aligned}$$

Consequently, for each partition $\alpha \leq T^n\gamma_T^- \vee \pi^{-1}\epsilon_Y \in \mathcal{R}^\sharp_{\pi^{-1}\epsilon_Y}$ we finally obtain that

$$\begin{aligned} H_\mu\left(\alpha \,|\, \zeta \vee \pi^{-1}\epsilon_Y\right) &= H_\mu\left(\alpha \,|\, \gamma \vee T^{-1}\zeta \vee \pi^{-1}\epsilon_Y\right) \\ &\leq H_\mu\left(\alpha \,|\, \gamma_T^- \vee \pi^{-1}\epsilon_Y\right) \\ &= H_\mu\left(T^n\gamma_T^- \,|\, \gamma_T^- \vee \pi^{-1}\epsilon_Y\right) - H_\mu\left(T^n\gamma_T^- \,|\, \alpha \vee \gamma_T^- \vee \pi^{-1}\epsilon_Y\right) \\ &\leq H_\mu\left(T^n\gamma_T^- \,|\, \zeta \vee \pi^{-1}\epsilon_Y\right) - H_\mu\left(T^n\gamma_T^- \,|\, \alpha \vee \zeta \vee \pi^{-1}\epsilon_Y\right) \\ &= H_\mu\left(\alpha \,|\, \zeta \vee \pi^{-1}\epsilon_Y\right) . \end{aligned}$$

□

Now we are able to show under which assumptions even unilateral relative generators exist.

Theorem 4.3.7. *Let $\mathfrak{Y} = (X, T; Y, S; \pi)$ be a fibred dynamical system endowed with an aperiodic endomorphism T and the property that*

$$h_\mu(T|S, \epsilon_X) = h_\mu(T|S) < \infty .$$

Then, there is an unilateral relative generator γ with finite entropy.

Proof. Let $\hat{\mathfrak{Y}} = (\hat{X}, \hat{T}; \hat{Y}, \hat{S}; \hat{\pi})$ be the natural extension of $\mathfrak{Y}$. Furthermore, in order to avoid confusions, let us denote the extension of a partition α of X by $\hat{\alpha}$. We then know that $\widehat{\epsilon_X}$ and $\widehat{\pi^{-1}\epsilon_Y}$ are invariant partitions of $\hat{X}$ (w.r.t. $\hat{T}$),

where the partition $\widehat{\epsilon_X}$ is even exhaustive. Moreover, these partitions satisfy the relations

$$\hat{T}^n\big(\widehat{\pi^{-1}\epsilon_Y}\big) \nearrow \hat{\pi}^{-1}\epsilon_{\hat{Y}} \quad \text{and} \quad \widehat{\pi^{-1}\epsilon_Y} \leq \widehat{\epsilon_X}\,.$$

Thus, due to proposition 4.3.5, we find a two-sided generator γ' with

$$\gamma' \leq \widehat{\epsilon_X} \quad \text{and} \quad H_{\hat{\mu}}(\gamma') < \infty\,.$$

Since γ' is coarser than $\widehat{\epsilon_X}$, it is also a partition of X and we write $\hat{\gamma'}$ instead of γ'.
Furthermore, by corollary 4.2.6 and corollary 4.2.10, we get that

$$h_{\hat{\mu}}\left(\hat{T}|\hat{S}_{\widehat{\pi^{-1}\epsilon_Y}}, \widehat{\epsilon_X}\right) = h_{\mu}\left(T|S,\epsilon_X\right) = h_{\mu}\left(T|S\right) = h_{\hat{\mu}}(\hat{T}|\hat{S}) < \infty\,.$$

According to theorem 1.2.1, we then find a partition $\hat{\beta}$ of $\hat{X}$ with

$$\hat{\beta} \vee \hat{T}^{-1}\widehat{\epsilon_X} \vee \widehat{\pi^{-1}\epsilon_Y} = \widehat{\epsilon_X} \vee \widehat{\pi^{-1}\epsilon_Y} \quad \text{and} \quad H_{\hat{\mu}}\left(\hat{\beta}\right) < \infty\,.$$

With the aid of the partitions $\hat{\gamma'}$ and $\hat{\beta}$, we now define a new partition $\hat{\gamma}$ of $\hat{X}$ by

$$\hat{\gamma} := \hat{(\gamma')} \vee \hat{(\beta)}\,.$$

Due to its definition, the partition $\hat{\gamma}$ is again a two-sided relative generator for $\hat{T}$ and fits the relation

$$\begin{aligned}\hat{\gamma} \vee \hat{T}^{-1}\widehat{\epsilon_X} \vee \hat{\pi}^{-1}\epsilon_{\hat{Y}} &= \hat{\gamma'} \vee \hat{\beta} \vee \hat{T}^{-1}\widehat{\epsilon_X} \vee \widehat{\pi^{-1}\epsilon_Y} \vee \hat{\pi}^{-1}\epsilon_{\hat{Y}} \\ &= \hat{\gamma'} \vee \widehat{\epsilon_X} \vee \hat{\pi}^{-1}\epsilon_{\hat{Y}} \\ &= \widehat{\epsilon_X} \vee \hat{\pi}^{-1}\epsilon_{\hat{Y}}\,.\end{aligned}$$

Moreover, by proposition 4.3.6, the partition $\hat{\gamma}$ has the property that

$$(\hat{\gamma})^-_{\hat{T}} \vee \hat{\pi}^{-1}\epsilon_{\hat{Y}} = \widehat{\epsilon_X} \vee \hat{\pi}^{-1}\epsilon_{\hat{Y}}\,.$$

In the following, we show that γ is an unilateral relative generator for T. For this, let us denote the first coordinate of an element $\hat{x} \in \hat{X}$ by $[\hat{x}]_0$, or of an element $\hat{y} \in \hat{Y}$ by $[\hat{y}]_0$, respectively. We then have that each element $\hat{x} = (x_0, x_1, \ldots) \in \hat{\pi}^{-1}\hat{y} \in \hat{\pi}^{-1}\epsilon_{\hat{Y}}$ where $\hat{y} = (y_0, y_1, \ldots)$ satisfies the relation

$$[\hat{\pi}\hat{x}]_0 = \pi x_0 = y_0\,.$$

Hence, we get that for each $\hat{y} \in \hat{Y}$

$$\left[\hat{\pi}^{-1}\hat{y}\right]_0 := \left\{x \in X :\ \exists\, \hat{x} \in \hat{\pi}^{-1}\hat{y} \quad \text{with} \quad [\hat{x}]_0 = x\right\} = \pi^{-1}y_0$$

holds. Thus, $[\hat{\pi}^{-1}\epsilon_{\hat{Y}}]_0 := \{[\hat{\pi}^{-1}\hat{y}]_0 : \hat{\pi}^{-1}\hat{y} \in \hat{\pi}^{-1}\epsilon_{\hat{Y}}\}$ is a partition of X with the property that

$$\left[\hat{\pi}^{-1}\epsilon_{\hat{Y}}\right]_0 = \pi^{-1}\epsilon_Y\,.$$

Furthermore, we also know that

$$[\widehat{\epsilon_X}]_0 = \epsilon_X \quad \text{and} \quad [\hat{\gamma}]_0 = \gamma\,.$$

Consequently, we finally have that

$$\epsilon_X = \left[\widehat{\epsilon_X} \vee \hat{\pi}^{-1}\epsilon_{\hat{Y}}\right]_0 = \left[(\hat{\gamma})^-_{\hat{T}} \vee \hat{\pi}^{-1}\epsilon_{\hat{Y}}\right]_0 = \gamma^-_T \vee \pi^{-1}\epsilon_Y\,.$$

Remark 4.6. *i*) For an invertible base S, theorem 4.3.7 provides a sufficient criterion for condition 5) of theorem 3.13 in [DGH00].
ii) The existence and the nature of finite relative generators is still an open problem (the work in [KW00] is based on a completely different notion of relative generators).

□

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Curriculum Vitae

Persönliche Daten	
	Oliver Schmitt geboren am 02.02.1973 in 66687 Wadern deutsche Staatsangehörigkeit
Schulausbildung	
1979 - 1983	Besuch der Grundschule in Scheuern, Landkreis St.Wendel
1983 - 1992	Besuch des Gymnasiums Johanneum in Homburg (Saar)
Studium	
Okt. 1992	Beginn des Mathematikstudiums mit Nebenfach Physik an der Universität Kaiserslautern
März 1995	Diplom-Vorprüfung
10/1997 - 2/1999	Wissenschaftliche Hilfskraft im Fachbereich Mathematik
Feb. 1999	Diplomprüfung
April 1999	Beginn des Promotionsstudiums bei Prof. Dr. M. Denker am Institut für Mathematische Stochastik der Georg-August-Universität Göttingen
seit April 1999	Stipendiat des Graduiertenkollegs „Strömungsinstabilitäten und Turbulenz“ der Universität Göttingen
April 2000	Dreiwöchiger Forschungsaufenthalt am Steklov Institut der Universität St. Petersburg (Russland)
10/2000 - 12/2000	Wissenschaftliche Hilfskraft am Institut für Mathematische Stochastik